Yang–Mills and the mass gap

Volume 3 Toward the mass gap — the questions that remain, and this book's results

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2026-10-01 · chapter Yang–Mills and the mass gap Part 16 Where things stand, and the questions that remain timeline and where things stand

How far the world has come — reading where things stand from a timeline and tables by dimension

> In this chapter, § 01– § 06 are established knowledge, § 07 is a map of open questions and § 08 is guide material.

What this chapter shows — A timeline of the main results from 1954 to 2026 (37 rows, including the four-dimensional construction published in 2026). A board showing, for each dimension, how far through the six steps there are theorems, and how it got filled in. A comparison with abelian groups and scalar fields. Where on the number line of β the lattice theorems lie.

Prerequisite chapters — 7-10 Where four dimensions stops. The six stages are in 7-01, and the content for each dimension in 7-02, 7-06 and 7-07.

Said first — This chapter rearranges known literature and shows no theorem. The existence of four-dimensional Yang–Mills (a theory satisfying the Wightman axioms, as a limit of Wilson's lattice gauge theory) was shown by the construction published in 2026, as a formalisation in Lean, and the committee assessed it as “proving” this Assessment (Part 6). The uniqueness of the infinite-volume limit and the mass gap have not been shown (16-02). The timeline covers what was checked in the main text or the abstract (some rows from the reference list of a review), and is not exhaustive. The articles on this site have not moved any cell of the board (the label Assessment means "according to the committee's assessment report"; for details, see “Labels” in 6-00).

The order of this chapter

1. Reading one result along six axes
2. Timeline I (1954–1999)
3. Timeline II (2000–2026)
4. The board: moving through the years — moving figure, computed
5. Tables by dimension
6. Comparison with neighbouring theories: the scope of "it has been done in two and three dimensions"

7. Where the lattice theorems lie on the number line of β — moving figure, computed (the β knob)
8. Detour: what the timeline leaves out

01

Reading one result along six axes

Every theorem about Yang–Mills comes with conditions on "where it holds". Drop the conditions, and a two-dimensional theorem looks like a statement about four dimensions, a lattice theorem like a statement about the continuum. This chapter gathers the conditions into six axes and calls them the **address**.

$$\text{address} = \big(d,\ \text{lattice or continuum},\ \text{volume},\ \text{coupling side},\ G,\ \text{matter fields}\big)\ \text{(1)}$$

d is the dimension of Euclideanised spacetime, and the coupling side is whether the lattice inverse coupling $\beta = 2N/g^2$ (table of notation) is small (strong coupling) or large (weak coupling). The address of the problem statement is "4, continuum, $\mathbb{R}^4$, any compact simple group, none", and there it is required to pass through all six stages. Known The continuum limit of the four-dimensional lattice is believed to lie on the $\beta \to \infty$ side (Chatterjee 2018 § 5). Physics

Example: Driver 1989 reaches the ultraviolet-limit step and the infinite-volume step at (2, continuum, $\mathbb{R}^2$, —, connected compact, none), and Shen–Zhu–Zhu 2022 gives exponential decay on the lattice at (any, lattice with fixed spacing, infinite, strong, $SU(N)$, none). The "Scope" in the timeline abbreviates this address; the year is the year of first public appearance (the arXiv year if there is one), and the year in parentheses is the year of publication. Rows that combine several papers give a range of years.

02

Timeline I (1954–1999): words, tools, and the first constructions

The first half-century had three strands. In physics, the theory was written down (1954), and asymptotic freedom was found in perturbation theory (1973). In the framework, the axioms (1973, 1975) and the lattice (1974) were put in place. In construction came the strongly coupled lattice (1978), ultraviolet estimates in three and four dimensions (1985, 1987–89), the two-dimensional continuum measure (1989), and in four dimensions an outline of a construction that removes the ultraviolet cutoff while keeping the infrared one (1993). Known The formulas of 1974 and 1973 are the following two (5-01, 4-06).

$$S_W = \beta \sum_P \Big(1 - \tfrac{1}{N} \operatorname{Re} \operatorname{tr} U_P\Big), \qquad \mu \frac{dg}{d\mu} = -b_0\, g^3 + \cdots, \qquad b_0 = \frac{11N}{3} \cdot \frac{1}{16\pi^2} \tag{2}$$

Example: for $SU(2)$, $\beta = 2.5$ is $g^2 = 2N/\beta = 1.6$. The sign $b_0 > 0$ is asymptotic freedom. The "Comparison" rows are neighbouring theories, abelian groups and scalar fields, and do not move cells. The physics rows continue in Part 17 (17-04, 17-11, 17-09; lattice numbers in 5-08).

Known Table 1 Timeline I (except rows with the Physics label). Pressing a row sets Figure 1 to that year.

Year	Event	Scope
1954	Yang–Mills: classical theory of non-abelian gauge fields	Classical
1973	Gross–Wilczek, Politzer: asymptotic freedom	Coefficient in perturbation theory
1973, 1975	Osterwalder–Schrader: Euclidean axioms and reconstruction of a Wightman theory (conditions revised in 1975)	Axioms
1974	Wilson: lattice gauge theory (a gauge-invariant lattice action)	All dimensions, lattice
1974	't Hooft: the large-N expansion	Physics
1976	Feldman–Osterwalder: the Wightman axioms and a mass gap for weakly coupled ϕ^4_3	Comparison, 3D
1978	Osterwalder–Seiler: reflection positivity and the transfer matrix on the lattice; infinite-volume limit and area law at strong coupling (convergence of the cluster expansion in outline)	Lattice, strong coupling
1978	Singer: a global continuous gauge fixing cannot in general be taken	Geometry
1979–81	Brydges–Fröhlich–Seiler: the two-dimensional abelian Higgs model up to the OS axioms	Comparison, 2D
1980, 1982	Guth, Fröhlich–Spencer: a non-confining phase at large β for four-dimensional lattice U(1)	Comparison, lattice
1981–82	Aizenman, Fröhlich: ϕ^4 is trivial for $d \geq 5$	Comparison
1983–87	Gross, King, Driver: continuum limit of U(1) (3D, 4D, including with Higgs)	Comparison
1985	Bałaban: ultraviolet stability on the three-dimensional lattice (torus; uniform upper and lower bounds on the partition function)	3D, lattice, finite volume
1987–89	Bałaban: ultraviolet stability on the four-dimensional lattice; Theorem 1 of the final paper assumes the effective coupling is small	4D, lattice, finite volume
1989	Driver, Gross–King–Sengupta: the continuum measure on the plane $\mathbb{R}^2$ (Driver also convergence of the lattice approximation)	2D, continuum, infinite volume

Year	Event	Scope
1992–93	Sengupta: the measure on closed surfaces	2D, continuum, finite volume
1993	Magnen–Rivasseau–Sénéor: the "foundations" for constructing the Schwinger functions of four-dimensional SU(2) without ultraviolet cutoff, with the infrared cutoff held fixed (in a regularised axial gauge) (no detailed proof)	4D, continuum
1994	Seiberg–Witten: gap and confinement in a supersymmetric theory	Physics
1997–98	Maldacena, Witten: holography, and models that see the gap as geometry	Physics
1999	Morningstar–Peardon: lattice numbers for SU(3) glueballs	Physics Numerical

03

Timeline II (2000–2026): after the problem statement

On 24 May 2000 the Clay Mathematics Institute announced the Millennium Prize Problems in Paris. The problem statement (Jaffe–Witten) describes the state of four dimensions as follows. Known

> 'at present we do not know any non-trivial relativistic field theory that satisfies the Wightman (or any other reasonable) axioms in four dimensions.' (JW § 6)

Since then, new constructions in two dimensions, the state space and dynamics in three dimensions, and new theorems on the lattice have been added, but a 2004 report writes 'The state of the art remains the works of Balaban and of Magnen, Rivasseau and Sénéor'. Known A 2018 review put the question on the lattice into a formula (the lattice correlation-length problem (Chatterjee's Problem 5.1); 5-09). $f_\beta(x)$ is the correlation of the Wilson loops of the plaquettes nearest to the origin and to the point x.

$$\lim_{|x|\to\infty} \frac{\log f_\beta(x)}{|x|} = -\frac{1}{\xi(\beta)}, \quad \xi(\beta) \in (0,\infty) \ \ (\text{for all } \beta > 0), \qquad \lim_{\beta\to\infty} \xi(\beta) = \infty \ (3)$$

Example: there are 17 rows from 2000 on (11 of them in the 2020s), and the rows that changed a colour on the board are the 2 rows of 2022 (Shen–Zhu–Zhu and Adhikari–Cao) and the 1 row of the construction published in 2026 (three cells). Computed The lattice correlation-length problem is a question on the lattice, and is a different thing from the requirements of the problem statement (continuum, $\mathbb{R}^4$, axioms, non-triviality) (0-03).

Known Table 2 Timeline II.

Year	Event	Scope
2000	The Clay prize problem (the problem statement of Jaffe–Witten)	Problem
2001 (2003)–10	Lévy: random holonomy fields on closed surfaces (non-trivial bundles in 2006)	2D, continuum
2004	Douglas's report: the state of the art is still Bałaban and MRS	Assessment
2013 (2014)	Hairer: regularity structures (renormalisation of stochastic PDEs)	Tool
2018	Chatterjee's review: the lattice correlation-length problem (Problem 5.1) and the problem of the continuum limit of Wilson loops (5.2)	Lattice formulation
2019 (2021)	Aizenman–Duminil-Copin: the limit of four-dimensional ϕ^4 is Gaussian	Comparison
2020 (2022)	Chandra–Chevyrev–Hairer–Shen (CCHS): the two-dimensional stochastic Yang–Mills heat flow and the space of gauge orbits	2D, T^2
2021 (2024)	Cao–Chatterjee: the three-dimensional state space (under the assumption that the theory behaves like a Gaussian free field [GFF; 1-09])	3D, conditional
2022 (2024)	CCHS: stochastic quantisation of three-dimensional Yang–Mills–Higgs (local solutions and a Markov process; the measure not yet reached)	3D, T^3
2022	Chevyrev's review: the three-dimensional measure is not constructed even on T^3	Assessment
2022 (2023)	Shen–Zhu–Zhu: uniqueness and a positive mass gap on the strongly coupled lattice, in an explicit range independent of N	Lattice, strong coupling
2022 (2025)	Adhikari–Cao: exponential decay of correlations at weak coupling on the four-dimensional lattice for finite groups (including non-abelian)	Lattice, weak coupling
2023 (2026)	Chevyrev–Shen: the measure on T^2 is the unique invariant measure of the Langevin dynamics, independent of the lattice action	2D, T^2
2024 (2026)	Chatterjee: a special scaling limit of SU(2) Yang–Mills–Higgs is a Gaussian field	Comparison, Higgs
2025	Cao–Nissim–Sheffield: the area law in the 't Hooft regime (in 't Hooft's convention, multiplying the action by βN, the side with β small independently of N; 7-10 § 05) (not peer-reviewed)	Lattice, strong coupling
2026	Chevyrev–Klose–Mohamed, Chhaibi et al.: representations of the two-dimensional measure, and a construction not via the lattice (not peer-reviewed)	2D

Year	Event	Scope
2026	The published construction (a formalisation in Lean; submitted 2026-10-28, published 2026-11-10 together with the committee's assessment report; the committee's statement is dated 2026-11-09): for any compact simple group, builds a theory satisfying the Wightman axioms as a limit of four-dimensional Wilson lattice gauge theory. In a periodic box it converges along the whole sequence of cutoffs, and it passes to infinite volume along a subsequence. The uniqueness of the infinite-volume limit and the mass gap are not shown Assessment (6-00, 16-02)	4D, continuum (the committee's assessment)

Words in Table 2: **stochastic quantisation** = a method that builds a measure as the invariant measure (a measure whose distribution does not change over time) of some stochastic time evolution. **Langevin dynamics** = a gradient flow with noise that leaves the measure invariant. **Stochastic Yang–Mills heat flow** = the flow in the direction that decreases the Yang–Mills action (heat flow), with noise added. **Regularity structures** = a framework that gives meaning to singular stochastic partial differential equations and renormalises them. All of these are in 7-09. **Markov process** = a stochastic process whose future distribution is determined by the present state alone (7-02).

04

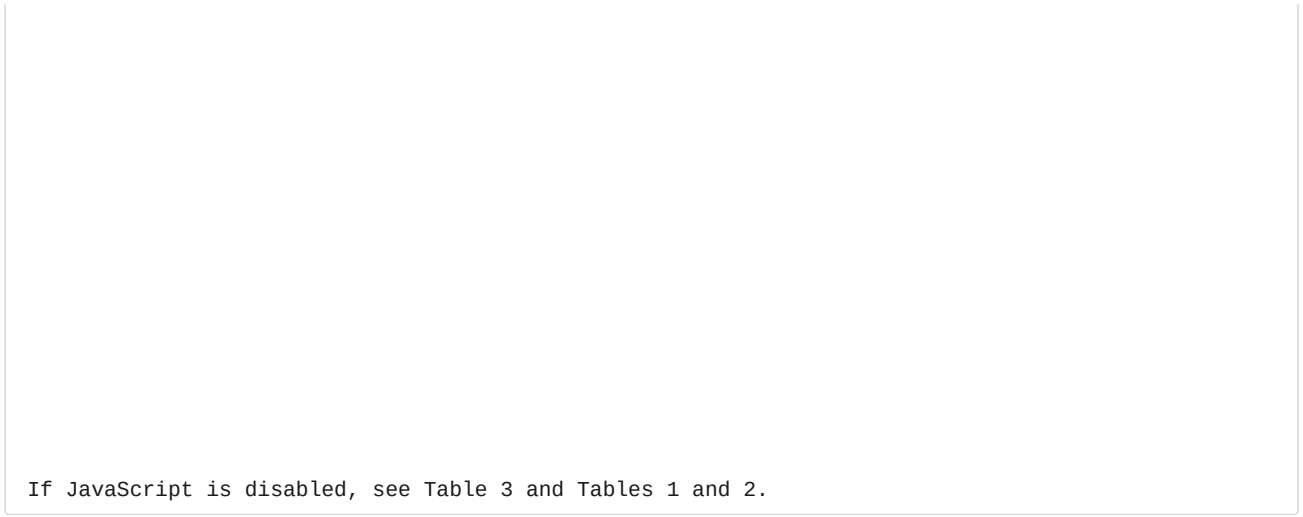
The board: moving through the years

We transfer the timeline onto cells. The rows are continuum non-abelian Yang–Mills (no matter fields) in $d = 2, 3, 4$ and the four-dimensional lattice, and the columns are the six steps (7-01). The regularisation step and the finite-volume step build the measure $\mu_{a,L}$ on the lattice, and close on the lattice (7-10 § 03). In three and four dimensions the finite-volume measure has not yet been built while staying in the continuum, so the finite-volume step there is written "Closed (lattice)". The construction published in 2026 does not build a continuum probability measure either (16-02 § 03), so the four-dimensional finite-volume cell stays "Closed (lattice)".

The names of the states are the same as in Table 2 of 7-01: **Closed** (there is a theorem), **Partial** (with assumptions, in outline, up to estimates of the partition function, or only part of the requirements; "with a hypothesis" in Table 2 of 7-10 also goes here), **Not yet**, and — (the question is hard to pose in that form; 7-02). The state in year y is fixed by the record of the last change (y_k, c_k, v_k) at or before y.

$$s_C(y) = v_k, \qquad k = \max\{\, k : \; y_k \le y, \; c_k = c \,\} \quad (4)$$

Figure 1 Moving the year with the knob fills the cells with the theorems up to that year (press a row of Table 1 or Table 2, or select a row and press Enter, to go to that year)



Computed The numbers in this figure were computed on this machine (formula (4)). How the cells are divided and how the states are decided is this chapter's choice.

Computed Table 3 The board in 2026 (state, and the year it reached that state). The three four-dimensional cells of 2026 are at the level of the committee's assessment. The three cells of the lattice row do not correspond to the step columns.

	1 Regularisation	2 Finite volume	3 Ultraviolet limit	4 Infinite volume	5 Axioms	6
2D continuum	Closed 1974	Closed 1974	Closed 1989	Closed 1989	Partial 1989	—
3D continuum	Closed 1974	Closed (lattice) 1974	Partial 1985	Not yet	Not yet	N
4D continuum	Closed 1974	Closed (lattice) 1974	Closed 2026 (Partial from 1989)	Partial 2026	Closed 2026	N
Lattice cells	**Strong coupling**		**Weak coupling, continuous group**		**Weak coupling, finit**	
4D lattice (fixed spacing)	Closed 2022 (Partial from 1978)		Not yet		Closed 2022	

Example: the board moved in six years, 1974, 1978, 1985, 1989, 2022 and 2026, and the continuum cells did not move from 1989 to 2025. Of the 21 cells, 12 are Closed, 3 Partial, 5 Not yet and 1 —. Computed The three cells of 2026 (the four-dimensional ultraviolet limit, infinite volume and axioms) are filled in with the published construction, at the level at which the committee assessed it as “proved in Lean” Assessment . The infinite-volume cell is made "Partial" because the limit is taken along a subsequence and uniqueness is not shown (16-02 § 02). Table 2 of 7-10 makes the infinite-volume step of the lattice "Closed" at strong coupling by OS 1978. The lattice cells of this chapter count the infinite-volume step and the uniform-gap step (gap) as one, so 1978 is "Partial". The axiom step in two dimensions goes as far as Euclidean invariance of gauge-invariant expectations (Corollary 6.7 of Driver 1989), with no claim in the form of the OS axioms, so it is "Partial". The uniform-gap step in two dimensions has no propagating degrees of freedom and the question is hard to pose, so it is "—" (both as in Table 2 of 7-01). Known

The strong-coupling lattice cell becomes "Partial" in 1978 and "Closed" in 2022. Osterwalder–Seiler 1978 gave the convergence of the cluster expansion in outline, and the gap (exponential clustering) follows from that convergence. The proof of uniqueness and a positive mass gap in an explicit range is Shen–Zhu–Zhu 2022 (both according to the introduction of SZZ; the abstract of OS 1978 states the existence and analyticity of the infinite-volume limit at strong coupling and Wilson's confinement estimate; 5-02 § 08). Known That the continuum cells did not move from 1989 to 2025 is not because there was no progress. Progress *inside* the cells, such as the characterisation of the two-dimensional measure by Langevin dynamics and the dynamics in three dimensions, did not show up on the coarse board (this chapter's reading; for the words of the literature step by step, see 7-10).

05

Tables by dimension

We rewrite the rows of the board in the language of addresses. The columns are the lattice (fixed spacing), the inside and the outside of the double limit (7-01), and the axioms and the gap. $\mu_{a,L}$ is the measure on a box with lattice spacing a and side L.

$$\mu = \lim_{L\to\infty} \underbrace{\lim_{a\to 0} \mu_{a,L}}_{\text{continuum, finite volume}} \tag{5}$$

Known Table 4 Where things stand, by dimension (non-abelian, no matter fields).

d	Lattice (a fixed)	Continuum, finite volume	Continuum, infinite volume	Axioms, gap
2	Exactly solvable (7-02)	Closed surfaces, T^2: Sengupta, Lévy, CCHS, Chevyrev–Shen	$\mathbb{R}^2$: Driver, Gross–King–Sengupta	The literature makes no claim (the observables are loops)
3	Infinite volume and exponential decay at strong coupling	Ultraviolet stability (estimates of the partition function). The measure not constructed; up to the state space and dynamics	Not yet	Not yet

d	Lattice (a fixed)	Continuum, finite volume	Continuum, infinite volume	Axioms, gap
4	Infinite volume and exponential decay at strong coupling. At weak coupling, finite groups only	The continuum limit in a periodic box (the published construction, 2026; correlation functions converge along the whole sequence of cutoffs; no continuum probability measure is built) Assessment . Before that, ultraviolet stability with a hypothesis (Bałaban) and an outline with an infrared cutoff (MRS)	Limits along subsequences and phase decomposition (the published construction). Convergence and uniqueness of the limit are not shown Assessment	The Wightman axioms phase by phase (the published construction) Assessment . The gap: not yet

That **two dimensions** is closed up to continuum and infinite volume is due to its exactly solvable structure (independence of plaquettes and the heat kernel of the group; 7-02). **Three dimensions** is said to be easier, but the continuum measure is 'open, even in finite volume' (7-06). In **four dimensions**, up to 2025 the ultraviolet-limit step was with a hypothesis and in outline in finite volume, and there were no theorems from the infinite-volume step to the uniform-gap step (7-07) Known . The construction published in 2026 filled the ultraviolet-limit step, the infinite-volume step (along a subsequence) and the axiom step Assessment (6-01). The uniform-gap step remains open (16-02 § 04).

Example: the four-dimensional "Continuum, finite volume" of Table 4 corresponds to the ultraviolet-limit step of Table 3 (Closed from 2026), and the "Lattice" column to the lattice row of Table 3. The outline of MRS removes the ultraviolet cutoff but keeps the infrared cutoff. The published construction takes the ultraviolet limit in a periodic box and then passes to infinite volume along a subsequence (8-05).

06

Comparison with neighbouring theories: the scope of "it has been done in two and three dimensions"

Saying "it has been done in two and three dimensions" is correct for non-abelian Yang–Mills only in two dimensions. What it fits up to three dimensions is abelian groups and scalar fields (16-08). We line up the "Comparison" rows of the timeline. None of them moves a cell of the board. Known

Known Table 5 How far neighbouring theories have got (the "Comparison" rows of the timeline).

Theory	d	How far it has got
U(1) Higgs (an abelian gauge field and a charged scalar field)	2, 3	In two dimensions up to the OS axioms (Brydges–Fröhlich–Seiler). The problem statement calls it "the only complete example of an interacting gauge theory satisfying the axioms". In three dimensions, the continuum infinite-volume limit (King)

Theory	d	How far it has got
ϕ^4_3 (weak coupling)	3	The Wightman axioms (JW § 6.2) and a mass gap (Feldman–Osterwalder)
Pure U(1) gauge theory	4	The limit from lattice to continuum (Driver 1987). The limit is expected to be the free electromagnetic field (the explanation in 7-10). The lattice has a non-confining phase at large β (Guth, Fröhlich–Spencer)
ϕ^4_4	4	The limit of lattice-regularised $\lambda\phi^4$ at infinite volume and zero lattice spacing is Gaussian (Aizenman–Duminil-Copin). Triviality for $d \geq 5$ is Aizenman, Fröhlich
Two-dimensional O(3) and O(4) sigma models (asymptotically free, with non-abelian global symmetry)	2	There is a manuscript with claims in `github.com/openai/math` (result group 215), made public on 2026-10-06. Exponential decay of correlations on the lattice at all temperatures (n ≥ 3, uniform over finite subgraphs) comes **with a Lean formalisation**. The continuum limit of O(3) (OS axioms, a unique vacuum, a positive mass gap, non-Gaussian) and the asymptotics of the lattice mass for O(4), m ∼ 32e^(π/4−1/2) √ β e^(−πβ) (the same constant as predicted by Hasenfratz–Maggiore–Niedermayer 1990), are **in a paper manuscript only, unverified**

Two-dimensional sigma models, unlike two-dimensional YM (7-02), have asymptotic freedom and dimensional transmutation, so they are the closer example of "a problem of the same shape" as four-dimensional YM. They have no gauge symmetry and no confinement. This row does not move the four-dimensional cells of the board (Table 3).

The two four-dimensional rows are both examples where the limit is a free (Gaussian) field. For asymptotically free (4-06) Yang–Mills a non-trivial limit is expected, but there is no proof (7-08). Physics Even in the same four dimensions, ϕ^4 becomes strongly coupled at short distances, while Yang–Mills becomes weakly coupled. The directions are opposite, so the triviality theorems for ϕ^4 do not apply to Yang–Mills. Example: comparing in three dimensions, ϕ^4_3 has passed all six steps (Table 2 of 7-01). Non-abelian Yang–Mills is Partial at the ultraviolet-limit step, and Not yet from the infinite-volume step on (Table 3).

07

Where the lattice theorems lie on the number line of β

At strong coupling the high-temperature expansion (expansion in powers of small β, also called the strong-coupling expansion; 5-02) works routinely, and gives exponential decay even for continuous Lie groups (according to the introductions of Adhikari–Cao and Shen–Zhu–Zhu; Osterwalder–Seiler 1978 abstract only). At weak coupling, uniqueness of the infinite-volume limit of lattice gauge theory is in general not known for $d > 2$ with large β (Chatterjee 2018 § 3). That is the side the continuum limit needs (5-05). Known

Converting the conditions of two theorems to one β

Converting the conditions of Shen–Zhu–Zhu, with weight $\exp(N\beta\,\mathrm{Re}\sum_P \mathrm{Tr}\,U_P)$, and of Adhikari–Cao, with weight $\exp(-\beta\sum_P \mathrm{Re}(\chi(1)-\chi(U_P)))$ (χ is the character of a representation), to the β_W of formula (2) gives

$$\beta_W = N^2\beta_{\mathrm{SZZ}} < \frac{N^2}{16(d-1)}, \qquad \beta_W = 2\beta_{\mathrm{AC}} \geq \frac{2\,(114+4\log|G|)}{\Delta_G}, \qquad \Delta_G = \min_{g\neq 1} \mathrm{Re}(\chi$$

(Δ_G is the notation of Adhikari–Cao, not the mass gap Δ_G of 16-05 and 16-08). It is related to the β of 7-10 § 05 (Chatterjee's convention $e^{-\beta\sum_p \mathrm{Re}\,\mathrm{tr}(I-U_p)}$) by $\beta_W = N\beta_{\mathrm{Ch}}$, and the $|\beta| < N/48$ of 7-10 is the same condition as $\beta_W < N^2/48$ ($d=4$) in this chapter. Adhikari–Cao allow any unitary representation ρ of G, with $\chi = \mathrm{tr}\,\rho$. Here we take ρ to be the inclusion $G \subset SU(2)$, which gives Wilson's model with link variables restricted to G ($\chi(1)=2$, so $\beta_W = 2\beta_{\mathrm{AC}}$). log is the natural logarithm.

The former is $\beta_W < 0.25,\ 0.125,\ 0.083$ for $d = 2, 3, 4$ with $SU(2)$, and 9/4 times that for $SU(3)$. Computed Example: $\beta_W < 1/12$ at $d=4$ is 1/27 of the $\beta_W = 2.25$–2.60 ($SU(2)$, Lucini–Teper 2001) Physics that lattice numbers use to extrapolate to the continuum, and the condition $\beta_W \geq 697$ for the binary icosahedral group is 268 times the upper end of that range. Computed

Figure 2 Choosing a finite group $G \subset SU(2)$ with the knob moves the position where the Adhikari–Cao theorem starts to apply (the left end of the green on the right). The green on the left is the strong-coupling theorem, and the dashed lines are lattice numbers

Computed The numbers in this figure were computed on this machine (the groups built by closure from generators with their orders checked, and formula (6) computed). The range of lattice numbers is from the literature.

Computed Table 6 Finite subgroups of $SU(2)$ (defining representation) and the condition of formula (6). The defining representation is the matrices themselves, and the character is the trace (1-02). The smallest angle is the θ of the eigenvalues $e^{\pm i\theta}$ of the element nearest the identity.

Group	Order	Smallest angle	Δ_G	$\beta_W \geq$
$\{\pm 1\}$	2	180°	4	58.4

Group	Order	Smallest angle	Δ_G	$\beta_W \geq$
Quaternion group Q_8	8	90°	2	122.3
2T (binary tetrahedral group)	24	60°	1	253.4
2O (binary octahedral group)	48	45°	0.5858	442.1
2I (binary icosahedral group)	120	36°	0.3820	697.2
$2D_{50}$ (binary dihedral group)	200	3.6°	0.00395	68512

How to read Table 6

(a) The thresholds are constants that appear in the proofs, not the positions of phase transitions. (b) The finite subgroups of $SU(2)$ are only the cyclic groups, the binary dihedral groups, and three exceptions of order 24, 48 and 120. Known The groups whose names carry "binary" are rotation groups in $SO(3)$ pulled back by the double cover $SU(2) \to SO(3)$ (1-02). The binary tetrahedral, octahedral and icosahedral groups are the inverse images of the rotation groups of the regular polyhedra, and the binary dihedral group $2D_n$ is the inverse image of the rotation group of the regular n-gon (the dihedral group of order $2n$); each has twice the order of the original. For $2D_n$ (order $4n$), $\Delta_G = 2 - 2\cos(\pi/n) \approx \pi^2/n^2$, so the condition recedes roughly in proportion to n^2. Computed (c) With the one-loop $\xi \propto e^{3\pi^2 \beta_W/11}$ ($SU(2)$), each reduction of the lattice spacing to 1/10 increases β_W by about 0.86 (a one-loop rough guide; for two-loop corrections and the fact that asymptotic scaling does not yet hold in this range of β, see 5-05). Physics Why results for finite groups do not carry over to continuous groups is in 5-10, and the strong-coupling expansion in 5-02.

The β knob — two shores and a band on one axis

On the number line of Figure 2 we overlay the correlation length ξ in lattice units. The left shore is the strong-coupling theorems (the condition (9) of SZZ, and the paper of this site Paper ; 23-01 The line of construction and the diffusive-relaxation-time assumption) and the leading strong-coupling term $\xi = -1/(4 \ln u)$; the right shore is the two-loop formula (16-05 (10), $C = 185$) and Bałaban's hypothesis. The hypothesis is that the effective coupling is small enough, and no value is written (7-07 § 04), so a rough guide reading $g^2 \leq \gamma$ as $\beta \geq 4/\gamma$ is moved with the γ knob. In the middle are the numerical window 2.25–2.60 (Lucini–Teper 2001) and the band 1.75–2.6 where neither leading term is within 5% for the plaquette (16-05 Figure 1). Physics

The lower panel is the range of β over which the statements of 23-01 The line of construction and the diffusive-relaxation-time assumption are needed (this book's arrangement). The role of crossing the band falls to the infinite-volume uniqueness statement, the physical-unit gap statement and the relaxation-time band at the

bookkeeping length, which are needed at each β. The reflection-positivity preservation statement holds at every β; the Wilson-loop limit statement, the rotation-recovery statement and the non-triviality statement are statements about the $\beta \to \infty$ limit; and the OS reconstruction statement is a theorem independent of β. Known Theorems exist only at the left end, and the smaller γ is made, the further the rough guide for the right shore recedes.

Figure 3 Moving the vertical line with the β knob shows below the ξ at that β (leading strong-coupling term, two loops, numerical) and what exists there. The γ knob moves the side of Bałaban's hypothesis (the rough guide $\beta \geq 4/\gamma$). Both axes logarithmic



Computed The curves are (9)(10) of 16-05 computed on the spot (Bessel functions by series; Table 7 gives the same values with scipy). The points and window values are Lucini–Teper 2001 Physics .

Computed Table 7 Markers of β in Figure 3 and ξ (in lattice units; the column "ξ numerical" is the $1/(am)$ of 16-05 Table 3).

β	ξ leading strong-coupling term	ξ two loops (C=185)	ξ numerical	What is at this β
0.08333	0.065	—	—	Edge of condition (9) of SZZ (β < 1/12)
0.2722	0.093	—	—	Edge of the condition of this site's paper (β < 0.2722)
1.5	0.234	0.127	—	Right end where the leading strong-coupling term is within 5% for the plaquette
1.75	0.266	0.233	—	Left end of the band (neither leading term within 5%)

β	ξ leading strong-coupling term	ξ two loops (C=185)	ξ numerical	What is at this β
2.25	0.333	0.806	0.72	Left end of the numerical window
2.3	0.340	0.914	0.92	
2.4	0.354	1.17	1.05	
2.475	0.365	1.42	1.33	
2.55	0.375	1.72	1.71	
2.6	0.383	1.95	1.95	Right end of the numerical window and of the band
2.75	0.405	2.85	—	Left end where the leading weak-coupling term is within 5% for the plaquette
3.243	—	10	—	ξ = 10 at two loops (rough guide for extrapolation)
4.136	—	100	—	ξ = 100 at two loops (rough guide for extrapolation)
5.022	—	1000	—	ξ = 1000 at two loops (rough guide for extrapolation)
8	—	2.49×10^6	—	Side of Bałaban's hypothesis if γ = 0.5 (rough guide β ≥ 4/γ)
20	—	1.8×10^20	—	Side of Bałaban's hypothesis if γ = 0.2 (rough guide β ≥ 4/γ)

08

Detour: what the timeline leaves out

A section you may skip.

One comes across postings claiming to have proved the Yang–Mills problem. This chapter puts a row in the timeline under two conditions.

$$\text{include} \iff (\text{text or abstract checked}) \wedge (\text{address (1) can be written}) \quad (7)$$

Preprints that have not been peer-reviewed are included with a mark. Postings claiming to have proved the prize problem itself are not included unless there is a record of verification by experts (there are also cases where arXiv moderators withdrew one on grounds of the standard of its content). Physical numbers and holographic models measure after putting that "the theory exists" (17-13), and do not move cells. Example: the 1999 row of lattice numbers does not change the colour of any cell. The study in Part 23 of this book (23-01) is also outside the board.

The construction published in 2026 is included because there are the committee's assessment report and statement.

What this chapter can say

	Content
What is said	The content of the 37 rows of the timeline (the kind of checking is in the table of sources) Known . The row of the construction published in 2026, and the three 2026 cells of the board, are at the level of the committee's assessment Assessment
What is said	The board and the counts of rows under this chapter's way of deciding the cells. The conversion of formula (6), and the orders, Δ_G and condition values of the finite subgroups Computed . The ξ of Figure 3 and Table 7 restate the formulas and numbers in the text, and the assignment of β ranges to the eight statements of construction is this book's arrangement
What is not said	That the timeline is exhaustive. Other ways of dividing the cells. The grounds for the 1978 judgement of the strong-coupling cell are statements in the introductions of Shen–Zhu–Zhu and Adhikari–Cao; the main text of Osterwalder–Seiler 1978 has not been read. That the limit of Driver 1987 is the free electromagnetic field (main text: bibliographic data only). Drawing conclusions about the relation between finite-group models and weakly coupled $SU(2)$ from the thresholds of Table 6. This site shows nothing about any of the steps in four dimensions from the ultraviolet-limit step to the uniform-gap step (what moved cells in 2026 is the published construction). Whether the proof of the published construction is correct (cited as the committee assessed it)

Sources and reproduction

Item	Kind	Source / tools
The quotation of the problem statement in § 03, ϕ^4_3 and "the only complete example" in Table 5	Text	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Math. Inst. (2000) § 6, § 6.2, § 6.4. Date and place of the announcement from claymath.org
The 2004 sentence	Text	M. R. Douglas, report for the Clay Math. Inst. (2004)
The continuum-limit side in § 01, the lattice correlation-length problem in § 03, the uniqueness sentence in § 07, the abelian-group rows of Table 5	Text § 3, § 4, § 5, § 6	S. Chatterjee, arXiv:1803.01950
The state of three dimensions	Text § 1	I. Chevyrev, arXiv:2202.13359 / Chevyrev–Shen, arXiv:2302.12160

Item	Kind	Source / tools
The weights and conditions of formula (6). The judgement of the strong-coupling cell (convergence of the cluster expansion in OS 1978 is in outline, and the gap follows from that convergence)	Text § 1	Shen–Zhu–Zhu, arXiv:2204.12737, CMP 400 (2023) 805–851 (equations (1.2)(1.3), introduction) / Adhikari–Cao, arXiv:2202.10375v3, Ann. Probab. 53 (2025) 140–174 (equations (1.2)(1.3), Theorem 1.1, introduction)
Driver, Bałaban, MRS	Opening of the text (material check)	Driver, CMP 123 (1989) 575–616 (Corollary 6.7; the text makes no claim about the axioms) / Bałaban, CMP 102 (1985) 255–275, 122 (1989) 355–392 / Magnen–Rivasseau–Sénéor, CMP 155 (1993) 325–383 (abstract and § I)
The range of years 1987–89 for Bałaban in four dimensions	Bibliographic data only (the 1987 and 1988 papers)	Counted from the four-dimensional paper CMP 109 (1987) 249–301 to CMP 122 (1989) 355–392 (the same counting as 7-07 and 7-01; the foundations of 1983–85 shared with three dimensions are not counted)
Other rows from 2000 on	Abstract (material check)	arXiv:2006.04987, 2201.03487, 2111.12813, math/0101239 (Lévy; submitted January 2001, Mem. AMS 166 no.790 (2003); non-trivial bundles in PTRF 136, 2006), 1303.5113, 1912.07973, 2401.10507, 2509.04688, 2607.22236, 2607.24640
The row of the construction published in 2026, and the 2026 cells of the board	Assessment	Code of the published construction `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, submitted 2026-10-28). The assessment report `yang-mills-group/YM`, `docs/assessment-1.pdf` (2026-11-10), opening and § 8. The committee's statement (the third statement on the page of the Kavli Institute for Theoretical Physics, 2026-11-09; signed by Michael R. Douglas, David J. Gross (chair), Arthur Jaffe, Igor R. Klebanov, Nikita Nekrasov). For how to read them, see 6-00
Range of lattice numbers	Text, Table 1 (material check)	Lucini–Teper, JHEP 0106 (2001) 050

Item	Kind	Source / tools
Original papers before 2000: Yang–Mills 1954, Gross–Wilczek / Politzer 1973, Osterwalder–Schrader 1973 / 1975, Wilson 1974, 't Hooft 1974, Feldman–Osterwalder 1976, Osterwalder–Seiler 1978, Singer 1978, Brydges–Fröhlich–Seiler 1979–81, Guth 1980, Fröhlich–Spencer 1982, Aizenman 1981–82, Fröhlich 1982, Gross 1983, King 1986, Driver 1987, Gross–King–Sengupta 1989, Sengupta 1992–93, Seiberg–Witten 1994, Maldacena 1997, Witten 1998, Morningstar–Peardon 1999	Bibliographic data only	The content is from the text and reference list of Jaffe–Witten, Chatterjee § 4 and § 6, and the statements in the introductions of Adhikari–Cao and Shen–Zhu–Zhu. Bibliographic examples: Osterwalder–Seiler, Ann. Phys. 110 (1978) 440–471 / Aizenman, PRL 47 (1981) 1, CMP 86 (1982) / Fröhlich, NPB 200 (1982) / Driver, CMP 110 (1987) 479–501 / Maldacena, hep-th/9711200 / Witten, ATMP 2, 505 / Sengupta, JFA 108, Ann. Phys. 221 / Singer, CMP 60 / Fröhlich–Spencer, CMP 83 / Morningstar–Peardon, PRD 60
Table 7, Figure 3	Computed	Python and SciPy (beta_axis.py; 16-05 (9)(10), $C = 185$). The numerical window is Lucini–Teper Table 1 (16-05 Table 3, 23-01 Table 2)
Classification of the finite subgroups of $SU(2)$	Known (bibliographic data not checked)	Standard classification. Orders checked by computing closures
Table 3, Figure 1, counts of rows (recounted with the three 2026 cells added) / Table 6, Figure 2, the numbers in § 07	Computed	Python on this machine (counted year by year with formula (4)) / Python and NumPy (groups built by closure from quaternion generators, with orders, determinants and unitarity checked; $\ln 10 \cdot 11/(3\pi^2) = 0.855$)

Next chapter to read: 16-02 Five open questions. The questions left by the published construction are read following the five in the assessment report. The places where the ways physics and mathematics dig come apart are in 16-05.

Five open questions

This chapter is a map of what is unsolved. § 01 is existing knowledge (quotations from the assessment report).

What this chapter shows — For the construction of the existence of four-dimensional Yang–Mills published in 2026, we read one by one the five questions that the committee's assessment report lists as "the main points this work does not address". For each question we set out (1) what is needed, as mathematics, to answer it, (2) what is present on the side of the construction's Lean code as preparation for it, and (3) what is absent. Present: polynomial estimates uniform in the phase, uniqueness of the vacuum in each component, and parallel transport along lattice paths. Absent: exponential decay, continuum Wilson loops, and boundary conditions other than periodic ones.

Chapters this one relies on — 6-04 What is not shown, 12-05 Arzelà–Ascoli and the properties that pass to the limit, 12-07 Phase decomposition, 4-11 The definition of the mass gap.

Said first — The results of the published construction are cited at the level of "the committee's assessment is that they are proved in Lean". What this book's study has checked goes only as far as reading the definitions and the forms of the statements in the source. "Absent" means "not found within the range of a search of names and documentation strings"; the possibility that the same content exists under another name cannot be excluded. This chapter is a map of questions, and answers none of them.

Order of this chapter

1. The five questions of the assessment report — a summary of § 8 of the report
2. Convergence and uniqueness — the ultraviolet along the whole sequence, infinite volume along a subsequence
3. A continuum probability measure — the table of correlation functions and the reconstruction space
4. A uniform mass gap — polynomial estimates, and uniqueness of the vacuum in each component

5. Wilson loops and the area law — as far as parallel transport along lattice paths
6. One-particle states — tools for single-insertion states
7. Summary (Table 1) — what is needed, what is present, what is absent
8. Notes on reading — what "absent" means, and how the questions relate

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Labels — Source = the source of the published construction was read / Inferred = this book's inference / Assessment = according to the committee's assessment report (for details, see “Labels” in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are labels used throughout the book; Reading is this book's reading of numbers, and Lean is a Lean theorem of this book.

Names of the sources: the code of the published construction is `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, submitted 2026-10-28), and the assessment report is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). Locations in the code are given as relative paths within that repository, with declaration names and line numbers.

01

The five questions of the assessment report

At the opening, the assessment report judges that the submission proves that, for any simple compact group, a relativistic quantum field theory satisfying the Wightman axioms can be constructed as a scaling limit of four-dimensional Wilson lattice gauge theory, and writes that it does not prove that the infinite-volume limit is unique, nor the mass gap Assessment (the summary at the opening of the report). The way of construction — correlation functions first take the limit in a finite physical volume, and pass to infinite volume along a subsequence obtained by compactness — is also explained in the same place Assessment . § 8 lists five main points the construction does not address Assessment . In summary:

1. **Convergence and uniqueness.** To show convergence and uniqueness of the infinite-volume limit, rather than finding a convergent subsequence.
2. **A continuum probability measure.** To define a probability measure of the continuum theory on a suitable space, and have the Schwinger functions of the construction arise from it.
3. **A uniform mass gap.** To show a uniform mass gap: exponential decay of the corresponding correlations, and a gap in the spectrum of the Osterwalder–Schrader

Hamiltonian.

4. **Wilson loops and the area law.** To build Wilson-loop observables and connect them with the local observables of the construction. To show the area law for Wilson loops.
5. **One-particle states.** To build one-particle states and analyse the energy–momentum spectrum. The report adds that this analysis has been carried out in many constructive models in two and three dimensions.

What the Jaffe–Witten problem statement (0-03) asks for is existence and a mass gap. Of the five, 3 corresponds to the latter half; 1, 2, 4 and 5 are questions for bringing the construction closer to "a complete description of Yang–Mills theory". Below we read each question in the order: what is needed as mathematics, what is present on the side of the code, and what is absent.

02

Convergence and uniqueness

What is needed. That the correlation functions converge along the whole sequence of increasing volumes (without taking a subsequence), and that the limit does not depend on how the sequence is chosen. In other words, that the infinite-volume limit is determined uniquely Proved here .

What is present.

- The ultraviolet limit (the limit sending the lattice spacing to 0) is taken along the whole sequence of cutoffs once the volume is fixed. The conclusion of the construction theorem `constructed` (`Construction.lean:15`) contains the position statement `FullVolume.PositionsUV` along the whole sequence (`Extraction.Sequence.full`) Source . In the public theorem `JointExistence` (`ChallengeSpec.lean:186`) as well, the convergence field `Converges` (`PublicWilson.lean:85-90`) requires, in each volume, convergence along the whole sequence of cutoffs; only the volume side is a subsequence Source . The strong form with kernels and differences (`PositionsUV … Sequence.full`) is present only in the construction theorem `constructed` Source .
- The infinite-volume limit is taken as a subsequence, by Arzelà–Ascoli from boundedness and equicontinuity (`Extraction.first_surface`, `YM/Extraction/Lipschitz.lean:166`; `crops`, `YM/Extraction/Held.lean:136`) Source . For how to read this, see 12-05.
- The uniqueness lemma for limits `Extraction.Order.unique` (`YM/Extraction/Orders.lean:70`) is the general fact about Hausdorff spaces that two limits along the same subsequence are equal Source . It does not say that the limit is

independent of the choice of the sequence of volumes. In the record of how declarations are used, it is not reached from any headline theorem (outside) Source .

What is absent. Estimates of the difference when the volume is changed, decay of correlations, and material for uniqueness of the limit do not enter where the infinite-volume limit is taken Source . The only inputs are upper bounds for boundedness and equicontinuity.

Uniqueness in the volume and the "uniqueness of the vacuum in each component" of the phase decomposition (§ 04) are different questions. The former is uniqueness of the limiting measure; the latter is uniqueness of the vacuum in each component within one limit.

03

A continuum probability measure

What is needed. A probability measure on a space of distributions (or a suitable quotient of the space of connections) whose moments are the Schwinger functions of the construction Proved here . It would also be the foundation for handling non-local observables such as Wilson loops.

What is present. The objects of the construction are a table of correlation functions indexed by labels of local fields (`Table`) and the Osterwalder–Schrader space built from it Inferred (for how to read this, see 12-08). On the lattice side, the Wilson measure at each lattice spacing and each volume is defined as a probability measure (`law` in `YM/Wilson/Law.lean:87`) Source .

What is absent. No definition of a probability measure of the continuum theory has been found Inferred (within the range of a name search). There is neither a statement taking the limit of the sequence of lattice measures as a measure, nor a definition giving a name to the limiting measure. For the effective density at a fixed coarse step, there is as far as convergence of that sequence (16-03).

04

A uniform mass gap

What is needed. That there is some $m > 0$ such that the spectrum of the reconstructed Hamiltonian H lies in $[m, \infty)$ away from the vacuum. In the language of correlations, the same thing is that two-point correlations decay exponentially in the distance at a rate at least m (4-11). If the limit splits into components (phases), that m must also be uniform over the components Proved here .

What is present.

- **Polynomial estimates uniform in the phase.** For a finite word w, for all phases i and $u \in (0,1]$, the spectral weight at energy $(0,u]$ is $\leq M\,u^2$ (`EveryPhase.light_bound`, `YM/EveryPhase/Light.lean:221`). For the heat semigroup, for all phases and $s \geq 1$, $0 \leq \langle A, (e^{-sH} - P_0)A\rangle \leq M\,s^{-2}$ (`EveryPhase.inverse_square`, `YM/EveryPhase/Decay.lean:173`; P_0 is the projection onto the zero-energy space) Source . The constant M does not depend on the phase.
- **Uniqueness of the vacuum in each component.** That, in each component of the phase decomposition, the zero-energy space consists only of multiples of the vacuum (`UniqueHeat`, `YM/Phase/Mixture.lean:35`) is stated for all components (`EveryPhase.every`, `YM/EveryPhase/Vacuum.lean:146`) Source . No speed (rate) is attached.

The polynomial estimates come from a far-distance estimate that comes from local commutativity (`EveryPhase.late`, `YM/EveryPhase/Light.lean:208`) Source . The estimate of the weight by u^2 is of a form compatible even with a theory whose spectrum continues down to 0, and does not contain a mass gap Inferred . In other words, the mass-gap question is to "make this u^2 weight 0 below some m" Inferred .

What is absent. No statement of exponential decay (clustering with a rate, a spectral gap) has been found Source (within the range of a name search).

05

Wilson loops and the area law

What is needed. (a) To define Wilson-loop observables (the trace of parallel transport along a closed curve) in the continuum theory. On the lattice they can be written as products of links along a path, but in the continuum the connection itself is a distribution, so arguments of regularisation and limits are needed. (b) To connect them with local fields (the basic observables of the construction). (c) That for large loops the expectation becomes exponentially small in the enclosed area (the area law) Proved here .

What is present. There are parallel transport along lattice paths, `Lattice.travel` (`YM/Lattice/Path.lean:105`), and lemmas for its concatenation, reversal and gauge transformation Source . These are the material for writing Wilson loops on the lattice. The predictor of the probability kernel of one step of renormalisation is also built from framed parallel transport (9-04) Source .

What is absent. Continuum loop observables, the area law, and the relation between loops and local fields have not been found Inferred (within the range of a name search). The lattice Wilson measure is defined only on the periodic torus Source , and other

boundary conditions such as 't Hooft twists, or a quantity corresponding to the vortex free energy, have not been found Inferred (within the range of a name search).

This book's study has the area law in the strong-coupling window on the lattice (20-06) and estimates of the vortex ratio with twisted boundary conditions (Part 21). Both are statements with the lattice spacing fixed, and do not answer (a)–(c) of this question. Their place is described in 16-04.

06

One-particle states

What is needed. A subspace with an isolated spectral shell of mass m ($p^2 = m^2$ in energy–momentum), and tools to build scattering states from it (of Haag–Ruelle type) Proved here . Usually the mass gap (§ 04) is presupposed.

What is present. There is a module dealing with single-insertion states and the spectral representation (`YM/TraceExtension/Damping.lean` , titled "Single insertion point states and spectral representation") Source . Its documentation says it "does not assume mass-gap or cone-support results" Source . Whether this amounts to preparation for one-particle analysis has not been checked Inferred .

What is absent. A definition of one-particle states, isolated shells and scattering states have not been found Inferred (within the range of a name search).

07

Summary (Table 1)

Question	What is needed	What is present on the side of the code	What is absent
Convergence and uniqueness	Convergence along the whole sequence of volumes, uniqueness of the limit	The ultraviolet (cutoff) along the whole sequence, also in the public theorem; infinite volume along an Arzelà–Ascoli subsequence Source . The strong form with kernels and differences only in `constructed` Source . Uniqueness on the same subsequence, `Order.unique` Source	Estimates of the difference when the volume changes, decay, material for uniqueness Source
A continuum probability measure	A measure whose moments are the Schwinger functions	The table of correlation functions and the reconstruction space Inferred . The lattice Wilson measure Source	A definition of the continuum measure Inferred (within the range of a name search)

Question	What is needed	What is present on the side of the code	What is absent
A uniform mass gap	Exponential decay, a spectral gap, a rate uniform over components	Polynomial estimates u^2, s^{-2} uniform in the phase Source . Uniqueness of the vacuum in each component (no rate) Source	A statement of exponential decay Source (within the range of a name search)
Wilson loops and the area law	Continuum loop observables, their relation to local fields, the area law	Parallel transport along lattice paths, `travel` Source	Continuum loops, the area law Inferred (within the range of a name search). Boundary conditions other than periodic Inferred (within the range of a name search)
One-particle states	Isolated shells, scattering states	Tools for single-insertion states and the spectral representation Source	A definition of one-particle states Inferred

Table 1 The five open questions, and the preparation on the side of the code. The wording of the questions follows § 8 of the assessment report Assessment .

08

Notes on reading

- **What "absent" means.** It means not found by searching names, documentation strings (moddoc) and the record of how declarations are used. The possibility that the same content exists under another name cannot be excluded.
- **The level of "present".** What is present is part of what "the committee assessed as proved in Lean", and this book's study has only read the forms of the statements. Which declarations enter the dependency chain of the public theorem is stated only for those checked by following the chain (6-02). For many of the declarations listed in this chapter, the chain has not been checked.
- **How the questions relate.** 3 (the mass gap) is tied to 1 (uniqueness): clustering with a rate is usually a tool for deriving uniqueness of the limit Inferred . 5 (one particle) usually presupposes 3. 4 (the area law) and 3 are different statements, and no general argument deriving one from the other is known Inferred (16-09).
- **This chapter does not answer.** For none of the questions does this book show an answer. The holdings of this book's study and their distance from the questions are described in 16-03 and 16-04. A map of the ideas this book's study has tried for the band is in 16-11.

What this chapter says

What is said	What is not said
Assessment The assessment report accepts the construction of existence, and writes that uniqueness in infinite volume and the mass gap are not proved. The five questions of § 8	Whether the judgements of the assessment report are right
Source Also in the public theorem, the ultraviolet (cutoff) is along the whole sequence and infinite volume along a subsequence (`Converges`; `first_surface`, `crops`). The strong form with kernels and differences is present only in the construction theorem `constructed`	A route to uniqueness in the volume
Source There are polynomial estimates uniform in the phase (`light_bound`, `inverse_square`), uniqueness of the vacuum in each component (`UniqueHeat`, `every`), lattice parallel transport (`travel`), and tools for single-insertion states (`TraceExtension/Damping.lean`)	Whether these enter the dependency chain of the public theorem (not checked)
Inferred A continuum probability measure, continuum Wilson loops, the area law, boundary conditions other than periodic, and one-particle states are not found. The u^2 estimate does not contain a mass gap	Excluding the possibility that they exist under other names

Sources and reproduction

Item	Kind	Source, tools
The five open questions, the judgement at the opening	Assessment	The assessment report (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), opening and § 8
Public theorem, construction theorem	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `ChallengeSpec.lean:186` `JointExistence`, `Construction.lean:15` `constructed`
The infinite-volume subsequence and the uniqueness lemma	Source	`YM/Extraction/Lipschitz.lean:166` `first_surface`, `YM/Extraction/Held.lean:136` `crops`, `YM/Extraction/Orders.lean:70` `unique`
Polynomial estimates, uniqueness of the vacuum	Source	`YM/EveryPhase/Light.lean:208` `late`, `:221` `light_bound`, `YM/EveryPhase/Decay.lean:173` `inverse_square`, `YM/Phase/Mixture.lean:35` `UniqueHeat`, `YM/EveryPhase/Vacuum.lean:146` `every`
Lattice parallel transport, the Wilson measure, single insertion	Source	`YM/Lattice/Path.lean:105` `travel`, `YM/Wilson/Law.lean:87` `law`, `YM/TraceExtension/Damping.lean` (documentation)
The problem statement	Cited	A. Jaffe, E. Witten (the official problem statement of the Clay Mathematics Institute). 0-03

Words used in this chapter — open question, infinite-volume limit, uniqueness, subsequence, probability measure, Schwinger function, mass gap, spectral gap, phase

decomposition, uniqueness of the vacuum, Wilson loop, area law, one-particle state. Their meanings are in the A-02 Glossary.

Next chapter to read: 16-03 The infrared edge of the published construction — reads what is and is not said where the construction stops (the step with lattice spacing 1).

The infrared edge of the published construction

This chapter is existing knowledge (a reading of the published construction). § 05 and § 06 are a map of what is unsolved.

What this chapter shows — We read where the sequence of renormalisation steps of the published construction ends, and what is said there. Every step is carried within the weak-coupling band; the coarsest step has lattice spacing 1, and the coupling there is matched to a large reference value $\bar{\beta}$. The effective density at that step converges uniformly along the whole sequence of cutoffs once the volume is fixed. On the other hand, its limiting measure is not defined as an object, and reflection positivity, the relation to coarse Wilson loops, estimates uniform in the volume, and renormalisation to coarser scales are not stated. We set out precisely that restating the remaining questions as "long-distance questions of the effective theory at the step with lattice spacing 1" is likely right in direction, but that some statements needed to assert this are missing.

Chapters this one relies on — 16-02 Five open questions, 8-02 Sequences of spacings and of volumes, 9-01 The block kernel, 11-10 Convergence at a fixed step, and the step that removes the step index.

Said first — The results of the published construction are cited at the level of "the committee's assessment is that they are proved in Lean"; what this book's study has checked goes only as far as reading the definitions and the forms of the statements in the source. "Not stated" means not found within the range searched. This chapter describes the form of the remaining questions; it does not show a route to answering them.

Order of this chapter

1. A map of the steps — every step at weak coupling, step 0 at spacing 1 — the band of couplings and the reference value
2. The effective density at step 0, and its convergence — uniform in the cutoff, along the whole sequence
3. Correlations are read from fine steps — step 0 is where units are fixed
4. What is not stated about the objects at step 0 — five things

5. Restating the remaining questions — what is right and what is missing — the direction, the points that are not precise, the missing statements
6. From weak coupling to strong coupling — correspondence with this book's "crossover band"
7. What this chapter does not say

Labels — Source = the source of the published construction was read / Inferred = this book's inference / Assessment = according to the committee's assessment report (for details, see “Labels” in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are labels used throughout the book; Reading is this book's reading of numbers, and Lean is a Lean theorem of this book. Sources: relative paths, declaration names and line numbers in `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0); the assessment report is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10).

01

A map of the steps — every step at weak coupling, step 0 at spacing 1

The construction starts from the Wilson measure at cutoff N (the finest step), applies one step of renormalisation by a probability kernel (9-01) N times, and coarsens through steps $N, N-1, \dots, 0$.

- **The lattice.** The lattice at step m is a periodic lattice of side $L_p \cdot D_m$, with lattice spacing $a_m = 1/D_m$ (`Schedule.side`, `YM/Lattice/Scale.lean:67`) Source . Step 0 has side L_p and lattice spacing 1, and is the coarsest step. The volume L_p is measured in the units of step 0 Source .
- **The coupling.** At every step $m \le N$, the coupling $\beta(N, p, m)$ of step m lies in the band $[\hat\beta_m/2,\ 2\hat\beta_m]$ (the field `Conv.band` of the induction hypothesis, `YM/RG/History.lean:218`) Source . $\hat\beta_m$ is the reference value $\bar\beta$ plus the one-loop running $s\log(1/a_m)$ (`Calendar.hatBeta`, `YM/RG/History.lean:79`; `scale`, `YM/Lattice/Scale.lean:115`), and at step 0, since $a_0 = 1$, $\hat\beta_0 = \bar\beta$ Source . s is the one-loop slope, with value $11C_g/(24\pi^2)$ (`slopeConstant`, `YM/Gauge/Colors.lean:184`) Source .
- **Matching at step 0.** For all N, p, $\beta(N, p, 0) = \bar\beta$ (`Selected.reaches`, `YM/RG/History.lean:239`) Source . The bare coupling at each cutoff is chosen by the

shooting method so as to satisfy this condition (10-04).

The size of the reference value. In the record of the construction (`Paid.bar`, `YM/Price/Intent.lean:148`), the reference value $\bar{\beta}$ is set equal to the barrier value `Bar.value`, and the barrier is subject to the condition "$x^{-1/2} e^{(\log\log x)^2} \le 1$ for all $x \ge \bar{\beta}$" (`Bar.threshold`, `YM/Price/Intent.lean:127`) Source. From this condition alone, $\bar{\beta}$ comes out at roughly 9×10^5 or more (the threshold being 8.97×10^5) Reading (a floating-point computation of the threshold $x \approx 8.97 \times 10^5$ beyond which the inequality holds for all x). The other condition is that the entry value `betaEntry` (same file, `:116`) is at most $\bar{\beta}$ (same file, `:126`), but the source gives no formula or number for the entry value, only its existence Source. Since the coupling at every step is at least $\hat{\beta}_m/2 \ge \bar{\beta}/2$ Source, it reads that **every step is within the weak-coupling band, and the strong-coupling region is never passed through** Inferred. This is a design choice of the construction, in a form that avoids the difficulty on the strong-coupling side (that there is no estimate that works with the same constants in every band) Inferred.

02

The effective density at step 0, and its convergence

The density. The effective density $\mathrm{eval}_{N,p,m}$ at step m, for cutoff N and volume p, is a function on configurations of the coarse lattice at step m; it is non-negative, and its integral against the reference measure (the product of Haar probability measures over links) equals the partition function Z_N (`History.partition_eq`, `YM/RG/History.lean:242`) Source. Successive steps are linked by the push-forward identity with the probability kernel that uses the coupling $\beta(N, p, m+1)$ of step $m+1$ (`Transfer`, `YM/RG/History.lean:223`) Source. In the language of measures, the normalised $\mathrm{eval}_{N,p,m}/Z_N$ is the density of the probability measure obtained by pushing the bare Wilson measure forward $N - m$ times by the kernel Inferred (combining the three identities; there is no statement in the form of an identity of measures).

Since the kernel depends on $\beta(N, p, m+1)$, this push-forward is not "the Wilson measure sent by a fixed map"; the map itself moves with N Source.

Uniform convergence in the cutoff. Fixing the step m and the volume p, and viewing the density coals_N, with the vacuum constant $e^{-v|\Lambda|}$ removed, as a sequence in N:

- It is uniformly bounded and uniformly Cauchy in the sup norm on the configuration space (`FullVolume.Retreat.Flux.fanned`, `YM/FullVolume/Fields/Cement.lean:50`) Source.

- Its integral (the normalisation constant) is also Cauchy (`Flux.lacks`, same file, `:56`) Source .
- Within the construction, this is obtained for every step $m \geq 0$ (including step 0) and every p, along the whole sequence of cutoffs (without subsequences) (`sees`, `YM/FullVolume/Fields/Filled.lean:30`) Source .

Uniform convergence of the normalised density eval/Z_N is not a named statement, but it follows in one line from the uniform Cauchy property of the numerator, the Cauchy property of the denominator, and a positive lower bound on the denominator independent of N (`lingers`, `YM/FullVolume/Fields/Till.lean:20`) Inferred . Since the configuration space is compact and the reference measure is a probability measure, convergence in the sup norm includes convergence in total variation, and is stronger than weak convergence Inferred .

03

Correlations are read from fine steps

Continuum correlations are read not from step 0 but from a fine step j. The difference between the value computed on the representation at step j using the step-j image of the observable and the true lattice correlation is bounded by a quantity independent of N (`Sever.echoes`, `YM/FullVolume/Fields/Peat.lean:42`), and becomes small as $j \to \infty$ under a condition on the separation of points Source . Combining this with the Cauchy property at a fixed step j (`latter`, `YM/FullVolume/Fields/Timber.lean:58`), the true lattice correlation becomes Cauchy in N (`arrives`, same file, `:74`) Source .

In this line of argument, step 0 is "where units are fixed by $\bar{\beta}$", and is not used in estimating correlations Inferred . For distances longer than step 0, the volume L_p is simply placed in the units of step 0, and the infinite-volume limit is taken by compactness along a subsequence (12-05) Source .

04

What is not stated about the objects at step 0

1. **The limiting measure.** The $N \to \infty$ limit of the measure at step 0 (or at any step) is not defined as an object. No definition giving a name to a limiting density or limiting representation has been found Source (within the range of a name search). What there is goes only as far as the materials of the representation (couplings, activities, path data) being Cauchy in N, or converging to some limit Source . That the limit has a representation of the same form, or that it is consistent across steps, is also not stated.

2. **Reflection positivity.** The statement of reflection positivity (`Slice.reflected_positive`, `YM/Slice/Reflection.lean:98`) is only about the bare Wilson measure Source . Reflection positivity of the pushed-forward measure at step 0 is not stated Source (within the range of a name search). Since the support of the kernel can be large enough to straddle the mirror plane (`fitsPatch`, `YM/RG/History.lean:110`), we infer that the push-forward does not automatically inherit reflection positivity Inferred .
3. **The relation to coarse Wilson loops.** The relation between Wilson loops of configurations at step 0 and Wilson loops of the fine field (or the continuum) is not stated Source (within the range of a name search). Since a coarse bond is a prediction from framed parallel transport multiplied by a fluctuation (`connector`, `YM/RG/Block.lean:39`), we infer that a coarse loop is close to "parallel transport along a thick path, with fluctuations" Inferred .
4. **Estimates uniform in the volume.** Estimates for the objects at step 0, such as the lower bound on the normalisation constant (`till`, `YM/FullVolume/Fields/Till.lean:9`) and the constant in the difference estimate (`refreeze`, `YM/FullVolume/Fields/Flight.lean:28`), contain exponentials of the number of lattice points $|\Lambda|$, and are not uniform in the volume p Source . What could be read as estimates uniform in the volume are things like the size of observables measured by the chessboard root on the side of the Wilson measure (`Extraction.Wcmain`, `YM/Extraction/Center.lean:38`; for how to read this, see 12-02), and these are not estimates of the measure at step 0 Source . The bookkeeping of one step itself (the norms of activities and the contraction constants) consists of pointwise sums and does not depend on the volume Source .
5. **Renormalisation to coarser scales.** Step numbers are natural numbers, and step 0 is the coarsest step. No theorem continuing renormalisation to scales coarser than step 0 has been found Source (within the range of a name search). The group of modules with ultraviolet extension in their names (`YM/Extension/UV/`) is a separate route on the ultraviolet side, not a continuation to the coarse side Source (documentation).

05

Restating the remaining questions — what is right and what is missing

Among the open questions of the assessment report, uniqueness in infinite volume, the mass gap and the area law (1, 3 and 4 of 16-02) are all questions about long-distance behaviour. We consider restating them as "long-distance questions, as $L_p \to \infty$, of the effective lattice theory at step 0 (with large coupling $\bar{\beta}$ and lattice spacing 1)".

Facts that can be read from the source Source .

- Every step is carried within the weak-coupling band, and at step 0, $\beta = \bar{\beta}$.
- The volume L_p is measured in the units of step 0.
- Behaviour at distances longer than lattice spacing 1 (exponential decay, the area law, uniqueness of the volume limit) is not dealt with in any theorem Source (within the range of a name search).

On the basis of these facts, restating the remaining questions as "long-distance questions of the theory at step 0 ($\bar{\beta}$ large, $L_p \to \infty$)" is likely right in direction Inferred. Since the coupling at that step is on the weak-coupling side, it is this book's inference that they can be further restated as "the question of following the effective lattice theory at spacing 1 from weak coupling to strong coupling" Inferred.

Neither can be asserted. The theory at step 0 is not a theory with the Wilson action but a theory with an effective action given by a representation made of a truncated action, activities and path data, and $\bar{\beta}$ is a coefficient in that representation. The representation is not unique Source. Reflection positivity of that measure is not stated either (§ 04). What can be written is only "the main part of the remaining questions is likely to be reducible to exponential decay at long distances in the effective theory at step 0" Inferred.

Points where this is not precise.

- (a) Continuum correlations are not functions of the measure at step 0 alone. The true correlation can be sent to step 0 as the push-forward of a density with the observable inserted, but to write it as an expectation under the measure at step 0, the image of the observable at step 0 is needed Source.
- (b) Convergence of the image of an observable at step m (`Rising.hail`, `YM/FullVolume/Fields/Capital.lean:57`) is stated only for m at or above a reference step for each observable. For observables whose reference step is above 0, the image at step 0 is not dealt with Source (a field of the hypotheses).
- (c) The estimate of the difference between a fine step and the true correlation (§ 03) is used only in the direction $j \to \infty$, and the error of reading out at step 0 is not small Source.

Statements missing for the restatement (the same root as § 04).

1. The existence and naming of the $N \to \infty$ limit of the measure at step 0. It is likely to follow from the convergence of § 02 Inferred, but there is no statement.
2. Reflection positivity of the pushed-forward measure at step 0 (or at any step). Because of the support of the kernel, it is not obvious Inferred.

3. The relation between Wilson loops of coarse configurations and Wilson loops of the fine field (or the continuum).
4. Estimates for the objects at step 0 that are uniform in the volume p.
5. The limit of the images of observables at step 0 (at every order), and an identity connecting expectations at step 0 with continuum correlations.

In particular, without the three of 1 (the limiting object), 4 (local estimates uniform in the volume) and 5 (images of observables at step 0), one cannot state that "the remaining questions are long-distance questions of the effective theory at step 0". With all five, the remaining questions could be stated as "questions about one named family of lattice measures". Until they are all in place, the restatement is a guide to direction.

06

From weak coupling to strong coupling

The coupling $\bar{\beta}$ at step 0 is large (weak coupling), and the lengths at which confinement and the mass gap become visible are expected to lie much further on, even in the units of that step Inferred . If the one-loop running is formally extended to the side coarser than step 0, the length at which the coupling becomes of order 1 is of order $e^{\bar{\beta}/s}$ in the units of step 0 Inferred (an extrapolation, not a statement of the construction). Getting there would need new steps flowing from step 0 further to the coarse side until the strong-coupling region is entered Inferred . What is needed at the entrance to them is 1–3 of § 05 (the limiting measure, reflection positivity, the relation to coarse Wilson loops) Inferred . This is a different set from the three needed to turn the restatement into a statement (1, 4 and 5 of § 05).

This book has called the intermediate region connecting weak and strong coupling "the crossover band" (26-00). If the remaining questions may be restated as above, they correspond to questions of crossing this band Inferred . The holdings of this book's study lie at the two ends of the band, and the middle remains open (16-04).

One caution. The renormalisation of this book's study (the push-forward by a deterministic block map, 19-04) and the push-forward of the published construction (a coupling-dependent probability kernel) are not the same, even if their objects are similar. Convergence of one does not give convergence of the other Inferred .

07

What this chapter does not say

- It shows nothing about the long-distance behaviour of the theory at step 0.

- It says neither that the five statements of § 05 hold nor that they do not. About 2, it only infers that it is not obvious.
- "Not stated" is the result within the range searched. The possibility that the same content exists under another name cannot be excluded.
- Whether the proof of the published construction is correct stays at the level of the committee's assessment.

What this chapter says

What is said	What is not said
Source The coupling at every step is in the band $[\hat{\beta}_m/2, 2\hat{\beta}_m]$; step 0 has lattice spacing 1 and $\beta = \bar{\beta}$ (`Conv.band`, `Selected.reaches`, `Schedule.side`)	An explicit value of $\bar{\beta}$ (from the condition, roughly 9×10^5 or more Reading ; the entry value is given only as existing)
Source The effective density at each step, including step 0, is uniformly Cauchy in the sup norm along the whole sequence of cutoffs with the volume fixed, and the normalisation constant is also Cauchy (`Flux.fanned`, `lacks`, `sees`)	Convergence uniform in the volume
Inferred Convergence of the normalised density in total variation follows in one line	Its form as a named statement
Source The limiting measure, its reflection positivity, the relation to coarse Wilson loops, estimates uniform in the volume for the objects at step 0, and renormalisation coarser than step 0 are not stated within the range searched	Excluding the possibility that they exist under other names
Source Every step is in the weak-coupling band, step 0 is at the reference value, the volume is in the units of step 0, and distances longer than spacing 1 are dealt with in no theorem (within the range of a name search). Inferred On that basis, restating the remaining questions as "long-distance questions of the theory at step 0" is likely right in direction, and they can be further restated as "the question of following it from weak coupling to strong coupling". Five statements are missing to make this a statement	The answer to that question

Sources and reproduction

Item	Kind	Source, tools
Lattices of the steps, the band of couplings, matching at step 0, the mass of the density, push-forward	Source	`yang-mills-group/OpenAI-YM-20261028`: `YM/Lattice/Scale.lean:67` `side`, `YM/RG/History.lean:218` `Conv.band`, `:223` `Transfer`, `:239` `Selected.reaches`, `:242` `partition_eq`, `:110` `fitsPatch`

Item	Kind	Source, tools
Uniform convergence of the density	Source	`YM/FullVolume/Fields/Cement.lean:50` `Flux.fanned`, `:56` `lacks`, `Filled.lean:30` `sees`, `Till.lean:9` `till`, `:20` `lingers`
Reading out from fine steps	Source	`YM/FullVolume/Fields/Peat.lean:42` `Sever.echoes`, `Timber.lean:58` `latter`, `:74` `arrives`, `Flight.lean:28` `refreeze`, `Capital.lean:57` `Rising.hail`
Reflection positivity, the chessboard root, prediction of coarse bonds	Source	`YM/Slice/Reflection.lean:98` `reflected_positive`, `YM/Extraction/Center.lean:38` `Wcmain`, `YM/RG/Block.lean:39` `connector`
Open questions	Assessment	The assessment report (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), § 8

Words used in this chapter — renormalisation step, lattice spacing, band of couplings, reference value, probability kernel, push-forward, effective density, uniformly Cauchy, total variation, reflection positivity, chessboard root, crossover band. Their meanings are in the A-02 Glossary.

Next chapter to read: 16-04 Where this book stands — sets out at which end, weak coupling or strong coupling, the holdings of this book's study lie, and guides the reading of Volume 3.

Where this book stands

> This chapter is a guide. § 02 is a map of what is unsolved.

What this chapter shows — We line up the theorems that this book's study has machine-checked in Lean along the axis of the coupling. The holdings lie at both ends. At the strong-coupling end there are the infinite-volume limit, the mass gap (in lattice units), the area law and an upper bound on the vortex free energy; as things that hold at any coupling there are the vortex floors and properties of the vortex ratio; and on the weak-coupling side there is the rarity of large plaquettes. The middle (the crossover band) remains open. Finally, we guide the reader through what each part of Volume 3 deals with.

Chapters this one relies on — 16-02 Five open questions, 16-03 The infrared edge of the published construction.

Said first — Every theorem listed here is a statement with the lattice spacing fixed, and a formalisation of a result of a known type. Neither the continuum limit nor the mass gap in the sense of the problem statement is shown. Statements of kinds that the published construction does not deal with are included, but this does not mean that any open question has been answered.

Order of this chapter

1. Holdings on the axis of the coupling — Table 1
2. The middle remains open — what is missing in the crossover band
3. Overlaps and differences with the published construction — the same direction, different kinds
4. How to read Volume 3 — one line per part, and three reading orders
5. What this chapter does not say

Labels — Source = the source of the published construction was read / Inferred = this book's inference / Assessment = according to the committee's assessment report (for details, see "Labels" in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are labels used throughout the book; Reading is this book's reading of numbers, and Lean is a Lean theorem of this book.

01

Holdings on the axis of the coupling

We view the coupling β of lattice gauge theory as a single axis. The side of small β is strong coupling, the region where confinement and the mass gap can be shown by expansions of known type (5-02). $\beta \to \infty$ is weak coupling, and the continuum limit is taken at this end (5-05). Placing the theorems of this book's study on this axis gives Table 1.

Place	Theorem	Range	Chapter
Strong-coupling end	Exponential decay of correlations (constants free of the volume)	$18 \tanh \lvert\beta\rvert < 1$, etc.	20-02
	The infinite-volume limit (no subsequence), translation invariance, clustering	$\beta \neq 0$, $18 \tanh \lvert\beta\rvert < 1$	20-03
	The mass gap of the vessel of reconstruction (lattice units, a lower bound)	The same window	20-04
	The area law for Wilson loops (finite volume, infinite volume)	In a finite box $\beta < 1.135 \cdot 10^{-3}$ (the version with constant 441); in infinite volume $\beta < 1.22 \cdot 10^{-4}$ (the version with constant 4096; also $\beta > 0$ and $18 \tanh \beta < 1$)	20-06
	An upper bound on the vortex free energy (the rate per unit area does not depend on the transverse size; the prefactor N^2 (the transverse area) remains)	$\beta < 1.22 \cdot 10^{-4}$ in the basic version (constant 4096), $\beta < 1.135 \cdot 10^{-3}$ in the sharp version (constant 441)	20-06

Place	Theorem	Range	Chapter
At any coupling	Reflection positivity (three forms)	All real β (depending on the form)	20-01
	The thin-vortex floor (the rate depends on the transverse size N)	All β	21-02
	The thick-vortex floor, the minimal action of a vortex	For each sufficiently large β (the threshold only as existing)	21-03, 21-05
	The sign of θ, pairs of Polyakov loops	Even vertical size M	21-06
	Continuity of the vortex ratio, the $\beta \to \infty$ limit, the largest solution of the matching, a lower bound on the deficit of the twisted box	With the lattice fixed	21-07
Weak-coupling side	Rarity of large plaquettes (uniform in the volume)	$\beta \geq 81$ (explicit threshold)	22-01

Table 1 Where the firm theorems of this book's study are placed. Each is a formalisation in this book's Lean, with the lattice spacing fixed, of a result of a known type Lean . The values of the windows are floating-point readings Reading .

Two notes on reading it.

- **The strong-coupling end is "the side where things are easier to show".** The mass gap and area law here are statements different from the mass gap of the continuum theory that the problem statement asks for, and they lie on the opposite side from the weak-coupling end where the continuum limit is needed.
- **What holds at any coupling is not uniform in the transverse size.** The rate of the vortex floor tends to 0 as the transverse size N is made large (the thin floor), or the threshold depends on N (the thick floor). Restricted to the strong-coupling window, there is a form whose rate does not depend on N. The prefactor N^2 (the transverse area) remains, and outside the window there is no uniform form (20-06).

02

The middle remains open

The strong-coupling window for the infinite-volume limit and the mass gap is $|\beta| < 1/19$ (≈ 0.053; an explicit sufficient condition for 18 $\tanh|\beta| < 1$, the window itself being $\operatorname{artanh}(1/18) \approx 0.0556$; 20-03), and the threshold for rarity is $\beta \geq 81$; the band of interest in physics (β around 2 to 3) falls in neither. We have called this middle "the crossover band" (26-00). What is missing to cross the band is set out in 19-07 as statements written with defined objects only (matching, convergence of the full sequence, non-degeneracy, equicontinuity, uniform decay, the band). Each of them asks for estimates **uniform** in the lattice spacing and the size of the box, and they do

not follow from the firm theorems of this book (each a statement at one lattice spacing). A map of the ideas this book's study has tried for the band is in 16-11, and a list of the models on which arguments about the band are tested is in 26-12.

Seen from the side of the published construction too, the remaining questions are in the same place. The construction carries every step at weak coupling and stops at the step with lattice spacing 1 (16-03). The main part of the remaining questions (the mass gap, the area law, uniqueness) is likely to be reducible to exponential decay at long distances in the effective theory at that step Inferred . However, the theory at step 0 is not one with the Wilson action but a theory with an effective action given by a representation (the reference coupling $\bar{\beta}$ is a coefficient in it, and the representation is not unique), and its reflection positivity is not stated either Source . To state this, three things are needed: the limiting object, local estimates uniform in the volume, and images of observables at step 0. Under this expectation, the questions gather around the band from both sides Inferred .

About the band, what this book's study has is a conditional implication and a list of premises (27-01, 28-04). It has not shown that the premises are satisfied.

03

Overlaps and differences with the published construction

Things in the same direction. Reflection positivity of the lattice Wilson measure and the Chessboard Estimate are present in both Source . The published construction has, for the original Wilson measure, reflection positivity in planes through lattice points and in planes through midpoints of edges, along all four axes ($\beta \geq 0$), and also the triangle inequality for the chessboard root Source (12-01, 12-02). The line of argument that takes subsequential limits and transfers properties to them is also of the same type.

Kinds of things not found in the published construction (within the range searched).

- **Rarity of large plaquettes.** The published construction uses the Chessboard Estimate to measure the size of observables, not to estimate the probability of sets of large plaquettes Inferred (22-00).
- **Twisted boundary conditions and the vortex free energy.** The Wilson measure of the published construction is defined only on the periodic torus, and no statement about twists has been found Inferred (within the range of a name search).
- **The mass gap and area law inside the strong-coupling window.** The published construction does not pass through the strong-coupling region (16-03 § 01), so statements of this kind appear to be absent Inferred (within the range of a name search).

This difference is not a comparison of which has got further. The published construction gives the existence of the four-dimensional continuum theory, as assessed by the committee Assessment . This book's study does not have that. Conversely, the three items above are statements with the lattice fixed, and answer none of the open questions of the assessment report (16-02). This book's strong-coupling results (the infinite-volume limit, the mass gap, the area law, the vortex floors) are statements at the **opposite end** (strong coupling, with the lattice spacing fixed) from the remaining questions (a uniform mass gap of the continuum theory, and so on). They deal with the same kinds of quantities in a different region, and we do not say that they give a route to the open questions.

04

How to read Volume 3

Part	Title	What it deals with
Part 16	Where things stand, and the questions that remain	The timeline, the open questions, the infrared edge of the published construction, where this book stands, the seams, tasks, misconceptions, the table of seven stages, the map of tools
Part 17	From the side of physics	Mechanisms of confinement, the string picture, large N, the mass gap as geometry. An arrangement of views that are not mathematical theorems
Part 18	A dictionary of two languages	Correspondence between the words of physics and the words of mathematics; this book's assumptions in the language of physics
Part 19	Objects, and the statements to be shown	Definitions of the lattice, the Wilson measure, twists, reflection and the block map (with Lean definitions), and the statements to be shown
Part 20	Firm theorems — the vessel and strong coupling	Reflection positivity, decay at strong coupling, the infinite-volume limit, reconstruction and the mass gap, the vessel with fermions, the area law
Part 21	Firm theorems — the vortex floor and Polyakov loops	Sign systems, the thin- and thick-vortex floors, the minimal action of a vortex, the sign of θ, the vortex ratio — continuity, limit, matching
Part 22	Firm theorems — the rarity of large plaquettes	The Chessboard Estimate, and rarity uniform in the volume at weak coupling
Parts 23 and 24	The first road — from the direction of time	The record of the study heading for the band from relaxation times and correlation lengths
Part 25	The second road — from the direction of space	The record of the study heading for the band from small twisted tori and confinement
Part 26	The crossover band	The correct form of the band, reduction, what cannot be crossed by structure alone, the comparison with renormalisation

Part	Title	What it deals with
Part 27	Closing — conditional statements	The conditional statement and the ledger of conditions, the index of questions and of what is not said
Part 28	The tree of premises	Three implications, the list of premises, what depends on design choices, the places that are open

Table 2 The parts of Volume 3.

Suggested reading orders.

- **If you want only what is firm.** Part 19 (objects) → Parts 20–22 (firm theorems). The last chapter of each part has a table of "what it says, and what it does not".
- **If you want the shape of the remaining questions.** This part (16-02 to 16-04) → 19-07 (the statements to be shown) → Part 26 (the band) → Part 28 (the tree of premises).
- **If you want to start from motivation.** Parts 17 and 18 (the physical view and the dictionary) → this part → Parts 20–22.

Parts 23–25 are records of the study along the two roads towards the band, and have not been rewritten. Most of their conclusions are gathered in Parts 26–28.

05

What this chapter does not say

- It does not claim to have shown anything about the mass gap proper (a mass gap of the continuum theory, uniform in the lattice spacing).
- Every theorem of Table 1 is a formalisation of a result of a known type, and no novelty is claimed.
- What is "not found" in the published construction is the result within the range searched; the possibility that it exists under another name cannot be excluded.
- It says neither that the band statement is true nor by which route it could be shown.

What this chapter says

What is said	What is not said
`Lean` The theorems of Table 1 (the strong-coupling end, any coupling, the weak-coupling side) are machine-checked in Lean. Details are in each chapter	Estimates uniform in the lattice spacing, the continuum limit, the band

What is said	What is not said
Source Reflection positivity and the chessboard are present in both. Inferred The published construction does not pass through the strong-coupling region (depending on the size of $\bar{\beta}$)	—
Inferred Rarity of large plaquettes and twisted boundary conditions are not found in the published construction (within the range of a name search). This book's strong-coupling results are statements at the opposite end from the remaining questions	Answers or routes to the remaining questions

Sources and reproduction

Item	Kind	Source, tools
The theorems of Table 1	Lean	The tables of sources in each chapter (of the types of Osterwalder–Seiler, Dobrushin, 't Hooft, Fröhlich–Israel–Lieb–Simon) and the Lean declarations
The infrared of the published construction and reflection positivity	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0). Locations are in 16-03 and 12-01
The assessment of the construction of existence, the open questions	Assessment	The assessment report (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), opening and § 8

Words used in this chapter — strong coupling, weak coupling, crossover band, area law, vortex floor, vortex free energy, rarity of large plaquettes, statements to be shown, tree of premises. Their meanings are in the A-02 Glossary.

Next chapter to read: 16-05 Where the two ways of digging part — maps where the two roads of this book's study are cut off. To read from the firm theorems, go to 19-00.

Where the two ways of digging part — seeing the mismatch of two tunnels in seven seams

This chapter is a map of open questions. § 05 is this book's own investigation.

What this chapter shows — For each of the seven seams, the sentence of physics, the mathematical statement and the partial bridges, and three types. Comparing, for a local quantity and for a mass, how far the two shores are from the numbers (calculations on this machine).

Prerequisite chapters — 7-11 Obstacles beyond divergence and 17-13 Where the way physics digs parts from mathematics. The tools are 4-10 (reflection positivity), 5-01 (the lattice) and 4-07 (dimensional transmutation).

Said first — This chapter names the seams; it fills none of them. The count of seven and the types are this book's arrangement. The record of this book's study has worked only around the seam of "there is one scale", and has not reached a theorem. This chapter shows neither the existence of the mass gap nor the construction of the continuum limit. The seam of existence has been bridged by the construction published in 2026 (the committee's assessment Assessment ; Part 6). The label Assessment means "according to the committee's assessment report" (“Labels” in 6-00).

The order of this chapter

1. Two ways of digging — from existence, or from values. A parallel table of walls (six obstacles and seven moves)
2. A map of the seams — seven seams and three types (Table 1)
3. The type in which a premise becomes a goal — existence, a single scale
4. The type of limits and uniformity — shores, finite and infinite, extra dimensions
5. Measuring between the shores — the plaquette and the mass (Figure 1, computed)
6. The type in which the same word points to different things — spacetime and space, particles
7. Overlaying the two lists — partial bridges

01

Two ways of digging

There are two directions in which to dig at this problem (0-02). Mathematics starts from "there is no theory yet" and passes through the six stages (7-01) in order; physics starts from "the theory exists" and computes quantities inside it. As a metaphor, they are two tunnels dug into the same mountain from both sides; more precisely, there are places where a sentence on one side cannot be handed over as it is to the other. The goal of mathematics has the following form (0-03) Known .

$$\forall G \ \exists \mathcal{T}_G : \quad \text{axioms} \ \wedge \ \text{non-trivial} \ \wedge \ \text{agrees with asymptotic freedom at short distances} \ \wedge \ \text{s}$$

G is a compact simple group, $\mathcal{T}_G$ the quantum Yang–Mills theory on $\mathbb{R}^4$, and H its Hamiltonian. The physical sentence "computing the quantity Q in $\mathcal{T}_G$ gives q" includes $\exists \mathcal{T}_G$ among its premises. We translate a sentence of physics P into a mathematical statement M, and call a place where the translation does not become a theorem a **seam**.

$$\text{seam} = (P, M) : \quad P \text{ is used in physics}, \quad M \text{ is not a theorem, or points at something other t}$$

Example: the $m_{0^{++}}/\sqrt{\sigma} = 3.844(61)$ of $SU(2)$ (Lucini–Teper 2001) Physics is a prediction in physics, and in mathematics a target that the constructed theory must hit (17-13 § 01).

A parallel table of walls — the six obstacles of mathematics and the seven moves of physics

The seams of Table 1 were made by pairing rows of the lists on the two sides. Before that, we give a parallel table of the rows by name. On the left are the six obstacles of 7-11 Table 4, and on the right the seven moves of 17-13 Table 5. The stock phrases are the expressions each side uses when pointing at that wall. Some rows exist on one side only.

Parallel table Names of the walls (the correspondence is this book's arrangement)

Name in mathematics (7-11)	Name in physics (17-13)	Stock phrase (mathematics / physics)	One side only?
Obstacle 1 No small parameter in the infrared	Moving to a neighbouring theory and coming back with a knob, Looking from the side of a small parameter	"No uniform estimate across the scale where the coupling is $O(1)$" / "Below Λ it is non-perturbative"	Both sides

Name in mathematics (7-11)	Name in physics (17-13)	Stock phrase (mathematics / physics)	One side only?
Obstacle 2 Invisible to perturbation theory	Looking from the side of a small parameter, Regarding as one	"All Taylor coefficients of $e^{-c/\alpha}$ are 0" / "$\Delta = c\Lambda$"	Both sides
Obstacle 3 Not collapsing to a free field	Measuring after putting that it exists	"Show that the limit is not a free field" / "The theory exists"	Both sides
Obstacle 4 The state space is curved	Looking after choosing a gauge	"There is no global gauge fixing" / "Compute with the gauge fixed"	Both sides ("Outside the table: gauge" in Table 4)
Obstacle 5 Few regularisations preserve reflection positivity	—	"A regularisation that preserves positivity" / —	**Mathematics only**. Numerical calculations in physics also use improved actions that lose positivity (the entry "improved action" in 18-04)
Obstacle 6 The structure of infinitely many degrees of freedom	—	"The representations are inequivalent (Haag's theorem)" / —	**Mathematics only**. Perturbative calculations in physics proceed using the interaction picture
—	Taking σ from the string picture (§ 03, § 04)	— / "A string is stretched, the area law"	**Physics only**. Confinement (the area law) is outside what the problem statement requires, an extension problem (17-02)

Name in mathematics (7-11)	Name in physics (17-13)	Stock phrase (mathematics / physics)	One side only?
— (outside the six: the ultraviolet side)	Extrapolating from finite β	"Construction of the continuum limit" / "Extrapolate in a^2"	Not among the six of 7-11. In mathematics it is a problem on the ultraviolet side (7-07, 7-10)

The four rows on both sides call the same wall by different words. The rows on one side only are places that, in that side's language, are not seen as walls. Positivity and equivalence of representations pass unasked in the procedures of physics, and the area law passes unasked in the mathematical problem statement. Rearranging this parallel table into seams gives Table 1, and lining it up with the partial bridges gives Table 4 of § 07.

02

A map of the seams — seven seams and three types

Table 1 lines up the seven seams. The order of mathematics (existence → axioms → gap) and the order of physics (values → mechanisms) run in opposite directions, so stages at the same position do not correspond. As a metaphor this is the mismatch of the tunnels; more precisely, a premise on one side becomes the goal of the last stage on the other.

Table 1 The seven seams (the division and the types are this book's arrangement)

Seam	Sentence of physics	Mathematical statement	Type
1 The premise of existence	Put that the theory exists, and compute quantities in it	Make the limit satisfying the axioms, not as a subsequence	Type in which a premise becomes a goal
2 A single scale	Mass and string tension are both constant multiples of one Λ	Formula (5) holds uniformly in β	Type in which a premise becomes a goal
3 Different shores	Compute on the side of g→0, small β, N→∞	Control every scale at finite N	Type of limits and uniformity
4 Finite and infinite	Measure in finite boxes, and extrapolate in a^2	A gap uniform in the volume (Chatterjee's lattice correlation-length problem, Problem 5.1)	Type of limits and uniformity
5 Spacetime and space	Think in a spacetime where particles fly and there is causality	Come back from a Euclidean measure by reflection positivity	Type in which the same word points to different things

Seam	Sentence of physics	Mathematical statement	Type
6 The word "particle"	The gap is the mass of the lightest glueball	A spectral gap (an isolated one-particle state is an extension problem)	Type in which the same word points to different things
7 Extra dimensions	Get the gap from the geometry of the cigar	The R→0 (λ→0) limit, and exchanging limits	Type of limits and uniformity

The type in which a premise becomes a goal is a seam where a premise of physics becomes a goal of mathematics; **the type of limits and uniformity** is a seam where physics computes at a particular value (a shore) of a parameter p, and mathematics needs the limit that moves p to the value p_* of the problem statement; **the type in which the same word points to different things** is a seam where the same word points at different objects. The type of limits and uniformity has the following form.

$$\text{physics}: \ F(p_{\text{shore}}) \text{ computed}, \qquad \text{mathematics}: \ \lim_{p \to p_*} F(p) \text{ exists, and } \inf_{p} \Delta(p) > 0 \text{ and}$$

Example: in the seam of finite and infinite, $p = L$ (the side of the box) with $p_* = \infty$; in the seam of extra dimensions, $p = R$ (the size of the extra dimensions) with $p_* = 0$.

The following sections are ordered by type (the type in which a premise becomes a goal is 1 and 2, the type of limits and uniformity is 3, 4 and 7, and the type in which the same word points to different things is 5 and 6).

03

The type in which a premise becomes a goal

The seam of the premise of existence

The calculations of physics stand on the existence of the theory (a measure and a Hilbert space), and mathematics has to make that existence. Partway through the making, neither mass nor string tension is defined. The problem statement writes 'one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time, nor even a precise definition' Known . Mathematics shows $\exists\,\mathcal{T}$ as the limit of approximations $\mu_{a,L}$ with lattice spacing a and side L.

$$\mu = \lim_{L \to \infty} \lim_{a \to 0} \mu_{a,L} \qquad \text{(not a subsequence: the limit itself exists and is unique)} \ (4)$$

The problem statement writes, in footnote 2 of § 6.5, that a subsequential limit obtained by compactness (weak existence) alone is not accepted as a solution (unless the properties of the limit are established by another method) Known . Example: in two dimensions the measure has been built up to $\mathbb{R}^2$ (7-02); in three dimensions, 'a construction of the 3D YM measure and a description of its gauge-invariant

observables, even on T^3, remains open' (Chevyrev 2022)[Known] ; and in four dimensions, up to 2025, the stage reached was ultraviolet stability in finite volume obtained with a hypothesis (7-07).

The construction published in 2026 built, for any compact simple group, a theory satisfying the Wightman axioms as a limit of four-dimensional Wilson lattice gauge theory. The committee's assessment is that this is “proved in Lean”[Assessment] (6-00). The inner limit $a \to 0$ of formula (4) is taken with the periodic box fixed, along the whole sequence of cutoffs, and the outer limit $L \to \infty$ is along a subsequence obtained by compactness (8-05, 12-05). The assessment report writes that convergence and uniqueness of the infinite-volume limit are not shown[Assessment] (16-02 § 02). This chapter does not judge what this construction amounts to in the sense of footnote 2. Seen from the side of physics, half of "the theory exists" in this seam has become a mathematical statement. What remains is that the quantities in that theory (the mass gap, the string tension) be determined as mathematics, and that is a question on the side of the seam of "there is one scale" and the seam of finite and infinite.

The seam of "there is one scale"

In physics every quantity with a dimension is taken to be a constant multiple of a single scale Λ (4-07) [Physics] . Translated into mathematics, for example for the mass $m(\beta)$ and string tension $\sigma(\beta)$ in lattice units ($\beta = 2N/g^2$ is the lattice inverse coupling, not an inverse temperature), it is that the following holds.

$$0 < c \le m(\beta)^2/\sigma(\beta) \le C < \infty \qquad (\beta \ge \beta_0 \text{ uniformly}) \quad (5)$$

This is peculiar to four dimensions because $[g^2] = (\text{mass})^{4-d}$ [Known] . For $d < 4$, if the theory exists, dimensional analysis gives a single scale; at $d = 4$, $\Lambda = \mu\, e^{-1/(2b_0 g^2(\mu))}$ is born as a quantum effect from the running of the coupling [Physics] . But dimensional analysis fixes only the form. For example at $d = 3$, g^2 has the dimension of mass, so one can write $m = c_1 g^2$ and $\sigma = c_2 g^4$, but this does not forbid the constants c_1, c_2 from going to 0 or ∞ in the limit. Example: examples where one scale breaks are three-dimensional compact $U(1)$ (said to have $\sigma/m^2 \to \infty$ in the continuum limit with the mass fixed in physical units; the abstract of Göpfert–Mack writes that in the continuum limit with m_D fixed the string tension in physical units is infinite; main text not read) and the supergravity regime ($m^2/\sigma \sim 1/\lambda$) [Physics] (17-12 § 08). The record of this book's study cut out a sentence close to (5) as the diffusive-relaxation-time assumption and split it into three chains (17-12, 23-01), but it has not become a theorem.

> 04

The type of limits and uniformity

The seam of the different shores

Theorems and calculations work at three shores: perturbation theory at $g \to 0$, the strong-coupling expansion at small β, and holography at $N \to \infty$ together with $\lambda = g^2 N \to \infty$. The theory of the problem statement is on none of these shores. Converting the lattice coupling to the 't Hooft coupling gives the following.

$$\lambda = g^2 N = 2N^2/\beta \quad (6)$$

On the strong-coupling side, the infinite-volume limit and exponential clustering (the gap) at sufficiently small β go back to Osterwalder–Seiler 1978 (range not made explicit; according to the introduction of Shen–Zhu–Zhu) Known . The condition of the theorem with an explicit range (Shen–Zhu–Zhu; arXiv 2022, CMP 2023) is $\beta < N^2/(16(d-1))$ in Wilson's normalisation (16-01) Known . Converted by (6), it is $\lambda > 32(d-1)$, or $\lambda > 96$ in four dimensions, independent of N Computed . The continuum limit is in the direction $\lambda \to 0$, and the lattice numbers are at $\beta = 2.25$–2.60 for $SU(2)$, which is $\lambda = 3.6$–3.1 Computed (a rough guide, since the conventions differ). Example: both strong shores are places where the cutoff and the physical scale are not separated (on the strongly coupled lattice the correlation length is shorter than the lattice spacing, and in supergravity the KK tower stays at the same height as the glueballs; 17-10).

The seam of finite and infinite

In a finite box at finite β the entries of the transfer matrix are positive, so the gap is always positive (17-13) Known . What is asked is that it does not collapse as the box is enlarged.

$$\text{for each } L\,,\ \Delta_L > 0 \qquad \text{and} \qquad \inf_L \Delta_L > 0 \qquad \text{are different claims} \quad (7)$$

On the latter, the problem statement writes 'no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume' Known , and the form on the lattice is Chatterjee's 2018 lattice correlation-length problem (16-08 § 08). Physics measures in finite boxes, checks volume effects with rules of thumb such as $L\sqrt{\sigma} \geq 3$ (L is the physical length of the side of the box), and extrapolates in the lattice spacing with a^2 (17-13 § 05) Physics . Example: for a strip of the two-dimensional Ising model, the gap closes as $1/L$ at the critical point, but it is positive for every L (2-09). $\inf_L$ cannot be read from numbers in finite boxes.

The seam of extra dimensions

In holography the gap comes from the geometry of the extra dimensions (the cigar where a circle shrinks) (17-09) Physics . To return to four dimensions one sends the size of the extra dimensions $R = 1/M_{\mathrm{KK}}$ to 0 with Λ fixed, but from the one-loop estimate

$$\Lambda/M_{\mathrm{KK}} \approx \exp\!\left(-24\pi^2/(11\lambda)\right) \qquad (\lambda \text{ is the value at } M_{\mathrm{KK}}\ ) \quad (8)$$

$R \to 0$ is the same direction as $\lambda \to 0$ (17-10) Physics . Supergravity can be used for $\lambda \gg 1$, and Csáki et al. write 'This is the opposite of the limit that is required for the supergravity description to be valid' Physics . Example: at $\lambda \approx 16.6$ converted from the Sakai–Sugimoto fit (a fitted value from a model with quarks, not a value for pure Yang–Mills; $N = 3$; can be off by a factor of about 2 by convention), (8) is 0.27, and the scales are not separated Computed . Within the range we searched, we found no justification for exchanging the limits.

05

Measuring between the shores — the plaquette and the mass

How far the shores are depends on the quantity. For the four-dimensional $SU(2)$ Wilson action $S = \sum_P \left(1 - \frac{1}{2}\operatorname{tr} U_P\right)$ with weight $e^{-\beta S}$ ($\beta = 4/g^2$; 5-01), we compare the leading terms of the two shores with the numbers, for a local quantity (the plaquette) and a long-distance quantity (the mass). Both leading terms are the standard lowest-order expansions (the strong-coupling expansion is treated in 5-02).

$$\left\langle \tfrac{1}{2}\operatorname{tr} U_P \right\rangle \approx u(\beta) = \frac{I_2(\beta)}{I_1(\beta)} \ \ (\beta \text{ small}), \qquad \left\langle \tfrac{1}{2}\operatorname{tr} U_P \right\rangle \approx 1 - \frac{3}{4\beta} \ \ (\beta \text{ large}) \quad (9)$$

What is inside the two leading terms

I_n is the modified Bessel function, defined by the following integral.

$$I_n(\beta) = \frac{1}{\pi}\int_0^{\pi} e^{\beta\cos\theta}\cos n\theta \, d\theta$$

The leading strong-coupling term u is the average of $\frac{1}{2}\operatorname{tr} U$ with weight $e^{\beta\cdot\frac{1}{2}\operatorname{tr} U}$, treating one plaquette as a $SU(2)$ variable unrelated to its surroundings. Under the Haar measure on the group (5-01 § 06) the density of $\frac{1}{2}\operatorname{tr} U = \cos\theta$ is proportional to $\sin^2\theta$ ($0 \le \theta \le \pi$), and this average is exactly I_2/I_1 Known (agreeing to 10 digits by numerical integration Computed). When β is small the ties with the surrounding plaquettes are weak, and this single-plaquette average is the leading term Known .

The leading weak-coupling term is a count by **equipartition**. Equipartition means that under the weight $e^{-\beta S}$, each degree of freedom entering S quadratically contributes $\frac{1}{2}$ to $\langle \beta S \rangle$ (from $\int x^2 e^{-\beta x^2}\,dx / \int e^{-\beta x^2}\,dx = 1/(2\beta)$). When β is large the plaquettes are pressed close to 1, and S can be approximated by a quadratic form in small fluctuations of the links. Per lattice point there are 4 links, and $SU(2)$ has 3 directions, making 12. Of these, the 3 gauge-transformation directions do not change the action and are not counted, leaving 9. There are 6 plaquettes per lattice point, so per plaquette $\langle 1 - \frac{1}{2}\operatorname{tr} UP \rangle \approx \frac{9/2}{6\beta} = \frac{3}{4\beta}$ Known .

The mass is the rate at which correlations decrease. At strong coupling, joining two distant plaquettes needs the space between them filled with plaquettes in the shape of a tube, and each lattice step the tube grows picks up u^4 for the 4 plaquettes of its side. Correlations decrease with distance t like $u^{4t} = e^{-(-4\ln u)t}$, so $am \approx -4 \ln u$ Known . On the weak-coupling side, expanding am as a power series in g^2 gives coefficient 0 at every order (because of the form e^{-1/g^2}), and only the form comes from asymptotic freedom Physics .

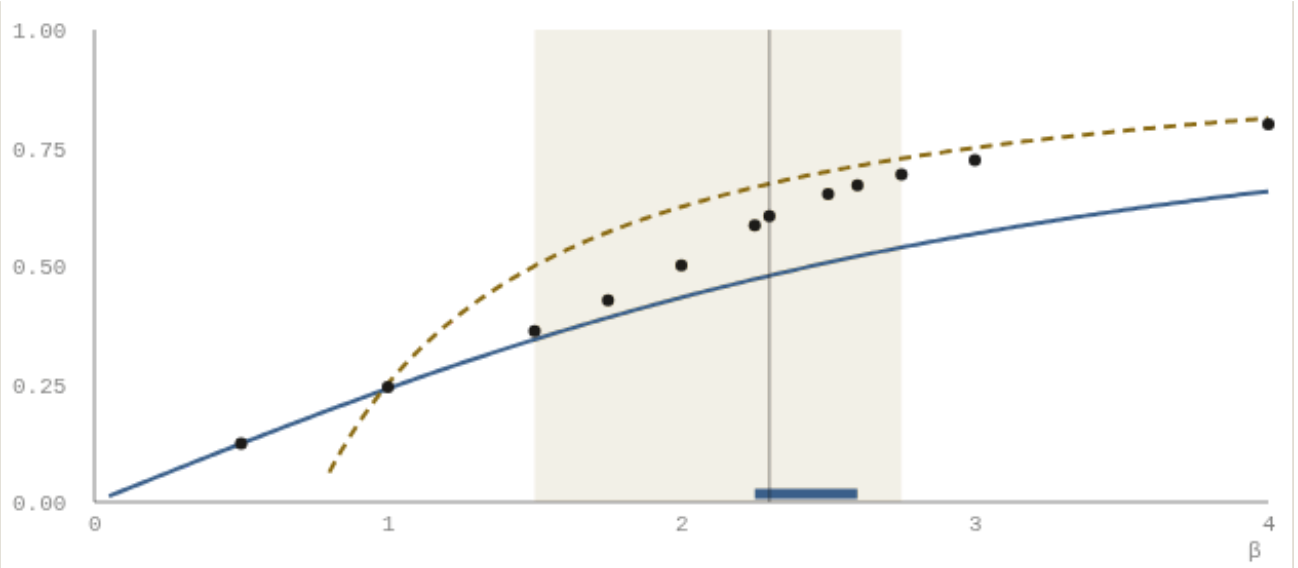
$$am \approx C\,(b_0 g^2)^{-51/121}\, e^{-3\pi^2\beta/11}, \qquad b_0 = \frac{22}{48\pi^2}$$ (10)

This is two-loop asymptotic scaling (4-06). The constant C is not fixed by asymptotic freedom, and it is the very information of the gap.

Setting them beside the numbers

The numbers are, for the plaquette, the heat-bath method on a 6^4 lattice on this machine Computed , and for the mass, Table 1 of Lucini–Teper 2001 Physics . The heat-bath method is a Monte Carlo method that repeats updates redrawing one link at a time directly from its conditional distribution with the surrounding links fixed (5-07).

Figure 1 Choosing the quantity with the buttons and moving β with the knob changes the readings at the vertical line. The solid line is the leading strong-coupling term, the dashed line the leading weak-coupling term (for the mass, (10) fitted at β = 2.6), and the points are the numbers. Only on the plaquette plot are drawn the shaded band (the interval containing the measured points β = 1.75–2.6 where neither leading term is within 5%; at its ends, 1.5 and 2.75, one of them is within 5%) and the thick line at the bottom (β = 2.25–2.60, where there are numbers for the mass)



β = 2.30, leading strong-coupling term 0.4793, leading weak-coupling term 0.6739, numerical 0.6051 (deviation −20.8%, +11.4%)

Computed The plaquette points (Table 2) and the curves ((9) and (10) computed on the spot) were computed on this machine. Physics The mass points are Table 1 of Lucini–Teper 2001.

Computed Table 2 $\langle \frac{1}{2} \operatorname{tr} UP \rangle$: heat-bath method on 6^4 (in parentheses a rough guide to the statistical error; extracted from the 15 points measured) and the two leading terms. Deviations are percentages of the numerical value.

β	Numerical	Strong coupling u	Deviation	Weak coupling	Deviation
0.50	0.1234(2)	0.1237	+0.3%	—	—
1.00	0.2438(3)	0.2402	−1.5%	0.2500	+2.5%
1.50	0.3616(4)	0.3441	−4.8%	0.5000	+38.3%
1.75	0.4271(4)	0.3906	−8.6%	0.5714	+33.8%
2.00	0.5010(7)	0.4331	−13.6%	0.6250	+24.7%
2.25	0.5856(7)	0.4720	−19.4%	0.6667	+13.8%
2.30	0.6051(9)	0.4793	−20.8%	0.6739	+11.4%
2.50	0.6526(5)	0.5072	−22.3%	0.7000	+7.3%
2.60	0.6705(4)	0.5203	−22.4%	0.7115	+6.1%
2.75	0.6935(4)	0.5391	−22.3%	0.7273	+4.9%
3.00	0.7235(2)	0.5679	−21.5%	0.7500	+3.7%
4.00	0.7997(1)	0.6580	−17.7%	0.8125	+1.6%

Table 3 Mass am (lattice units). Physics The numerical am column is Table 1 of Lucini–Teper (the master copy when this book cites the six SU(2) points; $a\sqrt{\sigma}$ and lattice sizes

are in 5-09 Table 1). Computed The columns $\xi = 1/(am)$, strong coupling and (10) are calculations on this machine, with (10) fitted to the point $\beta = 2.6$ ($C = 185$).

β	Numerical am	ξ	Strong coupling $-4\ln u$	Strong coupling / numerical	(10)
2.25	1.390(30)	0.72	3.004	2.16	1.241
2.30	1.090(33)	0.92	2.942	2.70	1.095
2.40	0.953(19)	1.05	2.825	2.96	0.851
2.475	0.754(10)	1.33	2.742	3.64	0.705
2.55	0.586(10)	1.71	2.663	4.55	0.583
2.60	0.514(8)	1.95	2.613	5.08	0.514

Example: for the plaquette, the leading strong-coupling term is within 5% of the numbers for $\beta \le 1.5$, and the leading weak-coupling term for $\beta \ge 2.75$ (the agreement at $\beta = 1$ is only where the curves cross); at the points with $1.75 \le \beta \le 2.6$ neither is within 5% Computed . The $\beta = 2.25$–2.60 used to extrapolate the continuum value of the mass is inside this band. For the mass, the leading strong-coupling term is 2.2–5.1 times the numbers, and (10) reproduces the other five points to within 11%, but C is a fitted number Computed . For a local quantity the numbers fill the space between the two leading terms; for a long-distance quantity the weak-coupling shore has no value (7-11 § 03), and the strong-coupling shore does not reach.

The physical continuum value 3.844(61) is the ratio $m/\sqrt{\sigma}$ of the six points in this band ($\xi \le 2$) extended to $a^2\sigma \to 0$ (Lucini–Teper Table 5; for how it is extended, see 17-13 § 05) Physics . This is an extrapolation to the continuum limit ($a \to 0$), and volume effects are checked separately with rules of thumb for finite boxes (the seam of finite and infinite).

06

The type in which the same word points to different things

The seam of spacetime and space

The intuition of physics lives on spacetime (Minkowski), where particles fly and there is causality. The construction of mathematics is a probability measure on Euclidean $\mathbb{R}^4$, with neither light cones nor causality (0-04). The way back is reflection positivity (4-10). With Θ the reflection of time, F, G functions on the positive-time side, and τ_t translation in time,

$$\langle F, G\rangle_{\mathcal{H}} := \langle \Theta F\, G\rangle, \qquad \langle F, F\rangle_{\mathcal{H}} \ge 0, \qquad \langle F, e^{-tH} G\rangle_{\mathcal{H}} = \langle \Theta F\, \tau_t G\rangle \quad (11)$$

the inequality in the middle is reflection positivity, and only with it can $\mathcal{H}$ and $H \geq 0$ be built (the Osterwalder–Schrader reconstruction) Known . "Time", "energy" and "particle" are words that acquire meaning through (11). That the Wilson action preserves reflection positivity is due to Osterwalder–Seiler 1978 Known . On the other hand, the lattice itself breaks rotational symmetry. The problem statement writes that when reflection positivity is lost through regularisation, there are few ways to recover it (quotation in 0-03 § 03) Known . Example: for the Pauli–Villars two-point function an F with $\langle F, F \rangle_{\mathcal{H}} < 0$ can be built (7-11 § 06).

The seam of the word "particle"

Physics counts glueball masses Physics . What the problem statement asks for is a spectral gap, and the existence of an isolated one-particle state at the bottom is set aside among the 'natural extensions' (0-03 § 09) Known .

$$\text{problem statement}: \ \operatorname{spec}(H) \cap (0, \Delta) = \varnothing, \qquad \text{physics}: \ m_{0^{++}} \text{ is an isolated eigenvalue of}$$

($\mathbf{P}$ is the momentum.) Example: the way correlations measured on the lattice decrease tells the position of the bottom, but not whether it is an isolated point or the edge of a continuous spectrum (16-08 § 05). The unit GeV does not come out of pure Yang–Mills either, and Athenodorou–Teper note 'Strictly speaking this is of course not possible: the pure gauge theory and QCD … are two different theories' Physics .

07

Overlaying the two lists — obstacles, moves, partial bridges

7-11 listed the obstacles on the side of mathematics, and 17-13 the moves of physics. The seams pair their rows. The partial bridges that already exist all have the following form.

$$A \implies (M \text{ , in part}), \qquad A \text{ in four dimensions, at weak coupling, in the continuum, in infin}$$

Example: in Shen–Zhu–Zhu, A is $\lambda > 96$ (an explicit range; below $N^2/48$ in Wilson's β, with the conversion in Table 4 of 5-01), the conclusion is a gap in infinite volume, and A does not hold in the direction of the continuum limit $\lambda \to 0$.

Known Table 4 The seams with the obstacles of 7-11, the moves of 17-13 and partial bridges (the correspondence is this book's arrangement)

Seam	Obstacle	Move	Partial bridge
1 Existence	3, 6	Putting that it exists	The measure on $\mathbb{R}^2$ (7-02). In four dimensions, the published construction (the continuum limit in a periodic box, infinite volume along a subsequence, the Wightman axioms; the committee's assessment Assessment , Part 6). There is no uniqueness of the infinite-volume limit
2 One scale	2	Regarding as one (scale)	Not found within the range we searched (17-12 is at the stage of cutting it out as an assumption)
3 Shores	1	Expansion, neighbouring theories	The infinite-volume limit, area law and exponential clustering at strong coupling (Osterwalder–Seiler 1978; range not made explicit). A gap and a log-Sobolev inequality in the explicit range $\lambda > 96$ (Shen–Zhu–Zhu)
4 Finite and infinite	1, 6	Extrapolation, the string picture	Exponential decay at weak coupling for finite groups (Adhikari–Cao). The lattice correlation-length problem for continuous non-abelian groups remains open. The published construction passes to infinite volume along a subsequence, but there is no gap uniform in the volume (16-02)
5 Spacetime and space	5	All the premises	On finite lattices, reflection positivity of the Wilson action and a positive transfer matrix (preserved in weak limits too). The recovery of rotational symmetry remains
6 Particles	—	Regarding as one (seeing the gap as the mass of a particle; 17-13 § 09)	Within the range we searched, not found in four dimensions
7 Extra dimensions	1	Neighbouring theories	The eigenvalue problem after fixing the background (17-09). No exchange of limits found
Outside the table: gauge	4	Choosing a gauge	A wall common to both sides. A conditional theorem stating confinement in terms of gauge-invariant quantities (Wilson loops) without choosing a gauge (Chatterjee 2021: if centre symmetry is unbroken there is confinement; 17-13 § 08, 16-08 § 06)

Apart from the bridge over the seam of existence, the partial bridges all lie on the side of the shores (strong coupling, low dimensions, finite lattices), and within the range we searched none extends towards the problem statement. The bridge over the seam of existence (the published construction), too, carries every step of renormalisation at weak coupling and stops at the step with lattice spacing 1. It does not enter the behaviour at long distances (a uniform gap, the area law) (16-03). Gauge is not put among the seven because local gauge fixing is a wall common to both mathematics (7-05) and physics. Chatterjee's 2021 theorem is one road that goes on in front of this wall without choosing a gauge. Topological sectors and θ (7-11) could also be counted as a

seam, in that the problem statement does not touch θ. What would need to be known to move which rows is lined up in 16-07, and the list of questions remaining after the published construction in 16-02.

What this chapter can say

	What is said	What is not said
Known	For each seam, the sentence of physics, the mathematical statement, and the current state and partial bridges as stated by the primary sources	That seven is exhaustive (the counting, the types and the correspondences of Table 4 are this book's arrangement)
Computed	The plaquette on 6^4, and the β where the two leading terms are within 5%. The ratio of the leading strong-coupling term to the masses of the primary source. That u equals the single-plaquette average I_2/I_1 (numerical integration). The conversions (6) and (8)	Infinite-volume values (finite-volume effects not estimated). The 5% line is this chapter's choice
Physics	The estimates (8) and (10), and the reading of the examples where one scale breaks	That asymptotic scaling and the exchange of limits are correct
Overall	—	This site fills none of the seams (the bridge over the seam of existence belongs to the published construction). It does not show the existence of the mass gap

Sources and reproduction

Item	Kind	Source / tools
Quotations from the problem statement, the lattice correlation-length problem, three dimensions	Text	Jaffe–Witten, Quantum Yang–Mills Theory, Clay Math. Inst. (2000) / Chatterjee, arXiv:1803.01950 (2018) / Chevyrev, arXiv:2202.13359 (2022)
The published construction in § 03	Assessment	The assessment report `yang-mills-group/YM`, `docs/assessment-1.pdf` (2026-11-10), opening, § 5 and § 8. The code is `yang-mills-group/OpenAI-YM-20261028` (submitted 2026-10-28)
Lattice numbers, the quotation in § 06	Text	Lucini–Teper, JHEP 0106 (2001) 050, hep-lat/0103027 (Tables 1 and 5) / Athenodorou–Teper, arXiv:2007.06422 (2020)

Item	Kind	Source / tools
Theorems on the strong-coupling side	Abstract, part of the text	Osterwalder–Seiler, Ann. Phys. 110 (1978) 440–471 (abstract; that the gap (exponential clustering) follows from convergence of the cluster expansion is according to the introduction of Shen–Zhu–Zhu: 'as explained in [OS78], the existence of a mass gap (exponential clustering) follows from convergence of the cluster expansion') / Adhikari–Cao, arXiv:2202.10375 (abstract) / Shen–Zhu–Zhu, arXiv:2204.12737, CMP 400 (2023) 805–851 (abstract, introduction and equation (1.2) (normalisation of the action) checked; the conversion $\beta = N^2\beta_{\mathrm{SZZ}}$ to Wilson's β is arithmetic on this machine)
The quotation in § 04, non-separation of KK, $m^2/\sigma \sim 1/\lambda$, $\lambda \approx 16.6$	Text	Csáki–Ooguri–Oz–Terning, hep-th/9806021 / Klebanov–Strassler, hep-th/0007191 / KMMW, hep-th/0311270 (equation (2.7)) / Sakai–Sugimoto, hep-th/0412141, hep-th/0507073 (the conversion to λ and substitution into (8) are arithmetic on this machine)
Centre symmetry, three-dimensional compact $U(1)$	Text, abstract	Chatterjee, arXiv:2006.16229 (text) / Göpfert–Mack, CMP 82 (1982) 545–606 (abstract (INSPIRE): Villain action, positive string tension at all couplings, the continuum limit with m_D fixed is a free scalar field; main text not read; the year is Crossref's year of publication, INSPIRE gives 1981)
(9), $-4\ln u$, the coefficients of (10), (11)	Known, bibliographic data only	(9) and $-4\ln u$ are the standard lowest-order strong-coupling expansion and equipartition (the strong-coupling expansion is treated in 5-02). Review: Drouffe–Zuber, Phys. Rep. 102 (1983) 1–119 (bibliographic data only) / the two-loop coefficients (4-06) / the reconstruction theorem (4-10)
Tables 2 and 3, Figure 1, (6), (8)	Computed	Python on this machine: plaquette_mc.py (numpy; 6^4, Kennedy–Pendleton heat-bath method, 100 sweeps of thermalisation and 300 of measurement, fixed seed, bin averages), shores.py (scipy), one_plaquette.py (scipy; numerical integration of $u = I_2/I_1$). The only input for the masses is Table 1 of Lucini–Teper

Next chapter to read: 16-06 Candidates for new concepts, which looks from the side of concepts at what seems to be needed to cross the seams. The rearrangement as tasks is in 16-07.

Candidates for new concepts — what plays the part of "Ricci flow"

> This chapter is a map of open questions. § 02, § 03 and § 04 are established knowledge.

What this chapter shows — The three components of Ricci flow (flow, monotone quantity, surgery). That in the two-dimensional σ model the renormalisation group at one loop is Ricci flow itself, and the scale at which the sphere shrinks away is Λ. Where the candidates for each component stop in four dimensions.

Prerequisite chapters — 7-09. We also use 7-05, 4-07 and 7-11. Geometric tools such as Ricci curvature and scalar curvature are defined on the spot with the example of the sphere.

Said first — This chapter lines up candidates; it cannot say which one will work. "What plays the part of Ricci flow is the renormalisation group" and "what plays the part of $R = T$ is a duality transformation" are analogies (only the one loop of the σ model is not an analogy but an agreement). This site shows neither the mass gap nor the construction of the continuum limit. The label Assessment means "according to the committee's assessment report" (“Labels” in 6-00).

The order of this chapter

1. Where the setters asked for "new ideas"
2. Ricci flow: moving a metric by its curvature
3. Perelman's components: flow, monotone quantity, surgery
4. Where the renormalisation group becomes Ricci flow: the two-dimensional σ model — the scale at which it shrinks away is Λ. Figure 1, Table 1. Detour: a flat circle does not shrink
5. What flows in four dimensions — the flow of the coupling and the gradient flow of the field
6. Monotone quantities: c in two dimensions and a in four — monotonicity comes from reflection positivity. Figure 2, Table 2
7. What plays the part of surgery

8. Another type: identification
9. The candidates on one sheet

01

Where the setters asked for "new ideas"

The Clay problem statement (Jaffe–Witten 2000), § 6, takes the construction on T^4 to be within reach as an extension of known methods, and then writes: "no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume. Nor do present methods suggest how to obtain the existence of the infinite volume limit $T^4 \to \mathbb{R}^4$". § 6.5 reads "New ideas are needed". The same § 6, however, also treats the construction on T^4 as unfinished. What can be said is only "the side where the setters asked for new ideas is the infrared" (7-10 § 01). Known

Later, the construction published in 2026 gave the continuum limit in a periodic box and an infinite-volume theory along a subsequence (satisfying the Wightman axioms). The committee's assessment is that this is “proved in Lean” Assessment (Part 6). A mass gap uniform in the volume and the uniqueness of the infinite-volume limit remain among the open questions listed in the assessment report (16-02). The infrared side, where the setters asked for new ideas, is still open.

We write "uniform in the volume" as a formula. For the gap Δ_L of a box of side L,

$$\Delta_L > 0 \ \ (\text{for every } L) \qquad \text{and} \qquad \inf_L \Delta_L > 0 \ \ (1)$$

are different claims. Example: for a free field of mass m the gap is at least m for every L, and the right-hand one also holds. If $\Delta_L = 1/L$ held, only the left-hand one would hold.

Among the new concepts that worked on other great problems, two types can be seen: a flow with a monotone quantity (Perelman, the Poincaré conjecture), and the identification of two objects (Wiles, Fermat's Last Theorem) (this book's arrangement).

02

Ricci flow: moving a metric by its curvature

A Riemannian metric g is an inner product on the tangent space at each point (1-05), with components written g_{ij}. The Ricci curvature Ric (components R_{ij}) is, for a direction v of length 1, the sum of the bending (sectional curvatures) of $n-1$ mutually

orthogonal planes containing v (n is the dimension of the manifold). Equivalently, it is $n-1$ times the average. Precisely, it is a contraction of the curvature tensor. We use only the following example.

$$\text{sphere of radius } r \text{ in } n \text{ dimensions}: \qquad \mathrm{Ric} = \frac{n-1}{r^2}\, g \quad (2)$$

On a sphere of radius r, the bending of every plane is $1/r^2$. Adding $n-1$ of them gives the coefficient $n-1$. The trace of Ric (the sum of the diagonal components in an orthonormal basis) is called the scalar curvature R. On this sphere $R = n(n-1)/r^2$.

Ricci flow is the equation that moves a metric by its Ricci curvature, and Perelman's paper starts from this equation, introduced by Hamilton (1982). Known

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) \quad (3)$$

Positively curved directions shrink. Along the flow the scalar curvature obeys $\partial_t R = \Delta R + 2|\mathrm{Ric}|^2$. Δ is the Laplacian on the manifold (not the Δ of the mass gap), which in flat coordinates is $\sum_i \partial^2/\partial x_i^2$ (a different symbol from the gap Δ_L of § 01). $|\mathrm{Ric}|^2$ is the sum of squares of the components of Ric in an orthonormal basis.

If the right-hand side were ΔR alone it would be the heat equation, and just as heat evens out temperature, it would shave the peaks of R and fill its valleys. Precisely, the non-negative term $2|\mathrm{Ric}|^2$ is added upward everywhere. For equations of heat-equation type, a solution cannot create a new minimum in the interior (the maximum principle). So the minimum of R does not decrease along the flow. Perelman's introduction, presenting Hamilton's work, writes this equation and conclusion in this form. Known

Example: a sphere of initial radius $r(0)$ stays round and only its radius changes. From $g = r^2 \hat{g}$ ($\hat{g}$ is the metric of the unit sphere) and (2),

$$\frac{d(r^2)}{dt} = -2(n-1), \qquad r(t)^2 = r(0)^2 - 2(n-1)\,t, \qquad t_* = \frac{r(0)^2}{2(n-1)} \quad (4)$$

and at time t_* (the extinction time) it shrinks to a point. For the unit two-sphere $t_* = 1/2$. For a general shape the curvature becomes infinite somewhere. Perelman writes "The most natural way of forming a singularity in finite time is by pinching an (almost) round cylindrical neck", and continues the flow by surgery that cuts the thin neck and puts small caps on it.

03

Perelman's components: flow, monotone quantity, surgery

We split Perelman's proof into three components. The first, the flow, is (3) of § 02. The second is a quantity that increases monotonically along the flow. We define the tools first. M is a closed manifold (a compact manifold without boundary), f an auxiliary function on M, and dV the volume element determined by the metric ($\int_M dV$ is the volume of M). ∇f is the gradient of f, and $|\nabla f|^2$ the square of its length. $\nabla_i \nabla_j f$ is the second derivative of f, which in flat coordinates is $\partial_i \partial_j f$.

$$\mathcal{F}(g, f) = \int_M \left(R + |\nabla f|^2\right) e^{-f}\, dV$$

When the metric is moved by (3), f is moved at the same time by $\partial_t f = -\Delta f + |\nabla f|^2 - R$. This way of moving preserves the measure $e^{-f} dV$ up to a change of coordinates, and the total integral $\int_M e^{-f} dV$ does not change. Then

$$\frac{d\mathcal{F}}{dt} = 2 \int_M \left| R_{ij} + \nabla_i \nabla_j f \right|^2 e^{-f}\, dV \;\geq 0 \quad (5)$$

holds (Perelman § 1.2). The right-hand side is the integral of a square, so it is not negative. Known

The flow behaves as if it climbs the slope of $\mathcal{F}$. Precisely, the flow $\partial_t g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f)$ with the measure $e^{-f} dV$ held fixed is a gradient flow in the direction that increases $\mathcal{F}$, and a change of coordinates takes it to the pair of (3) and the equation for f above (Perelman § 1.1–1.2). On the choice of the measure held fixed, Perelman writes "the choice of m is analogous to the choice of gauge". The m of the original is this measure, not the mass of § 06. Whatever the choice, the flow is the same up to a change of coordinates. Known

Further, seen up to changes of coordinates and of scale, the flow has no non-trivial periodic orbits (Perelman § 2, § 3). The trivial ones are called solitons (solutions that keep their shape under changes of coordinates and scale alone), and in this view they correspond to fixed points. Known Example: on a flat torus $R_{ij} = 0$, and taking f constant, the right-hand side of (5) is 0 and the flow stands still (a fixed point).

The third component is to classify in advance the shapes of the regions of large curvature, and to remove by surgery only what has been classified. What could play the same roles for Yang–Mills: for the flow, the renormalisation group and gradient flow (§ 05); for the monotone quantity, c and a (§ 06); for classification and surgery, large fields, instantons and topological sectors (§ 07). As another type there is identification (§ 08). Every one of these correspondences is an analogy.

04

Where the renormalisation group becomes Ricci flow: the two-dimensional σ model

Within one-loop perturbation theory, the place where the correspondence stops being an analogy is the two-dimensional non-linear σ model. The field ϕ is a map from the plane to a manifold M. In this section we write the metric of M as h (components h_{ij}).

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^2} h_{ij}(\phi)\,\partial_\mu\phi^i\,\partial_\mu\phi^j\,d^2x \quad (6)$$

The coupling constant is not a number but the metric h itself. The abstract of Friedan (1980) writes "the renormalization group acting on the infinite-dimensional space of Riemannian metrics". At one loop, for the scale $\ell = \ln(\mu_0/\mu)$ (larger means seen more coarsely),

$$\frac{dh_{ij}}{d\ell} = -\frac{1}{2\pi}\,R_{ij}(h) + \cdots \quad (7)$$

and setting $t = \ell/4\pi$ gives (3) itself. Physics "$\cdots$" stands for corrections at two loops and beyond. From two loops a term quadratic in the curvature is added (of the form $R_{iklm}\,R_j{}^{klm}$, where R_{iklm} are the components of the curvature tensor), and the flow departs from Ricci flow. The abstract of Friedan (1985) says the coefficients of the renormalisation group were computed in the two-loop approximation. Physics

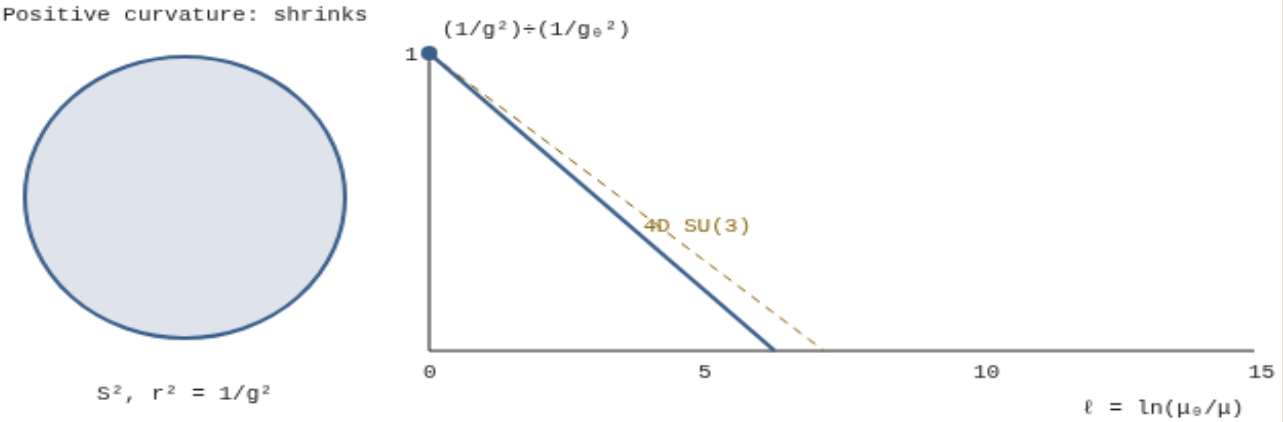
Example: taking M to be the sphere S^{N-1} of radius r gives the $O(N)$ model, with $r^2 = 1/g^2$. With $n = N-1$ in (2),

$$\frac{d}{d\ell}\Big(\frac{1}{g^2}\Big) = -\frac{N-2}{2\pi}, \qquad \ell_* = \frac{2\pi}{(N-2)\,g_0^2}, \qquad \Lambda = \mu_0\,e^{-\ell_*} \quad (8)$$

The more coarsely one looks, the more the sphere shrinks and the stronger the coupling becomes. This is the "asymptotic freedom" of the abstract of Polyakov (1975). The scale ℓ_* at which it shrinks away is 4π times the extinction time t_* of (4), and it determines the one-loop Λ (4-07). For four-dimensional $SU(N)$ too, $1/g^2$ decreases along the straight line $1/g_0^2 - 2b_0\ell$.

Perelman touches on this relation in section 2 of his introduction, writing "this connection between the Ricci flow and the RG flow suggests that Ricci flow must be gradient-like; the present work confirms this expectation". That the flow is gradient-like (moves along the slope of some quantity) was expected from the intuition of the renormalisation group, and (5) confirmed it: that is how Perelman puts it. Known

Figure 1 Change N and g_0^2 with the knobs, and advance ℓ with "Play". Left: a cross-section of the sphere whose radius squared is $1/g^2$. Right: the ratio of $1/g^2$. The dotted line is four-dimensional $SU(3)$



N = 3, g_0^2 = 1.0, ℓ = 0.00, 1/g² = 1.000, ℓ* = 6.28, Λ/μ₀ = 1.87×10⁻³, SU(3): ℓ* = 7.18

Computed The straight lines of the figure are formula (8) and $b_0 = 11N/(48\pi^2)$ drawn on the spot. Table 1 was computed on this machine.

Model	g_0^2	ℓ_*	$\Lambda/\mu 0$	Note
$O(2)$	Any	∞	–	The circle is flat and does not flow
$O(3)$	1	6.283	1.87×10⁻³	Ricci flow $t_* = 0.5$
Four-dimensional $SU(2)$	1 / 2	10.77 / 5.38	2.11×10⁻⁵ / 4.59×10⁻³	$b_0 = 0.04644$
Four-dimensional $SU(3)$	1 / 2	7.178 / 3.589	7.63×10⁻⁴ / 2.76×10⁻²	$b_0 = 0.06966$

Table 1 The scale at which the one-loop $1/g^2$ reaches 0. $\ell_* = 4\pi t_*$ was checked against formula (4). Computed

Before the straight line reaches 0 the coupling becomes large and the one-loop formula can no longer be used. As near a singularity of Ricci flow, what matters is what happens after the coupling becomes $O(1)$. The gap of the two-dimensional $O(N)$ model ($N \geq 3$) is a conclusion at the level of physics. Physics Douglas's report (2004) goes no further than citing an example in which a gap was shown for a regularised version of the $O(N)$ model (Kopper 1999). Its later status as a theorem has not been checked in this chapter.

Detour: a flat circle does not shrink

A subsection you may skip.

For $N = 2$ the target is a circle with curvature 0, and the coefficient $N - 2$ vanishes. The two-dimensional $O(2)$ model has a massless phase at low temperature. At high temperature correlations decay exponentially with distance (there is a mass); at low temperature they decay only as a power of the distance (massless). The transition between these two phases is called

the Kosterlitz–Thouless (KT) transition. Fröhlich–Spencer (1981) proved that this transition exists, for the two-dimensional lattice rotator model ($O(2)$), the Coulomb lattice gas and others. Known Set beside four-dimensional $U(1)$ (5-06), the picture "it shrinks because it is curved (non-abelian)" comes to mind, but it is an analogy.

05

What flows in four dimensions: the flow of the coupling and the gradient flow of the field

In four dimensions there are two different "flows". The **flow on the space of theories** (the renormalisation group, 7-05) has no picture of a sphere, but the one-loop $1/g^2$ is a straight line of the same form (Table 1). Bałaban's ultraviolet stability on T^4 comes with a hypothesis, and the limit is only along subsequences (7-07). The construction published in 2026 stacked one step of a coupling-dependent probability kernel into many steps and carried it as far as the continuum limit in a periodic box Assessment (Part 9 to Part 11). It carries every step within the weak-coupling band, however, and stops at the step with lattice spacing 1 (where the coupling has a large reference value) (16-03). Where the method of Bauerschmidt–Brydges–Slade reached $d = 4$ is the side where the coupling flows to 0 in the infrared (the $|\phi|^4$ model and the weakly self-avoiding walk), the exact opposite of the infrared of Yang–Mills. What is missing is "a controlled step after the coupling has reached $O(1)$" (this book's arrangement).

The **flow on the space of fields** (gradient flow) smooths a single configuration along the slope of the action. For an abelian field, following the gradient of the action $S = \frac{1}{4}\int F_{\mu\nu}F_{\mu\nu}\,d^4x$ backwards gives

$$\partial_t A_\mu = -\frac{\delta S}{\delta A_\mu} = \Delta A_\mu - \partial_\mu(\partial_\nu A_\nu), \qquad \frac{dS}{dt} = -\int \left|\frac{\delta S}{\delta A}\right|^2 \le 0 \quad (9)$$

(for a non-abelian field, replace ∂ by the covariant derivative). Since it is a gradient flow, the action always decreases. Example: after a Fourier transform, the component perpendicular to the wave number k decays as $e^{-k^2 t}$, and fluctuations with $|k| \gg 1/\sqrt{t}$ disappear. The component parallel to k (the gauge direction) does not move. In Ricci flow the same thing happens through invariance under changes of coordinates, and DeTurck (1983) fixed it by adding a term. The same device appears in the stochastic quantisation of Yang–Mills (7-09 § 07).

The abstract of Lüscher (2010) says that theory and numerics "suggest" that the flowed field is "a smooth renormalized field", and Carosso–Hasenfratz–Neil (2018) used it numerically as a continuous real-space renormalisation group. Waldron (2019) showed that, for smooth initial data on a closed four-dimensional manifold, Yang–Mills flow

has no finite-time singularity. Known The relation between the two flows is at the level of suggestion and numerics.

06

Monotone quantities: c in two dimensions and a in four

We prepare two tools. The stress tensor (energy–momentum tensor) $T_{\mu\nu}$ is the field that describes the flow of energy and momentum. Its trace $\Theta = T^{\mu}{}_{\mu}$ (in two dimensions $T11 + T22$) measures how much the action changes when the scale is changed slightly. For a theory that does not change under a change of scale (a fixed point), $\Theta = 0$. The central charge c is a number carried by the theory at a fixed point, and roughly counts the massless degrees of freedom (1 for one free scalar field). In this chapter we use only the name, and see its value through the sum rule (10) below.

In two dimensions there is the c-theorem of Zamolodchikov (1986), which constructs a function that decreases along the flow and equals the central charge at fixed points. Physics As a form we can compute, we look at the sum rule for correlations of Θ, cut off at distance r_0.

$$c(r_0) = 3\pi \int_{|x|>r_0} |x|^2 \, \langle \Theta(x)\Theta(0)\rangle \, d^2x \qquad (10)$$

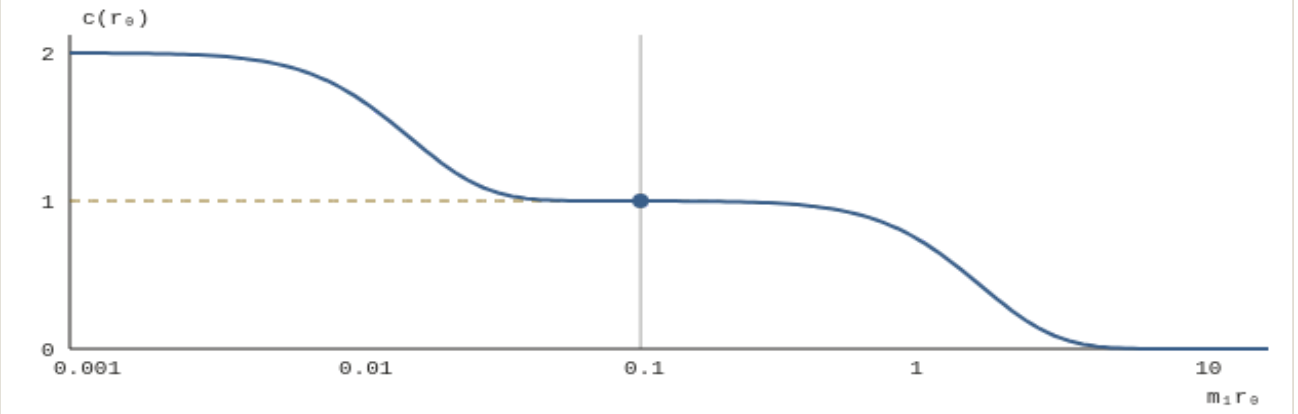
Here Θ is the trace of the stress tensor made directly from the action (in the normalisation $\Theta = 2\pi T^{\mu}{}_{\mu}$, the prefactor becomes $3/4\pi$). The content of the sum rule is that the value at $r_0 \to 0$ is the difference between the ultraviolet and infrared central charges. This sum rule is attributed to Cardy (1988). Physics

For a free scalar field of mass m, $\Theta = -m^2\phi^2$ and $\langle\Theta(x)\Theta(0)\rangle = 2m^4 G(x)^2$. $G(x) = K_0(m|x|)/2\pi$ is the propagator (1-11), and K_0 is a modified Bessel function (decaying like $e^{-m|x|}$ far away). Therefore

$$c(r_0) = 3\int_{mr_0}^{\infty} u^3 K_0(u)^2 \, du, \qquad c(0) = 1, \quad c(\infty) = 0 \qquad (11)$$

We checked $c(0) = 1$ by numerical integration as $1 - 8\times10^{-12}$. Computed Since the integrand is never negative, $c(r_0)$ decreases. In general, $\langle\Theta(x)\Theta(0)\rangle \geq 0$ for $x \neq 0$ follows from translation and rotation invariance and reflection positivity (4-10). A property that is easily lost under regularisation (7-11 the obstacle of positive regularisation) is where the monotonicity comes from. Known

Figure 2 Change the mass ratio m_2/m_1 and the scale r_0 with the knobs, and stretch r_0 with "Play". The solid line is $c(r_0)$ for two free fields, the dotted line for one



m₂/m₁ = 100 m₁r₀ = 0.10 c(r₀) = 0.999 (one field: 0.999)

Computed The curve is an interpolation of a table of 85 points from numerical integration of formula (11) on this machine (embedded with the trailing 0s omitted).

$mr0$	0.1	0.5	1	2	3	5
$c(r0)$	0.99946	0.93639	0.70823	0.25841	0.06861	0.00313

Table 2 $c(r_0)$ for one field. For two fields with mass ratio 100, at $m_1 r_0 = 10^{-3}, 10^{-2}, 0.1, 1$ it is 1.999, 1.708, 0.999, 0.708, staying on the step at 1 in between. Computed

The step shows that the flow passes near an intermediate fixed point (the scales at which only the light field looks massless); in an infrared with a gap, $c = 0$.

When a four-dimensional fixed point is placed on a curved background, the Euler density (a quadratic expression in the curvature whose integral is proportional to the Euler number of the manifold) appears with a coefficient in the expectation value of Θ. That coefficient is the Euler anomaly a (not the lattice spacing a; in this chapter only the name). The abstract of Komargodski–Schwimmer (2011) derives a monotonically decreasing function joining the ultraviolet and infrared a, "establishing the a-theorem". Physics The a of free fields is positive, and the contributions of a real scalar, a Weyl fermion and a gauge field are in the ratio $1 : \frac{11}{2} : 62$ (appendix A of the same paper). Known

The ultraviolet of pure Yang–Mills approaches free gluons, so $a > 0$. If there is a gap in the infrared, no massless degrees of freedom remain, so $a = 0$. Physics The two do not contradict each other, but $a_{\mathrm{UV}} > a_{\mathrm{IR}}$ does not force a gap. That there is a gap in the infrared is itself the unproved part of this book's subject. A separate "classification of infrared endpoints", corresponding to the classification of singularities, is needed (an analogy).

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07
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What plays the part of surgery: large fields and topological sectors

In four-dimensional Yang–Mills, the fact that the action does not change under changes of scale creates the same kind of difficulty as singularities in which curvature concentrates at a point. If $A_\lambda(x) = \lambda A(\lambda x)$, then $F_\lambda(x) = \lambda^2 F(\lambda x)$, and

$$\int_{\mathbb{R}^d} |F_\lambda|^2 \, d^d x = \lambda^{4-d} \int_{\mathbb{R}^d} |F|^2 \, d^d x \quad (12)$$

For $d = 4$ this does not depend on λ. Example: an instanton of size ρ has action $8\pi^2/g^2$ whatever ρ is (3-05), and as $\rho \to 0$ the action does not decrease even when the curvature concentrates at a point.

As a ruler for the size of the curvature we use $\|F\|_{L^p} = \left(\int |F|^p \, d^d x\right)^{1/p}$ ($p = 2$ is the L^2 of 1-10, and squared it is the action). The compactness of Uhlenbeck (1982) (from a sequence of connections whose curvature is bounded in L^p, one can extract, after changes of gauge, a subsequence converging in a weak sense) works with L^p estimates for $p > d/2$, and the L^2 of $d = 4$ (the action itself) lies exactly on the borderline. On the borderline, curvature can concentrate at finitely many points and small instantons can pinch off like bubbles (bubbling; in this chapter only the name). Known

On the quantum side, what could play the part of surgery are the large-field regions treated separately in Bałaban's method (7-05 § 04), small instantons, and topological sectors (7-11 Detour). None of them has reached the stage of "classify, then remove".

08

Another type: identification with an object in which the gap is visible

The Wiles type shows that two objects are the same, and brings in from one of them a property not visible in the other ($R = T$; a summary at the level of common knowledge). Jaffe–Witten § 6.6 writes that, since the gap is not visible classically, "a nonclassical change of variables or “duality transformation”" may be needed. Known

Examples where a gap was shown by this type start from an identity of Gaussian integrals. For a Gaussian field φ with covariance V,

$$\mathbb{E}\left[e^{\,i \sum_j q_j \varphi(x_j)}\right] = \exp\left(-\tfrac{1}{2} \sum_{j,k} q_j q_k \, V(x_j - x_k)\right) \quad (13)$$

The right-hand side is the weight of particles with charges q_j interacting through the potential V; the left-hand side is an expectation value of the field. Summing over the gas of charges, the left-hand side becomes a field theory containing $\cos\varphi$. This is called the sine-Gordon model (a field theory with a $\cos\varphi$ term in its action; in this chapter

only the name). Brydges–Federbush (1980) used this transformation to prove exponential decay of correlations (Debye screening) in the Coulomb gas at low density or high temperature. Known It is an abelian model of statistical mechanics.

Fragments for Yang–Mills itself are the gauge–string duality for $SO(N)$ at strong coupling and large N of Chatterjee (2019), the representation of Wilson loops as sums over surfaces of Cao–Park–Sheffield (17-04 § 09; the picture is in § 05), and "unbroken centre symmetry ⇒ confinement" of Chatterjee (2021) (an implication, not an equivalence). At the level of physics there are Seiberg–Witten (17-11) and holography (17-09). The problem asks for fixed N, so large N alone is not enough.

09

The candidates on one sheet

The proposal to give a symbol to $1/0$ is not included, since it carries little information (7-12 § 10).

First, how to read the table. The "Component" column says which of the three components of § 03 the candidate corresponds to. They divide into flow (i–iii), identification (v, vi) and surgery (vii); (iv) corresponds to none. "How far" is the range reached as a theorem; "Where it stops" is the first place not reached.

Lining up the places where they stop, they gather roughly in two. One is that small numbers disappear in the infrared, where the coupling becomes $O(1)$. (i) has no step there (the published construction, too, stops at the step with lattice spacing 1 just before it); the continuum limit obtained in (iii) is of a design that ties the coupling to 0; and (iv) is a power series, so it does not reach e^{-1/g^2}. The other is that they work only in a different limit, $N \to \infty$ or strong coupling: (v) and (vi) are of this kind. (i) tries to go from the ultraviolet side to the infrared and stops; (vi) has a picture of the infrared but stops without being able to return to finite N. They stop facing in opposite directions (this book's arrangement).

	Candidate	What it makes new	How far	Where it stops	Component
i	Constructive renormalisation group (Bałaban, MRS, BBS, the construction published in 2026)	Treat one step as a map on the space of theories	Published construction: the continuum limit in a periodic box and infinite volume along a subsequence, the Wightman axioms Assessment . Bałaban: ultraviolet stability on T^4 (with a hypothesis)	A step after the coupling has reached $O(1)$. The published construction stops at the step with lattice spacing 1 (where the coupling is large) (16-03)	Flow
ii	Stochastic quantisation and regularity structures (Hairer, CCHS)	The measure as the invariant measure of a dynamics, the space of orbits as the state space	The measure on T^2 is the unique invariant measure of the dynamics (Chevyrev–Shen). Local solutions for three-dimensional YMH	Subcritical $d < 4$ (7-09)	Flow
iii	New ways of taking limits (Chatterjee and others)	Decide first the space in which the limit lives	The gap of the strong-coupling lattice (in lattice units). The Gaussian limit of YMH (of a design that ties the coupling to 0)	A non-Gaussian limit	Flow
iv	Algebraic formulations (Connes–Kreimer, Costello)	Write renormalisation as an algebraic structure	A formulation within perturbation theory (at the level of formal power series)	Does not reach quantities invisible to perturbation theory, such as e^{-1/g^2}	—
v	Duality, change of variables (JW § 6.6)	Identify with an object in which the gap is visible	Debye screening (abelian). Fragments of gauge–string duality	No duality transformation that works for pure Yang–Mills was found, as far as we looked	Identification

	Candidate	What it makes new	How far	Where it stops	Component
vi	Large N and holography	See the gap in the geometry of extra dimensions	Level of physics Physics	$N \to \infty$, strong coupling	Identification
vii	Classical YM in geometric analysis (Uhlenbeck, Waldron)	Analysis in the critical dimension (bubbling, § 07)	The classical theory is within reach of existing methods (JW § 2)	Quantum configurations are expected not to be smooth	Surgery

This book's study treated in Part 23 (the diffusive-relaxation-time assumption, 23-01) is closest to type (i): it cuts out the physical premise "a single scale" as an inequality on the lattice (the judgement of closeness is this book's arrangement).

What this chapter can say

	Content
Said	What the setters asked new ideas for is the uniform gap and the infinite-volume limit. Perelman's monotonicity formula (5) (the formula when f is moved at the same time), and the statement that the gradient-like nature of the flow had been expected from the intuition of the renormalisation group (both from the original texts) Known
Said	The one-loop flow (8) of the $O(N)$ model agrees with the Ricci flow (4) of the sphere under $t = \ell/4\pi$, and the scale at which it shrinks away is the one-loop Λ (Table 1). From two loops it departs from Ricci flow Computed Physics
Said	For a free scalar field of mass m, $c(r_0)$ decreases monotonically from 1 to 0, and the monotonicity comes from reflection positivity Computed Known
Not said	Which candidate will work (the correspondences are analogies). The status as a theorem, after 2004, of the gap of the two-dimensional $O(N)$ model. Whether the infrared of pure Yang–Mills has a gap ($a = 0$ is under that condition). Whether a monotone quantity exists in four dimensions as a theorem. Exhaustive coverage from 2025 on

Sources and reproduction

Item	Kind	Source / tools
Quotations in § 01 and § 08	Text checked (verbatim comparison by the checked survey)	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Math. Inst. (2000)

Item	Kind	Source / tools
Formulas (3) and (5), the evolution equation of f, the evolution equation and minimum of R, section 2 of the introduction, § 1.1 and § 1.2, the periodic orbits and solitons of § 2 and § 3, the cutting of the neck	Text checked	G. Perelman, arXiv:math/0211159 (2002)
Introduction of Ricci flow, DeTurck, Uhlenbeck, the Wiles type	Bibliographic data only	R. S. Hamilton, J. Differential Geom. 17 (1982) 255–306. D. DeTurck, ibid. 18 (1983). K. Uhlenbeck, CMP 83 (1982) 11–29, 31–42. A. Wiles, Ann. of Math. 141 (1995) 443–551. Taylor–Wiles, ibid. 553–572
Renormalisation group of the σ model, asymptotic freedom	Abstract checked (the coefficient $(N-2)/2\pi$ is the standard one-loop count)	D. Friedan, PRL 45 (1980) 1057. A. M. Polyakov, Phys. Lett. B 59 (1975) 79
Two-loop coefficients (departure from Ricci flow)	Abstract checked (up to the two-loop calculation; the form of the term $R_{iklm}R_j{}^{klm}$ is the standard two-loop count)	D. Friedan, Ann. Phys. 163 (1985) 318
Status of the gap of the two-dimensional $O(N)$ model	Checked survey (Douglas verbatim from the text, Kopper bibliographic data only)	M. R. Douglas, Report on the Status of the Yang–Mills Millenium Prize Problem, Clay Math. Inst. (2004). C. Kopper, CMP 202 (1999) 89
KT transition, Debye screening, the a-theorem, gradient flow	Abstract checked	Fröhlich–Spencer, CMP 81 (1981) 527. Brydges–Federbush, CMP 73 (1980) 197. Komargodski–Schwimmer, JHEP 12 (2011) 099, arXiv:1107.3987. Lüscher, JHEP 08 (2010) 071. Carosso–Hasenfratz–Neil, arXiv:1806.01385. Waldron, Invent. Math. 217 (2019) 1069
The values of a for free fields ($1 : \frac{11}{2} : 62$, and that they are positive)	Text checked (appendix A)	Z. Komargodski, A. Schwimmer, arXiv:1107.3987

Item	Kind	Source / tools
The c-theorem, sum rule (10)	Bibliographic data only (sum rule (10) is attributed to Cardy 1988; the abstract goes only as far as a universal quantity of the second moment, and the form of the sum rule is unchecked. The normalisations of the coefficients 3π and $3/4\pi$, and $c = 1$ for a free field, were checked by calculation)	A. B. Zamolodchikov, JETP Lett. 43 (1986) 730. J. L. Cardy, PRL 60 (1988) 2709
Bauerschmidt–Brydges–Slade in § 05	Abstract checked	R. Bauerschmidt, D. C. Brydges, G. Slade, CMP 338 (2015) 169–193, arXiv:1403.7268 (the four-dimensional weakly self-avoiding walk). Same authors, arXiv:1403.7424 (the four-dimensional n-component $\lvert\phi\rvert^4$). Same authors, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474. D. C. Brydges, G. Slade, J. Stat. Phys. 159 (2015) 530, arXiv:1403.7255 (part IV of the method)
§ 09 table (ii), (iii)	Checked survey (Chevyrev–Shen from the text, Chatterjee from the abstract)	I. Chevyrev, H. Shen, Comm. Pure Appl. Math. (2026), arXiv:2302.12160. S. Chatterjee, A scaling limit of SU(2) lattice Yang–Mills–Higgs theory, Prob. Math. Phys. 7 (2026) 339–381, arXiv:2401.10507
The published construction in § 01, § 05 and row (i) of the table	Assessment	The assessment report `yang-mills-group/YM`, `docs/assessment-1.pdf` (2026-11-10), opening and § 8. The code is `yang-mills-group/OpenAI-YM-20261028` (submitted 2026-10-28). For how to read it, see Part 6 and 16-03
Candidates in § 05, § 08, § 09	Checked survey (abstracts; Chevyrev from the text)	Bałaban, CMP 122 (1989). Magnen–Rivasseau–Sénéor, CMP 155 (1993). Hairer, arXiv:1303.5113. CCHS, arXiv:2006.04987, 2201.03487. Chevyrev, arXiv:2202.13359. Connes–Kreimer, CMP 210 (2000). Costello, arXiv:0706.1533. Shen–Zhu–Zhu, CMP 400 (2023). Chatterjee, CMP 366 (2019), CMP 385 (2021). Cao–Park–Sheffield, arXiv:2307.06790
Table 1, Figure 1	Computed	Python on this machine: `sigma_rg.py` (formula (8) and four-dimensional one loop, the check of $\ell_* = 4\pi t_*$)

Item	Kind	Source / tools
Table 2, Figure 2, formula (11)	Computed	Python on this machine: `c_function.py` (scipy's quad; $\int_0^\infty u^3 K_0^2 du = 1/3$ checked to the order of 10^{-11})

Next chapter to read: 16-07 The list of open tasks (what would move if what were known).

The list of open tasks — joining the open questions by arrows of "if this were known, what would move"

In this chapter, § 01– § 08 are a map of open questions and § 09 is guide material.

What this chapter shows — Twelve open tasks and the closest results there are now. The three strengths of the arrows that join the tasks (theorem, roadmap, physical expectation). Checking, with a figure and a count, which requirements of the problem statement move when a task is assumed known.

Prerequisite chapters — 16-05 Where the two ways of digging part. We use the vocabulary of 7-10 (what is reached at each stage in four dimensions), 7-11 (the six obstacles) and 16-01 (the board).

Said first — This chapter lines up and joins the tasks; it settles none of them. The choice of arrows and the way of counting are this book's arrangement, and only arrows checked in primary sources are used. This site shows neither the existence of the mass gap nor the construction of the continuum limit. The tasks touched by the construction published in 2026 are written in the "closest result" column of Table 1 and at the end of § 02 (the committee's assessment Assessment ; for the meaning of the label Assessment , see "Labels" in 6-00). The list of questions that remain after the published construction is in 16-02.

The order of this chapter

1. Writing the tasks with "if ... then" — requirements and arrows
2. Twelve tasks — continuum and lattice (Table 1)
3. Arrows joined by theorems — reflection positivity, the uniform gap, the centre
4. Roadmap arrows — the order the setters and surveys write
5. Suppose it were known — a map of the tasks (Figure 1, Table 2)
6. Counting the map — tasks needed on every road (Table 3, computed)
7. For every group — arrows that vanish when the group is changed
8. Tasks outside the problem statement — one-particle states, general manifolds
9. The book's coordinate system — dictionary, seams, tasks and walls in nine cells (Figure 2, Table 4)

01

Writing the tasks with "if … then"

A **task** in this chapter is a statement that has not yet become a theorem. What moves when a task becomes known can be written as an arrow "if A is known, B can be said", and we divide its strength into three. **Theorem** (a proved implication, or a definition), **roadmap** (a prospect that the problem statement or a survey writes as "once this is known, the next step should be possible"), and **expectation** (an implication at the level of physics). As a metaphor it is a table matching keys to doors; precisely, it is a list of rules of the form "if all the premises are in place, the conclusion follows".

$$X_1 \wedge X_2 \wedge \cdots \wedge X_k \overset{\ell}{\Longrightarrow} Y, \qquad \ell \in \{0\ (\text{theorem}),\ 1\ (\text{roadmap}),\ 2\ (\text{expectation})\} \tag{1}$$

The destination is the problem statement (Jaffe–Witten 2000, below JW; 0-03). Fix one group G, and divide the demand into four **requirements** Known (a summary of JW § 4; the division into four is this book's arrangement).

$$\mathsf{P} = \bigwedge_{G} \mathsf{P}_G, \qquad \mathsf{P}_G = \mathsf{E} \wedge \mathsf{A} \wedge \mathsf{N} \wedge \mathsf{M} \tag{2}$$

The requirement of existence (E) is the existence of the theory on $\mathbb{R}^4$; the requirement of the axioms (A) is the axioms (at least as strong as those of the book of Streater–Wightman [45] and the papers of Osterwalder–Schrader [35] cited by JW); the requirement of non-triviality (N) is non-triviality, local fields and agreement with asymptotic freedom at short distances (JW § 4); and the requirement of the gap (M) is the gap $\Delta > 0$ and $m < \infty$.

We add one more requirement, outside the problem statement: the requirement of the string tension (C). The requirement of the string tension is that the string tension d of the continuum theory is not 0. d is the constant of the continuum string-tension problem of Chatterjee 2018 (Chatterjee's Problem 5.2), written σ in 7-10 § 04. The physical expectation is that d is positive.

Example: the Theorem from Uniform Decay to the Area Law of Chatterjee (2021) (Theorem 2.4 of Chatterjee 2021) is a theorem arrow. If a lattice gauge theory has uniform exponential decay under arbitrary boundary conditions (the definition of strong decay (Definition 2.3 of Chatterjee 2021); the constants depend only on the group, β (the coefficient of the coupling in the lattice action; not an inverse temperature) and the dimension of the lattice, not on the box or the boundary conditions), then centre symmetry is not broken. Further, rectangular Wilson loops in an irreducible representation that acts non-trivially on the centre obey the area law in every Gibbs measure (a distribution on the infinite lattice that follows the lattice

weights inside any box once the outside is fixed)Known (16-08 § 06). An example of a roadmap is JW § 6.5's 'Such a result presumably would enable the study of the limit as T4 → R4' (from the uniform gap to the infinite-volume limit).

02

Twelve tasks — six in the continuum and six on the lattice

The upper six of Table 1 (from the task of the T⁴ construction to the task of local fields and short distances) are tasks for the continuum theory, corresponding to the ultraviolet-limit step through the uniform-gap step of the six steps of construction (7-01) and to the additional conditions of the problem statement. The lower six (from the lattice correlation-length problem to the task of the three-dimensional measure) are lattice tasks, and tasks that turn lower dimensions and physical premises into mathematics. In this chapter limits are required along the whole sequence (this book's arrangement). JW's weak-existence footnote (footnote 2 of Jaffe–Witten; 0-03 § 08) writes that weak existence by compactness alone is not accepted as a solution of the existence part unless the properties of the limit (the gap and the axioms) are shown by other methodsKnown . The first three (in order, the task of the T⁴ construction, the task toward infinite volume and the task of a uniform gap) have the following form (notation as in 7-10).

$$\lim_{a\to 0}\langle W_{\gamma a}\rangle^{\mathrm{ren}}_{a,L}\ \text{exists along the whole sequence},\qquad \lim_{L\to\infty}\mu_L\ \text{exists},\qquad \inf_L \Delta_L > 0 \tag{3}$$

$W_{\gamma a}$ is the Wilson loop of a lattice path, $\langle\cdot\rangle^{\mathrm{ren}}$ is the expectation with a term proportional to the perimeter removed from its logarithm (formula (5.2) of Chatterjee 2018; 7-10 § 04), μ_L is the measure on the box T^4 of side L, and Δ_L its gap. Requiring the limit along the whole sequence in the task of the T⁴ construction and the task toward infinite volume is this book's arrangement.

Known Table 1 Twelve tasks

	Task	Statement	Closest result	What moves if known
1	Task of the T⁴ construction	Expectation values of gauge-invariant quantities converge as $a \to 0$ along the whole sequence (the whole sequence is this book's arrangement)	For correlation functions of local fields, the construction published in 2026 showed convergence along the whole sequence of cutoffs in a periodic boxAssessment (8-05). Not in the form of this row's statement (renormalised Wilson loops) (16-02 § 05). Before that, ultraviolet stability with a hypothesis and subsequences (Bałaban), an outline (MRS)	The ultraviolet-limit step, half of the requirement of existence

	Task	Statement	Closest result	What moves if known
2	Task toward infinite volume	The limit $L \to \infty$ exists along the whole sequence (likewise this book's arrangement)	What the published construction has goes only as far as limits along subsequences and phase decomposition Assessment . Convergence of the whole sequence and uniqueness are open questions of the assessment report (16-02 § 02). At the time of the problem statement, 'Nor do present methods suggest how to obtain the existence of the infinite volume limit' (JW § 6), a stage at which § 6.5 writes that the task of a uniform gap would open the way. On the lattice it is shown at strong coupling	The infinite-volume step, the rest of the requirement of existence
3	Task of a uniform gap	$\inf_L \Delta_L > 0$	'New ideas are needed' (JW § 6.5). What the published construction has goes only as far as polynomial estimates uniform in the phase; there is no exponential decay (16-02 § 04)	The uniform-gap step; through the roadmap, the task toward infinite volume
4	Task of the remaining axioms	Recovery of rotation symmetry, and the regularity needed for reconstruction	In the published construction, the Wightman axioms (including Euclidean invariance) for each phase Assessment (12-06, 12-08). Reflection positivity survives from the lattice. Rotations: Driver's Corollary 6.7 in two dimensions	The axiom step, the seam of spacetime and space
5	Task of non-triviality	The limit is not Gaussian	The scaling limit of lattice-cut four-dimensional ϕ^4 (including the Ising type) is Gaussian (Aizenman–Duminil-Copin 2021; a result in the opposite direction from non-triviality). In the published construction, the short-distance law of the two-point function of the trace field excludes descriptions of the type "polynomials in free fields of finitely many masses" Assessment (13-02). It does not exclude every description built from Gaussian fields	Half of the requirement of non-triviality

	Task	Statement	Closest result	What moves if known
6	Task of local fields and short distances	Local fields such as $\mathrm{Tr}\, F_{ij}F_{kl}(x)$. Agreement with asymptotic freedom at short distances	The level of perturbation theory. There is no natural one-to-one correspondence (JW's local-field footnote). The published construction has labels for local fields (8-03), and at short distances checks the one-loop running and the trace coefficient. Agreement with perturbation theory at all orders is not shown Assessment (6-04 § 06)	The rest of the requirement of non-triviality
7	Lattice correlation-length problem (Chatterjee's Problem 5.1)	$\xi(\beta) \in (0, \infty)$ (all β), $\xi \to \infty$	At strong coupling exponential decay (an upper bound on ξ) has been shown. For finite groups, exponential decay even at weak coupling (Adhikari–Cao; outside the scope of the lattice correlation-length problem, and without $\xi \to \infty$)	Only cells of the lattice
8	Continuum string-tension problem (Chatterjee's Problem 5.2)	String tension d $/= 0$ in the continuum limit of Wilson loops	Not yet. Whether a perimeter term is needed is also open. The published construction has no continuum Wilson loops either (16-02 § 05)	The requirement of the string tension
9	Task of strong decay	Uniform exponential decay under arbitrary boundary conditions (all β)	For small β it can be shown by the method of Dobrushin–Shlosman, as Chatterjee 2021 remarks. Whether the strong version is expected at large β is not clear (ibid.)	The lattice area-law problem
10	Lattice area-law problem (Chatterjee's Problem 4.1)	The area law for all β	Small β for every group (Osterwalder–Seiler 1978). Three-dimensional $U(1)$ for all β (Göpfert–Mack 1982)	Only lattice confinement
11	Task of one scale	$c\,\sigma(\beta) \le m(\beta)^2 \le C\,\sigma(\beta)$ for $\beta \ge \beta_0$ (constants $c, C > 0$ independent of β; m, σ are the mass and string tension in lattice units, and neither $m > 0$ nor $\sigma > 0$ is assumed)	No theorem. The record of this book's study is at the stage of placing it as an assumption in the form of 17-12 formula (3) (the diffusive-relaxation-time assumption is a separate formula (6), 17-12 § 05)	Joins the requirement of the string tension and the requirement of the gap at the level of expectation

	Task	Statement	Closest result	What moves if known
12	Task of the three-dimensional measure	Construction of the measure on T^3	'remains open' (Chevyrev 2022)	Only the 3D ultraviolet-limit step

For details: the task of the T^4 construction is in 7-07 (the published construction in 6-01), the task of non-triviality in 7-08, the lattice correlation-length problem and the lattice area-law problem in 16-01 § 07, and the task of the three-dimensional measure in 7-06. Example: what moves the four-dimensional cells of the board (16-01 Table 3) is the upper four of Table 1 (from the task of the T^4 construction to the task of the remaining axioms) and the lattice correlation-length problem; the three-dimensional row is moved by the task of the three-dimensional measure. The continuum string-tension problem, the task of strong decay, the lattice area-law problem and the task of one scale have no cells; they are tasks on the side of the seams. In the ledger of conditions (27-02), the lattice side of the task of a uniform gap corresponds to the row of uniform decay of block observables (correlations of block observables decay at a rate uniform in L and the lattice spacing; the premises are ultraviolet control, the small-T^2 gap and the band comparison (in the language of the gap)) Paper . The ultraviolet-control row of the ledger was shown, as a continuum limit in a periodic cubic box, by the published construction, while the form of this book's road remains open (27-02 Table 1). What remains on this book's road is the seam that puts the ultraviolet result into a form the infrared arguments can use (16-03, 16-04) Inferred . The arrow from the area law (the lattice area-law problem) to the task of a uniform gap does not hold in general. The area law is about the gap in the charged sector (electric flux $e \neq 0$), the task of a uniform gap about the gap in the neutral sector ($e = 0$), and the transfer matrix preserves this decomposition Paper . The known theorem goes the other way (uniform exponential decay ⇒ the area law; the Theorem from Uniform Decay to the Area Law of Chatterjee 2021 in § 01) Known .

After the published construction. Of the upper six of Table 1, the construction published in 2026 meets the task of the T^4 construction for correlation functions of local fields and the task of the remaining axioms for each phase, and touches the task toward infinite volume in the form of subsequences, and the task of non-triviality and the task of local fields and short distances in the form of checks at short distances Assessment . In the words of this chapter, the requirement of existence is not met in the form that requires the limit along the whole sequence (the volume side is along a subsequence), and nothing is given towards the requirement of the gap (the mass gap) Inferred . The figure of § 05 and the count of § 06 are a map of the logic between the tasks, and the rules do not change because of the published construction. The list of which tasks remain is placed in 16-02, following the five open questions of the assessment report, and where the construction stops is in 16-03.

03

Arrows joined by theorems

Not many arrows have a proof. We look at three in turn.

Reflection positivity survives in the limit

The first is reflection positivity. It holds on Wilson's lattice (Osterwalder–Seiler 1978), and since it is an inequality, if the sequence of measures μ_n **converges weakly** it survives in the limit Known (4-10). Converging weakly means that the expectation values of all bounded continuous quantities converge (0-03 § 08).

$$\langle \Theta F\, F \rangle_{\mu} = \lim_{n\to\infty} \langle \Theta F\, F \rangle_{\mu_n} \;\geq\; 0 \quad (4)$$

Θ is time reflection, and F a bounded continuous quantity determined by the side of positive times alone. Together with the task of the remaining axioms, the Osterwalder–Schrader reconstruction sets up a Hilbert space and $H \geq 0$ (the rule of the axioms; JW also writes 'Establishing a quantum mechanical Hilbert space is part of the solution to this Millennium problem').

A uniform gap survives in the limit

The second is that a uniform gap survives in the limit (the rule of the gap). Let τ_t be the operation of shifting by t in the time direction, and write ν_F for the spectral measure of the vector made by F with its vacuum component subtracted. ν_F is the same thing as the μ_F of 4-11 § 02. In this chapter we use μ for the sequence of measures, so we change the symbol.

$$\langle \Theta F\, \tau_t F \rangle - \left|\langle F \rangle\right|^2 = \int_{[0,\infty)} e^{-tE}\, \mathrm{d}\nu_F(E) \quad (5)$$

That the left-hand side is at most a constant times $e^{-\Delta t}$ for every bounded F is the same as $H \geq \Delta$ in the directions orthogonal to the vacuum Known (4-11 § 02). The integral also includes the point $E = 0$, so uniqueness of the vacuum comes out at the same time.

In the estimate box by box (7-10 formula (7)), if the rate of decay $\Delta_L \geq \Delta_*$ does not depend on L, the estimate survives in the limit and becomes a gap (the standard argument). The prefactor $\|A^* \Omega_L\|\, \|B\, \Omega_L\|$ in (7) is at most 1 if $|F| \leq 1$. This argument needs three premises. (i) The expectation values of the F used converge along the sequence $L \to \infty$ (part of the requirement of existence). (ii) The vectors of those F are dense in the reconstructed Hilbert space. (iii) Since T^4 is also periodic in the time direction, the term $e^{-\Delta_L (L-t)}$ added to the correlations in the box vanishes as $L \to \infty$

(it does if $\Delta_L \geq \Delta_*$). What the rule of the gap gives is only $\Delta > 0$; $m < \infty$ comes separately from the theory not being the vacuum alone (7-10 § 06).

Example: the decay of a single F tells us only the lower end of the support of ν_F. The $\xi(\beta)$ of the lattice correlation-length problem is the decay of plaquette correlations. Chatterjee defines ξ by the limit $|x| \to \infty$ taken in every direction, but what connects to the spectrum of the transfer matrix (5-03) is the decay in the time direction. Read through the decay in the time direction, what is learned is the lower end of H on the subspace made by the plaquette (this chapter's reading).

From centre symmetry to the area law

The third is the Theorem from Uniform Decay to the Area Law of Chatterjee (2021) from § 01 (the rule from decay to the area law). Its introduction also cites Durhuus–Fröhlich (1980), which derives confinement in a gauge theory from exponential decay in a non-linear σ model one dimension lower, and Fröhlich (1979), which carries confinement from a $\mathbb{Z}_n$ theory to an $SU(n)$ theory Known (bibliographic data only). By the title of Fröhlich's paper the destination is the lattice Higgs theory of $SU(n)$; the introduction of Chatterjee 2021 presents it as "the $SU(n)$ theory".

04

Roadmap arrows — the order the setters and surveys write

JW writes prospects for the order of the tasks. For the task of the T[4] construction it says there are ample signs that extending known methods would reach it, writing 'One presumably needs to revisit known results at a deep level, simplify the methods, and extend them'. For the task of a uniform gap it writes 'New ideas are needed', and goes on that this would make the task toward infinite volume possible. § 5 also writes that a uniform gap 'may play a fundamental role' in the proof of the infinite-volume limit Known . This is the rule of the problem setters' route.

$$\text{task of a uniform gap} \overset{1}{\Longrightarrow} \text{task toward infinite volume}: \qquad \inf_L \Delta_L \geq \Delta_* > 0 \quad \rightsquigarrow \quad \text{correla}$$

Precisely, the right-hand side of (6) is not the existence of the limit but the material of a uniform estimate for taking the limit, and how to assemble it is not written ('presumably').

Example: the two things that the rule of the problem setters' route joins (uniform decay and the infinite-volume limit) were obtained together from the cluster expansion in models with a small parameter. Weak-coupling ϕ^4_2 and ϕ^4_3 are examples Known (7-04 § 08). They did not pass in the order of deriving the infinite-volume limit from a uniform gap. Four-dimensional Yang–Mills has no such parameter at long

distances (7-11 the obstacle of the infrared parameter). On the lattice side, Chatterjee (2018) places the continuum string-tension problem as 'the most basic question that needs to be solved before making any further progress' on the road through Wilson loops Known . This is a sentence in the direction of necessity, "needed first", and not an arrow of sufficiency from the continuum string-tension problem to the requirements.

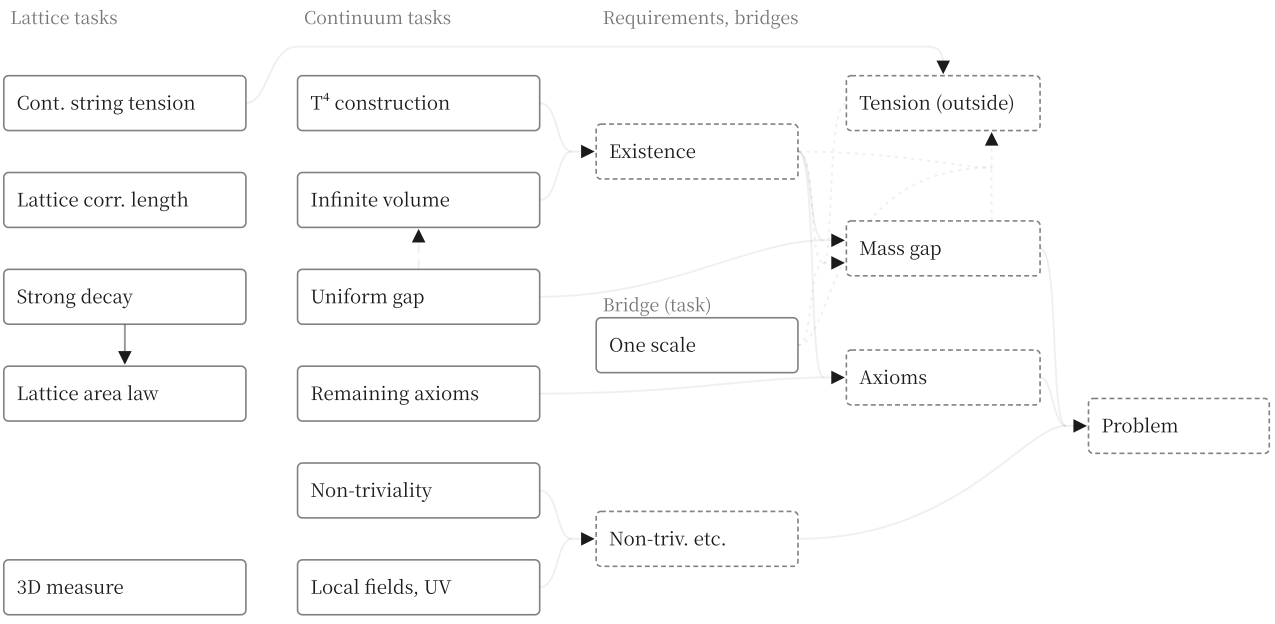
05

Suppose it were known — a map of the tasks

We put the rules of Table 2 into one figure. Pressing the box of a task makes that task "assumed known", and the conclusions of rules whose premises are all in place light up. What lights up is the smallest set that contains the conclusion whenever it contains all the premises (the closure).

$$\mathrm{cl}_\ell(S) = \bigcap \{ X \supseteq S : \text{ for every rule of level } \le \ell \text{, premises} \subseteq X \Rightarrow \text{conclusion} \in X \} \quad (7)$$

Figure 1 Pressing the box of a task makes it "assumed known", and the requirements that move and the cells of the board light up. The knob selects the strength of the arrows (theorems only / + roadmap / + expectation), and the button selects a group with trivial centre. Solid lines are theorems, dashed lines roadmaps, dotted lines expectations. G is fixed



Assumed known: none. To the problem statement: still T⁴ construction, Uniform gap, Remaining axioms, Non-triviality, Local fields, UV. Cells moved on the board: none (arrows: theorems, roadmaps, physical expectations)

Computed How this figure lights up is the closure (7) of the rules of Table 2, computed on the spot (the choice of arrows is this book's arrangement).

Known Table 2 Rules (if all the premises are in place, the conclusion). The upper seven (from the rule of existence to the rule of the string tension) are theorems or definitions, the rule of the problem setters' route is a roadmap, and the rule from the

string tension to the gap and the rule from the gap to the string tension are physical expectations Physics . The choice of rules is this book's arrangement. The requirement of the string tension is that of § 01: the continuum string tension d is not 0 (outside the problem statement)

Rule	Premises ⇒ conclusion	Strength	Grounds
Rule of existence	Task of the T^4 construction ∧ task toward infinite volume ⇒ requirement of existence	Definition	JW § 6.5
Rule of the axioms	Requirement of existence ∧ task of the remaining axioms ⇒ requirement of the axioms	Theorem	Formula (4) and OS reconstruction
Rule of the gap	Requirement of existence ∧ task of a uniform gap ⇒ requirement of the gap	Theorem	Formula (5) and 7-10 formula (7) (the standard argument; premises in § 03)
Rule of non-triviality	Task of non-triviality ∧ task of local fields and short distances ⇒ requirement of non-triviality	Definition	JW § 4
Rule of the problem statement	Requirement of the axioms ∧ requirement of non-triviality ∧ requirement of the gap ⇒ requirement of the problem statement (P_G)	Definition	Formula (2)
Rule from decay to the area law	Task of strong decay ⇒ lattice area-law problem	Theorem	Chatterjee 2021, Theorem from Uniform Decay to the Area Law
Rule of the string tension	Continuum string-tension problem ⇒ requirement of the string tension	Definition	Chatterjee 2018 § 5 (the requirement of the string tension is exactly the $d \neq 0$ of the continuum string-tension problem)
Rule of the problem setters' route	Task of a uniform gap ⇒ task toward infinite volume	Roadmap	JW § 5, § 6.5
Rule from the string tension to the gap	Requirement of existence ∧ requirement of the string tension ∧ task of one scale ⇒ requirement of the gap	Expectation	A single scale (17-12)
Rule from the gap to the string tension	Requirement of existence ∧ requirement of the gap ∧ task of one scale ⇒ requirement of the string tension	Expectation	The same sentence in the reverse direction

Example: even if the "Lattice tasks" button makes the lattice correlation-length problem, the continuum string-tension problem, the task of strong decay, the lattice area-law problem and the task of the three-dimensional measure assumed known,

only the requirement of the string tension lights up, and not a single requirement. This restates in the language of arrows the "showing it on the lattice is enough" of 16-08 § 08.

06

Counting the map — tasks needed on every road

Among the sets of tasks that reach the requirement of the problem statement, we call those that no longer reach it when any one task is removed **minimal sets**. We examined the 4096 subsets at three levels and counted all the minimal sets Computed .

$$\mathcal{S}\ell = \{ S : \ \mathsf{P}G \in \mathrm{cl}\ell(S), \quad S \text{ minus any one element gives } \mathsf{P}G \notin \mathrm{cl}\ell \} \tag{8}$$

With theorems only, there is one set, the upper six of Table 1 (from the task of the T^4 construction to the task of local fields and short distances); adding roadmaps, one set, {the task of the T^4 construction, the task of a uniform gap, the task of the remaining axioms, the task of non-triviality, the task of local fields and short distances}; adding physical expectations too, {the task of the T^4 construction, the task toward infinite volume, the task of the remaining axioms, the task of non-triviality, the task of local fields and short distances, the continuum string-tension problem, the task of one scale} is added, making two sets. The same held for groups with trivial centre.

Computed Table 3 For each task, whether it is in every minimal set for the requirement of the problem statement (required), in some, or in none

Task	Theorems only	+ Roadmap	+ Physical expectation	What it moves alone (up to physical
Task of the T^4 construction, task of the remaining axioms, task of non-triviality, task of local fields and short distances	Required	Required	Required	Nothing
Task toward infinite volume	Required	—	In some	Nothing
Task of a uniform gap	Required	Required	In some	Task toward infinite volume (roadr

Task	Theorems only	+ Roadmap	+ Physical expectation	What it moves alone (up to physical
Continuum string-tension problem	—	—	In some	Requirement of the string tension
Task of one scale	—	—	In some	Nothing
Lattice correlation-length problem, task of strong decay, lattice area-law problem, task of the three-dimensional measure	—	—	—	The task of strong decay alone give with non-trivial centre)

Example: (a) the task of the T^4 construction, the task of the remaining axioms, the task of non-triviality and the task of local fields and short distances are needed on every road. (b) At the level of physical expectation, instead of the task of a uniform gap, "the continuum string-tension problem and the task of one scale" reach the mass gap. The physical sentence that sees mass and string tension as two faces of one scale (formula (3) of 17-12) takes over the role of the task of a uniform gap. (c) The lattice correlation-length problem, the task of strong decay, the lattice area-law problem and the task of the three-dimensional measure are in no minimal set. The count uses only arrows checked in primary sources, so if an arrow from the lattice to a continuum requirement becomes a theorem, it must be redone.

07

For every group — arrows that vanish when the group is changed

The problem statement asks for the requirement of the problem statement for all compact simple groups G (formula (2)). The candidate Lie algebras are four series and five exceptions, and with each Lie algebra come groups in forms divided by the centre Known (1-04 Table 6). No general theorem carrying a result for one group to another was found in what this chapter read. Some arrows vanish depending on the group. The rule from decay to the area law uses that a finite-dimensional irreducible unitary representation π acts non-trivially on the centre, that is, that there is an element z of the centre with $\pi(z) \neq I$. An element of the centre commutes with every $\pi(g)$, so in an irreducible π it acts as a scalar by Schur's lemma (1-04 § 06).

$$\exists\, z \in Z(G): \qquad \pi(z) = \omega I, \quad \omega \neq 1 \qquad (\pi \text{ irreducible}) \ (9)$$

Example: in the fundamental representation of $SU(N)$, $z = e^{2\pi i/N} I$ and $\omega = e^{2\pi i/N}$ (1-04 § 07). For a group with trivial centre there is no representation satisfying (9). Besides the simply connected G_2, F_4 and E_8, this includes the forms divided by the centre $SO(3) = SU(2)/\mathbb{Z}_2$ and $PSU(N) = SU(N)/\mathbb{Z}_N$; every group divided by its centre has trivial centre Known . The reason in one line: for a connected Lie group, the centre is the kernel of the adjoint representation (1-04 § 06). $G/Z(G)$ is the same group as the image $\mathrm{Ad}(G) \subset GL(\mathfrak{g})$ of the adjoint representation, and its adjoint representation is $\mathrm{Ad}(g)$ itself on $\mathfrak{g}$ (for a simple Lie algebra ad is injective). The kernel is the elements with $\mathrm{Ad}(g) = I$, that is, the identity alone. The "group with trivial centre" of Figure 1 is this case, and the arrow from the task of strong decay to the lattice area-law problem vanishes. Chatterjee notes that for groups with trivial centre (for example $SO(3)$) it is believed they may not confine at weak coupling Physics . The problem statement asks for a gap for such groups too, so a road through confinement cannot be used for every group. On the other hand, the one-loop coefficient $16\pi^2 b_0 = \frac{11}{3} C_A$ is positive for every group (4-06 § 02), and perturbative asymptotic freedom does not depend on the group Known (0-03).

Replacing by SU(N). Moving the rows of this book's ledger of conditions (27-02) one by one to $SU(N)$, they fall into three kinds Paper . (i) Rows independent of the group: the uniform gap of the double well (the double well), the positivity of the 't Hooft tension and the reflection positivity of z (algebra), the uniform acceptance of insertion (an identity), and the limit to R^4. (ii) Rows that move as they are into the language of the centre $\mathbb{Z}_N$: in the twisted part of the small-T^2 gap, the Rigidity Lemma for Anticommuting Pairs of 25-01 (anticommuting pairs) is replaced by the uniqueness of the clock and shift pair (if the twist is primitive it is unique up to conjugation and its centraliser is $\mathbb{Z}_N$; González-Arroyo's review hep-th/9807108 § 4.2–4.3 Known), and the adjoint gap of the twisted T^2 becomes $4\sin^2(\pi/(NL))$ (for $SU(3)$, $L = 3$–6, it agrees with the computation of eigenvalues, and the value is determined by NL alone Computed). The source of the base point at small T^2 is for $SU(N)$, and the reflection positivity of z and frame decorrelation are in the general-group form of Osterwalder–Seiler. The one-loop $\beta - \beta'$ is $11N^2 \ln 2/(12\pi^2)$ (0.2575 for $N = 2$, 0.5794 for $N = 3$) Paper . The lemma of the colour average goes through with the $n = N^2 - 1$ colours of $\mathfrak{su}(N)$, and the constant 5 becomes $n(n+2)/3$ (including the existence of a weighted spherical 4-design, Lean: ColorAverageN (colorAverage, design_identity); the application to $\mathfrak{su}(N) \cong \mathbb{R}^n$ is Paper). The $SU(N)$ version of the Tomboulis–Yaffe inequality is Theorem 2 of Kanazawa arXiv:0808.3442 (unpublished) Known (its application to the slabs of this book is not done). (iii) Rows that need redoing: the ultraviolet parts that use properties of $SU(2) = S^3$ and $-1 \in$ the Weyl group (the

evenness of class functions) — that the conditional law of one link is one-parameter (15-07 (5)), the upper bound that drops the negative terms of the image sum of the heat kernel (the Heat-Kernel Fluctuation Rarity Lemma of 15-08), and the evenness step (in the hierarchical model, for $SU(3)$ it breaks through the cubic invariant $\operatorname{tr} X^3$) — and the numerics of the band (26-09 has only $SU(2)$ and $U(1)$). So to move this book's road to $SU(N)$, one must add, to the rows open for $SU(2)$ (the renormalisation comparison and others), the group versions of these three lemmas and the redoing of the numerics. Existence on the ultraviolet side is supplied by the published construction for any compact simple group Assessment . This book's ultraviolet parts are read not as parts for existence but as parts for the seam of this book's road Inferred (16-04). For groups with trivial centre (G_2, F_4, E_8 and groups divided by their centre) there is no tool of twists, and for $Sp(n)$ ($n \geq 2$) the centraliser of the pair that absorbs the twist is continuous, so a single twist of T^2 does not remove the zero modes Paper (one example). The road to all compact simple groups is open within the scope of this book.

08

Tasks outside the problem statement

JW § 5 lists 'natural extensions': isolated one-particle states (an upper gap in addition to the gap), confinement, other four-dimensional gauge theories with fields added that keep asymptotic freedom and their dynamics, and the extension from $\mathbb{R}^4$ to general four-dimensional manifolds Known . None of these is a requirement, and in Figure 1 only the requirement of the string tension was placed outside. Isolated one-particle states can be written through the spectrum of the mass operator $\sqrt{H^2 - \mathbf{P}^2}$ ($\mathbf{P}$ is the momentum) (16-05 the seam of the word "particle").

$$\operatorname{spec} \sqrt{H^2 - \mathbf{P}^2} \cap (0,\, m + \delta) = \{m\} \qquad (\text{some } \delta > 0) \ (10)$$

Example: for a free field of mass m, the one-particle states of Fock space (2-06) sit at m and two or more particles fill $[2m, \infty)$; the spectrum is $\{0\} \cup \{m\} \cup [2m, \infty)$, and the upper gap is m Known . JW § 5 also writes that the clustering that follows from the gap (7-10 § 08) may carry results on $\mathbb{R}^4$ to general four-dimensional manifolds ('may make it possible').

09

The book's coordinate system — dictionary, seams, tasks and walls on one grid

We use the two-column grades of the dictionary (18-01 § 05) as common coordinates, and plot four lists on nine cells. (a) is "if both exist, are they the same", (b) is "does it

exist in four dimensions". The grades of the dictionary rows are as in 18-01; the grades of seams, tasks and walls are this book's arrangement. The grounds are as follows.

Seams: take over the grade of the dictionary row corresponding to the pair of sentences in 16-05 Table 1 (1→1, 2→13, 3→9, 4→8, 5→2, 6→5, 7→14). Tasks: (a) is the strength of the rules leaving the task in Table 2 (theorem or definition = Settled, conditional on the group = Conditional, expectation = Unsettled; the lattice correlation-length problem, the lattice area-law problem and the task of the three-dimensional measure, which have no rule, are unrated), (b) is the "Closest result" of Table 1 (Conditional if there is a version with limited scope). Walls: the grade of the dictionary row named by the column of 23-02 Roads tried on the four-dimensional lattice, and walls (the relaxation-time upper-bound assumption, the relaxation-time lower-bound assumption, η, chain → 13; the physical-unit gap statement → 8; limit → 7. One step and the summability-of-shortfalls assumption have no row in the dictionary and are unrated).

Example. The thirteen walls that bear on the diffusive-relaxation-time assumption gather in the same "Unsettled, None" cell as dictionary row 13 (Λ, "a single scale"), the seam of "there is one scale" and the task of one scale. Although the types of wall (23-02 Roads tried on the four-dimensional lattice, and walls) divide into principle, structure, tool, assumption and measurement, on the dictionary they all bear on the same single pair. Only tasks enter "Settled, Conditional": those with a version of limited scope whose arrows to the requirements are theorems or definitions (the task of the T^4 construction, the task toward infinite volume, the task of the remaining axioms).

Figure 2 Choose a list with the buttons, and its rows are plotted on the nine cells of (a)×(b). "Overlay" plots all four at once. Below the frame: unrated

(a)	(b) Exists in four dimensions: Constructed	Cond.	None
Settled	#3 #10		#1 #2
Cond.	#12	#11	#4 #5 #6 #7 #8
Unsettled		#9	#13 #14

Unrated

```
Dictionary : Settled / Constructed 2, Settled / None 2, Conditional / Constructed 1, Conditional / Conditional 1, Conditional / None 5, Unsettled / Conditional 1, Unsettled / None 2
```

Computed The assignment of grades was made on this machine by the rules above from 18-01 Table 2, 16-05 Table 1 and Tables 1 and 2 of this chapter, and 23-02 Table 3, and the counts (14, 7, 12, 21) were checked.

Table 4 The rows that fall in each of the nine cells and in unrated (numbers are the rows of each table; walls are numbered as in 23-02 Table 3; in the figure, dictionary rows carry # and seams carry S).

(a) / (b)	Dictionary (18-01 Table 2)	Seams (16-05 Table 1)	Tasks (Table 1)	Walls (23-02 Table 3)
Settled / Constructed	3, 10			
Settled / Conditional			T^4 construction, infinite volume, remaining axioms	
Settled / None	1, 2	1, 5	uniform gap, non-triviality, local fields and short distances, continuum string tension	
Conditional / Constructed	12			
Conditional / Conditional	11		strong decay	
Conditional / None	4, 5, 6, 7, 8	4, 6		2, 9, 21
Unsettled / Constructed				
Unsettled / Conditional	9	3		
Unsettled / None	13, 14	2, 7	one scale	1, 3, 4, 5, 7, 10, 11, 12, 14, 16, 17, 18, 20
Unrated			lattice correlation length, lattice area law, three-dimensional measure	6, 8, 13, 15, 19

What this chapter can say

	Said	Not said
Known	The statements and current status of the tasks (within the primary sources). The implications of the rule of the axioms, the rule of the gap and the rule from decay to the area law (the rule of the gap is the standard argument, with the premises (i)–(iii) of § 03) and JW's roadmap sentences	That twelve exhausts them. The choice of arrows, and requiring limits along the whole sequence (this book's arrangement). Arrows not checked are not counted
Computed	The minimal sets under the rules of Table 2, and Table 3	Counts under other rules. Writing the rule of the gap out in detail for observables of four-dimensional Yang–Mills
Physics	That, if the rule from the string tension to the gap and the rule from the gap to the string tension are admitted, the continuum string-tension problem and the task of one scale take over the role of the task of a uniform gap	That the rule from the string tension to the gap and the rule from the gap to the string tension are theorems
Arrangement	The plots of Figure 2 and Table 4 come out mechanically from each table under the rules of § 09	The choice of rules itself (the grades of seams, tasks and walls are not theorems)
Overall	—	This site has settled none of the tasks. It does not show the existence of the mass gap. What the published construction reached, as written in Table 1, is at the level of the committee's assessment Assessment ; whether that proof is correct is outside this chapter

Sources and reproduction

Item	Kind	Source / tools
Quotations from the problem statement (§ 4, § 5, § 6, § 6.5, footnotes 1 and 2), current status of the task toward infinite volume, the task of a uniform gap and the task of local fields and short distances	Text	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Math. Inst. (2000)
Reference [45] for the axioms (the requirement of the axioms in formula (2))	Bibliographic data only	R. Streater, A. Wightman, PCT, Spin and Statistics and All That, W. A. Benjamin (1964) (bibliographic data from the reference list of JW)

Item	Kind	Source / tools
Problems 4.1, 5.1, 5.2 (the lattice area-law problem, the lattice correlation-length problem, the continuum string-tension problem) and the sentences of §4, §5, §6, the definition of the requirement of the string tension	Text	S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950 (2018)
Theorems 2.2 and 2.4, the definition of strong decay and its remark (the task of strong decay), the remark on $SO(3)$, the presentation in the introduction	Text	S. Chatterjee, A probabilistic mechanism for quark confinement, arXiv:2006.16229, Comm. Math. Phys. 385 (2021) 1007–1039
Source of the remark on $SO(3)$ / small β for the task of strong decay	Bibliographic data only	J. Greensite, B. Lautrup, Phys. Rev. Lett. 47 (1981) 9–11 / R. L. Dobrushin, S. B. Shlosman, Constructive criterion for the uniqueness of Gibbs field, in Statistical Physics and Dynamical Systems, Birkhäuser (1985) 347–370 (bibliographic data for both from the reference list of Chatterjee 2021)
Confinement from the σ model / $\mathbb{Z}_n$ to $SU(n)$ Higgs	Bibliographic data only	B. Durhuus, J. Fröhlich, CMP 75 (1980) 103–151 / J. Fröhlich, Phys. Lett. B 83 (1979) 195–198 (titles and the presentation in Chatterjee 2021)
Reflection positivity, reconstruction, small β for the lattice area-law problem	Abstract, bibliographic data only	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471 (abstract: existence and analyticity of the infinite-volume limit at strong coupling, Wilson's confinement bound) / K. Osterwalder, R. Schrader, CMP 31 (1973) 83–112; 42 (1975) 281–305
What the published construction reached, in Table 1	Assessment	Assessment report `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10), opening, §5, §7, §8. The code is `yang-mills-group/OpenAI-YM-20261028` (submitted 2026-10-28). How to read it: Parts 6, 8, 12 and 13
Current status of the task of the T^4 construction	Abstract and preface / text pp.325–328 (7-07)	T. Bałaban, Comm. Math. Phys. 122 (1989) 355–392 / J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383
The task toward infinite volume and the lattice correlation-length problem at strong coupling, the lattice correlation-length problem for finite groups	Text §1 (16-01)	H. Shen, R. Zhu, X. Zhu, arXiv:2204.12737, Comm. Math. Phys. 400 (2023) 805–851 / A. Adhikari, S. Cao, arXiv:2202.10375, Ann. Probab. 53 (2025) 140–174
Driver's Corollary 6.7 for the task of the remaining axioms	Introduction of the text	B. K. Driver, CMP 123 (1989) 575–616

Item	Kind	Source / tools
Current status of the task of non-triviality	Text (7-08)	M. Aizenman, H. Duminil-Copin, Marginal triviality of the scaling limits of critical 4D Ising and ϕ^4_4 models, Ann. of Math. 194 (2021) 163, arXiv:1912.07973
Three-dimensional $U(1)$ for the lattice area-law problem	Abstract	M. Göpfert, G. Mack, Proof of confinement of static quarks in 3-dimensional U(1) lattice gauge theory for all values of the coupling constant, CMP 82 (1982) 545–606 (abstract (INSPIRE): Villain action, positive string tension at all couplings, the continuum limit with m_D fixed is a free scalar field. Text not read. The year is Crossref's year of issue; INSPIRE gives 1981)
Statement and current status of the task of one scale	Record of this book's study	Formulas (3) and (6) of 17-12 and its source table
Current status of the task of the three-dimensional measure	Text § 1.1 (7-06)	I. Chevyrev, Stochastic quantisation of Yang–Mills, J. Math. Phys. 63 (2022) 091101, arXiv:2202.13359
Chapters of this book cited in the text	Known	0-03 § 08 (the local-field footnote, the weak-existence footnote, weak convergence), 1-04 § 06, § 07 (Schur's lemma, the centre), 4-06 § 02 (b_0), 4-10, 4-11 § 02 (reconstruction and μ_F), 7-04 § 08 (the cluster expansion), 7-10 § 04, § 06, § 07 (ren, $m < \infty$, formula (7))
Formulas (5) and (10)	Known	The spectral theorem and reconstruction (4-10, 4-11), Fock space (2-06)
Table 3, Figure 1	Computed	Python on this machine: graph.py (closures and minimal sets over 4096 subsets × three levels × with and without centre). The figure uses the same rules
Table 4, Figure 2	Computed	Python (coord.py; reads 18-01 Table 2, 16-05 Table 1, Tables 1 and 2, and 23-02 Table 3, makes the assignment and checks the number of rows)

Next chapter to read: 16-08 Common misconceptions, which takes ways of speaking that mistake the tasks back to the words of the problem statement. The tasks on the side of the record of this book's study are in 23-15.

Common misconceptions — taking half-right ways of speaking back to the words of the problem statement

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — For nine commonly seen ways of speaking, the correct core and the slip. How the state of proof changes with group and dimension (Figure 1). That the decay of correlations does not go as far as saying whether there is a particle (Figure 2).

Prerequisite chapters — 0-03 (faster with 4-11, 5-02, 7-11 and 17-01; the words used are defined one line each at the end of §01).

Said first — What is corrected goes only as far as the level of ways of speaking. This chapter shows neither the existence of the mass gap nor the construction of the continuum limit. The ways of speaking that changed with the four-dimensional construction published in 2026 (the committee's assessment Assessment ; for the meaning of the labels see "Labels" in 6-00) are written in §02, §03 and §08.

The order of this chapter

1. How to read the misconceptions — taking them back to the one sentence of the problem statement, and the words used in this chapter
2. "The heart of the difficulty is infinity" — it remains even when the infinities are removed
3. "Two and three dimensions are already done" — for non-abelian groups, only two
4. "There is a gap because it is non-abelian" — the theorems divide by group and dimension (Figure 1)
5. "The mass gap is the mass of the lightest particle" — decay does not speak of particles (Figure 2)
6. "Show confinement and the gap follows" — different claims
7. "It has been confirmed by lattice numerics" — predictions that assume existence

8. "Showing it on the lattice is enough", "Build it in a finite box and the rest is technique" — the lattice correlation-length problem (Chatterjee's Problem 5.1) and T^4
9. "The mass gap comes from the uncertainty principle" — the theorem only converts a length into a mass
10. Detour: other ways of speaking one comes across — 1/0, holography, triviality, QCD

01

How to read the misconceptions

Many of the commonly seen ways of speaking are half right, and to that extent the slip is harder to see. This chapter divides each item into **the way of speaking**, **the correct core**, **the slip** and **a precise way of saying it**. The standard is the one sentence of the problem statement (0-03); lining up its conditions,

$$\forall G \;\; \exists \mathcal{T}_G : \quad \textbf{axioms} \;\wedge\; \textbf{non-trivial} \;\wedge\; \textbf{agrees with asymptotic freedom at short distances} \;\wedge$$

We read each word in one line Known .

- G: a compact simple group. All of them, from $SU(2)$ and $SU(3)$ to the exceptional group G_2 (0-03).
- $\mathcal{T}_G$: quantum Yang–Mills theory on $\mathbb{R}^4$. A theory not on a lattice, nor in a finite box, but on the infinitely extended continuous four dimensions (0-03).
- Axioms: the list of conditions a quantum field theory must satisfy (the Wightman axioms or the Osterwalder–Schrader axioms) (0-03).
- Non-trivial: not collapsed to a Gaussian theory without interaction (a free field) (0-03).
- Agrees with asymptotic freedom at short distances: the behaviour at short distances matches the prediction of perturbation theory (asymptotic freedom; (4) of § 04) (0-03).
- $\Delta_G > 0$: the spectrum of the Hamiltonian H has no points in the interval $(0, \Delta_G)$ above the vacuum energy 0 (0-03).

Most slips happen by replacing one word of (1) with a nearby word. Examples: "Δ" → "the mass of a particle" (§ 05), "existence" → "the numbers appear to converge" (§ 07), "$\mathbb{R}^4$" → "the lattice" (§ 08).

Words used in this chapter

We borrow words from Parts 4 and 5 in advance. Here we give only one-line definitions, and leave the details to the chapters on the right Known .

Word	Meaning in this chapter	Details
β (lattice coupling)	The number multiplying the lattice action (2), with $\beta \propto 1/g^2$. Small β is strong coupling, large β weak coupling, and the continuum limit lies on the side of large β	5-01
Exponential decay, correlation length	The correlation of two observables decreases with distance r like $e^{-r/\xi}$. ξ is called the correlation length. The gap measured in units of the lattice spacing corresponds to $1/\xi$	2-08, 4-11
Wilson loop	The quantity W_γ obtained by multiplying the group elements of the links along a closed path γ and taking the trace of the representation matrix	5-01
Area law	For large rectangular loops, $\langle W_\gamma \rangle$ decreases like $e^{-\sigma \times \text{area}}$. σ is called the string tension. The lattice sign of confinement	5-02
Perimeter law	$\langle W_\gamma \rangle$ decreases only to the extent of $e^{-c \times \text{perimeter}}$. Slower than the area law	5-02
Centre of a group	The set of elements that commute with every element of the group. For $SU(N)$ the N elements $e^{2\pi i k/N} I$; for $SO(3)$ and G_2 the identity alone	1-04
Casimir number C_A	A number determined by each group, entering the coefficient of (4). N for $SU(N)$, 0 for $U(1)$	1-04
Spectral measure	The ρ of (5). The weight of how much of the energy-E component the state $O\Omega$ has	1-10
Transfer matrix	The operator T that advances one step in the time direction of the lattice. With $T = e^{-H}$ it gives the Hamiltonian H on the lattice	5-03
Reflection positivity	Positivity with respect to a mirror that flips time. With it the transfer matrix is a positive operator, and the Hilbert space and the Hamiltonian come back	5-03
Boundary condition	A placement of group elements fixed in advance on the links at the edge of a finite box	(7) of § 06
Infinite-volume limit	The limit of the measures as the box is extended to the whole of $\mathbb{Z}^d$ (a subsequential limit is allowed). The limiting measure is called a Gibbs measure	5-04
Phase transition	A point at which, as β is varied, the properties of the infinite system change as if with a kink	2-08

02

"The heart of the difficulty is infinity"

The way of speaking: physics proceeds by handling the infinities along the way cleverly, but mathematics does not allow that. That is why it is hard.

The correct core: the continuum theory needs renormalisation (4-04) to control quantities that diverge as the lattice spacing $a \to 0$. Bałaban's work is aimed mainly at this ultraviolet control (7-07) Known .

The slip: having granted that a construction on T^4 can be expected from extensions of known methods, the setters write 'no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume' (problem statement § 6) Known . On a finite periodic lattice with the lattice spacing fixed, the theory is a finite-dimensional integral and there are no infinities (7-11).

$$\langle F \rangle = \frac{1}{Z} \int F(U)\, e^{-\beta S_W(U)} \prod_{\ell} dU_\ell, \qquad 0 < Z \le 1 \tag{2}$$

dU_ℓ is the Haar measure of weight 1, and $S_W \ge 0$ the Wilson action (5-01). Example: for $SU(3)$ on a lattice of side 32 there are $4 \cdot 32^4$ links, each with 8 variables (the dimension of $SU(3)$), and the integral has $4 \cdot 32^4 \times 8 = 33{,}554{,}432$ variables Computed . Many, but finite.

What is hard is the theory with the box extended to $\mathbb{Z}^4$. Lecture material (Chatterjee 2020) wrote 'In 3D and 4D, mass gap has not been established at large β for any theory.' Known . Since then, for finite groups, Adhikari–Cao have shown exponential decay at large β (§ 04) Known . For continuous non-abelian groups, the gap at large β in three and four dimensions has still not been shown Known . What is missing is not the value of an infinity but an estimate uniform in the box and in β. The gap itself is also thought to have the form $e^{-1/(2b_0 g^2)}$, invisible to perturbation theory (0-02) Physics .

The ultraviolet side now: until 2025, "the ultraviolet is already settled" was also an overstatement. Bałaban's four-dimensional theorem (1989) comes with a hypothesis, and the theorem removing the hypothesis 'has not been published yet'. The limit of gauge-invariant observables, too, was at the level of taking subsequences by compactness (preface of Magnen–Rivasseau–Sénéor 1993; 7-07 § 05) Known . The construction published in 2026 takes the continuum limit along the whole sequence of cutoffs in a periodic box, passes to infinite volume along a subsequence, and builds a theory satisfying the Wightman axioms. The committee assesses this as "proved in Lean" Assessment (Part 6). Even so, one cannot go as far as saying "the ultraviolet is settled, so only the infrared remains". The uniqueness of the infinite-volume limit has not been shown, and the construction carries every step at weak coupling and stops at the step with lattice spacing 1. The seam that would put the result of that step into a

form the infrared arguments can use (the object of the limit, local estimates uniform in the volume, the image of observables) has not yet been made into a statement (16-03) Inferred .

A precise way of saying it

The ultraviolet side was supplied by the construction published in 2026, as a continuum limit in a periodic box (the committee's assessment). What the setters asked new ideas for is the gap uniform in the volume and the limit $T^4 \to \mathbb{R}^4$, that is, the infrared side, and that is still open. The seam connecting the two also remains.

03

"Two and three dimensions are already done"

The way of speaking: lower the dimension and it is done; only four dimensions remain.

The correct core: φ^4 has been constructed in two and three dimensions (problem statement § 6.2), and the two-dimensional abelian Higgs model has been shown up to the axioms. The problem statement calls it 'the only complete example of an interacting gauge theory satisfying the axioms' Known .

The slip: for non-abelian Yang–Mills the story is different. The two-dimensional construction (7-02) relies on an exactly solvable structure, the plaquettes becoming independent once the gauge is fixed, and there are no propagating degrees of freedom. In three dimensions, 'a construction of the 3D YM measure … even on T3, remains open' (Chevyrev 2022), and Bałaban (1985) goes as far as uniform estimates of the partition function on finite lattices (7-06) Known . The core of the difference is the dimension of the coupling constant.

$$[g^2] = (\text{mass})^{4-d} \quad (3)$$

Example: for $d = 3$, $g^2\ell$ (ℓ a length) is the dimensionless coupling, and it gets smaller at shorter distances. Interaction weakening at short distances is the foothold for the construction. For $d = 4$ this foothold survives only as a slow logarithmic change. Precisely, for $d = 4$, g^2 becomes dimensionless, and the dependence on scale is only the logarithm from the one-loop running ((4) of § 04) Known (a table by dimension is in 16-01).

A precise way of saying it

For non-abelian groups, two dimensions has been constructed (relying on an exactly solvable structure). In three dimensions the continuum measure has not been constructed even in finite volume. Four dimensions is the critical dimension, where the coupling constant becomes dimensionless, and the construction published in 2026 built a theory satisfying the Wightman axioms as a limit of Wilson's lattice (the committee's assessment). Even in four dimensions, the mass gap and the uniqueness of the infinite-volume limit have not been shown.

Even in two dimensions, for the continuum limit and mass gap of asymptotically free sigma models there was no claimed proof up to October 2026 (a manuscript with a claim made public on 2026-10-06 is unverified; 16-01 Table 5).

04

"There is a gap because it is non-abelian"

The way of speaking: the photon ($U(1)$) is massless, but gluons are non-abelian and interact with themselves, so there is a gap.

The correct core: in four-dimensional $U(1)$ lattice theory, the area law breaks down at large β (Guth 1980; the complete proof is Fröhlich–Spencer 1982), and the theory is in a massless phase (Fröhlich–Spencer 1982)Known . Both are said to be results mainly for Villain-type actions (actions of a different form from the Wilson action of (2)). The one-loop coefficient is also determined by the group.

$$\mu \frac{dg}{d\mu} = -b_0\, g^3 + \cdots, \qquad b_0 = \frac{11}{3} \cdot \frac{C_A}{16\pi^2} \quad (4)$$

The Casimir number C_A of the adjoint representation is N for $SU(N)$ and 0 for $U(1)$, and a theory without matter fields is asymptotically free only when non-abelian (4-06) Known .

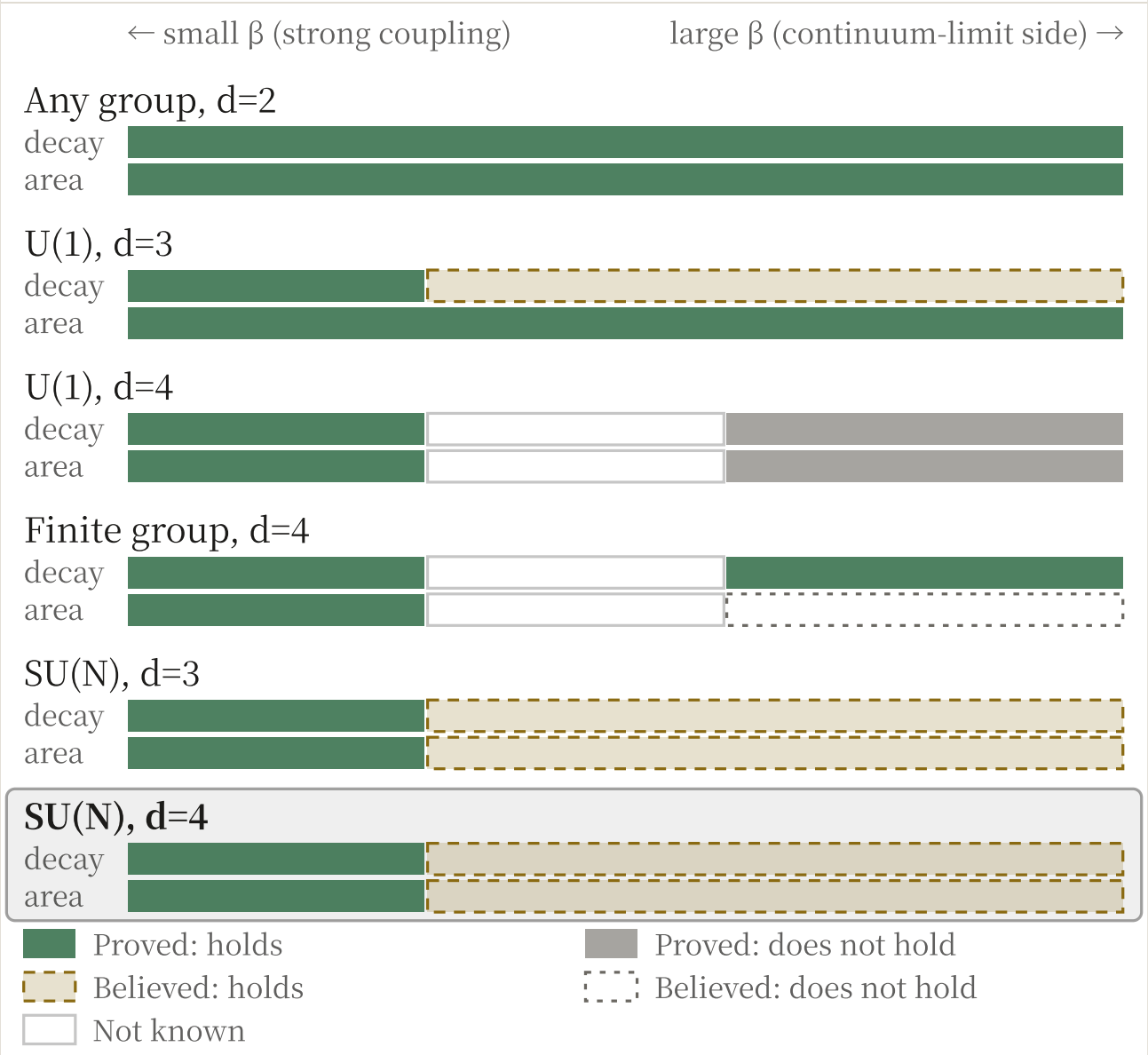
The slip: the lattice theorems divide by group and dimension (Figure 1, Table 1). At strong coupling (small β), exponential decay holds for every group, and the area law also holds for loops in representations that act non-trivially on the centre (Osterwalder–Seiler 1978)Known . For groups with trivial centre (G_2, $SO(3)$, finite groups with trivial centre), large loops are said to follow the perimeter law even at strong coupling, through screening by gluons Physics .

The area law at strong coupling is not evidence of confinement, because it holds at small β even for four-dimensional $U(1)$, which does not confine (Chatterjee 2018). Example: three-dimensional $U(1)$ is abelian, yet the area law holds for every β (Göpfert–Mack 1982; said to be for a Villain-type action)Known . Four-dimensional

finite groups (including non-abelian ones) have exponential decay even at large β (Adhikari–Cao) Known , but are believed not to confine Physics .

What matters is the following. For continuous non-abelian groups it is believed that there is no phase transition at finite β and that the correlation length diverges as $\beta \to \infty$ Physics . In that limit the gap measured in units of the lattice spacing, $1/\xi$, approaches 0, but physics holds that the gap in physical units (for example $m/\sqrt{\sigma}$) stays positive Physics . This is what needs to be shown (5-06, 5-10). An argument that does not distinguish groups would wrongly conclude that four-dimensional $U(1)$ also has a gap.

Figure 1 Choose a group and dimension with the buttons or a row of Table 1; that row is highlighted, and its content and sources appear below. In each row the upper band is exponential decay, the lower band the area law. Horizontal: β (the divisions are schematic)



SU(N), d=4: Theorems only on the strong-coupling side (Osterwalder–Seiler, Shen–Zhu–Zhu).

Known Table 1 The content of Figure 1 (theorems from primary sources and prospects that are "believed". The columns are for the side of large β). At small β, exponential decay is a theorem in every row, and the area law is a theorem for loops in representations that act non-trivially on the centre (in the row of finite groups, for groups and representations whose centre is non-trivial). Behaviour at intermediate β differs from group to group, and four-dimensional $\mathbb{Z}_N$ (for large N) is said to have a massless phase (note in the table of sources).

Group, dimension	Exponential decay (gap)	Area law (confinement)
Any group, $d = 2$	Yes	Yes (all β)
$U(1)$, $d = 3$	Said not to have been shown as of the lecture material (2020) (in physics it exists)	Yes (Göpfert–Mack; said to be for a Villain-type action)
$U(1)$, $d = 4$	No (massless phase; Fröhlich–Spencer)	No (the area law breaks down; Guth, Fröhlich–Spencer)
Finite group, $d = 4$	Yes (Adhikari–Cao)	Believed not to hold
$SU(N)$, $d = 3$	Unproved (believed to hold)	Same
$SU(N)$, $d = 4$	Unproved (believed to hold)	Same

A precise way of saying it

In four dimensions $U(1)$ is massless at weak coupling and non-abelian groups are asymptotically free; in that sense the group matters. But "there is a gap because it is non-abelian" is not a line of proof. For finite groups exponential decay has been shown even at weak coupling, and the weak-coupling side of continuous non-abelian groups is unproved in three and four dimensions.

05

"The mass gap is the mass of the lightest particle"

The way of speaking: the mass gap is the mass of the lightest glueball (a bound state of gluons), about 1.7 GeV.

The correct core: physics holds that there is an isolated one-particle state (the 0^{++} glueball) at the lower end of the gap, and measures it on the lattice. For pure $SU(3)$, $r_0 m = 4.21(11)(4)$ (Morningstar–Peardon 1999) (5-08) Physics .

There are three **slips**. First: what the problem statement asks for is a gap in the spectrum, and isolated one-particle states are one of the 'natural extensions' (0-03) Known . Second: what the lattice measures is the decay of correlations. Consider the

state $O\Omega$ obtained by applying a gauge-invariant operator O to the vacuum Ω. ρ is the weight (spectral measure) of how much of the energy-E component this state has. The connected correlation of O with itself separated by imaginary time τ can be written with ρ as follows (0-04 § 07, 4-11) Known .

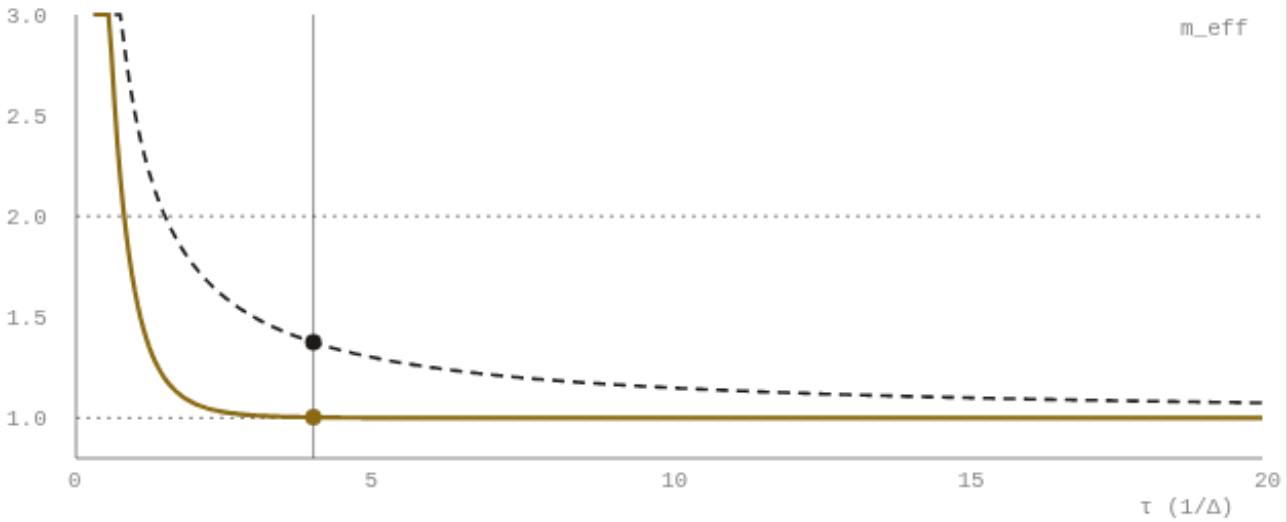
$$C(\tau) = \int_{[\Delta,\infty)} e^{-E\tau}\, d\rho(E), \qquad m_{\mathrm{eff}}(\tau) = -\frac{d}{d\tau}\log C(\tau) \quad (5)$$

m_{eff} approaches the lower end of the support of ρ as $\tau \to \infty$, but does not tell whether that lower end is an isolated point or the edge of a continuous spectrum. We compare two models ($\Delta = 1$).

A (with particle) : $d\rho = Z\,\delta(E-1)\,dE + c\,(E-2)_+^{1/2}\,dE$, B (no particle) : $d\rho = (E-1$

$(x)_+^{1/2}$ is $\sqrt{x}$ for $x > 0$ and 0 otherwise. From $\int_{E_0}^{\infty}(E-E_0)^{1/2}e^{-E\tau}dE = \Gamma(\tfrac{3}{2})e^{-E0\tau}\tau$ $^{-3/2}$, B has $m_{\mathrm{eff}} = 1 + \frac{3}{2\tau}$. Example: A comes within 1% at $\tau \approx 3.2$ ($c = 1$) to 4.2 ($c = 5$), while B only comes within it at $\tau = 150$ (Table 2). Fitting a single exponential in an intermediate window, B shows a "particle" heavier than Δ Computed . Reading the flat part as the mass of one particle is an interpretation at the level of physics Physics .

Figure 2 Changing the weight c of the continuous part of A (Z = 1) and the time τ changes the curves of the effective mass (solid A: with particle; dashed B: without particle) and the readings. Dotted lines are 1 and 2



c = 1.0, τ = 4.0: A 1.0028, B 1.3750 (Δ = 1)

Computed The numbers in this figure were computed on this machine (the closed form of (6) computed on the spot).

Computed Table 2 $m_{\mathrm{eff}}(\tau)$ ($\Delta = 1$, $Z = 1$). The last row is the first τ within 1%. The difference between the closed form and numerical integration is below 4×10^{-5} for $\tau \le 16$.

τ	A ($c=1$)	A ($c=5$)	B
1	1.6147	2.5495	2.5000
2	1.0712	1.3061	1.7500
4	1.0028	1.0138	1.3750
8	1.0000	1.0001	1.1875
16	1.0000	1.0000	1.0938
Within 1%	3.15	4.23	150

Third: the unit GeV does not come out of pure Yang–Mills itself; it borrows the scale r_0 of lattice QCD with quarks. The lattice papers also state that pure gauge theory and QCD are different theories (Athenodorou–Teper) Physics . "Gluons acquire a mass" is also a different matter: the mass term $m^2 \operatorname{tr} A_\mu A_\mu$ is not gauge-invariant Known .

A precise way of saying it

The mass gap is a gap in the spectrum. Physics holds that there is an isolated particle (a glueball) at its lower end, and measures its mass. What the decay of correlations tells is only the position of the lower end.

06

"Show confinement and the gap follows"

The way of speaking: confinement (that a quark cannot be taken out alone) and the gap are the same thing; show one and the other follows.

The correct core: physics sees the string tension σ and the gap m as two faces of one scale (17-01). Precisely, physics holds that the ratio $m/\sqrt{\sigma}$ approaches a finite positive number independent of β and the lattice spacing Physics . On the lattice there are also conditional theorems. Chatterjee (2021), for G a closed connected subgroup of $U(n)$, assumed decay in the following strong form Known .

$$\exists\, K_1, K_2 > 0 \;\; \forall B, \delta, f, g: \quad \left|\langle fg\rangle_{B,\delta} - \langle f\rangle_{B,\delta}\langle g\rangle_{B,\delta}\right| \le K_1 e^{-K_2 \operatorname{dist}(f,g)} \quad (7)$$

B is a cubic box, δ the boundary condition of the box, f, g local functions taking values in $[-1,1]$ (functions determined only by the variables near a single link), and $\operatorname{dist}(f,g)$ the distance between them. The constants K_1, K_2 do not depend on B, δ, f or g, and are determined by G, β and d alone. This order is the point: if the constants were allowed to differ from box to box and from boundary condition to boundary condition, the assumption would say almost nothing.

The Theorem from Uniform Decay to the Area Law (Theorem 2.4 of Chatterjee 2021) is that if (7) holds, centre symmetry is not broken, and the area law holds for rectangular loops in representations that act non-trivially on the centre Known . The author writes that at small β (7) can be shown for almost any lattice gauge theory, while also writing that it is not clear whether the decay believed in four dimensions includes this strong form.

The slip: (a) the problem statement does not ask for confinement; it only lists it among the 'natural extensions' Known . (b) "Any compact simple group" includes G_2 and others with trivial centre Known , and such groups are believed possibly not to confine at weak coupling (the remark on $SO(3)$ in the same paper) Physics . The problem statement asks for a gap there too. (c) Example: for three-dimensional $U(1)$ the area law has been shown (Göpfert–Mack; said to be for a Villain-type action). On the other hand, the gap at large β is said not to have been shown as of the lecture material (2020). For finite groups, exponential decay at large β is a theorem (Adhikari–Cao) Known , and they are believed not to confine Physics (Figure 1).

However, the author cautions that the "decay" spoken of for theories such as finite groups refers to the decay of two-point correlations under a particular boundary condition, which differs from the strong form of (7) (uniform under arbitrary boundary conditions) (the remark after Theorem 2.4 of Chatterjee 2021) Known . So the existence of theories that decay but do not confine does not contradict the Theorem from Uniform Decay to the Area Law.

A precise way of saying it

Confinement and the gap are different claims (17-02 § 02), and in the four-dimensional continuum theory neither implication is a theorem. On the lattice, for closed connected subgroups of $U(n)$, the area law for loops in representations that act non-trivially on the centre follows from the strong form of decay (7). What the problem statement asks for is the gap alone.

07

"It has been confirmed by lattice numerics"

The way of speaking: large-scale computations give glueball masses precisely, so the existence of the gap is known.

The correct core: example: for $SU(2)$, $m_{0^{++}}/\sqrt{\sigma}$ is 3.844(61) (Lucini–Teper 2001) and 3.781(23) (Athenodorou–Teper 2021), in good agreement Physics .

The slip: what is being confirmed is consistency under the assumption that it exists (17-13). In a finite box at finite β the gap is always positive (17-13) Known , so a positive number coming out does not test the claim. The value in the continuum is

$$\frac{m(a)}{\sqrt{\sigma(a)}} = \frac{m(0)}{\sqrt{\sigma(0)}} + c\,a^2\sigma \quad (8)$$

an extrapolation that assumes this form Physics , and the σ in the denominator passes through a correction that treats the flux tube as a Nambu–Goto string (17-13).

A precise way of saying it

Lattice numbers are precise predictions under the assumption that the theory exists, and strong physical evidence for the gap. They are not part of a mathematical proof, but they are a target the constructed theory must hit.

08

"Showing it on the lattice is enough", "Build it in a finite box and the rest is technique"

The way of speaking (lattice): show on the lattice that the correlation length is finite for every β, and that solves the prize problem.

The correct core: the lattice correlation-length problem of Chatterjee (2018) is a lattice version of the uniform gap: for any infinite-volume limit with a four-dimensional non-abelian compact Lie group $G \subseteq U(N)$ ($N \geq 2$), show the following about the length $\xi(\beta)$ over which plaquette correlations decay Known .

$$0 < \xi(\beta) < \infty \quad (\text{for all } \beta > 0), \qquad \lim_{\beta\to\infty} \xi(\beta) = \infty \quad (9)$$

The slip: the continuum limit, the axioms on $\mathbb{R}^4$, non-triviality and agreement at short distances are not in (9), and (9) alone does not solve the prize problem. The converse implication is not obvious either (5-09) Known .

The way of speaking (finite box): if it can be built on a finite box T^4 (the continuous four-dimensional torus), all that remains is to enlarge the box.

The slip: the problem statement says that construction on a compact space alone 'would represent a major breakthrough', and places the wall of the uniform gap beyond it Known . Example: even on a periodic lattice box (the lattice T^4), reflection positivity gives a transfer matrix, and the gap in a finite box is positive Known . The gap in a finite box is a quantity one naturally has. What is needed is $T^4 \to \mathbb{R}^4$ and a gap that does not collapse there (7-10). The construction published in 2026 gave a construction in a

periodic box and an infinite-volume theory along a subsequence Assessment , but what it has for correlations goes only as far as polynomial estimates uniform in the phase; there is no exponential decay (16-02 § 04). It is a current example of where "the rest is technique" in this way of speaking does not hold.

A precise way of saying it

(9) is a lattice version of the uniform gap, but it does not reach the requirements of the prize (continuum, $\mathbb{R}^4$, axioms, non-triviality). A construction on T^4 is also a big step, and beyond it remain the uniform gap and the infinite-volume limit.

09

"The mass gap comes from the uncertainty principle"

The way of speaking: squeeze a particle into a small region, and by the uncertainty principle its momentum spreads, so the energy does not go down. The mass gap is that lowest energy, and its origin is the uncertainty principle.

The correct core: the uncertainty relation is a theorem Known . For a state ψ in a Hilbert space and self-adjoint operators A, B,

$$\Delta_\psi A \, \Delta_\psi B \;\geq\; \tfrac{1}{2}\,\big|\langle \psi, [A,B]\psi\rangle\big| \qquad (10)$$

holds (Robertson 1929, Schrödinger 1930), and the proof is a single Cauchy–Schwarz inequality. Applied to $A = x,\ B = p$ it gives $\Delta x\,\Delta p \geq \hbar/2$ (Kennard 1927); equality holds for Gaussians, the ground state of the harmonic oscillator being an example (2-06 § 03). In the language of Fourier analysis, it is the theorem that a function and its transform cannot both be made narrow (1-11; there are also the strong forms of Hardy and Beurling). The kinetic energy of a state localised in a region of size R does not go below about $\hbar^2/(2mR^2)$, or, relativistically, about $\hbar c/R$. Physics In the picture of physics, a glueball is a lump of colour electric flux of size about $1/\Lambda$, and this count is why its mass is about Λ (17-01–17-03).

The slip: (a) what the theorem says is "given a length R, an energy $\hbar c/R$ is guaranteed"; it does not supply R itself. A free massless field (the photon, U(1)) also obeys (10), but its gap is 0. The gap of a free field of mass m is just the m put in by hand (2-06 § 08). The Yang–Mills action has neither a mass term nor a ruler of length (3-02). The length $1/\Lambda$ comes from a single scale being born out of a dimensionless coupling through a logarithmic flow (dimensional transmutation, 4-07), and the localisation comes from colour electric flux being squeezed into a closed tube (17-02, 26-03). Neither is a theorem in the four-dimensional continuum theory. (b) In the language of the Euclidean measure, uncertainty is not a principle but a duality. Exponential decay of

correlations and the absence of light modes are two faces of the same thing (4-11), and the content of (10) is already inside the definition of the gap. It is a restatement, not a cause. On this book's road the gap is the Poincaré constant of the vacuum measure (24-05), and uncertainty is dissolved inside that inequality. (c) In units with $\hbar = c = 1$, what (10) gives is $E \gtrsim 1/R$, and for the gap to be positive one needs R to be bounded, that is, that there is only one scale (17-12).

A precise way of saying it

The uncertainty relation is a theorem: given a length R, it guarantees an energy $1/R$. What the mass gap needs is the R side. That a single scale is born and that excitations are localised at that scale has not been shown for continuum non-abelian groups in four dimensions.

10

Detour: other ways of speaking one comes across

A section you may skip.

Four more commonly seen ways of speaking, corrected briefly.

- "Give infinity a symbol (1/0) and it is settled". The work of assigning values to divergences is already in order as mathematics inside perturbation theory, and the problem still remains (7-12).
- "Holography has shown the gap". What can be computed are models on the side of $N \to \infty$ and strong coupling, and the correspondence itself is a conjecture of physics (17-10) Physics .
- "φ^4_4 is trivial, so it collapses" / "It is asymptotically free, so non-triviality is automatic". Both are wrong; non-triviality is a condition to be shown separately (7-08).
- "It is a problem about QCD". The object is pure Yang–Mills (0-03).

What this chapter can say

Said	Not said
Known The separation of the correct core and the slip of nine ways of speaking, against the problem statement and theorems in primary sources	The existence of the mass gap. Unsolved, and this chapter shows nothing about it (the existence of the continuum limit was shown by the published construction and assessed by the committee Assessment ; whether that proof is correct is outside this chapter)

Said	Not said
Known That the lattice theorems divide by group and dimension (Figure 1, Table 1; the dates of the table are as in the table of sources)	Physics The gap and confinement at large β for continuous non-abelian groups in four dimensions. Unproved
Computed That the decay of correlations alone does not decide whether there is a particle (model (6), Table 2)	Physics That the continuum values of lattice numbers (values passed through the extrapolation (8)) are the values of the constructed theory
Known That the area law follows from decay in the strong form of (7) (closed connected subgroups of $U(n)$, representations acting non-trivially on the centre)	The current status of the gap of three-dimensional $U(1)$ at large β (according to the lecture material (2020)), and the phase of $\mathbb{Z}_N$ at intermediate β (unchecked)

Sources and reproduction

Item	Kind	Source / tools
Number of variables (§ 02), Table 2, Figure 2	Computed	python3 (Table 2 cross-checked by numerical integration). Figure 2 computes (6) in the browser
Problem statement, lecture material, (7), (9)	Known	Jaffe–Witten, the Clay problem description (2000). Chatterjee's slides "Yang–Mills on the lattice" (2020; the quotation in § 02 is a statement as of 2020). Chatterjee, arXiv:1803.01950 (§ 4, the lattice correlation-length problem). Chatterjee, arXiv:2006.16229, Comm. Math. Phys. 385 (2021) 1007–1039 (Definition 2.3, Theorems 2.2 and 2.4 and remarks). Text checked
Ultraviolet, three dimensions, (4)	Known	Bałaban, CMP 122 (1989) 355–392; CMP 102 (1985) 255–275. Magnen–Rivasseau–Sénéor, CMP 155 (1993) 325–383 (the quotation from the preface follows 7-07). Chevyrev, arXiv:2202.13359 (opening checked). (4) is from the PDG 2024 review of QCD
$U(1)$	Known	Guth, Phys. Rev. D 21 (1980) 2291–2307. For Guth, bibliographic data only; the content follows Chatterjee 2018 § 4. Göpfert–Mack, CMP 82 (1982) 545–606 (abstract (INSPIRE): Villain action, positive string tension at all couplings, the continuum limit at fixed mD is a free scalar field. Text not read. The year is Crossref's year of publication; INSPIRE gives 1981). Fröhlich–Spencer, CMP 83 (1982) 411–454 (bibliographic data checked; the content follows Chatterjee 2018 § 4). The type of action (Villain type) of Guth and Fröhlich–Spencer, and the massless phase of $\mathbb{Z}_N$ at intermediate β, are unchecked

Item	Kind	Source / tools
Strong coupling, finite groups	Known	Osterwalder–Seiler, Ann. Phys. 110 (1978) 440–471 (abstract: existence and analyticity of the infinite-volume limit at strong coupling, Wilson's confinement estimate. The condition attached to the strong-coupling area law ⟨representations acting non-trivially on the centre⟩ and the perimeter law for groups with trivial centre are unchecked in the text). Shen–Zhu–Zhu, CMP 400 (2023) (abstract checked). Adhikari–Cao, arXiv:2202.10375, Ann. Probab. 53(1) (2025) (Theorem 1.1 checked)
Lattice numbers	Physics	Morningstar–Peardon, PRD 60 (1999). Lucini–Teper, JHEP 06 (2001). Athenodorou–Teper, JHEP 11 (2020); 12 (2021) (tables checked)

Next chapter to read: 16-09 The stages a proof must contain. To read from the side of this book's study, 23-01 The line of construction and the diffusive-relaxation-time assumption.

The stages a proof must contain — the table of seven stages

> This chapter is a map of open questions. § 02, § 05 and § 06 are guide material.

What this chapter shows — The seven stages that a proof of the mass gap of four-dimensional Yang–Mills must contain, whichever road it starts from (the vessel, the ultraviolet, the gap on a small T^2, confinement on a small T^2, the crossover band, R^4, general groups). For each stage, what is known in the world, what this book has shown, and what has not yet been shown (within what we surveyed), set out with labels in one table. That the mass gap and confinement (the area law) are different statements. A figure of the two roads meeting at the band. An index of which cell of the table each chapter from Part 23 on fills, and the correspondence with the six continuum tasks of 16-07. The stages taken on by the construction published in 2026 (part of the vessel, existence on the ultraviolet side, existence and the axioms for R^4, existence for all groups), and the stages that remain.

Prerequisite chapters — 16-07 The list of open tasks (the twelve tasks and the three strengths of arrows) and 16-08 § 06 (confinement and the gap are different claims). The words for the stages follow 7-10 (what is reached at each stage in four dimensions). The questions that remain after the published construction are in 16-02, and where this book's holdings stand is in 16-04.

Said first — This chapter lays out stages; it settles none of them. The division into the seven stages a proof must contain is an arrangement often used in constructive field theory, cut again to fit the work of Parts 25 and 26 of this book. The marks ■◐□ in the table are marks of the state of the record, not a graduation of how much is proved (§ 02). This site shows neither the existence of the mass gap nor the construction of the continuum limit. The label Assessment means "according to the committee's assessment report" ("Labels" in 6-00).

The order of this chapter

1. The table of the seven stages a proof must contain — known, this book and unknown for each stage (Table 1)
2. How to read ■◐□ — marks of the state of the record, not a graduation

3. The mass gap and confinement are different statements — the road to each, and the missing arrow between them
4. The two roads meet at the band — the flow of the logic (Figure 1)
5. Which chapter fills which cell — an index of Parts 23–27 (Table 2)
6. Correspondence with the tasks of 16-07 — the six continuum tasks and the seven stages a proof must contain (Table 3)

01

The table of the seven stages a proof must contain — known, this book and unknown for each stage

We split the way from the lattice to the theory on $\mathbb{R}^4$ and the mass gap into seven stages. In order: (1) the vessel (the measure on the lattice, and the mechanism that assembles a quantum theory from it), (2) the ultraviolet (uniform control at all scales that does not break down as the lattice spacing a is sent to 0; as existence it was shown by the published construction, and what remains on this book's road is the seam), (3) the gap on a small T^2, (4) confinement on a small T^2, (5) the crossover band (the band for short), (6) the theory on R^4, (7) general groups.

The "small T^2" of the two small-T^2 stages (rows 3 and 4 of Table 1) is the case where four-dimensional space is written as $\mathbb{R}^2 \times T^2$ and the side L of the T^2 is taken much smaller than the physical length $1/\Lambda$ (Λ is the mass scale that the theory generates by itself, 4-07). Putting 't Hooft's twist (a boundary condition that glues opposite sides shifted by an element of the centre of the group) on the T^2 removes the zero modes (directions in which one can move without raising the action), and there is a prospect of keeping the coupling weak at scales up to L. It is a waypoint for splitting the problem into the stages up to there (the two small-T^2 stages) and the stage of widening the T^2 (the band stage). The content is treated in Part 25 (25-00).

The band of the band stage is the intermediate scale $L \sim 1/\Lambda$ at which the coupling passes from weak to strong. In four dimensions no tool has crossed this scale non-perturbatively (26-00). In the language of the space road, a statement that crosses the band takes the following form.

$$\ln \frac{1}{\theta(L,L)} \;\geq\; c\,L^2 \qquad (\text{for all } L, \text{ the same } c>0) \quad (1)$$

$\theta(L,L)$ is the sector bias when one vortex winding the T^2 is put into a container of side L (a quantity that moves away from 1 the "cheaper" the vortex is). (1) is an area law in the

form of 't Hooft's criterion; that "θ moves away from 1" alone is not enough is treated in 26-01.

After the construction published in 2026. The published construction takes on part of the vessel stage (reflection positivity of the original Wilson measure, the Chessboard Estimate, reconstruction; 12-01, 12-02, 12-08), existence for the ultraviolet stage (the continuum limit in a periodic box; Parts 8–11), existence and the axioms for the R^4 stage (infinite volume along a subsequence, phase decomposition, the Wightman axioms for each phase; 12-05, 12-07), and existence for the general-group stage (any compact simple group). The committee assesses this as "proved in Lean" Assessment . We have written what it takes on into the "known in the world" column of Table 1. What it does not take on is the two small-T^2 stages (twisted boundary conditions are not treated), the band stage, a uniform mass gap, and uniqueness of the infinite-volume limit (16-02). For the ultraviolet stage, even after existence has been shown, there remains the seam that puts the ultraviolet result into a form the infrared arguments can use (16-03) Inferred .

Table 1 The seven stages a proof must contain (to the right of the stage name is the mark: ■ closed in the record / ◐ partial / □ open. Labels are per claim within the row. "Chapters" is the same as the index of Table 2)

Stage and content	Known in the world	Shown in this book (with labels)	Unknown, and chapters that fill it
1 Vessel ◐ The lattice, reflection positivity, block observables, compactness. From decay on a dense set to a gap of the transfer matrix	Cited Reflection positivity and the transfer matrix (Osterwalder–Seiler 1978, 5-03), reconstruction (4-10). Assessment Shown by the published construction: reflection positivity of the original Wilson measure for planes through lattice points and planes through midpoints of edges, the Chessboard Estimate, reconstruction (Part 12)	Proved here Reflection positivity of z (positivity, under reflections through lattice points, of the ensemble summed over twists; the algebra of the kernel is in Lean: ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`), finite sums with non-negative weights). Lean: GapFromDecay (`GapFromDecay.gap_of_decay_dense`) (an abstract lemma for a bounded self-adjoint operator, with uniform decay on a dense set placed as a hypothesis)	Writing out the whole construction of the vessel (at present a strand) Chapters: 5-03, 26-04, 26-05

Stage and content	Known in the world	Shown in this book (with labels)	Unknown, and chapters that fill it
2 Ultraviolet ◐ Multiscale analysis uniform in a, from the lattice spacing a up to the scale L (locality of the effective action, rarity of large fields, flow of the coupling). Existence is from the published construction; what remains is the seam	Assessment Shown by the published construction: in a periodic box, the continuum limit along the whole sequence of cutoffs (Parts 8–11). It carries every step at weak coupling and stops at the step with lattice spacing 1 (16-03). Cited Before that, Bałaban's series (CMP, 1980s; parts remain at the central stage that the original sources call "unpublished", 7-07)	Proved here Only parts at the one-loop level (the perturbative level). The record so far placed the whole of ultraviolet control as an assumption, Assumption	The seam: fixing the theory at the step with spacing 1 as an object, local estimates uniform in the volume, the bridge for observables (writing continuum correlations as expectations at the step with spacing 1). On this book's road, in addition: the translation into quantities fixed by the measure, the construction with twisted boundary conditions and in elongated boxes, and how reflection positivity is carried Inferred Chapters: 16-03, 7-07, 25-00, 26-11, 27-02 ultraviolet control
3 Gap on a small T^2 ◐ The twist removes the zero modes, and the coupling stays weak up to the size L of the T^2. The gap of the transfer matrix $\geq c/L$ is uniform in a	Cited 't Hooft's twisted boundary conditions and flux (1979, 1981)	Proved here The twisted part: uniqueness of the vacuum that absorbs the twist, the gap $4\sin^2(\pi/2N)$ of the two-dimensional lattice Laplacian, and $S_{\min} > 0$ in the (1,1) sector (SU(2), lattice, N fixed). The form uniform in a holds only under the assumption of an ultraviolet construction in a box with twisted boundary conditions	A lower bound on the gap uniform in a, without that assumption. The published construction treats only periodic cubic boxes, so the construction in a twisted box becomes the first question of this book's road Inferred Chapters: 25-00, 25-01, 27-02 small-T^2 gap
4 Confinement on a small T^2 ◐ Bounds on containers and on insertions reduce confinement to "the insertion of one vortex" and "independence between regions"	Cited The container identity (Mack–Petkova 1980)	Proved here The reduction (on a finite configuration space, with positive weights and a bijection that flips the sector. Lean: Containers (`Containers.container_bound`), InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`), the algebra of finite sums). Proved here Slab confinement (with the lattice spacing and the T^2 fixed, an area law at every β. In the continuum limit the tension is 0)	Two inputs: uniform acceptance of insertions, and frame decorrelation Assumption Chapters: 25-03, 25-04, 25-05, 25-06

Stage and content	Known in the world	Shown in this book (with labels)	Unknown, and chapters that fill it
5 Band ☐ That (1) holds with the same c as the T^2 is widened. Crossing the intermediate scale $L \sim 1/\Lambda$ non-perturbatively	Cited 'New ideas are needed' (Jaffe–Witten § 6.5, the task of a uniform gap). The published construction gives no foothold inside the band (it carries every step at weak coupling) Inferred . The inequality of Tomboulis–Yaffe (1985) (from 't Hooft's criterion to the area law)	Proved here The correct form ($\theta \leq \theta_0$ is not enough). Proved here A conditional reduction to the lateral-widening inequality (assuming three things: the base point, frame decorrelation and lateral widening. The algebra of the iteration is in Lean: Doubling (`Doubling.iterate`)). Proved here The negative lemma that structural inputs alone cannot cross. Proved here The renormalisation comparison at the one-loop level ($\beta - \beta' = 0.258$). Reading Numerical signs (26-09)	The renormalisation comparison, non-perturbatively. The band itself. Stating the two ends of the band in the same quantity and the same class of actions (the strong side is β of the Wilson action in this book's Lean window; the weak side is the coefficients of the effective action at the entrance to the one-step estimate of the published construction) Inferred Chapters: 26-01–26-07, 26-09
6 R^4 ◐ The infinite-volume limit, the OS axioms, recovery of rotation invariance, non-triviality	Cited Reflection positivity survives weak limits; OS reconstruction (16-07 § 03). Assessment Shown by the published construction: the infinite-volume limit along a subsequence, phase decomposition, the Wightman axioms for each phase (Part 12). Checks at short distances (Part 13)	Assumption Only the strand that, under the band stage, most of it is the standard argument	Uniqueness of the infinite-volume limit (probably not needed for the mass gap of a single theory Inferred). If decay is obtained with twisted boundary conditions or in a pure phase, identifying the theory built from that state Inferred Chapters: 16-07, 27-01, 27-02 separate box
7 General groups ☐ SU(N), and groups without centre (G_2 and others)	Cited For groups with trivial centre, the arrows that use the centre disappear (16-07 § 07). Assessment Existence was shown by the published construction for any compact simple group	None (Parts 25 and 26 of this book are written for SU(2))	Everything on the side of the mass gap. The tools of vortices cannot be used for groups without centre Chapters: 16-07 § 07, 27-02 separate box

What to read from Table 1. (a) The "□" stages are the band and the general-group stage, which changes the group. The "◐" of the ultraviolet stage means that the continuum limit in a periodic cubic box was shown by the published construction, but there is no construction in twisted or elongated boxes and no statement of the seam (the same mark as 27-02 Table 1). What is open now at the ultraviolet stage is not existence but the seam; the band remains a long-standing open problem of the field. (b) The "◐" of the two small-T^2 stages is in both cases a "◐" under the assumption of an ultraviolet construction in a box with twisted boundary conditions. As long as that construction is missing, the marks of these two stages do not go up. (c) Of the claims written "Proved here" in this book, most of the chapters that place the paper proofs in the book are chapters still to be written (Parts 25 and 26). Until they are placed, the rows of the ledger (27-02) do not disappear.

02

How to read ■◖□ — marks of the state of the record, not a graduation

The marks of Table 1 mark how the record of this book's work treats each stage. ■ means closed in the record (there is a paper proof, and where there is a Lean entry its arithmetic is closed too), ◐ means part is closed and the rest stays a strand or a numerical sign, and □ means open.

These marks are not a graduation of how much is proved. There are three reasons. First, ■ says only that something is closed within the conditions written in the row, not the whole stage. Second, a ◐ stage does not necessarily have "less remaining" than a □ stage. The ◐ of the small-T^2 gap stage says that the twisted part is closed, but the rest depends on the seam of the ultraviolet stage (the construction in a twisted box; open). Third, the book's labels (proved here, cited, computer-assisted, reading, assumption) attach to each claim, while the marks attach to each stage. It is normal for "Proved here" and "Assumption" to stand side by side in the same stage. We do not add the marks up and read them as "so many of the seven stages a proof must contain".

Example. The reflection positivity of z in the vessel stage is closed in the record (a claim corresponding to ■). Even so, the mark of the vessel stage is ◐. This is because writing out the whole construction of the vessel (from block observables to the gap of the transfer matrix) remains a strand.

03

The mass gap and confinement are different statements — the road to each, and the missing arrow between them

What the problem statement asks for is the mass gap (that the spectrum of the Hamiltonian H of the quantum theory is separated by $\Delta > 0$ above the vacuum, 4-11); it does not ask for confinement (the area law of Wilson loops). As in 16-08 § 06,

confinement and the gap are different claims, and in the four-dimensional continuum theory neither implication is a theorem Cited . Physics sees the two as two faces of one scale (the ratio $m/\sqrt{\sigma}$ approaches a finite positive number), but that is a statement at the level of expectation Reading (physics). The implication from the area law to the mass gap does not hold as a general implication, because the area law is a statement about a gap in the charged sector, and the mass gap about a gap in the neutral sector (16-07 § 02).

The two roads of this book arrive at different statements.

- **The road to the gap** (the small-T^2 gap stage + the transfer matrix): carry the gap of the transfer matrix on a small T^2 (the small-T^2 gap stage) through the band to $\mathbb{R}^4$ with the same constant. If it can be carried, the gap survives in the limit (the rule of the gap of 16-07 § 03, with premises (i)–(iii)). The road of Part 23 (from the direction of time) also heads to the same statement of the gap, through the diffusive-relaxation-time assumption and the one-sided bridge.
- **The road to confinement** (the small-T^2 confinement stage, the band stage, the R^4 stage): make confinement on a small T^2 uniform in the size of the T^2 through the band, and turn it into an area law in the theory on $\mathbb{R}^4$ (the R^4 stage).

There is no known arrow joining the two roads, that is, no known road "from the area law to the mass gap". Even the statement of what must be added to get a gap from the area law is not yet settled (the last arrow of 27-01, the separate box of 27-02). So even if the road to confinement were closed entirely, that alone would not reach the requirement of the problem statement. This chapter sets the two roads side by side, but does not count the arrow from confinement to the gap as a "road that joins them".

A precise way of saying it

The mass gap and confinement (the area law) are different statements. The space road of this book heads separately to the two statements: from the small-T^2 gap stage through the transfer matrix to the gap, and through the small-T^2 confinement stage, the band stage and the R^4 stage to confinement. There is no general theorem that gives a gap from the area law, and this book does not show one either. This book's strong-coupling area law (20-06) is a statement inside the strong-coupling window with the lattice spacing fixed, and sits at the opposite end from the questions that remain.

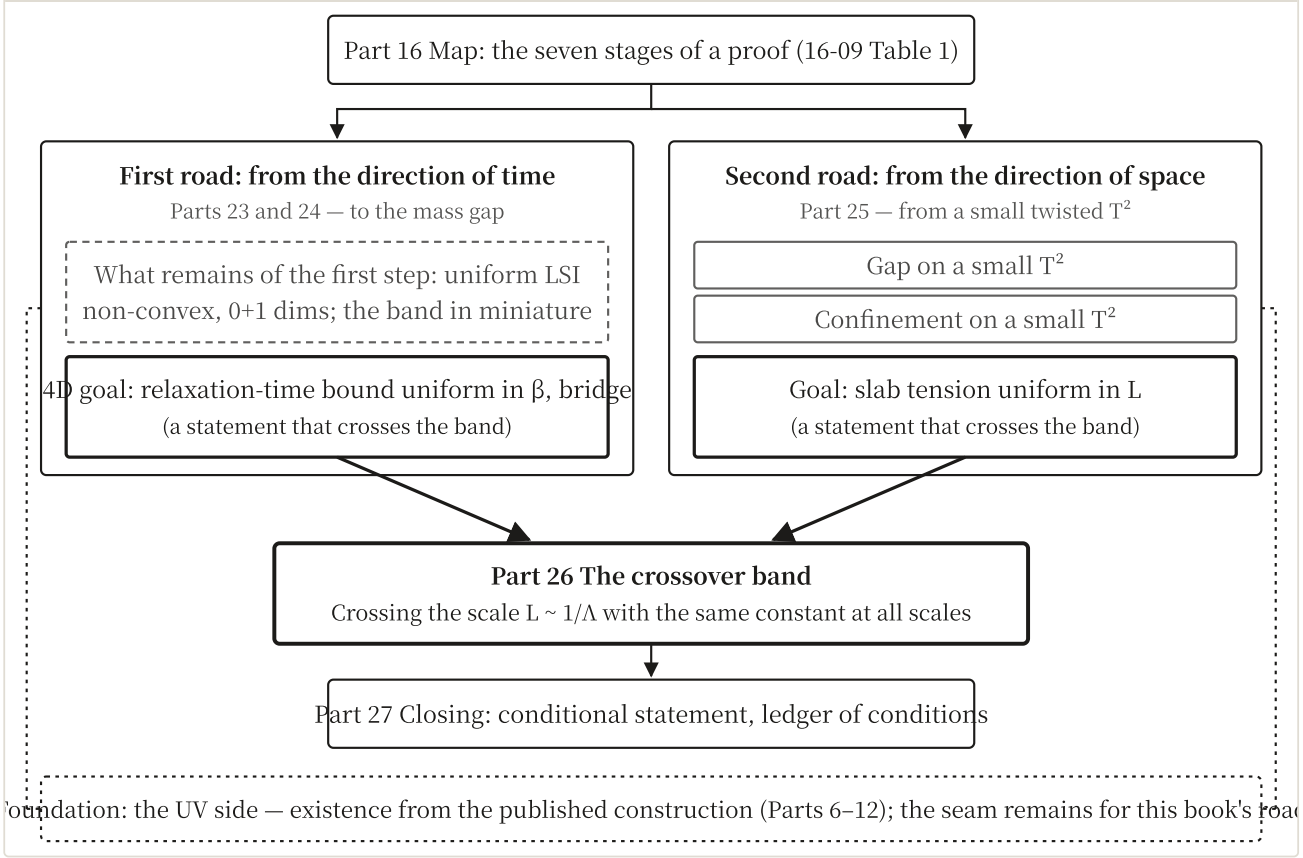
04

The two roads meet at the band — the flow of the logic

Figure 1 is the flow of the logic from Part 16 to Part 27. The first road comes from the direction of time (Parts 23 and 24), the second from the direction of space (Part 25), and at the level of the four-dimensional goal both arrive at a statement of one type: "the same

constant at all scales". That statement is the statement that crosses the band, and Part 26 treats it.

Figure 1 The two roads meet at the band. The column of the first road is split into two rows, and the arrow flowing into the band comes only from the "four-dimensional goal" row (what remains of the first step is a miniature of the band, not the band itself). The ultraviolet side at the bottom (existence from the published construction; what remains on this book's road is the seam) lies under both roads



What to read from Figure 1. (a) "What remains of the first step" on the first road (a uniform logarithmic Sobolev inequality for a measure that contains non-convex parts, the path measure of the 0+1-dimensional double well, 23-15 Table 1) is a wall of the same type as the band (there are different tools at the two ends, and the bound on the worst configuration stops in the middle), but it is not the band. The length of the crossover in the first step is fixed by a classical quantity, the first conjugate point of the Jacobi equation at the top; it does not depend on $\hbar$ and is finite in lattice units. The length of the crossover in four dimensions, $1/\Lambda$, is generated quantum mechanically (4-07) and diverges in lattice units. This difference is treated in 26-07 together with the proof of the computation of the conjugate point on the first-step side (until the proof is placed, Assumption). (b) What flows into the band is, on both roads, the "four-dimensional goal" row. On the first road it is the upper side of the diffusive-relaxation-time assumption uniform in $\beta \to \infty$ (the relaxation-time upper-bound assumption) and the one-sided bridge (the physical-unit target level of 23-15 § 07); on the second road it is the slab tension uniform in L. (c) The ultraviolet side lies outside the band. Existence was supplied by the published construction Assessment . This book's two roads proceed by

placing as an assumption the seam that puts the ultraviolet result into a form usable for their own objects (twisted boxes, slab tension, path measures)Inferred .

The meeting in Figure 1 means "arriving at a statement of the same type at the level of the goal"; it does not mean that the present walls of the two roads are in the same place. The "band" of the band part and the "windows" of Parts 23 and 24 (the site window, the window in $\hbar$, the time window) are different things (the note on words in 26-00).

05

Which chapter fills which cell — an index of Parts 23–27

Table 2 is an index of which cell of which stage of Table 1 the chapters from Part 23 on supply material to. The stages taken on by the published construction are read in Parts 6–13 of Volume 2, and this book's firm theorems are in Parts 19–22 (16-04 § 04). A "cell" is a column of Table 1 (known, this book, unknown). Parts 25–27 have chapters still being written, and the links are made by chapter number.

Table 2 The chapters of Parts 23–27, and the stages and cells of Table 1 (this book's arrangement)

Chapter	Stage	Cell	In a word
23-00–23-17	5 (time side)	This book, unknown	The first road. The statement of the four-dimensional goal (the physical-unit target level), and the list of rows of the first step (0+1 dimensions, a miniature of the band)
24-00–24-08	5 (time side), 6	This book, unknown	The first road (second half). Repairing the form of the assumption, and making time continuous. 24-07 is cited from the band part
25-00	2, 3	Known, unknown	Why go by way of a small twisted T^2
25-01	3	Known, this book	Twist and flux, and the three facts of the twisted part
25-02	4, 5	Known, this book	Vortices and fractional instantons, the form of the string tension
25-03, 25-04	4	Known, this book	The Container Identity and slab confinement
25-05, 25-06	4	This book	Insertion identities, driven insertion and the Flow Lemma
25-07	5	This book	Why fixed maps fail (at the one-loop level)
25-08	4	This book (Reading)	The measured record of the space road
25-09	5	Unknown	Where this road stops = widening the T^2
26-00–26-02	5	This book, unknown	Confluence, the correct form of the band, reduction to a single inequality
26-03, 26-04, 26-05	1, 5	This book	Where non-abelianness enters, structure alone cannot cross, the chessboard

Chapter	Stage	Cell	In a word
26-06, 26-07	5	This book, unknown	The renormalisation comparison, the band seen from the side of time
26-08, 26-09, 26-10	5	Known, this book (Reading), unknown	Roads tried and stopped, the measured record, candidate viewpoints
26-11	2	Known, this book, unknown	Ultraviolet control: what closing in one step means, the five stages of the series and where they stopped, a draft statement
27-00–27-04	All	Unknown (the column of conditions)	The conditional statement, the ledger of conditions, the index of questions and of what is not claimed
A-06, A-07	All	—	The index of names, the drafts of papers that came out of this book

06

Correspondence with the tasks of 16-07 — the six continuum tasks and the seven stages a proof must contain

The six continuum tasks of 16-07 Table 1 split the tasks needed for the continuum theory according to the requirements of the problem statement. The seven stages a proof must contain in Table 1 split the stages of one road that starts from the lattice. Since the axes of the splitting differ, they do not correspond one to one. Table 3 is the correspondence, and it is this book's arrangement.

Table 3 The seven stages a proof must contain and the tasks of 16-07 (this book's arrangement. The strengths of the arrows are as in 16-07)

Stage	Tasks of 16-07	Note
1 Vessel	Part of the task of the remaining axioms	Reflection positivity and the transfer matrix. The rest of the task of the remaining axioms (recovery of rotations) is in the R^4 stage. The published construction satisfied them for each phase Assessment
2 Ultraviolet	The task of the T^4 construction, the task of non-triviality, the task of local fields and short distances	Existence of the continuum limit, non-triviality, and agreement with asymptotic freedom at short distances are all tasks on the ultraviolet side. Existence, for local fields, was shown by the published construction; non-triviality and short distances are at the level of checks (Part 13) Assessment
3, 4 Small T^2	A partial problem of the task of the T^4 construction	Construction and bounds on a small volume. The task of the T^4 construction is a task on T^4, while the two small-T^2 stages are tasks on the small T^2 of $\mathbb{R}^2 \times T^2$, so they are not the same task
5 Band	The task of a uniform gap	The uniform gap $\inf_L \Delta_L > 0$. The task for which the setters write 'New ideas are needed'

Stage	Tasks of 16-07	Note
6 R^4	The task toward infinite volume, the task of the remaining axioms	The infinite-volume limit, and the remaining axioms
7 General groups	16-07 § 07	For groups with trivial centre, the arrows that use the centre disappear
(confinement side)	The continuum string-tension problem, the lattice area-law problem	The continuum string tension and the lattice area law. Outside the requirements of the problem statement (C of 16-07); it corresponds to where the small-T^2 confinement stage and the band stage arrive

What to read from Table 3. In the counting of 16-07 (16-07 § 06), if the arrows of the roadmap are admitted, the minimal set needed was one: {the task of the T^4 construction, the task of a uniform gap, the task of the remaining axioms, the task of non-triviality, the task of local fields and short distances}. In the language of the seven stages a proof must contain, it is the ultraviolet stage (the task of the T^4 construction, the task of non-triviality, the task of local fields and short distances), the band stage (the task of a uniform gap), and the axioms of the vessel stage and the R^4 stage (the task of the remaining axioms). After the published construction, the ultraviolet side has been filled as existence (for local fields), and the task of the remaining axioms has also been met for each phase Assessment . In either way of splitting, what remains is the band (the task of a uniform gap) and the seam that hands the ultraviolet result over to the infrared arguments Inferred (16-02, 16-03).

The list of premises is in 28-04 (a list written before the publication; the premises on the ultraviolet side have, as existence, been taken up by the published construction).

What this chapter can say

What is said	What is not said
Cited The "known in the world" column of each stage (the extent of the original sources is in each chapter's source table). That confinement and the gap are different claims	That the seven stages a proof must contain exhaust it. The division into stages is this book's arrangement
Proved here The claims in the "this book" column of Table 1 (within the conditions written in the row. The chapters that place the paper proofs are in Table 2)	Any of: the seam of the ultraviolet stage, the band stage, the mass gap of the general-group stage. The form of the two small-T^2 stages uniform in a (outside the assumption of an ultraviolet construction in a twisted box)
The arithmetic of the theorem names in the Lean entries (finite sums and lemmas on abstract operators. The physical measure is not formalised)	That the Lean entries checked the claims of the stages themselves
The meeting in Figure 1, in the sense of arriving at a statement of the same type at the level of the four-dimensional goal	That the present walls of the two roads are the same. The arrow from the area law to the gap

What is said	What is not said
Assessment The "shown by the published construction" entries of Table 1 are at the level of the committee's assessment	Whether the proof of the published construction is correct (outside this chapter)
Overall	This site shows neither the existence of the mass gap nor the construction of the continuum limit (the existence of the continuum limit belongs to the published construction)

Sources and reproduction

Item	Kind	Source / tools
Division into stages	This book	The stages of constructive field theory (7-01, 7-10), cut again to fit the material of Parts 25 and 26 (the table of stages, the list of tasks up to a proof, the list of assumptions)
Known for the vessel stage	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471 (abstract; source table of 16-07)
Known for the ultraviolet-control stage	Cited	T. Bałaban, Comm. Math. Phys. 122 (1989) 355–392 and the rest of the series (source table of 7-07)
The twist of the small-T^2 gap stage, the Container Identity of the small-T^2 confinement stage	Cited	't Hooft's twisted boundary conditions and flux (1979, 1981), the Container Identity of Mack–Petkova (1980). Bibliographic data are in the source tables of 25-01 and 25-03
Known for the band stage	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000) § 6.5. E. T. Tomboulis, L. G. Yaffe, Comm. Math. Phys. 100 (1985) (bibliographic data in the source table of 26-01)
"Shown by the published construction" in Table 1	Assessment	Assessment report `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10), opening, § 5, § 6, § 8. The code is `yang-mills-group/OpenAI-YM-20261028` (submitted 2026-10-28). The ultraviolet seam is in 16-03
Known for the R^4 stage and the general-group stage, the tasks of Table 3	This book	16-07 Table 1, Table 2, § 07
Confinement and the gap	This book	16-08 § 06, 17-02

Item	Kind	Source / tools
The "this book" column of Table 1	Proved here	The chapters of Parts 25 and 26 (Table 2), and the draft papers 'Exact centre-vortex insertion identities in the twist-summed ensemble, and bounded, averaged and driven confinement bounds', 'Where the twisted-slab route to confinement stops: a corrected window target, a reduction to transverse widening, and SU(2) versus U(1) numerics', 'Fractional instantons confine on the lattice: a rigorous semiclassical bound for SU(2) gauge theory on a twisted small torus' (all drafts of 2026-10, A-07)
Lean	Lean	GapFromDecay (`GapFromDecay.gap_of_decay_dense`), ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`), Containers (`Containers.container_bound`), InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`), Doubling (`Doubling.iterate`). What the bundles guarantee is only the algebra of finite sums and inequalities for abstract operators (A-04)
Figure 1	This book	A hand-drawn figure (no numerical computation)

Words used in this chapter — the seven stages a proof must contain (vessel, ultraviolet, gap on a small T^2, confinement on a small T^2, band, R^4, general groups), the crossover band (the band), 't Hooft's twist, the sector bias θ, the marks of the state of the record (■◐□), the road to the gap and the road to confinement, the four-dimensional goal, a miniature of the band. Their meanings are in A-02 Glossary.

Next chapter to read: 16-10 A map of the tools. To read from the side of this book's study, 23-00 Entrance — the first road (from the direction of time). If you read from the list of conditions, 27-02 The ledger of conditions.

A map of the tools — where the known tools used by this book live

> This chapter is a guide (a map of the tools). It lists the homes of the known tools used by this book's study, and where each stops.

What this chapter shows — The twenty-two tools that the work from Part 23 on uses (the Poincaré inequality, Bakry–Émery, log-Sobolev, reflection positivity, the strong-coupling expansion, the renormalisation step, Migdal–Kadanoff, one-loop perturbation theory, interval arithmetic, Lean and others) in one table. For each tool, with labels: (1) its home — the chapter where its definition and known results are first set out (Parts 1–5 and Parts 7, 17 and 18), and the source; (2) how this book uses it — in which chapter from Part 23 on, and for what; (3) where it stopped when carried over to the group, and cautions. Also which chapter to read first when the same topic appears in two or more parts, three reading orders, the dependencies between the tools, and a list of the limits of the tools.

Prerequisite chapters — 16-09 The stages a proof must contain — the table of seven stages (the stages and the labels) and 0-01 § 03, the key to the labels.

Said first — This is a guide chapter and makes no new claims. The claims in each cell of the table are copied from the chapters listed there, with those chapters' labels (the conditions are in the text of each chapter). The column "where it stopped" is not a claim that the tool cannot be used; it records where the way this book used it did not reach, within this book's study. This site shows neither the existence of the mass gap nor the construction of the continuum limit. Where the tools used by the construction published in 2026 live is written in (6) (the label Assessment means "according to the committee's assessment report"; see "Labels" in 6-00).

The order of this chapter

1. The role of this chapter — in five lines
2. The table of tools — home, use, where it stopped (Table 1), and how topics found in two places are divided (Table 2)
3. Three reading orders — from the tools, from the results, from the map
4. Dependencies between the tools — tools needed first, and tools used later (Table 3)
5. What cannot be said — a list of the limits of the tools

00

The role of this chapter

(1) The chapters from Part 23 on take tools from Parts 1–5 and Parts 7, 17 and 18 into "lattice gauge theory on a group" and use them there. This chapter lets you look up, in one table, where the definition and the known results of each tool are.

(2) When the same tool appears in two or more parts (for example, Bakry–Émery in 5-11, 24-06 and Part 23; Migdal–Kadanoff in 18-04, 26-08 and 15-04), it divides them into "home" and "use", and fixes one chapter to read first.

(3) It lists, from the side of the tools, where they stopped when carried over to the group. In which part's chapter each stopped can be looked up in the third column.

(4) The labels are the same five classes as in 16-09. Proved here = this book has set out a proof on paper, Cited = known, checked in the source, Computer-assisted = computer-assisted (interval arithmetic), Reading = a reading of numbers or a physical picture, Assumption = an assumption. The correspondence with the labels of the chapters of Parts 1–5 and Parts 7, 17 and 18 (Lean, paper, computed, known, physics) is in 0-01 § 03.

(5) There are three ways to read (§ 02). Whether you enter from the tools, from the results or from the map, coming back to this table takes you to the home chapter.

(6) The construction published in 2026 assembled, from Table 1, reflection positivity (row 5), the renormalisation-group step (row 10), the one-loop coefficient (row 12) and Lean (row 21), together with tools not in the table (a coupling-dependent probability kernel, matching of the coupling by a shooting method, a multi-step ledger, Arzelà–Ascoli compactness, phase decomposition, reconstruction), and built a continuum limit in a periodic box and an infinite-volume theory along a subsequence. The committee assesses this as "proved in Lean" Assessment . Where those tools live is Volume 2: the kernel of one step in Part 9, the matching of the coupling in 10-04, the multi-step estimates in Part 11, and reflection positivity, the chessboard, compactness, phase decomposition and reconstruction in Part 12. The comparison with this book's tools is in Part 14.

01

The table of tools — home, use, where it stopped

Table 1 has one tool per row. The first column, "home", is the chapter where the definition and known results of the tool are first set out; it is the place a second-year student reads first. The second column gives the chapters from Part 23 on that use the tool, and what for. The third column gives where the tool stopped when carried over to a lattice on the group $SU(2)$, and cautions for reading. Numbers and conditions are written on the same row; the detailed conditions are in the chapters listed.

Table 1 The tools used by this book's work (home, use, where it stopped; labels are per claim within a cell)

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
1 The Poincaré inequality and the gap The constant C_P that bounds the variance by the squared gradient. The gap is $1/C_P$	1-10 § 08 (on the line, $V'' \geq \kappa > 0$ gives $C_P \leq 1/\kappa$) Cited . The reading as a relaxation time is in 2-10	23-06 (the condition of local Poincaré within a phase). 24-05 (on the lattice with continuous time, the gap $\Delta = \hbar^2/(2C_P(\nu))$, with ν the vacuum measure) Proved here	The gap in Langevin time and the gap of the transfer matrix are different quantities (18-03). The identity of 24-05 is a rewording; there is no tool that bounds C_P of the vacuum measure uniformly in the coupling
2 The Bakry–Émery criterion Curvature + second derivative of the potential $\geq K > 0$ gives a gap $\geq K$	5-11 § 02 (the criterion and the outline of its proof) Cited . 5-04 § 08 uses only the conclusion	5-11 § 05– § 07 (integrate a perfect family first; curvature lower bound $2 - 27\beta^2$) Proved here . 23-02 (families on the four-dimensional lattice). 24-06 § 02	On one link the criterion speaks only for $\kappa < 2$, while the true gap grows with κ Reading . On the whole lattice $\beta < 1/12$ Cited ; integrating first, $\beta < \sqrt{2/27} \approx 0.272$ (the strong-coupling side, lattice spacing fixed). The curvature of the group does not move with β, so the threshold cannot exceed $O(1)$. At weak-coupling scales convexity on the group itself fails (the lemma on the ceiling of convexity on the group, in 24-06 § 03) Proved here

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
3 The log-Sobolev inequality Bounds the entropy by the gradient. Stronger than Poincaré	5-11 § 02 (the criterion also gives log-Sobolev) Cited . The language of measures is in 1-08	24-06 § 06 (tools that pay with rarity instead of convexity; the entropy currency). 23-10 the type of mistaking the currency	The arrangement "Poincaré is the currency of exit speed, and only the entropy currency can pay for rarity" is a picture Reading . "The class of measures that pay with rarity" is a candidate rule and promises no prospect of closing the third step Assumption
4 Brascamp–Lieb and the hierarchical induction An induction whose entry is that the conditional measure at each scale is uniformly convex	No home in Parts 1–5 or Parts 7, 17 and 18. The definition and the conditions of the source are in 24-06 § 02 (Bauerschmidt–Bodineau) Cited	24-06 § 05 (carrying the induction over to the group: what applies, what changes, what does not)	The convexity assumption cannot hold, in principle, on the group at weak-coupling scales Proved here . The non-convex region is the reverse of flat ϕ^4 (on the group, the hemisphere far from the identity), and the wall is only pushed into a region of probability $e^{-c\beta}$ Proved here . Results for the hierarchical model imply nothing for four-dimensional Yang–Mills
5 Reflection positivity and the transfer matrix	5-03 (Osterwalder–Seiler), 4-10 (reconstruction) Cited . On the side of the published construction, 12-01 (two kinds of planes), 12-02 (the chessboard) Assessment	26-04 (positivity of the twist-summed ensemble z under reflections through lattice points) Proved here . 26-05 (input to the chessboard estimate). 16-09 the vessel stage	Transverse reflections break with the twist (the lemma on non-positivity of transverse reflections in 26-04) Proved here . The structural inputs (reflection positivity, duality, the transfer matrix, the centre identities) hold unchanged for $\mathbb{Z}_2$ and $U(1)$, and this set alone does not give the widening inequality (the negative lemma; its last step inherits the Reading of 26-03)
6 The Kogut–Susskind Hamiltonian lattice Only space is a lattice; time is continuous	5-03, 4-01 Cited	24-05 § 04 (a road that treats the gap directly). In the colour singlet the Poincaré constant of the vacuum measure halves at leading order Proved here	Local-to-global gap theorems do not apply as they stand on this lattice (reasons in 24-05 § 04). For the Kogut–Susskind form used in 24-05, read 5-03 first
7 The strong-coupling and cluster expansions Add weak couplings as corrections, and guarantee convergence by a criterion	5-02, 5-04, 7-04 (the Kotecký–Preiss, Dobrushin and Fernández–Procacci conditions) Cited	26-06 (the strong-coupling end). 15-04 § 05 (the exit of the chain in the hierarchical model = the strong-coupling decision region) Computer-assisted	They reach only the strong-coupling shore, with the lattice spacing fixed. They do not reach the band (intermediate coupling). How the strong-coupling ranges compare is in 5-11 § 07. This book's Lean theorems at strong coupling (the infinite-volume limit, the mass gap, the area law) are in 20-02–20-06, and are statements inside the strong-coupling window with the lattice spacing fixed. They sit at the opposite end from the questions that remain
8 The character expansion and class functions Expand a weight on the group in characters of irreducible representations. Conjugation-invariant functions = class functions	5-02 (the character expansion), 7-02 (the heat kernel in two dimensions), 1-04 Cited	15-04 § 01 (the one-link heat-kernel class is a semigroup and closes with one parameter) Proved here . 15-06 (couplings by loop shape and the attracting class)	It closes because in this model one weight returns to one weight in one step. Multi-link couplings, gauge re-fixing and large-field remainders are absent by construction, so nothing is implied for one step of the four-dimensional hypercubic lattice ("Said first" of 15-04)

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
9 Dobrushin's influence coefficient A criterion of the type: if the total influence of the neighbours on one site is below 1, correlations decay	7-04 (Dobrushin's condition, $4/27$ in the gas example) Cited	26-07 § 03 (one site at the top in the first step). 26-05 § 04. 26-10 mixing of the box fluxes	At the top in the first step, the conditional one-site measure is $\frac{15}{4}$-convex and the coupling to a neighbour is 2, so the influence from two neighbours is $\frac{16}{15} > 1$, back to the deficit $\frac{15}{4}$ versus 4 Proved here . With the frame fixed, the continuous field follows the boundary and the influence coefficient is close to 1 Reading (physics; the original papers not checked)
10 The renormalisation-group step (blocking) Integrate out the fine variables to make the action at the coarse scale	7-05 (the sequence of steps, blocking, small and large fields) Cited . What has been reached is in 7-07. The one step of the published construction (a coupling-dependent probability kernel) is in 9-01–9-07, and the many steps in Part 11 Assessment	15-03 (the decomposition statement, the ledger of transverse contraction). 15-05. 15-07 (background-dependent fluctuations η)	The numbers of 15-05 are for a step seen in the marginal of one coarse plaquette, not a true blocking step. The true two steps do not match two iterated steps (the contraction λ is 3.4–3.7 times larger) Reading . Fixing all coarse plaquettes at once makes the contraction a matrix, exceeding $1/16$ in coarse directions (Bundle Network Theorem; abelian configurations, tree level) Proved here . The multi-step induction for η has not been written Assumption
11 Migdal–Kadanoff and the hierarchical model A recursion that approximates blocking by a map on one weight	18-04 Figure 1 (the approximate recursion and its character) Cited	15-04 § 02– § 05 (pulled to an invariant curve at rate $1/16$ per step, its position set by the balance of source and contraction, $1/3600$: leading saddle-point term Proved here ; invariance of the tube for $\beta \in [2.46, 64]$ Computer-assisted). 26-08 road 30	The MK flow goes to strong coupling whether the group is abelian or not, and does not distinguish $U(1)$ from $SU(2)$ (Kanazawa) Cited . Tomboulis's interpolation relies on a common t^* for the untwisted and twisted sectors; this reduces to inequality (3.5) of Ito–Seiler, not shown outside strong coupling Cited . The hierarchical model confines even for $U(1)$ and is no test of non-abelianness
12 One-loop perturbation theory and the coefficient 11/3 Gaussian fluctuations around a background field. $\frac{11}{3} = 4 - \frac{1}{3}$	4-03, 4-06 (asymptotic freedom and the split of the coefficient) Cited	25-07 (isometric maps give only $\beta' \geq \beta + 0.023$; maps that reshape the fluctuations reproduce $\beta - \beta' = 0.258$) Proved here . 26-03, 26-06, 15-06	All are statements at the one-loop (Gaussian) level. Controlling the error of the reshaping maps at intermediate coupling and large fields is open. The $\frac{11}{3}$ at the head of those tables repeats 4-06; read 4-06 first for the split
13 Dimensional transmutation and Λ A mass scale Λ arises from a dimensionless coupling	4-07 Cited	24-07 (because it is four dimensions). 26-00, 26-07 (the position of the band, $L \sim 1/\Lambda$)	Placing the band at $L \sim 1/\Lambda$ is a physical reading of scales Reading . Most of 24-07 is an arrangement of known material and can be read with Parts 4 and 7 alone

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
14 The Gibbs variational principle and relative entropy Transporting a measure by a map, the remainder in the bound on the free-energy difference is a relative entropy. Crooks and Jarzynski are of this type	2-08 (free energy, relative entropy) Cited	25-06 (dynamically driven insertion). 25-07 the transport identity (the principle of direction) Proved here	In a sandwich by fixed maps, the remaining relative entropy grows with the area of the vortex sheet, and the form $e^{-\sigma A}$ does not come out Reading (physics; the stopped road "the deterministic sandwich" of 26-08). What can be used is the qualitative target $\Delta F(L,L)\to 0$
15 Twisted boundary conditions, the centre and 't Hooft flux	3-05 (topology), 5-01 (the lattice), 1-04 (the centre) Cited . The definition of the twist itself is in 25-01 Cited	25-01 (three facts, including uniqueness of the vacuum that absorbs the twist) Proved here . 25-04, 26-01, 26-04	For groups with trivial centre (G_2 and others) the arrows that use the centre disappear (16-09 the general-group stage) Cited . Parts 25 and 26 are written for $SU(2)$
16 Centre vortices and fractional instantons	17-02 (pictures of confinement), 3-05 (instantons) Cited	25-02 (definition and action, an upper bound on the minimal lattice action). 26-03 (a picture of where non-abelianness acts)	The dilute-gas count is a physical reading Reading . For the vortices of 26-03, read the definitions of 25-02 first
17 The container identity and the Tomboulis–Yaffe inequality Fixing the frame makes containers independent. From 't Hooft's criterion to the area law	The basis is 5-02, 5-04 (sums over surfaces). The identity is in 25-03 § 03 (Mack–Petkova), the inequality in 26-01 Cited	25-03–25-05 (rewriting in the sector-summed ensemble, nesting) Proved here . 26-02 (reduction to a single inequality)	Placing containers of fixed length independently gives, in four dimensions, only $V(R)\gtrsim R/\ln R$ (Kovács–Tomboulis) Cited . The area law needs one container of the size of the loop, which is the same condition as crossing the band. That the sector bias θ moves away from 1 is not enough (26-01) Proved here
18 The flux network — Thomson's principle, Kirchhoff's laws, orthogonal projection	1-01 (linear maps and projections) Cited . Standard facts of electrical circuits are in the source table of 15-05	15-05 Unit Bundle Theorem (the minimiser is fixed by a projection; contraction $\lambda\le \max f^2$, $1/(16d-8)\le\lambda\le 1/16$) Proved here	Statements at tree level (leading term as $\beta\to\infty$) restricted to abelian configurations. They do not include non-abelian configurations, the one-loop source, large fields or gauge re-fixing
19 The BCH formula and quaternions The commutator series of $\log(e^X e^Y)$. $SU(2)$ is the quaternions of length 1	1-03 (BCH up to third order), 1-02, 1-04 Cited	5-11 (the quaternion inner product and one-link integration). 15-06 (commutators as images under BCH). 15-07 (the group-geometry part = the third-order BCH coefficient $1/9$; leading Gaussian term, two links) Proved here	For $SU(2)$ the group-geometry part rides on the background itself and does not contract, and its budget diverges whatever threshold is chosen (arithmetic treating the one-loop flow as exact) Proved here . For $U(1)$ the third-order BCH term is identically 0

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
20 Computer assistance — interval arithmetic, certificates, the discipline of double precision Hold every number as an interval containing the true value, rounding outward	No home in Parts 1–5 or Parts 7, 17 and 18. The general theory is in 23-08 § 01– § 03 (rounding conventions, holding intervals, positive and negative controls) Cited	23-07, 23-08 (the chain condition $q < 1$ at every point of the $\hbar$ window) Computer-assisted . 15-04 § 05 (the tube of the hierarchical model) Computer-assisted . 23-15	Inequalities have been enclosed by interval arithmetic in the first-step window and in the hierarchical model, a map on one weight (plus the upper bounds on the action of explicit configurations in 25-02 Computer-assisted , upper bounds only). Even in the hierarchical model, $\beta > 64$ (the limit of zero lattice spacing) is not closed. The road of checking finitely many L by interval arithmetic is out of computational reach in four dimensions (26-08 road 3) Reading . Computer assistance says nothing outside the window (23-08 § 10)
21 Formalisation in Lean Statements as types, proofs as terms, checked by a small kernel	A-04 (kernel checking, the criterion of the three standard axioms) Cited . How to read the Lean of the published construction is in 6-00	5-11 (the curvature lower bound and the combinatorics of families). 23-13 (the bundles of the first step). The Lean column of Table 1 of 16-09	What is guaranteed is only the arrow from the stated hypotheses to the stated conclusion. Physical measures are not formalised. The interval-arithmetic layer lies outside Lean (23-13 § 04). The general theory is A-04; the list of bundles and examples is 23-13
22 Reading Monte Carlo Generate lattice configurations at random and measure averages	5-07, 5-08 Cited	25-08, 26-09 (the measured record) Reading . The numbers of 15-05 and 15-07 Reading	All measured numbers carry the "Reading" label and are not used as grounds for theorems. Lattice size, β, seeds and errors are on the same row in each chapter. There are examples where the numbers move when the way of measuring changes (the true two steps and two iterated steps in 15-05)

What to read from Table 1. (a) Two tools have no home in Parts 1–5 or Parts 7, 17 and 18 (the hierarchical induction and computer assistance). For both, chapters from Part 23 on (24-06 § 02, 23-08 § 01– § 03) double as the home. (b) The ways of stopping in the third column fall into types. The curvature and convexity tools (rows 2–4, 9) stop at the closed shape of the group and at the single site at the top; the renormalisation tools (rows 10, 11, 19) stop where one step is stacked into many; the structural tools (rows 5, 17) stop where they cannot be told apart from $U(1)$. (c) Computer assistance and Lean (rows 20, 21) both say nothing outside what was written.

How topics found in two or more places are divided

When the same tool appears in several parts, read the "home" of Table 2 first and the "use" after the home. The home chapter alone has the definitions and the known results.

Table 2 Topics found in two or more places (home = read first, use = read later)

Topic	Home	Use	Row of Table 1
The Bakry–Émery criterion	5-11 (the criterion and the one-link example)	24-06 § 02– § 03 (why it fails on the group at weak coupling), 23-02, 23-06	2
Migdal–Kadanoff	18-04 Figure 1 (the character of the approximation)	15-04 (invariant curve and interval arithmetic), 26-08 road 30 (Tomboulis's claim and its critiques)	11
The Kogut–Susskind lattice	5-03	24-05 (the gap and the vacuum measure)	6
Reflection positivity and the transfer matrix	5-03	26-04, 26-05 (checked for $\mathbb{Z}_2$ and $U(1)$)	5
The one-loop coefficient $\frac{11}{3}$	4-06	25-07, 26-03, 26-06	12

Topic	Home	Use	Row of Table 1
Dimensional transmutation	4-07	24-07, 26-00, 26-07	13
Vortices (centre vortices, fractional instantons)	The picture in 17-02; definition and action in 25-02	26-03 (a picture)	16
Renormalisation steps and Bałaban's series	7-05, 7-07 (what has been reached)	15-02 (inside the series and the BBS chain), 15-03	10
Interval arithmetic	23-08 § 01– § 03	23-07, 23-15, 15-04	20
Lean	A-04 (the general theory)	23-13 (the bundles of the first step and two examples of the types)	21

02

Three reading orders — from the tools, from the results, from the map

In all three, going back to Table 1 along the way to look up the home chapter lets you read without missing prerequisites. They can be used alongside the reading orders of 0-01 § 07.

Three reading orders (read in the order of the arrows; [ ] are rows of Table 1)

Way in	Reading order	What you know when you finish
From the tools readers from the mathematical side	1-10 → 5-03 → 5-11 → 7-04 → 7-05 → 18-04 → 23-08 § 01– § 03 → A-04 → 24-06 → 15-04 → 15-05 [1, 2, 5, 7, 10, 11, 20, 21, 3, 4, 18]	What each tool needs, and where on the lattice on the group it stopped. The three types of stopping ((b) after Table 1)
From the results from this book's conditional statement	16-09 → 27-01 → 27-02 → 25-04 → 26-02 → 15-03, going back from the "use" column of Table 1 to the home chapters as you read	Which tools each arrow of the conditional statement relies on. How the rows of the ledger of conditions correspond to the tools
From the map seeing the whole of what is open first	0-02 → 16-05 → 16-07 → 16-09 → this chapter → 26-08 → 15-08 → 27-04	Which of the seven stages are open, which tools stopped where, and what this book does not claim

03

Dependencies between the tools — tools needed first, and tools used later

Table 3 connects the tools of Table 1 by "tools needed first". The numbers are rows of Table 1. A tool needed first may have a later row number, but there are no cycles. Following this column from the left fixes the order in which to read the home chapters.

Table 3 Dependencies of the tools (this book's arrangement; "used together" is where they are combined from Part 23 on)

Tool (row of Table 1)	Tools needed first	Where used together
1 Poincaré inequality	— (linear algebra and spectra of 1-10)	The form of the conclusions of 2 and 3. The identity of 24-05
2 Bakry–Émery	1, 19 (the curvature of the group comes from $SU(2)$ being the three-sphere)	Gives 3. Entry to 4. Combined with 21 in 5-11
3 log-Sobolev	2	The tools of 24-06 that pay with rarity
4 Hierarchical induction	2, 10	Compared with 11 in 24-06 § 05
5 Reflection positivity	— (the axioms of 4-10)	The chessboard of 26-05. With 15 and 17, the negative lemma of 26-04
6 Kogut–Susskind lattice	5	Combined with 1 in 24-05
7 Strong-coupling expansion	8	The exit of the chain in 15-04. A criterion of the same type as 9

Tool (row of Table 1)	Tools needed first	Where used together
8 Character expansion and class functions	19	Combined with 10 and 11 in 15-04 and 15-06
9 Dobrushin's influence coefficient	7	Set beside 2 in 26-07 (back to the same deficit)
10 Renormalisation step	12	The frame for 11 and 18. The decomposition of 15-03
11 Migdal–Kadanoff	10, 8	Combined with 20 in 15-04. Road 30 of 26-08
12 One loop	— (perturbation theory of 4-03)	Combined with 14 in 25-07. 26-06, 15-06
13 Dimensional transmutation	12	The reading of the band's position (26-07)
14 Gibbs variational principle	— (2-08)	Combined with 12 in 25-06 and 25-07
15 Twist, centre, flux	— (3-05, 5-01)	The stage for 16 and 17. 26-04
16 Vortices	15	25-02, 26-03
17 Containers and Tomboulis–Yaffe	15, 7	25-03–25-05, 26-02
18 Flux network	10, 11 (comparison with $1/16$)	the Unit Bundle Theorem and the Bundle Network Theorem of 15-05
19 BCH and quaternions	— (1-03, 1-04)	The group-geometry part of 15-07
20 Interval arithmetic	— (23-08 § 01– § 03)	In 23-07, 23-08 and 15-04, encloses inequalities for maps such as 11
21 Lean	— (A-04)	Checks the algebraic steps of 2, 5, 17, 19 and others. The output of 20 is placed as a hypothesis
22 Monte Carlo	— (5-07)	The numbers of 25-08, 26-09, 15-05, 15-07 (Reading)

What to read from Table 3. The rows with no tool needed first (1, 5, 12, 14, 15, 19, 20, 21, 22) are the homes a second-year student can read first. Of these, 20 and 21 have their homes in Part 23 and the appendices, so read them first following the division of Table 2.

04

What cannot be said — a list of the limits of the tools

The limits of each tool, taken one line at a time from the third column of Table 1. What is not claimed, part by part, is in 27-04 What is not claimed (a cross-index); the limits of the ultraviolet tools of Part 15 are in 15-09 What cannot be said.

1. The curvature tools (Bakry–Émery, Brascamp–Lieb, Dobrushin) work only on the strong-coupling side with the lattice spacing fixed, and in the first-step window. At weak-coupling scales convexity on the group itself fails (24-06 § 03), and at the top in the first step they return to the same deficit $\frac{15}{4}$ versus 4 (26-07 § 03).
2. The strong-coupling and cluster expansions are tools of the strong-coupling shore and do not cross the band (intermediate coupling).
3. The statements about one renormalisation step (15-04, 15-05, 15-06) are each restricted to a map on one weight, the marginal of one coarse plaquette, or tree level on abelian configurations; none of them stacks a true step of the four-dimensional hypercubic lattice into many (15-09).
4. The Migdal–Kadanoff flow and the hierarchical model do not distinguish $U(1)$ from $SU(2)$. Closing there is no test of non-abelianness.
5. The structural inputs alone (reflection positivity, duality, the transfer matrix, the centre identities) do not give the widening inequality (the negative lemma of 26-04, in the form "this set of inputs alone").

6. Comparisons by one-loop tools and by the Gibbs variational principle are statements at the one-loop (Gaussian) level, not non-perturbative bounds.
7. Interval arithmetic says nothing outside the intervals written. Lean guarantees only the arrow from the stated hypotheses to the stated conclusion, and physical measures are not formalised (23-13, A-04).
8. Monte Carlo numbers are readings and are not used as grounds for theorems.
9. Whatever tools in this table are combined, this book's study shows neither the existence of the mass gap nor the construction of the continuum limit (27-04). The existence of the continuum limit was shown by the published construction, with the set of tools in (6)Assessment . The existence of the mass gap is in neither (16-02).

What this chapter can say

Can say	Cannot say
Cited The known results in the "home" column of Table 1 (the extent of the sources is in each chapter's source table)	That the twenty-two of Table 1 exhaust the tools of this book. The division into tools is this book's arrangement
Proved here Computer-assisted Reading Assumption The claims in the second and third columns of Table 1 (with the labels and conditions of the chapters listed)	That "where it stopped" in the third column is a limit of the tool in principle (it records the range of this book's use)
The division of Table 2 and the dependencies of Table 3 (this book's arrangement)	That the dependency table exhausts all logical implications
Overall	This site shows neither the existence of the mass gap nor the construction of the continuum limit

Sources and reproduction

Item	Kind	Source, tools
The claims and numbers in each row of Table 1	As labelled in each chapter	The text and source tables of the chapters listed on the row (this chapter copies them and does no new computation)
The Bakry–Émery criterion	Cited	D. Bakry, M. Émery, Diffusions hypercontractives, Séminaire de Probabilités XIX (1985) 177–206 (bibliographic data in the source table of 5-11)
The whole-lattice threshold $1/12$	Cited	H. Shen, R. Zhu, X. Zhu, Commun. Math. Phys. 400 (2023) 805–851 (the source table of 5-11)
The assumption of the hierarchical induction	Cited	Bauerschmidt–Bodineau (the source table of 24-06)
Tomboulis's interpolation and its critiques	Cited	E. T. Tomboulis, arXiv:0707.2179. K. R. Ito, E. Seiler, arXiv:0711.4930 ((3.5)). T. Kanazawa, arXiv:0805.2742 (the source table of 26-08)
The container identity and $R/\ln R$	Cited	Mack–Petkova (1980), Kovács–Tomboulis hep-lat/0108017 (the source table of 25-03)
Thomson's principle, Kirchhoff's laws	Cited	Standard facts of electrical circuits (the source table of 15-05)
Dobrushin's condition	Cited	The source table of 7-04. The original papers of the criteria in 26-05 are not checked
Table 2, Table 3, the three reading orders	This book	Arranged from the prerequisite paragraphs and the cross-references in the text of the chapters (no numerical computation)

Words used in this chapter — home (the chapter where a tool's definition and known results are first set out), use, where it stopped, the Bakry–Émery criterion, the log-Sobolev inequality, the hierarchical model, reflection positivity, class function, influence coefficient, blocking, Migdal–Kadanoff, the flux network, BCH, interval arithmetic, certificate. Their meanings are in A-02 Glossary.

Next chapter to read: 16-11 A map of ideas for crossing the band — living ideas, broken ideas, mathematics that seems to be needed. To read alongside it, 23-00 Entrance — the first road (from the direction of time). If you read from the tools, 5-11 Detour: Bakry–Émery and one-link integration.

A map of ideas for crossing the band — living ideas, broken ideas, mathematics that seems to be needed

> This chapter is a map of what is unsolved. The derivations of § 02 and § 03 are this book's own study.

What this chapter shows — The ideas that this book's study tried for crossing the crossover band (26-00), on a single map. First we set out the two ends of the band and a rough order of magnitude for the distance between them, and split the core into two jobs (extending the control of the published construction, keeping the sum of errors finite, to where the coupling is of order a number; and a statement for the last stretch from there to strong coupling). Next we derive why a chain of lossy inequalities cannot give a conclusion uniform in the cutoff. Then we see that the facts specific to SU(2) in this book's Lean are all facts about the action of configurations (facts on the side of energy), that what the band needs is a bound that counts fluctuations and the number of arrangements from below, uniformly in scale (the shortfall on the side of entropy), and which direction the bounds that can be written in variational form go. On that basis we sort the ideas into three tables, living, broken and undecidable, put the smallest next step in the form of questions, and list by name the mathematics that seems to be needed. Widening the test opponents to seven, and the small-lattice computations, are in 26-12.

Prerequisite chapters — 16-03 The infrared edge of the published construction, 16-04 Where this book stands, 26-00 Entrance — confluence, 26-06 The renormalisation comparison, A-08 § 05 The infrared side.

Said first — This chapter does not give a road across the band. There is no proof. What there is: a map of ideas, facts checked by hand counts and small-lattice computations, and a sketch of "this type of argument cannot cross". The observations written here (the two-way split of the core, the lossy chain, the shortfall on the side of entropy) may be known to specialists. This book writes them not as new results but as a sketch map used by this book's study. The numbers for the distance (the number of doublings, the ratio of scales) are rough orders of magnitude, not quantitative claims.

The order of this chapter

1. The two ends of the band, and the core split in two — a rough order of magnitude for the distance; the ultraviolet extension and the statement for the last stretch to strong coupling
2. A lossy chain gives no conclusion uniform in the cutoff — the rate is uniform, the prefactor grows
3. Facts on the side of energy, and the shortfall on the side of entropy — the direction of variational bounds — the facts specific to SU(2) in this book, what the vortex floors count, what is missing
4. The list of ideas — living (Table 1), broken (Table 2), undecidable (Table 3)
5. The smallest next step — in the form of questions
6. Mathematics that seems to be needed — seven items
7. What this chapter does not say

Labels — Proved here = a hand count whose derivation is placed in this chapter / Cited = a fact from the literature, checked at the relevant place in the original or a review / Reading = a reading of this book's small-lattice numbers (lattice, sweeps and seeds given) / Assumption = a conjecture or an expectation / Lean = a theorem in this book's Lean (only inside the strong-coupling window, with the lattice spacing fixed). For details, see "Labels" in 6-00. Statements in the literature that we have not checked are written as "a statement in the literature (not checked by this book)", and are not used as grounds for claims.

Notation follows this book's conventions. The one-loop slope is $s = 11C_g/(24\pi^2)$, for SU(2) $s = 11/(3\pi^2) \approx 0.3715$ (in Parts 25 and 26 the same quantity is written $8b_0$; the $s(N)$ of 26-06 is a different quantity). We write $\bar{\beta}$ for the reference coupling of the published construction (a coefficient of the effective action), and $\beta = 4/g^2$ for the coupling of the SU(2) Wilson action.

01

The two ends of the band, and the core split in two

The two ends. The strong end is this book's Lean window: the infinite-volume limit and the mass gap for $|\beta| < 1/19$ (an explicit sufficient condition for $18 \tanh |\beta| < 1$; the window itself is $|\beta| < 0.0556$; 20-03), and the area law for $\beta < 1.1 \times 10^{-3}$ in a finite box and $\beta < 1.2 \times 10^{-4}$ in infinite volume Lean (20-06; lattice spacing fixed, Wilson action

only). The weak end is the threshold required by the one-step estimate of the published construction; it is a lower bound on a coefficient of the effective action, not a Wilson β. The value of the threshold itself is not written as a number in the source; from just one of the conditions imposed on it, the threshold comes out as at least 9×10^5 ($e^{13.7065}$) (16-03 § 01). The two ends are stated in different quantities and in different classes of actions.

A rough order of magnitude for the distance. In the one-loop formula, each doubling of the lattice spacing lowers the coupling by $s \ln 2 \approx 0.2575$. If this one-step pace is extended straight into the region where it cannot be trusted (coupling of order 1), and moreover the coefficient of the effective action and the Wilson β are put on the same axis, the number of doublings from 9×10^5 to $1/19$ is

$$k \;\approx\; \frac{9 \times 10^5 - 1/19}{0.2575} \;\approx\; 3.5 \times 10^6, \qquad 2^k \;\approx\; 10^{1.05 \times 10^6} \quad (1)$$

Proved here (the division and the computation of $k \log_{10} 2$). This is a **rough order of magnitude**. The one-loop formula cannot be used where the coupling is of order 1, and the two axes are coefficients of different actions, so (1) is not a quantitative claim. What can be said goes only as far as the expectation that the weak end and the strong end are separated by an astronomical ratio of scales in units of step 0 Assumption .

Splitting the core in two. We view the core of the band as split into the following two jobs Assumption .

1. **The ultraviolet extension.** The job of extending the control of the published construction from the present threshold (at least 9×10^5) down to the value β_* where the coupling is of order a number, while keeping the sum of the errors over the steps finite. It is the same kind of job as the published construction, but the whole amount remains (the order of magnitude of (1)).
2. **The statement for the last stretch to strong coupling.** A statement with few steps, from β_* to the strong-coupling window. It is a non-perturbative statement where the coupling is of order 1 (it is different from the "Corollary Crossing the Band in Finitely Many Steps" of 26-06).

Where the boundary β_* is placed changes how the difficulty is divided. If the threshold is left at its present value, the "box of physical size of order $1/\Lambda$" that the second statement treats has a side of the order of (1) in units of step 0, and is no longer a statement about a small box. The second becomes a statement about a small box only when the first job has brought the threshold down to order a number Assumption . Note that in a periodic box, the published construction has the result that, with the volume fixed, the effective density at step 0 converges along the whole sequence of cutoffs (16-

03 § 02; the limiting measure itself is not defined as an object). There is no quantitative information at the size $1/\Lambda$.

This view comes with another reading. On a line of fixed physical size (fixed ratio of the side of the box to the correlation length), one step of the renormalisation comparison (26-06) is a statement that "compares boxes of the same physical size with the lattice spacing halved". So the sequence of doubling steps is at the same time a sequence of steps running into the infrared and a sequence of statements of ultraviolet uniformity (control of the continuum limit) Assumption . The conclusion of § 02 agrees with this reading.

02

A lossy chain gives no conclusion uniform in the cutoff

Consider the renormalisation comparison (26-06) in a form that allows a loss of a factor $K \geq 1$ at each step. With $\Delta F(N, M, \beta)$ the vortex free energy of a slab of width N and length M (19-02) and $f(\beta) = \beta - c$ ($c > 0$), suppose that

$$\Delta F(2N, 2M, \beta) \;\leq\; K\, \Delta F(N, M, f(\beta)) \qquad (N \geq N_1,\; M \geq M_0,\; \beta_s \leq \beta \leq \beta_0) \quad (2)$$

holds at every β the chain passes through (from the edge β_s of the strong-coupling window up to the starting β_0) (an assumption; it has not been shown). Here β_s is the edge of the window in which the estimate exponential in the area, with prefactor the transverse area, can be used (the finite-box window of 20-06, $\beta < 1.1 \times 10^{-3}$). It is a different window from the $1/19$ of § 01 (the window for the infinite-volume limit and the mass gap).

Derivation Proved here . Chain (2) k_0 times. k_0 is the number of steps, starting from β, until the coupling reaches the edge β_s of the strong-coupling window, and $k_0 \approx (\beta - \beta_s)/c$. For any $N' \geq N_1$, $M' \geq M_0$,

$$\Delta F(2^{k_0} N',\, 2^{k_0} M',\, \beta) \;\leq\; K^{k_0}\, \Delta F(N', M', \beta_s).$$

Insert the area-exponential estimate at the edge of the window, $\Delta F(N', M', \beta_s) \leq C\, N'^2 e^{-\sigma_s M'^2}$ (of the type of 20-06; the prefactor is the transverse area and σ_s the rate at the edge of the window), and set $L := 2^{k_0} M'$, $\xi := 2^{k_0}$. Then

$$\Delta F(\xi N',\, L,\, \beta) \;\leq\; C\, N'^2\, K^{k_0}\, \exp\!\Big(-\sigma_s \frac{L^2}{\xi^2}\Big) \quad (3)$$

Three things can be read from this.

- **The rate is uniform.** The rate per unit area is σ_s/ξ^2 and does not depend on K. If ξ is read as "the ratio of scales between the lattice at β and the lattice at the edge of the window" (of the order of the correlation length), the rate in physical units is σ_s and does not depend on the cutoff.
- **The prefactor grows.** $K^{k0} = \xi^{\log_2 K}$ (an equality), so the prefactor diverges as a power of the correlation length (the cutoff seen in lattice units). The right-hand side of (3) becomes smaller than 1 only from $\sigma_s (L/\xi)^2 \gtrsim k_0 \ln K + \ln(CN'^2)$, which in physical length L/ξ is of order $\sqrt{k_0 \ln K/\sigma_s}$; since k_0 is proportional to β, this stretches like $\sqrt{\beta}$ (of order $\sqrt{\ln \xi}$).
- **The condition for a uniform statement.** For the prefactor not to depend on the cutoff, the product of the per-step losses K_i must be bounded, that is, $\sum_i \ln K_i < \infty$ ($K_i = 1 + \varepsilon_i$, $\sum_i \varepsilon_i < \infty$).

Reading. A chain whose loss is a constant $K > 1$ uniform over the steps can carry **confinement at each** β (exponential in the area at fixed lattice spacing), but cannot carry **constants uniform in the cutoff** Proved here . A uniform statement needs the sum of the losses to be finite, and this is not a matter of cleverness with inequalities but a job of the kind of controlling the continuum limit (the first job of § 01) Assumption .

Seen additively, the distinction lies in the leading term. On the weak-coupling side, the vortex free energy is semiclassically of the type $\Delta F \approx \pi^2 \beta_{\text{eff}}(L)$ (π^2 is the lower bound of the sector action of the continuum theory; 25-02). If at one loop β_{eff} decreases by $s \ln 2$ per doubling, the change per doubling is

$$\Delta F(2L) - \Delta F(L) \;\approx\; -\pi^2 s \ln 2 \;=\; -\tfrac{11}{3} \ln 2 \;\approx\; -2.54 \quad (4)$$

Proved here ($\pi^2 \cdot 11/(3\pi^2) = 11/3$; at the one-loop level). In the Coulomb phase of U(1), the quantity of the same type does not run. So "the difference on the weak side is small, $O(1/\beta)$" is a relative way of speaking; on an additive ruler, the distinction is $O(1)$ per step. Subtracting values in this book's measured record (Tables 8 and 10 of 26-09, 1–3 seeds), SU(2) $4^4 \to 8^4$ gives 1.61, 1.88, 1.80 at $\beta = 2.7, 2.9, 3.0$ Reading (with a scatter of ± 0.3–0.7 on each value). The recorded values are $\Delta F(L) - \Delta F(2L)$ (the decrease), and compared with the size 2.54 of (4) they are of the same order.

However, the K-fold loss of (2) multiplies $\Delta F \approx \pi^2 \beta$, so $(K-1)\pi^2\beta$ eats up the 2.54 of (4) in a single step. Looking at the approximate formula, the lossless additive comparison $\Delta F(2L, \beta) \le \Delta F(L, \beta - c')$ is false in small boxes because of lattice artefacts: the minimal sector action $S(L)$ seen by cooling increases with L (7.84 at 2×2, 9.80 at 4×4, approaching the continuum $\pi^2 \approx 9.87$ from below; 25-02 Table 1) Reading , and the first term of $\Delta F(2L) - \Delta F(L) \approx \beta\,(S(2L) - S(L)) - 2.54$ is positive, so

the lossless form can only be written with a lower bound of order $\sqrt{\beta}$ on the side of the box Proved here . That the recorded decrease $2^4 \to 4^4$ moves away from 2.54 as $1.83,\ 0.85,\ 0.55$ at $\beta = 2.5,\ 2.75,\ 3.0$ agrees with the direction of this artefact Reading .

03

Facts on the side of energy, and the shortfall on the side of entropy — the direction of variational bounds

1. The facts specific to SU(2) in this book Assumption (within what we read of this book's Lean theorems (A-04 Table 4g and the pages 21-05, 21-07, 20-06); we have not checked by carrying the proof of each theorem over to U(1)). The ones that can clearly be read as depending on the non-commutativity of SU(2) and not carrying over to U(1) as they stand are the following four.

- With a twist in one direction only as background, there is a configuration of action 0 (the twisted vacuum) Lean (`ThetaLimit.flat_exists_10`, 21-07). This is because the quaternions **i** and **j** anticommute. In an abelian group, a single twist cannot be satisfied with action 0 Proved here (multiplying the products over the faces of a closed surface leaves only the sign of the twist).
- The gauge transformations that fix the twisted vacuum are only ± 1 Lean (`TwistBlocks.stab_vacT`, `TwistBlocks.rigidT_all`, the detour of 21-05).
- A box with twists in two directions at once has no configuration of action 0 (`TwistFlat.no_flat_twist11`, 21-07), and the minimal action of a vortex is positive, $S_N > 0$ (`VortexPos.SN_pos`, 21-05) Lean .
- For $N = 2, 3, 4$, a thick vortex is cheaper than a thin one: $S_N < 2N^2$ Lean (computer certificates; `VortexCert.SN_two_lt` and others, 21-05).

All of these are **facts about the action of configurations (facts on the side of energy)**. They point both ways: the first and fourth say "the twist goes in for free" and "a vortex can be made cheap", while the third says "a vortex is not free".

2. What this book's vortex floors count. The thin-vortex floor and the thick-vortex floor (21-02, 21-03) are upper bounds on the vortex ratio θ, that is, bounds in the direction of confinement, with rate $-\ln\tanh(N^2\beta)$ for the thin vortex and $e^{-\beta(S_N+\varepsilon)}$ for the thick vortex Lean (the thick-vortex floor is a statement for sufficiently large β and, for each β, sufficiently large M; the threshold is not explicit). This rate can be read as coming from the action of a single vortex (its classical value) and the number of positions at which the vortex pierces the longitudinal face (the factor M^2) Assumption . It is a value with the transverse size N fixed, not uniform in N, and the decrease at large β stays the exponential of the classical action. The upper-bound certificate $S_N <$

$2N^2$ is used in the direction of improving this rate (`TwistBlocks.thick_vortex_floor_two` and others, 21-05).

3. What is missing Assumption . What the band needs is that the rate is preserved when the transverse size is widened beyond the physical length (26-01, 26-02). In the picture of physics, what acts there is that the fluctuations of vortices and the number of their arrangements (entropy) run with scale (the reading of the margin $5/3 = 11/3 - 2$ in 26-00 Figure 2). In U(1) there is no such running. The vortex floors count positions too, but that count is with the transverse size fixed. However many facts about the action of configurations are piled up, this running is not expected to come out. What is needed is a bound that counts fluctuations and the number of arrangements from below, uniformly in scale. This book calls this **the shortfall on the side of entropy**.

4. The direction of variational bounds Proved here . Written with the partition functions Y_0, Y_1 of electric flux $e \in \{0, 1\}$, the untwisted and twisted partition functions are $Z_{n34=0} = Y_0 + Y_1$ and $Z_{n34=1} = Y_0 - Y_1$, and the vortex ratio is $\theta = Y_1/Y_0$ (26-06 § 02, 26-04). Confinement needs an upper bound on θ, that is, a **lower bound** on the energy of the flux sector. What a trial state (a variational bound) gives is an **upper bound** on the energy, that is, a lower bound on Y_1 and a lower bound on θ, the opposite of the direction confinement needs. On the magnetic side, a lower bound on the twisted partition function $Z_{n34=1}$ (an ensemble of trial configurations and Jensen's inequality) gives an upper bound on θ, which is the direction of confinement. Only on the magnetic side can the direction of confinement be written in variational form. We do not, however, say that a proof for the band is limited to this type alone.

04

The list of ideas

Each row gives the reason and the range checked. The seven test opponents, and the type of argument each one knocks out, are collected in 26-12; here we give only names and conclusions. Three terms first: a **mixed action** is the SU(2) action made by adding an adjoint-representation term to the Wilson action (the fundamental-representation term); a **first-order transition** is a transition in which the plaquette average or the like jumps as a function of the coupling; a **critical endpoint** is the point where a line of first-order transitions ends. A **flat connection** (a flat configuration, as 21-07 calls it) is a configuration in which the holonomy of every face equals the identity (or the sign of the twist when there is a twist), and a **frozen phase** is the phase of a gauge theory with a finite group, on the large-β side, that clings to the neighbourhood of flat connections.

Table 1 Living ideas (none of them crosses the band. "Living" means that there is not yet a reason to break it)

Name	Content	Reason and range checked
Seven test opponents	Test arguments about the band not only against weak-coupling U(1) but against seven opponents	With U(1) alone, the types of argument that can be knocked out are coarse. Finite non-abelian groups, the limit of the adjoint Higgs model, and the mixed action of SU(2) itself knock out arguments of other types. The counts and the traces of the first-order line are in 26-12 § 05 Reading ; the properties of the opponents checked in the literature are in the source table of 26-12 Cited
Splitting the core in two	The ultraviolet extension (bringing the threshold down to β_* while keeping the sum of errors finite), and the statement for the last stretch to strong coupling	§ 01, § 02. The count of the prefactor Proved here . Where the boundary is placed is Assumption
Mixing statement for a box of physical size	With the side of the box taken at a physical length C/Λ, the total variation of the law of gauge-invariant observables inside, when the frame (the links on the boundary of the box) is changed, is below a threshold, uniformly in β	Can be written independently of the group, the centre and the class of action, and is a form that should fail both in the Coulomb phase of U(1) and at a point of the endpoint type of the mixed action (if the gap closes at the endpoint; for SU(2) not checked by this book) Assumption . But it is a restatement of the mass gap (in a strong form), not a road. The direction of the input on the strong side (finite-box mixing ⇒ uniform exponential decay) is a combination of theorems of known types Assumption (not written out in the form for gauge theory)
Looking at the side of energy and the side of entropy separately	The facts specific to SU(2) in this book are facts about the action of configurations (the side of energy), and they point both ways. What the band needs is a bound that counts fluctuations and the number of arrangements from below, uniformly in scale (named by the picture of § 03: the shortfall on the side of entropy)	§ 03. Within the list we read Assumption ; the direction of variational bounds Proved here
Three classes by level of value	The quantity for a symmetric box with one twist: at weak coupling, proportional to β for U(1), constant in the frozen phase of finite non-abelian groups, of the type $\frac{3}{2}\ln\beta$ for SU(2)	Counts for finite groups Reading ; the power for SU(2) on 1^4 Proved here Reading (26-12 § 05). A distinction at the level of values; it does not cross the band

Name	Content	Reason and range checked
Stating the quantifiers explicitly	In the renormalisation comparison, where U(1) fails changes depending on whether the aspect ratio is fixed, or the width is fixed and the length extended	The form with fixed aspect ratio holds with a uniform constant even for U(1) on the weak side Proved here Reading ; extending the aspect ratio makes the constant grow (26-12 § 02)

Table 2 Broken ideas ("this type of argument does not go through, because ...")

Idea	Why it does not go through
Setting a measure-defined coupling by the ratio of the twisted to the untwisted partition function	The untwisted periodic box has a continuous family of flat connections, whose contribution acts logarithmically near the centre, so the denominator of the ratio is contaminated (the power on 1^4 in 26-12 § 05) Proved here Reading . This book's ΔF, a ratio of two twisted sectors, suffices
Crossing the band by an open-and-closed dichotomy ("if the vortex ratio in some box is below a threshold, it decays exponentially in the area from then on", plus connectedness of the set of β with decay)	Showing "the closed side is empty" takes in the whole of the running (the steps of the order of (1)), and the open-and-closed form returns to a restatement of "there is no transition" Proved here (logic)
Getting a conclusion uniform in the cutoff from a chain with a loss $K > 1$ uniform over the steps	The prefactor grows like $\xi^{\log 2\, K}$ (§ 02) Proved here . No harm as far as confinement at each β
Using "no transition on the β axis / no fixed point of the step map" in a class of effective actions	Within the class of SU(2) fundamental + adjoint mixed actions there is a line of first-order transitions and a critical endpoint (26-12 § 02) Cited Reading . That both sides of the line are confining phases is an inference from the line ending at the endpoint and connecting smoothly to the confining phase on the Wilson axis Assumption . It is a fact specific to the Wilson axis, and does not extend to the class of effective actions at step 0 of the published construction
Arguments whose only input is "non-commutative as an algebra", or only "the group is continuous and non-abelian, centre symmetry, positivity of the weights"	The former fails for finite non-abelian groups. The latter fails for the adjoint Higgs family: these are the same for every J in the family, and $J = \infty$ becomes the U(1) Wilson action Proved here , which has a non-confining phase Cited (Fröhlich–Spencer; shown mainly in the Villain approximation, with the text stating that all the results can be extended to the Wilson action; the extension relies on techniques of other papers) (26-12 § 03)
Using convexity at a single scale (of the Bakry–Émery type) at weak coupling	The curvature of the Haar measure is $O(1)$ and the Hessian of the action is $O(\beta)$, so the input at a single scale is relatively $O(1/\beta)$ and vanishes at weak coupling Proved here (a count of orders). The form that accumulates step by step is in Table 3

Idea	Why it does not go through
Getting the direction of confinement from a variational bound on the electric side (trial states)	What a trial state gives is an upper bound on the energy of the flux sector, the wrong direction (§ 03) Proved here
Taking a global log-Sobolev inequality on the torus as a gap quantity that works for every group	If the statement in the literature (not checked by this book) that the weak-coupling phase of Z_2 has a gap is right, then there the mixing between the holonomy sectors is slow and the global form is expected to fail, so it would not work as a gap quantity Assumption . Recast as local mixing with boundary conditions (spatial mixing) and kept in Table 1
Taking reflection positivity of the push-forward measure and a fixed block map as non-abelian-specific parts usable for the band	The push-forward identities hold just the same when the U(1) Wilson measure is pushed forward by the same map. It is a vessel, not an input that distinguishes groups Proved here
Convexity of the slab tension in the logarithm of the scale	For twisted slabs, in the semiclassical dilute region the logarithmic slope of $N^2\sigma_N$ is expected to be $11/3$, and 2 at large N, so the slope decreases Assumption (based on the picture of 26-00 Figure 2)
Using the a-theorem or trace-anomaly sum rules on the continuum side to say that the infrared is not a free photon	The value of a is consistent with decreasing from the ultraviolet (three gluons) to the infrared (one photon), so it does not exclude a free photon Assumption

Table 3 Undecidable ideas

Idea	Present state
Termwise sign inequalities in the twisted sector ($\partial\Delta F/\partial\beta_p \geq 0$ for the per-face couplings β_p)	All 48 positive for SU(2) on 1^4 Reading (26-12 § 05). For the finite group Q_8 there are points where a similar monotonicity breaks (21-01: monotonicity in β, three dimensions, no background) Reading . The lattice is a single point, so we cannot say it "holds"
A finite-box criterion for the total variation under a change of frame (on gauge-invariant observables)	The proofs of this book's Lean strong-coupling window use Dobrushin's condition on all links without fixing a gauge (a reading of the proofs of 20-02 and 20-03). A criterion that captures, outside the window, decay present only in the gauge-invariant part has no precedent in this book. In an inner region that does not touch the frame, this distinction may not be needed (item 5 of § 05, not checked by computation)
Filling the gap between physical strong coupling ($\beta \lesssim 2.2$) and the proof window ($1/19$) with finite-box criteria and certificates from the loop equations	There is a method that gets upper and lower bounds on lattice plaquette averages from the loop equations and positivity (in the large-N limit) Cited . The scale needed for small boxes of SU(2) is unknown

Idea	Present state
Seeing the generator of the heat bath as a sum of projections, and borrowing the criterion from quantum spin systems "if the gap of a finite chain exceeds a threshold, there is a gap in infinite size too"	The criterion itself exists for one-dimensional translation-invariant systems (and the two-dimensional square lattice) Cited . Whether the form of the threshold fits for projections on links in four dimensions has not been checked
Using the binary icosahedral group for a finite version of the band	There is a count of flat connections Reading . The value of β at which it freezes, and whether it imitates the behaviour of SU(2), are statements in the literature (not checked by this book)
Curvature that accumulates step by step	In the hierarchical model the one-step pace $(1-b^{-2})/3$ (0.25 for $b=2$) comes from the curvature factor of the group Proved here (note below). Most of the $11/3$ in four dimensions is not expected to come from curvature, so this is not expected to be the core Assumption
From tension to gap, by soft arguments within a class of actions	For SU(3) there is a report that the scalar mass in lattice units drops as the endpoint of the first-order line of the mixed action is approached Cited . For SU(2) not checked by this book, and the position of the endpoint also varies in the literature (an apparent endpoint on small lattices, and a revision on large lattices; 26-12 § 02). It concerns fixed lattice spacing and says nothing about the gap of the continuum theory
Using "the rate is uniformly positive" in a class of effective actions	Across a first-order line the tension may only jump and stay positive, so the first-order line of the mixed action does not break this. What would break it is the gap closing at the endpoint, which for SU(2) is not checked by this book
With pure gauge fields and a local action, the 't Hooft loop cannot have an area law (so for SU(2) the gap and confinement are hard to separate)	A conjecture Assumption
The trajectory of renormalised actions passes far from the endpoint of the mixed action	A statement in the literature (not checked by this book)
Vortex insertions adapted to the configuration, with relative entropy tending to 0 (a lower bound on the side of entropy)	Only a draft of the form. The first hurdle is to reproduce the one-loop $11/3$ Assumption
A monotonicity formula with the trace anomaly as its deficit	Only a draft of the form Assumption

Note (one step of curvature accumulating step by step) Proved here . Approximating the SU(2) plaquette weight by the heat kernel $K_t(\theta)$ (θ the angle of the plaquette holonomy), for small t one has $K_t(\theta) \propto \frac{\theta}{\sin\theta}\, e^{-\theta^2/(4t)}$. Bundling n plaquettes (the n-th power of the weight) gives $K_t^n \propto (\theta/\sin\theta)^{n-1} K_{t/n}$, and putting $\ln(\theta/\sin\theta) \approx \theta^2/6$ into the exponent, the Wilson β ($\beta = 1/(2t)$ from $1-\cos\theta \approx \theta^2/2$) becomes $n\beta - (n-1)/3$.

In the four-dimensional hierarchy with $b = 2$ (bundling $n = b^2 = 4$ plaquettes, and convolving b^2 plaquettes within a face, $t \to b^2 t$), one step gives $\beta \mapsto \beta - (1 - b^{-2})/3 = \beta - 0.25$. We read its closeness to the one-loop step $s \ln 2 \approx 0.2575$ as a coincidence Assumption .

05

The smallest next step

We write it in the form of questions. None of them has been answered.

1. **Filling in the sieve table.** For each "—" cell of 26-12 Table 2: in that opponent model, is the statement true, false, or undefined? Can the candidates for the core be re-selected by the criterion of keeping only statements that are false for weak-coupling U(1) and also false at a point of the endpoint type of the mixed action (if the gap closes at the endpoint; for SU(2) not checked by this book)?
2. **The minimal form of the ultraviolet extension.** In a box with background twist $n_{12} = 1$, does $\Delta F(N, M, \beta)$ with the ratio of the side of the box to the correlation length fixed converge as $\beta \to \infty$? The first step is reading: in the published construction's proof of convergence of the whole sequence in a periodic box, where does it depend on the boundary conditions?
3. **Testing the sign inequality.** For SU(2), background $n_{12} = 1$ and non-negative per-face couplings, is $\partial \Delta F / \partial \beta_p \geq 0$? After 1^4, elongated boxes (because the breaking for finite groups appears in geometries long in the transverse direction), then the closed formula in two dimensions and 2^4. Lift the smallest example that breaks for Q_8 to SU(2), and see where the breaking disappears.
4. **The additive comparison on the weak side.** Can $\Delta F(2L, \beta) \leq \Delta F(L, \beta - c')$ $(0 < c' < 0.2575)$ for $\beta \geq \beta_0$, $L \geq N_1(\beta)$ (N_1 of order $\sqrt{\beta}$) be written as a proposition at the one-loop level? The margin per doubling is the 2.54 of (4).
5. **A model case of a finite-box criterion on the strong side** Assumption (not checked by computation). In a two-dimensional SU(2) box, is the total variation, under a change of frame, of the law of an inner region not touching the frame bounded for all β by $C(\beta)\,(I_2(\beta)/I_1(\beta))^n$, with n the number of plaquettes in between (I_ν the modified Bessel functions)? And when the law of the inner links is invariant under gauge transformations at the inner points, is its total variation the same whether measured on all links or on the gauge-invariant part? Both are proposals not checked within this book's study; if checked, they would likely give the smallest model case of "the box criterion works even at β where the single-link criterion fails".

6. **A single-plaquette ruler** Assumption (not checked by computation). Measuring disorder by the ratio of fundamental-representation coefficients of a single plaquette, SU(2)'s $I_2(\beta)/I_1(\beta)$ matches U(1)'s value at its transition point, $I_1(1.01)/I_0(1.01) \approx 0.450$, at $\beta \approx 2.105$ (a numerical value of a ratio of Bessel functions, recomputed with this book's script). Whether it is meaningful to compare the two groups with this ruler, and whether its closeness to where SU(2) crosses over is not a coincidence, has not been checked.

06

Mathematics that seems to be needed

All of these are this book's conjectures Assumption . The names were chosen to describe the content.

- **Control of the continuum limit in twisted boxes.** Comparisons of scales, with a finite sum of errors, for quantities fixed by the measure and in twisted (framed) boxes. An extension of the type of the published construction, and the tool for the first job of § 01.
- **Coupling the laws of gauge-invariant observables.** A way of coupling conditional laws of gauge-invariant functions rather than link by link, and a finite-box criterion of Dobrushin–Shlosman type on top of it.
- **A method for computing lower bounds on the side of entropy.** Insertions adapted to the configuration, with relative entropy tending to 0. The first hurdle is to reproduce the one-loop $11/3$. A tool to fill the shortfall on the side of entropy of § 03.
- **Orderings of actions in the twisted sector.** A non-abelian inequality of Ginibre type saying that ΔF increases with the per-face couplings, or an ordering between actions that would take its place.
- **Expansion-free tools where the coupling is of order 1.** The combination of finite-box criteria and computational certificates.
- **A monotonicity formula with the trace anomaly as its deficit.** A quantum version of the classical monotonicity that becomes exactly trivial in four dimensions.
- **A stand-in for the vortex free energy for groups without centre.** The response to boundary fields and the like. For G_2 the tools of vortices cannot be used (26-12 § 02).

07

What this chapter does not say

- It does not give a road across the band. None of the ideas crosses the band. There is no proof.
- The number of doublings and the ratio of scales in (1) are rough orders of magnitude, not quantitative claims. They were computed from the lower bound that the threshold is at least 9×10^5.
- It does not say that "the remaining question is a question about the lattice theory at lattice spacing 1" (16-03 § 05).
- The discussion of the endpoint of the mixed action concerns fixed lattice spacing; it does not mean that the continuum SU(2) theory has no mass gap.
- The enumeration of the facts specific to SU(2) in this book (§ 03) is within what we read of the list of Lean theorems; it was not checked by carrying the proofs over to U(1).
- "Broken" means that arguments of that type do not go through; it does not mean that the goal itself is false.
- It does not say that the observations written here are new.

What this chapter has established

Established	Not established
Lean The strong end: the infinite-volume limit, the mass gap and the area law in the strong-coupling window (20-06). Facts specific to SU(2): the vacuum with a twist in one direction, its stabiliser ± 1, no configuration of action 0 in a box with two twists, $S_N > 0$, $S_N < 2N^2$ ($N = 2, 3, 4$) (21-05, 21-07)	Outside the window. The same statements in a class of effective actions. A rate uniform in the transverse size
Proved here The division of (1) (as a rough order of magnitude). (3): the prefactor is exactly $K^{k0} = \xi^{\log 2\, K}$, the rate σ_s/ξ^2 does not depend on K, and uniformity needs $\sum \ln K_i < \infty$. The one-loop 2.54 per doubling of (4). The direction of variational bounds. The hierarchical step 0.25	The quantitative meaning of (1). That (2) holds. The existence of a comparison with a finite sum of losses
Reading The recorded decrease $4^4 \to 8^4$: 1.61, 1.88, 1.80. The minimal sector action 7.84, 9.80. Cited The literature rows of Tables 2 and 3 (the source table of 26-12)	Conclusions about large lattices from values on small lattices
Assumption The two-way split of the core, the shortfall on the side of entropy, the division into Tables 1–3, § 05 and § 06	A road across the band

Sources and reproduction

Item	Kind	Source, tool
Theorems at the strong end, facts specific to SU(2) (§ 03)	Lean	The declarations of 20-03, 20-04, 20-06, 21-05, 21-07 (`InfMain2.wilson_infvol_main2`, `AreaLaw.area_law_wilson`, `AreaLaw.area_law_infvol`, `ThetaLimit.flat_exists_10`, `TwistFlat.no_flat_twist11`, `TwistBlocks.stab_vacT`, `TwistBlocks.rigidT_all`, `VortexPos.SN_pos`, `VortexCert.SN_two_lt`, `TwistBlocks.thick_vortex_floor_two` and others). The statements were checked against A-04 Table 4g and the pages 21-05, 21-07 and 20-06
Lower bound on the threshold of the published construction	Proved here	16-03 § 01 (from one of the conditions imposed on the threshold; a floating-point computation)
The division of (1)	Proved here	A script of this book's study (floating point; rerun for this chapter)
Differences of the recorded ΔF, minimal sector action	Reading	26-09 Tables 8 and 10, 25-02 Table 1
The non-confining phase of the U(1) Wilson action	Cited	J. Fröhlich, T. Spencer, “Massless phases and symmetry restoration in abelian gauge theories and spin systems”, Commun. Math. Phys. 83 (1982) 411 (checked against the abstract and § 1 of the preprint IHES/P/81/40; shown mainly in the Villain approximation, stating that all the results can be extended to models with the Wilson action; the extension relies on techniques of other papers)
Bounds on plaquette averages from the loop equations and positivity	Cited	V. Kazakov, Z. Zheng, arXiv:2203.11360 (the large-N limit, $D = 2, 3, 4$; checked against the abstract)
Threshold for the gap of finite chains	Cited	D. Gosset, E. Mozgunov, “Local gap threshold for frustration-free spin systems”, J. Math. Phys. 57 (2016) 091901, arXiv:1512.00088 (checked against the abstract; one dimension and the two-dimensional square lattice. The earlier threshold is due to Knabe, according to the same source)
The first-order line and endpoint of the mixed action, the revision of the endpoint's position, the scalar mass near the endpoint (SU(3))	Cited	G. Bhanot, M. Creutz, Phys. Rev. D 24 (1981) 3212. R. V. Gavai, Nucl. Phys. B 474 (1996) 446. U. M. Heller, Phys. Lett. B 362 (1995) 123 (for details, see the sources of 26-12)

Item	Kind	Source, tool
Statements in the literature in Tables 1–3 that have not been checked (the gap in the weak-coupling phase of Z_2, the freezing β of the binary icosahedral group, the position of the renormalised trajectory, the scalar mass at the SU(2) endpoint, the equivalence of Stroock–Zegarlinski type)	—	Statements in the literature (not checked by this book). Not used as grounds for claims in the text

Words used in this chapter — the crossover band, the two-way split of the core, the ultraviolet extension, the statement for the last stretch to strong coupling, the lossy chain, the prefactor, facts on the side of energy, the shortfall on the side of entropy, the mixing statement for a box of physical size, the seven test opponents, the three classes by level of value, mixed action, first-order transition, critical endpoint, flat connection, frozen phase, control of the continuum limit in twisted boxes, coupling the laws of gauge-invariant observables. Their meanings are in A-02 Glossary.

Next chapter to read: 17-01 How physics sees it. The table of opponents against which arguments about the band are tested is in 26-12 (after reading Part 26).

2026-10-01 · chapter Yang–Mills and the mass gap Part 17 From the side of physics asymptotic freedom, dimensional transmutation, confinement

How physics sees it — freedom at short distances, a single scale, the tube of confinement, and the premise that "the theory exists"

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — We split the picture physics draws of pure Yang–Mills theory into three parts, asymptotic freedom, dimensional transmutation and confinement, and look at each with formulas and numbers. We read how the ratios of the lattice glueball masses connect with "there is a single scale". We check that the whole picture rests on the premise that "the theory exists", and that confinement and the mass gap are different claims in mathematics.

Prerequisite chapters — 4-07 Dimensional transmutation and Λ (and 4-06 before it). Easier with: 3-02 (gauge fields), 5-01 (the lattice and β), 4-11 (the definition of the gap). The vocabulary of Wilson loops and the lattice is defined on the spot as far as needed. Within Volume 3, this chapter is read after Part 16 (where things stand, and the questions that remain).

Said first — This is a chapter of pictures at the level of physics. "Confinement", "a single scale" and "the glueball mass" are not theorems of the four-dimensional continuum theory (label Physics). This book does not itself show either the mass gap or the construction of the continuum limit. For the construction of the continuum limit, Volume 2 reads the one published on 2026-11-10 (a theory satisfying the Wightman axioms; it does not show the mass gap). The lattice numbers are values from primary sources; only the ratios and conversions were recomputed on this terminal.

The order of this chapter

1. The question of physics and the question of mathematics — how it is, rather than whether it exists
2. Short distances: the coupling becomes weak — the minus sign of the β function

3. A single scale is born: dimensional transmutation — every mass is a constant multiple of Λ
4. Long distances: Wilson loops and the area law — the definition of confinement, and its linear case
5. The flux tube and the Lüscher term — the picture of the tube, and the correction at long distances
6. The physical face of the gap: glueballs — the lattice ratios $m/\sqrt{\sigma}$
7. Detour: where the unit MeV comes from — borrowing a unit
8. The premise that "the theory exists" — at which stage of the numbers it enters
9. Confinement and the gap are different claims — examples with only one of them, and the implication on the lattice
10. The tools of physics, and the shore of each — a map of Part 17
11. Detour: why the gap has a minimum — the picture of being squeezed from both sides — on the large side anti-screening, on the small side uncertainty

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01

The question of physics and the question of mathematics

The prize problem asks "does the theory exist, and is there a gap in the spectrum?" (0-03). The questions of physics are "what is the mass of the lightest particle?" and "how does the force change as a pair of quarks is pulled apart?". Physics takes the theory as already existing and asks about its contents.

Jaffe and Witten list three properties that QCD (Yang–Mills theory with quarks) must have at the quantum level Known . (1) The mass gap: 'every excitation of the vacuum has energy at least Δ'. This is why the nuclear force is strong yet reaches only short distances. (2) Quark confinement: physical particle states are invariant under SU(3) transformations, and a single quark is not seen. (3) Chiral symmetry breaking.

Pure Yang–Mills has no quarks, so (3) does not arise. (1) is the body of the prize problem, and (2) is set apart in § 5 of the problem statement as a "natural extension". The picture of physics tries to explain (1) and (2) in one sheet, and is made of three parts: the coupling weakens at short distances (§ 02), a single scale is born (§ 03), and at long distances the flux is confined to a tube (§ 04, § 05).

02

Short distances: the coupling becomes weak

Under renormalisation, the coupling g changes its value with the scale μ at which we look (an energy; as a distance, roughly $1/\mu$) (4-04). The β function that fixes how it changes has the following form in the one-loop computation for pure SU(N) Known .

$$\beta(g) = \mu\frac{dg}{d\mu} = -b_0\, g^3 + O(g^5), \qquad b_0 = \frac{11N}{48\pi^2} \quad (1)$$

Because of the minus sign, raising μ (looking at shorter distances) makes g smaller. This is **asymptotic freedom** (4-06). The PDG review writes, with $\alpha = g^2/4\pi$, $\mu^2 d\alpha/d\mu^2 = -b'_0\alpha^2 - \cdots$ and $b'_0 = 11N/(12\pi)$ (the same thing Computed), and says 'The minus sign in Eq. (9.3) is the origin of "asymptotic freedom"'. With the first term alone, $d(1/g^2)/d\ln\mu = 2b_0$ is constant, so

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + 2b_0 \ln\frac{\mu}{\mu_0} \quad (2)$$

With SU(3) and $g^2(\mu_0) = 1$, we get $g^2 = 0.757$ at $\mu = 10\mu_0$, and still 0.342 at $10^6\mu_0$ Computed . The weakening is logarithmic. In the words of the problem statement, 'at short distances the field displays quantum behavior very similar to its classical behavior'. But (1) is a perturbative computation. That the continuum theory really behaves like this at short distances is one of the properties the problem statement requires of a solution.

03

A single scale is born: dimensional transmutation

Reading (2) in the direction of lowering μ, $1/g^2$ becomes 0 at a certain scale Λ.

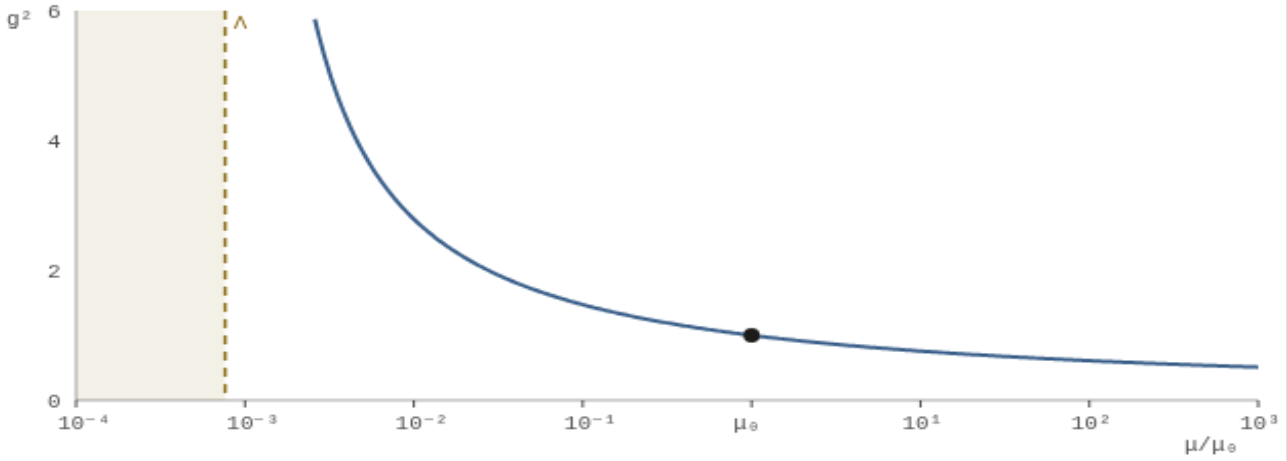
$$\Lambda = \mu\, \exp\!\Big(-\frac{1}{2b_0\, g^2(\mu)}\Big) \quad (3)$$

Substituting (2), the $\ln(\mu/\mu_0)$ cancels and Λ does not depend on μ (at three points along the flow it agrees up to double-precision rounding Computed). Classical four-dimensional Yang–Mills has no constant with the dimension of mass (g is dimensionless, 1-12). Even so, in the quantum theory a scale Λ appears. This is **dimensional transmutation** (4-07). The PDG explains Λ as 'a constant of integration, which corresponds to the scale where the perturbatively-defined coupling would diverge'. What breaks at $\mu = \Lambda$ is the one-loop formula, not the theory. From here physics concludes that every quantity with a dimension is a constant multiple of Λ Physics .

$$m_0{}^{++} = c_1\Lambda, \qquad \sqrt{\sigma} = c_2\Lambda, \qquad \Delta = c_3\Lambda \quad (4)$$

$m_0{}^{++}$ is the mass of the lightest glueball (§ 06), σ the string tension (§ 04), and c_i pure numbers fixed by N. This "single scale" looks obvious in physics as dimensional analysis, and is not proved in mathematics ("the diffusive-relaxation-time assumption" in the record of this book's study; 17-12, 23-01). Also, $\exp(-1/(2b_0 g^2))$ has all Taylor coefficients 0 at $g = 0$. If (4) is granted, Δ too appears at no order of the perturbation series. This runs the same way as the problem statement's 'the mass gap is not classically visible' (4-12).

Figure 1 Changing, with the knobs, the coupling $g^2(\mu_0)$ at the reference scale μ_0 and N moves the running coupling $g^2(\mu)$ (solid line) and the scale Λ where it diverges (dashed line). For SU(3), changing $g^2(\mu_0)$ from 0.5 to 1 moves Λ/μ_0 by a factor of about 1300 (three orders of magnitude)




```
SU(3), g²(μ₀) = 1.00 → Λ/μ₀ = 7.63×10⁻⁴ | increasing g² by 1% multiplies Λ by 1.074
```

Table 1 The one-loop Λ/μ_0, and the factor by which Λ changes when g^2 is increased by 1%

Group	$g^2(\mu_0)$	Λ/μ_0	Factor
SU(2)	0.5	4.45×10^{-10}	1.238
SU(2)	1.5	7.63×10^{-4}	1.074
SU(3)	1.0	7.63×10^{-4}	1.074
SU(3)	3.0	9.14×10^{-2}	1.024

Computed The numbers in this figure were computed on this machine. The curves are drawn on the spot from formulas (2) (3), and Table 1 evaluates the same formulas in Python 3 (for SU(3) at $g^2 = 0.5$, $\Lambda/\mu 0 = 5.83 \times 10^{-7}$).

The smaller the coupling, the exponentially smaller Λ is, and the more sensitive to changes. SU(2) at $g^2 = 1.5$ and SU(3) at $g^2 = 1.0$ give the same value because at one loop only Ng^2 (the 't Hooft coupling, 17-04) matters. The numerical value of Λ changes by a fixed factor depending on how g is defined, but the form of (4) is kept.

04

Long distances: Wilson loops and the area law

To measure the force in a theory without quarks, we place a pair of static colour charges from outside (a colour charge is a charge that acts as a source of the gauge field; 3-02). In Euclidean spacetime (4-02, One sign is different) take a rectangle of width R and length T; the trace of the matrix obtained by parallel transport once around it along the gauge field (the holonomy, 1-07) we call the **Wilson loop** $W_{R,T}$. The rectangle is the process "create the pair, leave it for a time T, annihilate it", and the expectation value decreases roughly as $e^{-V(R)T}$, so we define the potential by the following (Wilson 1974; the notation follows Chatterjee 2018 § 4) Known .

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \log\langle W_{R,T}\rangle \quad (5)$$

If $V(R) \to \infty$ as $R \to \infty$, the pair cannot move apart beyond a fixed distance. This is **confinement** (the definition in Chatterjee 2018 § 4). In particular, if $\langle W_{R,T}\rangle$ decreases as $e^{-\sigma RT}$ in the area RT (the **area law**), then $V(R) \approx \sigma R$ grows linearly. The coefficient σ in this linear case is the **string tension**. That it grows linearly in four-dimensional non-abelian Yang–Mills stands at the level of 'it is believed that V(R) grows like a multiple of R' (ibid. § 4) Physics .

On the lattice, which replaces spacetime by a mesh of points (5-01), the strength of the coupling is expressed by the **lattice inverse coupling** β (for the standard Wilson action $\beta = 2N/g^2$; a large β is weak coupling; this is not the β function of § 02). On the side where β is small (strong coupling) the area law is proved, but it holds for every group including `U(1)`, so it is no evidence of non-abelian confinement. The weak-coupling side for non-abelian groups in four dimensions 'remains elusive' (Chatterjee 2018; 5-02, 5-06) Known .

05

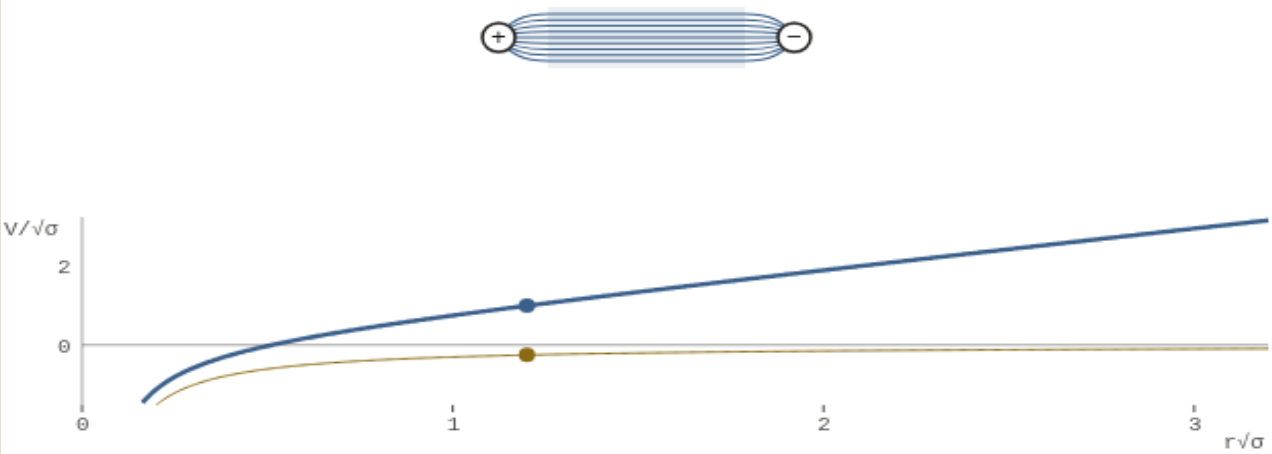
The flux tube and the Lüscher term

The picture of physics is the **flux tube**. The lines of force of electromagnetism spread out, and pulling the charges apart the energy approaches a constant value ($V \sim -c/R$). Under confinement the lines of force are squeezed into a tube of fixed thickness, and the energy is proportional to the length. As a metaphor, it is a string whose tension does not change as it is stretched. More precisely, the tube is not a fact derived from the theory but a picture that fits the lattice numbers; the effective theory of the fluctuations of the tube (17-03) takes the existence of the tube as its starting point and gives the correction at long distances as follows Physics .

$$V(r) = \sigma r + c_0 - \frac{\pi}{12\,r} + \cdots \quad (6)$$

$-\pi/(12r)$ is the Lüscher term, and its coefficient is held not to depend on the details of the theory ($-\pi(d-2)/24$ in d dimensions). c_0 is a non-universal constant. Putting (6) into the lattice length standard r_0 ($r_0^2 F(r_0) = 1.65$ for $F = dV/dr$, Sommer 1994) gives $r_0\sqrt{\sigma} = \sqrt{1.65 - \pi/12} = 1.178$, which differs by 1.6% from the lattice continuum value $1.160(6)$ (Athenodorou–Teper 2020) Computed . At r_0 the tube is still at short distance, so this is a rough guide that extends formula (6) to that distance. It agrees in outline, but the difference is about 3 times the lattice error, and it does not agree in detail.

Figure 2 Changing the distance r between the charges with the knob moves the lines of force above and the point on the energy V(r) below. The buttons switch between "electromagnetism (spreading)" and "confinement (tube)"



r√σ = 1.20 (about 0.49 fm), confinement: V/√σ = 0.98, electromagnetic sample: −0.25

r√σ	r (fm)	Confinement V/√σ	Electromagnetic sample V/√σ
0.3	0.12	−0.57	−1.00
0.51	0.21	0.00	−0.59
3.0	1.22	2.91	−0.10

Computed The numbers in this figure were computed on this machine. Confinement is $\sigma r - \pi/(12r)$ of (6) (without c_0), electromagnetism is the sample $-0.3/r$, and fm is converted with $\sqrt{\sigma} = 485$ MeV (§07). 0.51 is $\sqrt{\pi/12}$, where the two terms balance.

In QCD with quarks, signs that a long tube breaks have been reported on the lattice (SESAM 2005) Physics . Pure Yang–Mills has no quarks to create a pair midway and break the tube.

06

The physical face of the gap: glueballs

The particles of pure Yang–Mills are thought to be **glueballs**, colourless bound states made only of gluons. States are classified by J^{PC} (spin, parity, charge conjugation). 0^{++} is a state of spin 0 that changes sign neither under spatial inversion (parity) nor under the exchange of particle and antiparticle (charge conjugation). In physics the mass of the lightest 0^{++} is often called "the mass gap" Physics . Chatterjee 2018 explains the correlation length (2-08) as 'the reciprocal of the mass of the lightest glueball', and writes that the existence 'has not yet been experimentally verified'.

The gap of the problem statement is defined as a gap in the spectrum of the Hamiltonian (the problem statement calls its supremum 'the mass'), and does not presuppose the existence of particles. If the lowest excitation is an isolated one-particle state, the gap coincides with the mass of the lightest glueball (4-11). A proof that an isolated one-particle state exists is counted by the problem statement on the side of the "natural extensions".

Lattice computations replace spacetime by a mesh with lattice spacing a (the distance between lattice points) and produce the dimensionless ratio $m/\sqrt{\sigma}$.

Table 2 Glueball masses (continuum limit), and the ratios computed on this machine

J^{PC}	SU(2) $m/\sqrt{\sigma}$	SU(3) $m/\sqrt{\sigma}$	SU(∞) $m/\sqrt{\sigma}$	$m/m_{0^{++}}$: SU(2), SU(3), SU(∞)
0^{++}	3.781(23)	3.405(21)	3.072(14)	1, 1, 1
2^{++}	5.349(20)	4.894(22)	4.582(14)	1.415(10), 1.437(11), 1.492(8)
0^{-+}	6.017(61)	5.276(45)	4.711(25)	1.591(19), 1.549(16), 1.534(11)
0^{++*}	6.126(38)	5.855(41)	–	1.620(14), 1.720(16), –

Computed The errors of the ratios are a naive propagation treating them as independent, a rough guide that ignores correlations. SU(∞) is the value extrapolated in $1/N^2$. On the lattice rotation symmetry is reduced to the rotations of a cube, and spin 2 splits into two sets (E, T_2) that are measured separately. The 2^{++} of SU(∞) is the E value (4.578(11) for T_2; 5-08). 0^{++*} is an excited state.

For every N, 0^{++} is the lightest, and the ratios change only slowly with N (17-04). On the other hand, $m_{0^{++}}/\sqrt{\sigma}$ for SU(3) is 3.347(68) (Meyer 2005), 3.405(21) (Athenodorou–Teper 2020) and 3.607(87) (Lucini–Teper 2001), scattered more widely than the quoted errors. σ is obtained by subtracting the fluctuation correction from the mass of a closed tube. Formula (4) of Lucini–Teper is $am_l = a^2\sigma L - \pi/(3L)$, where m_l is the mass of a flux tube closed once around a periodic box, and L is the side of the box counted in lattice points. The authors note: 'in using eqn(4) we are assuming that we have linear confinement'. The continuum limit is an extrapolation assuming $m(a)/\sqrt{\sigma(a)} = m(0)/\sqrt{\sigma(0)} + c\,a^2\sigma$ (5-08).

07

Detour: where the unit MeV comes from

A section you may skip.

When we say "the lightest glueball is about 1.7 GeV", where does the MeV come from? Pure Yang–Mills has only one scale, Λ, and the unit is borrowed from outside. Athenodorou–Teper 2020 take $r_0\sqrt{\sigma} = 1.160(6)$ from pure Yang–Mills and $r_0 = 0.472(5)$ fm from lattice QCD with quarks, obtain $\sqrt{\sigma} = 485(6)$ MeV, and give 0^{++} as 1653(26) MeV. The recomputation on this machine gives $\sqrt{\sigma} = 484.96 \pm 5.7$ MeV and $0^{++} = 3.405 \times 485 = 1651 \pm 22$ MeV Computed . The paper states that the errors of its Table 18 are statistical only, while our ± 22 is a rough guide that includes the error of $\sqrt{\sigma}$; the meanings differ. The authors add: 'Strictly speaking this is of course not possible: the pure gauge theory and QCD … are two different theories'. What has meaning within pure Yang–Mills is the ratio 3.405.

08

The premise that "the theory exists"

The parts and numbers so far rest on the premise that four-dimensional pure Yang–Mills theory really does exist, as one theory. We list the stages of a lattice computation at which the premise enters.

Stage	What is done	Premise
Finite lattice	A finite-dimensional integral evaluated by the Monte Carlo method (5-07)	None (a well-defined probability measure)
Extracting σ	Subtracting the fluctuation correction of the tube	Linear confinement and the universality of the fluctuations
Infinite volume	Using only boxes with side at least $3/\sqrt{\sigma}$ (Lucini–Teper)	That the values do not change beyond that
Continuum limit	Extrapolating to $a \to 0$ in the form $c\,a^2\sigma$	That the limit exists, is unique, and is approached in this form

For physics this premise is natural, and the problem statement too writes that experiment and computers 'have given strong encouragement'. But the same paragraph continues 'there does not exist a convincing, whether or not mathematically complete, theoretical computation demonstrating any of the three properties in QCD', and the mathematical side is at the stage of 'nor even a precise definition' Known .

Put as a metaphor, physics draws a detailed map of land it is convinced exists, and mathematics is asked to show that the land exists. Precisely, the task is, for every compact simple group, to construct a non-trivial quantum Yang–Mills theory on $\mathbb{R}^4$ (with axioms at least as strong as those of Wightman or Osterwalder–Schrader; 4-09, 4-10) and to show $\Delta > 0$. A commonly taken road is to construct a Euclidean probability

measure with a regularisation and take the limit. A footnote of the problem statement says that weak existence, where only a subsequence converges, is not accepted unless the properties of the limit are shown separately. The numbers of physics, however precise, are "predictions made on the assumption of existence" (5-09, 7-10).

For the construction published on 2026-11-10, the committee's assessment is that it is proved in Lean that a theory satisfying the Wightman axioms can be constructed as a limit of Wilson's lattice gauge theory (Volume 2, 6-01). However, the infinite-volume limit is taken along a subsequence, and its uniqueness and the mass gap are not shown (6-04). The premise of the "continuum limit" row in the table above (that the limit exists and is unique) is still not fully met.

09

Confinement and the gap are different claims

In the picture of physics σ and m are two faces of the same Λ (formula (4)), and confinement and the gap are spoken of together. In mathematics they are separate claims.

There are examples where only one of them holds. For lattice gauge theory with finite groups, exponential decay of correlations at weak coupling is proved, while the absence of confinement is at the level of 'believed' (5-10). For compact `U(1)` in three dimensions the area law is proved at every coupling, and the continuum limit is a free scalar field (abstract of Göpfert–Mack 1982). In the converse direction, exponential decay under arbitrary boundary conditions on the lattice gives the area law for Wilson loops in representations on which the centre of the group acts non-trivially (Chatterjee 2021) `Known` . The table and the precise statements are in 17-02 § 02 (the master copy in this book).

This converse implication has a form close to "gap ⇒ confinement", but its hypothesis is a strong form of decay on the lattice. There is no theorem deriving confinement from a gap in the sense of the problem statement ($\Delta > 0$ in the continuum and in infinite volume). The variables physics mainly looks at (the tube and σ) and the target of the problem statement (Δ) are slightly offset.

10

The tools of physics, and the shore of each

Each tool of physics has a "shore" where its computations are under control. More precisely, a shore is a region near some limit ($g \to 0$, lattice $\beta \to 0$, $N \to \infty$ and so on) in which the errors of the computation can be controlled. The remaining chapters of Part 17 dig into the rows of the table.

Tool	Shore where the computation works	Level	Chapter
Perturbation theory	$g \to 0$. The gap does not show (granting (4))	Known	4-03–4-07
Strong-coupling expansion	The lattice inverse coupling β is small (strong coupling). Area law for every group including U(1)	Known	5-02
Lattice numerics	Finite a and boxes, and their extrapolation	Physics	5-08
Mechanisms of confinement, strings	Gauge-fixed variables / a long tube (confinement assumed)	Physics	17-02, 17-03
Large N, holography	$N \to \infty$ (holography also $g^2N \to \infty$)	Physics	17-04, 17-08–17-10
Supersymmetry	Supersymmetric theories with matter fields	Physics	17-11

The theory of the problem statement — any compact simple group, finite N, no matter, the continuum $\mathbb{R}^4$ — is on none of the shores. The pictures of physics fill the space between the shores with the premise that "one theory connects them" (§ 08). The places where that premise leaves mathematics are gathered in 17-13 and 16-05.

11

Detour: why the gap has a minimum — the picture of being squeezed from both sides

This section may be skipped. It reassembles three parts of this chapter (§ 02 asymptotic freedom, § 03 dimensional transmutation, § 04 confinement) towards a single question: "why does the lightest excitation have a non-zero mass?" Physics This is a picture, not a proof. None of its steps is a theorem in the four-dimensional continuum theory (27-02).

In one sentence. Because the field binds itself with its own charge, it cannot become a long wave, only a short lump. The minimum appears where the size of an excitation is squeezed from both sides.

The large side is closed off. A massless free field (light) has no gap because a wave can be made however long the wavelength, and its energy $E = |p|$ goes to zero. A long wave needs the field to keep its direction aligned over long distances. In Yang–Mills the mediating field itself carries colour charge, so the fluctuations of the vacuum do not screen the field but anti-screen it, and the coupling grows stronger as the distance grows (§ 02, 4-06). Beyond the distance $1/\Lambda$ where the coupling becomes of order one, the field cannot keep its direction and falls apart. So waves with wavelength longer than $1/\Lambda$ cannot survive as states. The colour electric flux cannot spread out, and is bundled into a closed tube (§ 05).

The small side is closed off too. In a region smaller than $1/\Lambda$ the coupling is weak and the field is nearly free, but an excitation localised to size R cannot, by the uncertainty relation, have energy below about $\hbar c/R$ (2-05, 16-08 § 09). If $R < 1/\Lambda$ then $E > \Lambda$.

Squeezed from both sides. What keeps it from growing large is anti-screening, and what keeps it from becoming small is uncertainty, so the lightest excitation is a closed ring of electric flux of size about $1/\Lambda$. Its mass is of order Λ, and that is the gap (the glueball of § 06). In one word, the origin is self-interaction; in one word, the reason for the minimum is the squeeze from both sides.

The difference from U(1). The photon carries no charge, so the vacuum only screens, and the coupling stays weak at long distances. The electric flux spreads out, long waves are allowed, and the gap is zero (5-06). Non-commutativity takes effect at a single point, "whether the field acts on itself", and the one-loop $11/3 = -1/3 + 4$ is its face (26-03).

In the language of the strong-coupling lattice. On the Hamiltonian lattice (Kogut–Susskind), the Hamiltonian is the sum of an electric term $\frac{g^2}{2}\sum E^2$ and a magnetic term $\frac{1}{g^2}\sum(1-\operatorname{tr}U_\square)$, and at strong coupling the electric term wins. Electric flux sits on single links, and each link costs of order g^2. The smallest gauge-invariant excitation is a ring of electric flux going round one face, and its cost is a finite positive number. This is the gap at strong coupling, and at strong coupling with the lattice spacing fixed it is a theorem (5-02, 5-04). What asymptotic freedom says is that "at the distance $1/\Lambda$ the effective coupling is strong", so this picture becomes, as it is, the picture at long distances. The question that remains is that this picture does not break down as the lattice spacing goes to zero; that is the ultraviolet control row of the ledger (as existence, shown by the published construction; the form of this book's road is open) and the renormalisation comparison row (27-02, A-08).

What this chapter can say

What is said	What is not said
Known The one-loop b_0, the solution (2), and within it that Λ does not depend on μ (as a perturbative computation). The definition of the gap in the problem statement, and that the gap coincides with the mass of an isolated one-particle state if there is one. Physics A single scale, the area law and the tube, the Lüscher term, the values of Table 2.	None is a theorem of the four-dimensional continuum theory (the construction published on 2026-11-10 gives the existence of the theory, but does not show confinement or the mass gap; 6-04). Table 2 is an extrapolation presupposing that the limit exists, and σ passes through the tube correction.
Known The area law at strong coupling (every group), the decay of correlations at weak coupling for finite groups, centre symmetry and its implication for confinement on the lattice, the area law at every coupling for U(1) in three dimensions.	The area law on the weak-coupling side for non-abelian groups in four dimensions. Deriving confinement from a gap in the sense of the problem statement. That finite groups do not confine (the level at which Chatterjee 2021 writes 'believed').
Computed The ratios of Tables 1 and 2, Figures 1 and 2, the estimate 1.178 of $r_0\sqrt{\sigma}$, the recomputation in MeV.	Ratios and conversions of primary-source values and evaluations of one-loop formulas, not measurements. The errors of the ratios are rough guides ignoring correlations. 1.178 is a rough guide extending a long-distance formula to short distances.

What is said	What is not said
Physics The squeeze picture of § 11 (on the large side anti-screening, on the small side uncertainty, the minimum a ring of size $1/\Lambda$). That on the strong-coupling lattice the cost of a ring round one face is positive.	No step of the picture is a theorem in the four-dimensional continuum. That anti-screening removes long waves, that the ring is the smallest excitation, that the picture survives as the lattice spacing goes to zero.

Sources and reproduction

Item	Kind	Source / tools
The three properties, the short-distance requirement, the definition of the gap and 'the mass', the requirements of the problem (group, non-triviality, $\mathbb{R}^4$, axioms), the classification of the extension problems, the quotations from the text, the footnote excluding weak existence	Known	A. Jaffe, E. Witten, "Quantum Yang–Mills Theory" (the problem statement of the Clay Mathematics Institute) § 1, § 2, § 4, § 5, § 6.6, footnote 2. Text checked.
The running of α, b_0', the minus sign, the explanation of Λ	Known	J. Huston, K. Rabbertz, G. Zanderighi, "Quantum Chromodynamics", Particle Data Group, Phys. Rev. D 110 (2024) 030001, § 9.1.1. Text checked.
The original papers on asymptotic freedom	Known	D. J. Gross, F. Wilczek, PRL 30 (1973) 1343. H. D. Politzer, PRL 30 (1973) 1346. Bibliographic data only (from the reference list of the problem statement).
Formula (5), the definition of confinement, the statements and quotations about the area law, strong coupling, the correlation length and glueballs	Known	S. Chatterjee, "Yang–Mills for probabilists", arXiv:1803.01950 § 4, § 5. Text checked. K. G. Wilson, PRD 10 (1974) 2445: bibliographic data only. Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 and Göpfert–Mack, CMP 82 (1982) 545: abstracts (INSPIRE; the former on the infinite-volume limit at strong coupling and Wilson's confinement estimate, the latter on the Villain action and the description of the continuum limit; texts not read. The year of Göpfert–Mack is the Crossref publication year; INSPIRE gives 1981).
The definitions of the centre and centre symmetry, centre symmetry and confinement, 'believed' for finite groups	Known	S. Chatterjee, CMP (2021), arXiv:2006.16229. Definition 2.1, Theorems 2.2 and 2.4 and the quotation 'believed' checked in the text.
Decay of correlations at weak coupling for finite groups (four dimensions)	Known	A. Adhikari, S. Cao, arXiv:2202.10375, Ann. Probab. 53 (2025) 140–174 (bibliographic data from a search). Abstract and the opening of the text (restriction to four dimensions) checked.

Item	Kind	Source / tools
Table 2, $r_0\sqrt{\sigma}$, r_0, 485(6) MeV, 1653(26) MeV and the note on its error, the note on the conversion	Physics	A. Athenodorou, M. Teper, JHEP 11 (2020) 172, arXiv:2007.06422 / JHEP 12 (2021) 082, arXiv:2106.00364. Numbers and quotations checked in the text (Tables 17, 18, 23, 31, 34, formulas (8)(9)).
3.607(87), the extraction of σ (formula (4)) and the form of the extrapolation, the coefficient π/12 for Wilson loops	Physics	B. Lucini, M. Teper, JHEP 0106 (2001) 050, hep-lat/0103027 Table 5, § 4. Text checked.
The Lüscher term, the definition of r_0, 3.347(68)	Physics	M. Lüscher, NPB 180 (1981) 317, R. Sommer, NPB 411 (1994) 839, H. B. Meyer, hep-lat/0508002: bibliographic data only (Meyer's value via the table of Gürsoy–Kiritsis–Nitti, arXiv:0707.1349).
Signs of tube breaking in QCD	Physics	G. S. Bali et al. (SESAM), PRD 71 (2005) 114513, hep-lat/0505012. Abstract checked.
Table 1, Figure 1, the ratios of Table 2, the estimate of $r_0\sqrt{\sigma}$, the table of Figure 2, the recomputation in MeV	Computed	Python 3 on this machine (standard library, double precision). Errors propagated treating them as independent. $\hbar c = 197.3269804$ MeV·fm.
The picture of § 11 (squeeze)	Physics	Put into words by one of this book's authors (Shiori) in a dialogue on 2026-10-29. The primary sources for the parts are Gross–Wilczek and Politzer (1973, row above), J. Kogut, L. Susskind, PRD 11 (1975) 395 (the Hamiltonian lattice and electric flux at strong coupling; bibliographic data only), K. G. Wilson, PRD 10 (1974) 2445.

Next chapter to read: 17-02 Mechanisms of confinement — on to the explanations from physics of why the tube of § 05 forms (dual superconductivity, monopoles, centre vortices, Gribov–Zwanziger).

Mechanisms of confinement — reading four pictures by what each casts in the leading role, and where each is cut off from mathematics

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — The leading roles and the support of the four pictures of confinement (dual superconductivity, abelian projection, centre vortices, Gribov–Zwanziger), and where each is cut off as a mathematical statement. That confinement and the gap are different sentences.

Prerequisite chapters — 17-01. Also 5-02 (Wilson loops), 4-08 (Gribov copies), 1-04 (the centre of a group).

Said first — All four pictures are physical pictures, and there is no theorem deriving confinement in the four-dimensional non-abelian theory. This chapter shows neither confinement nor the gap. The model of Figure 2 is a toy.

The order of this chapter

1. What we want to explain — the area law
2. Confinement and the gap are different sentences — the area law and exponential decay, a conditional implication
3. Dual superconductivity — turning the magnetic flux tube inside out (Figure 1)
4. Abelian projection and monopoles — singularities of diagonalisation / Detour: Elitzur's theorem
5. Centre vortices — membranes that multiply by a centre element
6. A model of a vortex gas — the area law and the perimeter law (Figure 2)
7. The Gribov region and the horizon — from the gauge-fixed side
8. The propagator and positivity — a correlation that would not go negative for a particle (Figure 3)
9. Setting the four pictures side by side — leading roles, support, what is missing

01

What we want to explain

"Confinement" is a word that comes from the observation that quarks cannot be extracted on their own. Pure Yang–Mills theory has no quarks, so we place two infinitely heavy test charges and ask about the energy of pulling them apart (17-01). On the lattice, the expectation of the trace of the matrix obtained by multiplying link variables along a closed path C is the Wilson loop (5-01); for an $R \times T$ rectangle $W \approx e^{-V(R)T}$, where $V(R)$ is the energy of the pair of charges.

$$W(C) = \left\langle \frac{1}{N} \operatorname{tr} \prod_{\ell \in C} U_\ell \right\rangle, \qquad V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln W(R,T) \quad (1)$$

There are two ways a large loop can decrease.

$$W(R,T) \approx e^{-\sigma RT} \ \text{(area law)}, \qquad W(R,T) \approx e^{-c\,(2R+2T)} \ \text{(perimeter law)} \quad (2)$$

Under the area law $V(R) \approx \sigma R$, and the energy grows in proportion to the distance. σ is the **string tension**. Under the perimeter law V is bounded. Example: with $\sigma = c = 0.05$ and $R = T = 10$, $-\ln W$ is 5 and 2; with sides of 20 it is 20 and 4. To tell them apart we use the Creutz ratio, which is σ for a pure area law and 0 for the perimeter law Known .

$$\chi(R) = -\ln \frac{W(R,R)\,W(R-1,R-1)}{W(R,R-1)\,W(R-1,R)} \quad (3)$$

To ask about the "mechanism" is to ask **what in the vacuum** produces the area law. The area law at small β (the lattice inverse coupling; for $SU(N)$, $\beta = 2N/g^2$, smaller meaning stronger coupling) is a theorem (5-02), but it holds for every group including `U(1)`, so it is no explanation for the non-abelian case Known . The question remains on the side of large β, heading towards the continuum limit.

02

Confinement and the gap are different sentences

The mass gap is exponential decay of correlations (4-11), confinement is the area law, and as logic they are different claims. What the Clay problem statement requires is $\Delta > 0$; confinement is set apart as a "natural extension" Known .

Table 1 Presence of confinement (area law) and of the gap (exponential decay), and the level

Model	Area law	Exponential decay	Level

Model	Area law	Exponential decay	Level
Any group, small β	Yes	Yes	Theorem (5-02)
4-dimensional `U(1)`, large β	No	No (photon)	Theorem (5-06)
Finite group, large β	Believed not	Yes (in a weak sense)	Decay is a theorem (5-10)
3-dimensional compact `U(1)`	Yes	In physics the photon acquires a mass	Area law is a theorem (§ 03)
4-dimensional `SU(N)`, continuum limit	Expected	Expected	Unproved

On the lattice there is an implication in one direction. Chatterjee (2021) showed the following Known . If the representation forming the Wilson loop acts non-trivially on the centre of the group (§ 05) (the fundamental representation of `SU(N)` does), and centre symmetry is unbroken (that is, in slab-shaped regions of finite thickness, the measure is invariant under centre transformations whatever boundary condition is taken; Definition 2.1 of that paper), then for any measure on the infinite-volume lattice $\mathbb{Z}^d$ the following holds for some function V with $V(R) \to \infty$ as $R \to \infty$.

$$|\langle W(R \times T) \rangle| \leq e^{-V(R)\,T} \tag{4}$$

This means that V is unbounded, not that $\sigma > 0$. Assuming a strong form of the gap, exponential decay under arbitrary boundary conditions, yields even the area law (the Theorem from Uniform Decay to the Area Law; Theorem 2.4 of the same paper). The decay shown for finite groups is decay of two-point correlations under a specific boundary condition, weaker than the "decay under arbitrary boundary conditions" that this theorem assumes. So it does not contradict the finite-group row of Table 1 (§ 2.4 of the same paper). There is no theorem deriving confinement from a gap in the Clay sense.

03

Dual superconductivity: turning the magnetic flux tube inside out

A superconductor expels magnetic fields (the Meissner effect), and a magnetic field entering from the surface decays over the penetration depth λ_L. If forced through, it is bundled into thin tubes (Abrikosov vortices), and the flux of one tube is an integer multiple of $h/2e$ Known .

$$\nabla^2 \mathbf{B} = \mathbf{B}/\lambda_L^2, \qquad B(x) = B_0\, e^{-x/\lambda L} \tag{5}$$

A tube with energy density $B^2/2$, cross-section S and flux $\Phi = BS$ carries $E/\ell = \Phi^2/(2S)$ per unit length. In vacuum S can be widened to bring this close to 0, but in a

superconductor widening costs something, and E/ℓ stays at a positive constant. If a pair of monopoles were embedded, the energy should be proportional to the distance (a schematic estimate).

Maxwell's equations with a magnetic current j^m added are symmetric under exchanging electric and magnetic ($\tilde{F}$ is the dual; only here do we use real-time notation)Known .

$$\partial^\mu F_{\mu\nu} = j^e_\nu, \quad \partial^\mu \tilde{F}_{\mu\nu} = j^m_\nu; \qquad \mathbf{E} \to \mathbf{B},\ \mathbf{B} \to -\mathbf{E},\ j^e \to j^m,\ j^m \to -j^e \tag{6}$$

If the vacuum is a **condensate of colour-magnetic monopoles**, what is expelled is the colour electric field, the colour electric flux forms a tube, and quarks are confined. This is the picture of dual superconductivity, said to have been proposed independently by 't Hooft (a 1975 talk) and Mandelstam (1976)Physics . Nambu (1974) is regarded as its forerunner. More precisely, the monopoles are not fields appearing in the action but objects defined only once a gauge is chosen (§ 04), and there is as yet no way to write "condensation" as a statement about the measure.

Figure 1 Left: in vacuum the lines of force spread out. Right: in a vacuum where monopoles have condensed, the electric flux forms a tube (schematic).



Vacuum (E/ℓ → 0) Condensed vacuum (E/ℓ = σ)

Di Giacomo et al. (2000, SU(2)) measured an order parameter for monopole condensation on the lattice and reported that "confinement appears as a transition to a dual superconductor"Physics . The context is the deconfinement transition at finite temperature, and the mechanism was not shown. Neither step of "condensation ⇒ area law ⇒ gap" is proved for the four-dimensional non-abelian case.

Examples where the mechanism can be computed in an abelian model

In three-dimensional compact U(1) monopoles act as instantons; Polyakov (1977) obtained the area law semiclassically, and Göpfert–Mack proved $\sigma > 0$ at every coupling (on the lattice with the Villain action)Known . But the abstract of the same work also states that the continuum limit is a free scalar field and that the string tension in physical units goes to ∞ (text not read). From this mechanism "the masses and the string tension are fixed by a single scale" (17-12) does not follow. In four dimensions, for an $\mathcal{N} = 2$ supersymmetric theory broken to $\mathcal{N} = 1$ by a mass term,

Seiberg–Witten (1994) described confinement by monopole condensation (17-11) Physics .

04

Abelian projection and monopoles

't Hooft (1981) found monopoles, absent from the action, as **singularities of the choice of gauge**. Choose a quantity $X(x)$ that transforms as $X \mapsto gXg^{-1}$ (for example F_{12}) and diagonalise it at each point.

$$g(x)\,X(x)\,g(x)^{-1} = \operatorname{diag}\big(\varepsilon_1(x),\dots,\varepsilon_N(x)\big) \tag{7}$$

ε_i are the eigenvalues. The remaining symmetry is $U(1)^{N-1}$, and the diagonal gluons are the "photons". At points where eigenvalues coincide the diagonalisation is not determined, and these are the monopoles Physics . Example: in `SU(2)` with $X = \sum_a X^a \sigma^a/2$ the eigenvalues are $\pm|\boldsymbol{X}|/2$, and they coincide at $\boldsymbol{X}=0$, which is 3 real conditions. In three dimensions this is a point, in four a line (a world line) (the same for general N, the detour of 1-01). If $\boldsymbol{X} \approx \underline{x}$ near the zero, then $\boldsymbol{X}/|\boldsymbol{X}|$ covers the sphere surrounding the zero once, and that number of times is the magnetic charge Known .

On the lattice, the maximal abelian gauge (MAG, Kronfeld et al. 1987) brings the links close to diagonal.

$$R[U] = \sum_{x,\mu} \operatorname{tr}\left[\,U_\mu(x)\,\sigma^3\,U_\mu(x)^\dagger\,\sigma^3\,\right] \;\to\; \max \tag{8}$$

No method is known for reliably finding the global maximum, and in practice local maxima (Gribov copies) are used. Suzuki–Yotsuyanagi (1990) reported that the string tension of the projected links accounts for most of the original (abelian dominance), and Greensite's review (2003), citing $(92 \pm 4)\%$ for `SU(2)` (Bali et al. 1996), writes that "abelian dominance is not yet a proof of confinement by monopoles" Physics .

Detour: Elitzur's theorem

A subsection you may skip.

A gauge transformation g at the starting point of a link U_ℓ alone gives $U_\ell \to gU_\ell$ (the other links touching that point change too, but the action and the measure are invariant). So $\langle U_\ell\rangle = g\langle U_\ell\rangle$ holds for every g, and averaging over the Haar measure (5-01) gives

$$\langle U_\ell\rangle = \Big(\int_{SU(N)} g\,dg\Big)\langle U_\ell\rangle = 0 \tag{9}$$

Elitzur's theorem (1975) states that, even in infinite volume, the expectation of a local quantity that is not gauge invariant stays 0 Known . Unlike a global symmetry (2-09), a gauge transformation at one point meets no barrier. A monopole defined by fixing a gauge is a function of the configuration once the gauge-fixing procedure is included. But since the maximum is not unique (Gribov copies), it is not determined as a single function, nor is it a local quantity.

05

Centre vortices: membranes that multiply a loop by a centre element

The **centre** of SU(N) is $Z_N = \{e^{2\pi i k/N} I\}$, which commutes with every element. A centre element z acts in each representation as z^k, and this $k \pmod N$ is the **N-ality**. The fundamental representation has 1 and the adjoint 0 Known .

The starting point is the lattice observation that the string tension of large loops is fixed by the N-ality alone, and is 0 for the adjoint (screened by the gluons themselves) (Greensite's review § 4.3) Physics . A centre vortex is a line in three dimensions and a surface in four, and multiplies a loop linked with it by one centre element (-1 for SU(2)). Adjoint loops do not notice the vortex. More precisely, the thinnest vortex on the lattice is a configuration in which the links piercing some region are all multiplied by a centre element, and only the plaquettes on the rim of the region change (inside, the centre elements cancel). This rim is the vortex, and only loops linked with the rim pick up the centre element. Divide the plane into squares of area $\Sigma_{\min}$; if the $Z(C_i)$ picked up by the loops of the squares are nearly independent, a large loop splits into a product and obeys the area law (formulas (94)–(96) of the same review).

$$\langle W_k(C)\rangle \approx e^{-\sigma(k)\,\Sigma}, \qquad \sigma(k) = -\ln\langle Z(C_i)^k\rangle / \Sigma_{\min} \quad (10)$$

Confinement becomes the question of whether the vortex surfaces spread through the volume (percolate). Numerically, Del Debbio–Faber–Greensite–Olejník (1997) reported that projected vortices carry most of the string tension (centre dominance), and de Forcrand–D'Elia (1999) that removing the vortices removes confinement Physics . As a theorem, Tomboulis–Yaffe (1985, SU(2)) bounded the Wilson loop from above by a function of the vortex free energy Known . The vortex free energy is $-\ln$ of the ratio of the partition function $\mathcal{Z}_{\mathrm{tw}}$ with a centre twist put into the boundary condition in one direction to the partition function $\mathcal{Z}$ without it, that is $F_v = -\ln(\mathcal{Z}_{\mathrm{tw}}/\mathcal{Z})$, and it is gauge invariant. In the same family of gauge-invariant quantities is the 't Hooft loop (an operator inserting a centre vortex along a loop). Formula (4) is also a theorem bounding Wilson loops from centre symmetry. Tomboulis (2007) claimed a proof at every coupling for SU(2), and Ito–Seiler point out a gap in the argument.

Vortices defined by projection depend on the gauge. The hypotheses of the conditional theorems concentrate on a single point, that vortices condense at weak coupling in four dimensions (gauge-invariantly, decay of the vortex free energy), and this is not proved.

For how this book uses this, see 26-08 § 03 (where the average over vortex containers and the Migdal–Kadanoff interpolation stopped in the literature).

06

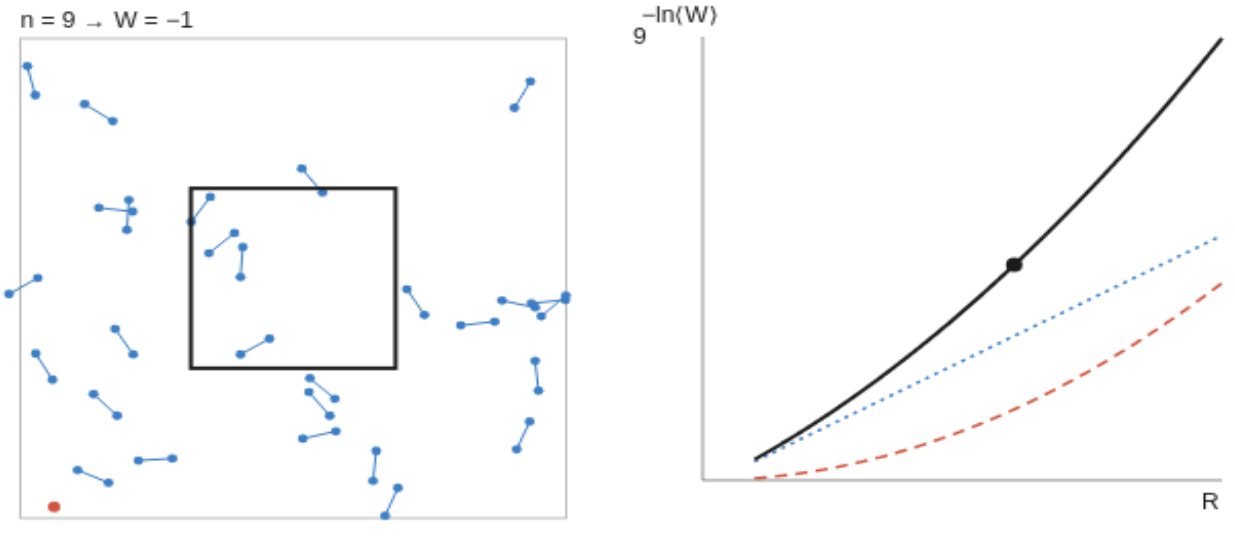
A model of a vortex gas: counting the area law and the perimeter law

Scatter over a plane the points where `SU(2)` vortices pierce it, and with n the number of points inside the loop set $W = (-1)^n$. There are two kinds of points. (a) Independent points of density ρ_f (cross-sections of extended vortices). (b) Pairs of points at distance d, with point density ρ_p (small vortices piercing the plane twice). A pair changes the sign only when it straddles the edge. For an $I \times J$ rectangle the following holds exactly (averaging over directions the area of the symmetric difference of two translated rectangles) Computed .

$$-\ln\langle W(I,J)\rangle = 2\rho_f\, IJ + 2\rho_p\Big[\frac{2d}{\pi}(I+J) - \frac{d^2}{\pi}\Big] \qquad (I,J \geq d) \quad (11)$$

The first term comes from the number n of independent points following a Poisson distribution with mean $\mu = \rho_f IJ$, so that $\mathbb{E}[(-1)^n] = \sum_n e^{-\mu}\mu^n(-1)^n/n! = e^{-2\mu}$ (1-08). The first term is the area law and the second the perimeter law, and the Creutz ratio is $2\rho_f$. However many small vortices there are, they contribute nothing to the string tension.

Figure 2 Changing R, ρ_f and ρ_p changes the sample sheet on the left (red = independent, blue = pairs) and, on the right, −ln⟨W⟩ (solid line, formula (11)), the area term (red dashed) and the perimeter term (blue dotted).



```
R = 6、ρf = 0.020、ρp = 0.20 : −ln⟨W⟩ = 1.44 (area) + 2.93 (perimeter) = 4.37. Creutz ratio = 0.040.
This sheet has n = 9
```

Computed The curves are computed on the spot from formula (11), and the points on the left are one random sample. Formula (11) was checked on this machine against a Monte Carlo (MC) of one million samples (Table 2).

Table 2 −ln⟨W(R,R)⟩: formula (11) and MC (lattice spacing 1, d = 1; in brackets the MC standard error in the last digit; "—" means it cannot be measured)

R	Independent 0.05	MC	Pairs 0.2	MC	Mixed	MC
2	0.400	0.399(1)	0.891	0.890(2)	1.051	1.052(3)
4	1.600	1.596(5)	1.910	1.920(7)	2.550	2.546(13)
6	3.600	3.574(36)	2.929	2.929(19)	4.369	4.255(70)
8	6.400	—	3.947	3.921(50)	6.507	—
Creutz ratio	0.100		0.000		0.040	

Mixed means $\rho_f = 0.02$, $\rho_p = 0.2$. The MC standard error grows as $-\ln W$ grows, and is at most 0.02 for $-\ln W \le 3$. Mixed at $R = 6$ has error 0.070, and its difference 0.11 from the exact formula is about 1.6 standard errors. $e^{-6.4} \approx 0.0017$ is about the size of the noise 10^{-3} of an average of ± 1, and cannot be measured. This difficulty is the same as in real lattice computations (5-07). For `SU(3)`, multiplying by $e^{\pm 2\pi i/3}$ at each piercing gives the string tension $1.5\rho_f$ for $k = 1, 2$ and 0 for the adjoint Computed .

For how this book uses this, see 25-03 § 03 (the identity by which fixing the frame makes the container signs independent).

07

The Gribov region and the horizon: from the gauge-fixed side

The fourth picture reads confinement from the shape of the gauge-fixed propagator. The Landau gauge surface $\partial_\mu A_\mu = 0$ can meet a single orbit more than once (4-08), so Gribov (1978) restricted the integral to the region Ω where the Faddeev–Popov operator $\mathcal{M}$ is positive. Its boundary is the Gribov horizon.

$$\mathcal{M}^{ab} = -\partial_\mu\left(\partial_\mu \delta^{ab} + g f^{acb} A^c_\mu\right), \qquad \Omega = \{A : \ \partial_\mu A^a_\mu = 0,\ \mathcal{M} > 0\} \quad (12)$$

"$\mathcal{M} > 0$" means that the eigenvalues of $\mathcal{M}$ (the spectrum of a self-adjoint operator, 1-10) are all positive. At $A = 0$, $\mathcal{M} = -\partial^2$, which is positive apart from constants. As A grows the smallest eigenvalue goes down, and where it reaches 0 is the horizon. On the horizon an infinitesimal gauge transformation in the direction of the zero eigenvalue preserves the gauge condition, so copies arbitrarily close appear.

What is certain as mathematics goes only as far as Dell'Antonio–Zwanziger (1991): "every orbit passes inside the horizon" Known . Copies remain inside Ω too, and no method is known for handling a copy-free region analytically (Zwanziger claims that the copies inside do not affect expectation values Physics).

08

The propagator and positivity

Zwanziger (1989) wrote the restriction to Ω as a local action. The propagator of its tree approximation (lowest order, without loop contributions, 4-03) is on the left below, and the version with condensates added (RGZ, Dudal et al. 2008) on the right Physics .

$$D_{\mathrm{GZ}}=\frac{p^2}{p^4+\lambda^4},\qquad D_{\mathrm{RGZ}}=\frac{p^2+M^2}{p^4+(M^2+m^2)\,p^2+\lambda^4+M^2m^2} \tag{13}$$

$\lambda^4=2g^2N\gamma^4$, and γ is fixed by the horizon condition. Here λ is a mass scale, different both from the penetration depth λ_L of § 03 and from the large-N $\lambda=g^2N$. M^2 and m^2 are mass parameters added in RGZ. The GZ form is 0 at $p=0$ and its poles are complex, $p^2=\pm i\lambda^2$, quite unlike $1/(p^2+m_0^2)$ for a free particle of mass m_0.

If the gluon were a physical particle, the propagator could be written as a superposition with positive density ρ (the Källén–Lehmann representation), and the correlation in the time direction $C(t)$ could not be negative (formulas (159)(160) of Dudal et al. 2008, 4-10). ω is the mass being superposed.

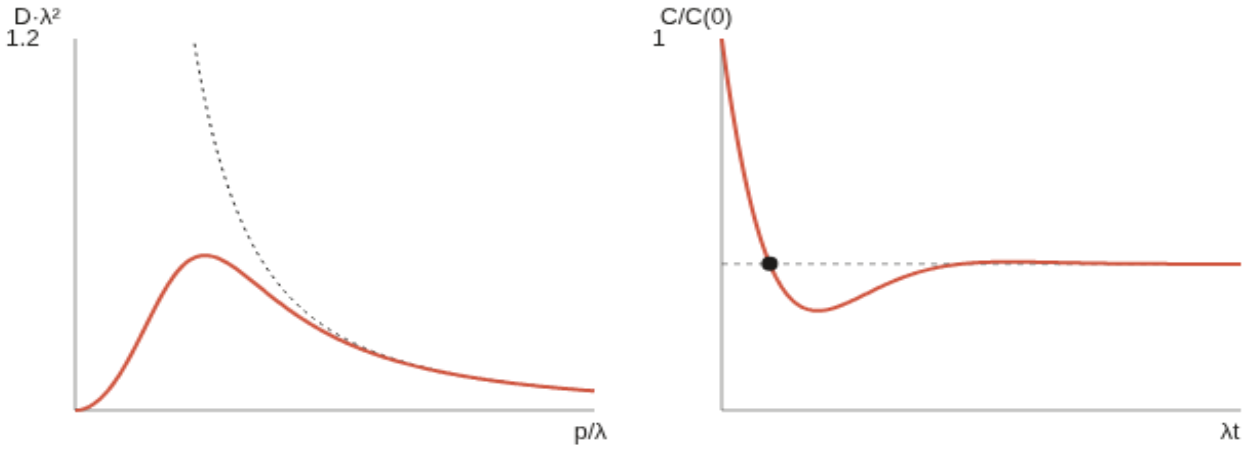
$$D(p^2)=\int_0^\infty d\omega^2\,\frac{\rho(\omega^2)}{p^2+\omega^2},\qquad C(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ipt}D(p^2)\,dp=\int_0^\infty d\omega\,\rho(\omega^2)\,e^{-\omega t} \tag{14}$$

At $m=0$, computing by complex integration (residues) gives a closed form (it can also be checked by numerical integration; bottom of Figure 3). For $M^2<2\lambda^2$, setting $\cos\varphi=M^2/(2\lambda^2)$ Computed

$$C(t)=\frac{e^{-\lambda t\cos(\varphi/2)}}{2\lambda\sin\varphi}\,\sin\!\left(\frac{3\varphi}{2}+\lambda t\sin\frac{\varphi}{2}\right) \tag{15}$$

and it first becomes negative at $t_*=(\pi-3\varphi/2)/(\lambda\sin\frac{\varphi}{2})$. For GZ, $t_*=\pi/(2\sqrt{2}\,\lambda)\approx 1.111/\lambda$. For $M^2>2\lambda^2$ it splits into real poles, $D=A/(p^2+r_1)+B/(p^2+r_2)$ $(0<r_1<r_2)$, and although B is negative (ρ is not positive), $C(t)\geq e^{-\sqrt{r_2}\,t}C(0)>0$ and it does not become negative. This test works in one direction only.

Figure 3 Changing M^2/λ^2 (m = 0) changes $D(p^2)$ on the left (dotted line $1/p^2$) and $C(t)/C(0)$ on the right. 0 is GZ; the black dot is the first zero.



M²/λ² = 0.0 (GZ) : D(0)·λ² = 0.0。The poles are complex conjugates. Negative from t* = 1.111/λ.

Computed The curves are computed on the spot from formulas (13)(15) and the partial fractions for real roots. The closed formula was checked on this machine against numerical integration, with differences at most 2×10^{-11}.

Table 3 Tree approximation of RGZ at m = 0 (λ = 1)

M²	D(0)	Poles	C(0)	Zero t*
0 (GZ)	0	Complex conjugate	0.354	1.111
1.0	1.0	Complex conjugate	0.577	3.142
1.9	1.9	Complex conjugate	0.734	16.86
2.5	2.5	Real (B = −0.333)	0.825	No
4.0	4.0	Real (B = −0.077)	1.021	No

The lattice did not agree with the original prediction $D(0) = 0$. Cucchieri–Mendes (SU(2), 128^4, about $(27\ \text{fm})^4$; 1 fm = 10^{-15} m) report in their abstract that at zero momentum it "seems to show a finite non-zero value", and Bogolubsky et al. (SU(3)) point the same way Physics . RGZ is built to reproduce this Physics . BRST symmetry (the symmetry that remains after gauge fixing, 4-08) is softly broken in GZ. Soft breaking means that the breaking term has low dimension and has no effect at short distances. This breaking is not a price paid by RGZ but a property of the horizon term of GZ itself Physics .

Seam

The propagator depends on the gauge. The target of the problem statement is correlations of gauge-invariant local fields such as $\mathrm{tr}\, F_{\mu\nu}F_{\rho\sigma}$, and there is no theorem deriving their gap from the infrared shape of the propagator. Nor can a continuous global gauge fixing be made (Singer, 3-04).

09

Setting the four pictures side by side

Table 4 The leading role, support and what is missing in the four pictures

Picture	Leading role	Gauge dependence	Support	What mathematics can say	What
Dual superconductivity	Monopole condensation	Through the definition of monopoles	Numbers for the order parameter	Area law for 3-dimensional compact `U(1)`	Cond ⇒ ar gap
Abelian projection	MAG monopoles	Strong	Abelian dominance	— (Elitzur's theorem constrains it)	A def a uni deter gaug invar quan
Centre vortices	Vortices / their free energy	The former only	Centre dominance, vortex removal	Tomboulis–Yaffe, formula (4)	Vorte cond at we coup dime
GZ	Propagator	By definition	Lattice propagators	Orbits pass inside the horizon	A bri gaug invar corre

Every leading variable becomes visible only once a gauge is chosen, and the gauge-invariant core is only the vortex free energy and the inequalities around it. What every picture explains is σ; Δ is merely expected as a consequence (§ 02). None of the pictures yields "a single scale". The seams are sorted out in 16-05, obstacles other than divergence in 7-11.

What this chapter can say

What is said	What is not said
Known The area law at small β. On the lattice, a strong form of the gap ⇒ the area law	The area law in the continuum limit of the four-dimensional non-abelian theory. An implication between the Clay gap and confinement
Computed Agreement of the exact formula (11) for the vortex gas with MC, Creutz ratio $= 2\rho_f$	That this model describes the Yang–Mills vacuum (the distribution of vortices is assumed)
Computed The closed form (15) of $C(t)$ in the tree approximation of RGZ ($m = 0$)	Deriving the gap of gauge-invariant correlations from the propagator

What is said	What is not said
Physics Lattice numbers for monopole condensation, abelian dominance, centre dominance, vortex removal and $D(0) \not= 0$	Deciding which mechanism is right (the numbers depend on the gauge)

Sources and reproduction

Item	Kind	Source / tools
Formulas (1)(3)(6)(8)(10), N-ality, $(92 \pm 4)\%$, attribution of dual superconductivity	Review	Greensite, Prog. Part. Nucl. Phys. 51 (2003) 1, hep-lat/0301023 (text)
Wilson loops	Primary	Wilson, PRD 10 (1974) 2445 (abstract)
Confinement is a "natural extension" (§ 02)	Primary	Jaffe–Witten, the Clay problem statement (text)
Dual superconductivity	Primary	't Hooft, EPS 1975 talk (published 1976) (bibliographic data only). Mandelstam, Phys. Rep. 23 (1976) 245 (bibliographic data only). Nambu, PRD 10 (1974) 4262 (bibliographic data only)
3-dimensional compact `U(1)`	Primary	Polyakov, NPB 120 (1977) 429 (bibliographic data only). Göpfert–Mack, CMP 82 (1982) 545 (abstract; the description of the continuum limit checked in the INSPIRE abstract, text not read; the year is the Crossref publication year, INSPIRE gives 1981)
Lattice numbers for monopole condensation, SW	Primary	Di Giacomo et al., PRD 61 (2000) 034503 (abstract). Seiberg–Witten, NPB 426 (1994) 19 (abstract)
Abelian projection, MAG, abelian dominance	Primary	't Hooft, NPB 190 (1981) 455 (abstract). Kronfeld–Schierholz–Wiese, NPB 293 (1987) 461 (bibliographic data only). Kronfeld et al., PLB 198 (1987) 516 (bibliographic data only). Suzuki–Yotsuyanagi, PRD 42 (1990) 4257 (bibliographic data only). Bali et al., PRD 54 (1996) 2863 (bibliographic data only)
Elitzur's theorem	Primary	Elitzur, PRD 12 (1975) 3978 (bibliographic data only)
Centre vortices	Primary	Del Debbio et al., PRD 55 (1997) 2298 (abstract). de Forcrand–D'Elia, PRL 82 (1999) 4582 (abstract). Tomboulis, arXiv:0707.2179 (abstract). Ito–Seiler, arXiv:0711.4930 (abstract). Tomboulis–Yaffe, CMP 100 (1985) 313 (bibliographic data only)

Item	Kind	Source / tools
Formula (4), Table 1	Primary	Chatterjee, arXiv:2006.16229 (text; Definition 2.1, Theorems 2.2 and 2.4, § 2.4). Chatterjee, arXiv:1803.01950 (text). Fröhlich–Spencer, CMP 83 (1982) 411 (bibliographic data only). Adhikari–Cao, arXiv:2202.10375 (abstract). Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 (bibliographic data only). Shen–Zhu–Zhu, arXiv:2204.12737 (abstract)
Formulas (12)–(14)	Primary	Dudal et al., PRD 78 (2008) 065047, arXiv:0806.4348 (text; formulas (2)(3)(58)(59)(159)(160)(207)). Gribov, NPB 139 (1978) 1 (bibliographic data only). Zwanziger, NPB 323 (1989) 513 (bibliographic data only)
The horizon, copies inside, lattice propagators	Primary	Dell'Antonio–Zwanziger, CMP 138 (1991) 291 (bibliographic data only). Zwanziger, PRD 69 (2004) 016002 (abstract). Cucchieri–Mendes, arXiv:0710.0412, PoS LAT2007:297 (abstract). Bogolubsky et al., arXiv:0901.0736 (abstract)
Formulas (11)(15), Tables 2 and 3	Computed	python3 (numpy, scipy): direction average of the area of symmetric differences and MC of one million samples (with standard errors) / checking the closed form against numerical integration, the minimum of $C(t)/C(0)$

Next chapter to read: 17-03 The string picture — treating the squeezed tube as a single string and looking at its fluctuations.

The string picture — treating the tube of confinement as a single string, a universal correction and a shape come out of its fluctuations

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — We follow how the Lüscher term $-\pi(d-2)/(24R)$ comes out of the zero-point energy of the transverse fluctuations of the tube, and why its coefficient does not depend on the details of the theory. With formulas, moving figures and computations we look at the thickness of the tube, the Nambu–Goto levels, the "rigid side" and the "rough side" of the lattice string, and the shape selected by a tension that depends on direction (the Wulff shape).

Prerequisite chapters — 17-01 How physics sees it (Wilson loops, the area law, the string tension σ). The zero-point energy (2-06) and the strong-coupling expansion (5-02) are written on the spot as far as needed.

Said first — This is a chapter at the level of physics Physics . The string picture takes as its starting point that a long tube exists (confinement), and in four-dimensional non-abelian Yang–Mills that is not a theorem. This book does not itself show confinement, the mass gap or the construction of the continuum limit (the published construction of the continuum limit is read in Volume 2).

The order of this chapter

1. Treating the tube as a single string — the weight of the area
2. The Nambu–Goto action — expanding the fluctuations
3. The zero-point energy and the Lüscher term — open strings and closed strings
4. Detour: 1+2+3+… and −1/12 — the part that does not depend on how we cut off
5. The tube thickens logarithmically — a fluctuating string
6. The Nambu–Goto levels and short tubes — the range of universality
7. The lattice string is rigid, and roughens somewhere — the side with theorems
8. The Wulff shape — a tension depending on direction, and the shape

9. The droplet of the two-dimensional Ising model — an exactly known tension, and from a square to a circle
10. What the string picture hands to mathematics, and what it does not — what is fixed under an assumption, and the assumption itself

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01

Treating the tube as a single string

The flux tube of 17-01 § 05, if its thickness is much smaller than the distance R of the pair, is from afar a single line. Extended in the direction of Euclidean time, the line sweeps out a surface (the **world sheet**). The string picture is an effective theory that describes long distances by the fluctuations of this surface Physics . To a surface S bounded by the rectangle $R \times T$ of the Wilson loop we give a weight fixed by its area $A(S)$.

$$\langle W_{R,T} \rangle \approx \sum_{S:\ \partial S = \text{rectangle}} e^{-\sigma A(S)} \quad (1)$$

With the minimal surface alone this is $e^{-\sigma RT}$, the area law. The question is what the fluctuations add. As a metaphor, it is the trembling of a stretched membrane. More precisely, (1) is not a formula derived from Yang–Mills but a starting point under the assumption that "a long tube exists, and its low excitations are only transverse fluctuations" (Athenodorou–Bringoltz–Teper 2011 § 2; ABT below).

At strong coupling on the lattice (5-02) a formula close to (1) holds, and the weight u of one plaquette plays the role of $e^{-\sigma 0}$ Known . But the area law also comes out for `U(1)`, so it is no evidence of non-abelian confinement (Chatterjee 2018 § 4, 5-06).

02

The Nambu–Goto action

An action proportional to the area is called the **Nambu–Goto action** (the name is said to be after Nambu and Goto). We describe the surface by coordinates t, x and a perpendicular displacement $h(t,x) \in \mathbb{R}^{d-2}$ (static gauge; d is the dimension of spacetime). The area of a surface $X(t,x)$ is given by the surface-area formula learnt in multivariable calculus, $\iint \sqrt{|X_t|^2 |X_x|^2 - (X_t \cdot X_x)^2}\, dt\, dx$ ($X_t = \partial_t X$, $X_x = \partial_x X$). Putting in $X = (t, x, h)$ gives the following.

$$A = \int_0^T dt \int_0^R dx \, \sqrt{1 + |\partial_t h|^2 + |\partial_x h|^2 + |\partial_t h|^2 |\partial_x h|^2 - (\partial_t h \cdot \partial_x h)^2} \tag{2}$$

If the fluctuations are gentle, expanding with $\sqrt{1+z} \approx 1 + z/2$ gives

$$\sigma A = \sigma R T + \frac{\sigma}{2} \iint \left(|\partial_t h|^2 + |\partial_x h|^2 \right) dt \, dx + O\big((\partial h)^4\big) \tag{3}$$

The second term is a massless free field with $d-2$ components, of the same form as the vibrating string of 2-02 (with $h = 0$ at the ends). For a long string the higher terms are suppressed by powers of $1/(\sigma R^2)$ (ABT § 2.1).

Example: the increase in length of a ripple $h = A_0 \sin(\pi x/R)$ is $\pi^2 A_0^2/(4R)$ in the quadratic approximation. With $R = 1$ fm and $A_0 = 0.2$ fm this is 0.099 fm (0.092 fm exactly). Multiplying by $\sigma = 1192$ MeV/fm ($\sqrt{\sigma} = 485$ MeV, 17-01 § 07) gives about 110 MeV Computed .

03

The zero-point energy and the Lüscher term

The second term of (3) is the action of a Euclideanised free field. When T is long, $\langle W_{R,T} \rangle \approx e^{-V(R)T}$, so the ground-state energy of the Hamiltonian of the same field gives $V(R)$ (4-02). The modes of this field are $k_n = n\pi/R$ ($n = 1, 2, \ldots$). Each mode is a harmonic oscillator of frequency k_n, and even in its lowest state it has the zero-point energy $k_n/2$ (2-06; $\hbar = c = 1$). Summing over the $d-2$ components gives the following.

$$E_0(R) = \sigma R + \frac{d-2}{2} \sum_{n=1}^{\infty} \frac{n\pi}{R} \tag{4}$$

The sum diverges. Fluctuations finer than a length a are outside the effective theory, so we cut off with $e^{-k_n a}$.

$$\sum_{n \geq 1} n \, e^{-\varepsilon n} = \frac{e^{-\varepsilon}}{(1 - e^{-\varepsilon})^2} = \frac{1}{\varepsilon^2} - \frac{1}{12} + \frac{\varepsilon^2}{240} - \cdots, \qquad \varepsilon = \frac{\pi a}{R} \tag{5}$$

Putting (5) into (4), the first term is $(d-2)R/(2\pi a^2)$, proportional to R, and is absorbed into the tension. What remains is the static potential.

$$V(R) = \sigma R + c_0 - \frac{\pi (d-2)}{24 R} + \cdots \tag{6}$$

c_0 is a non-universal constant. At $d = 4$ this is $-\pi/(12R)$, the **Lüscher term** of 17-01 formula (6) (Lüscher–Symanzik–Weisz 1980, Lüscher 1981). It is a coefficient derived

within the effective theory of long tubes, not a mathematical theorem Physics . The $1/R$ term is a finite part that does not diverge, so its coefficient does not depend on a. With $R = 1$ and $a = 0.2,\ 0.1,\ 0.05$, the remainder is $-0.2567,\ -0.2605,\ -0.2615$, approaching $-\pi/12$ Computed .

Closed strings and the Casimir energy of a cylinder

For a tube wound around a direction of period L (flux going round a periodic boundary), $k_n = 2\pi n/L$ occurs in both directions, positive and negative, and the same computation gives the following (ABT formula (9)).

$$E_0(L) = \sigma L - \frac{\pi(d-2)}{6L} + \cdots \quad (7)$$

This is the Casimir energy of a free field on a cylinder. Lucini–Teper 2001 extract the lattice σ with formula (4) of their paper, $am_l = a^2\sigma L - \pi/(3L)$, and note: 'in using eqn(4) we are assuming that we have linear confinement' (5-08). A rough guide: at $R = 1$ fm, $\pi/(12R) \approx 52$ MeV is 4.3% of σR, and at $L\sqrt{\sigma} = 3$ (the lower limit of Lucini–Teper; about 1.22 fm converting with $\sqrt{\sigma} = 485$ MeV, while Lucini–Teper themselves write $\simeq$1.3 fm) the correction in (7) is 11.6% of the leading term Computed .

04

Detour: 1+2+3+… and −1/12

A section you may skip.

The sum in the slogan "$1 + 2 + 3 + \cdots = -1/12$" diverges. The precise content is the following formula. For a smooth cutoff f ($f(0) = 1$, decreasing fast), as $\varepsilon \to 0$,

$$\sum_{n \geq 1} n\, f(\varepsilon n) = \frac{C_f}{\varepsilon^2} - \frac{1}{12} + O(\varepsilon^2), \qquad C_f = \int_0^\infty x f(x)\, dx \quad (8)$$

The coefficient C_f of the divergent term changes with the cutoff, and $-1/12$ does not. The reason is the **Euler–Maclaurin formula**. It writes the difference between a sum and an integral in terms of derivatives at the end point, and when g and its derivatives decrease fast at infinity it takes the following form ($B_2 = 1/6$ is a Bernoulli number).

$$\sum_{n \geq 0} g(n) = \int_0^\infty g(x)\, dx + \tfrac{1}{2} g(0) - \tfrac{B_2}{2!}\, g'(0) - \cdots$$

For $g(x) = x f(\varepsilon x)$ we have $g(0) = 0$, $g'(0) = f(0) = 1$, and the integral is C_f/ε^2, so the constant term is $-B_2/2 = -1/12$. The remaining terms are powers ε^2 and higher. The analytic continuation of the zeta function, $\zeta(-1) = -1/12$, is the same number Known .

Computed Table 1 The sum minus C_f/ε^2 (exponential $f=e^{-x}$, $C_f=1$; Gaussian $f=e^{-x^2}$, $C_f=1/2$)

ε	Exponential	Gaussian
0.1	−0.0832917	−0.0834169
0.01	−0.0833329	−0.0833342
→ 0	−0.0833333	−0.0833333

The part that changes with the cutoff is absorbed into the tension, and only $-1/12$ remains in the coefficient of $1/R$ (for divergent series see also 4-12, 7-12).

05

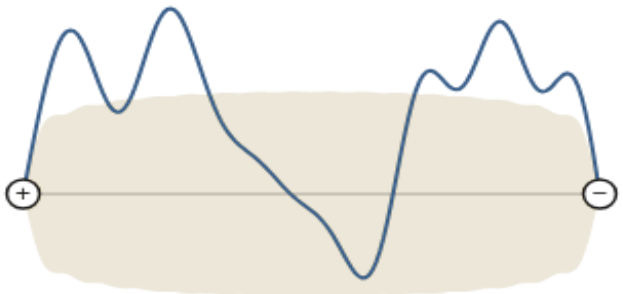
The tube thickens logarithmically

The modes q_n of (3) are oscillators with mass σ and frequency k_n, and the variance in the ground state is $1/(2\sigma k_n)$. At the midpoint of $h=\sum_n \sqrt{2/R}\, q_n \sin k_n x$ only odd n survive, and $\langle h^2\rangle = \sum_{n\ \mathrm{odd}} 1/(\sigma\pi n)$. Summing up to $n \lesssim R/a$ gives a logarithm (Lüscher–Münster–Weisz 1981; Gliozzi–Pepe–Wiese 2010 formula (3)). It is a formula derived within the effective theory of long tubes Physics .

$$w^2\left(\frac{R}{2}\right) = \left\langle |h(R/2)|^2 \right\rangle \approx \frac{d-2}{2\pi\sigma}\ln\frac{R}{R_0} \quad (9)$$

What is universal is the coefficient of $\ln R$; R_0 is not universal. The thickness grows without bound, but $w/R \to 0$, so from afar it stays a thin line.

Figure 1 Changing the distance R with the knob changes the number of modes on the string and the width of the fluctuation (the band). "Play" makes each mode oscillate with its zero-point size



1/√σ

R√σ = 4.00 (1.63 fm), modes: 16, midpoint w²σ = 0.644 (ln N+γ+ln2)/(2π) = 0.643

Computed The numbers in this figure were computed on this machine. Cutoff $a = 0.25/\sqrt{\sigma}$, the band is $\pm w$, one transverse component. The oscillation is a classical picture, not the quantum fluctuation itself.

Table 2 Variance of the free field at the midpoint of a lattice strip (width R, h=0 at both ends, infinite time)

(one component, in units of $1/\sigma$)

R (lattice spacings)	$\sigma\langle h^2\rangle$	Increase when R is doubled
8	0.516070	0.111626
1024	1.288650	0.110318
Prediction of (9)	—	ln2/(2π) = 0.110318

Computed Summed exactly with the closed form $1/\sqrt{m^2(m^2+4)}$ for each mode. The increase agrees with (9) to 6 digits.

At $d=4$ the coefficient $1/(\pi\sigma)$ is 0.053 fm^2, and doubling R increases w^2 by only 0.037 fm^2 Computed . The logarithmic growth has been confirmed on the lattice for `SU(2)` in (2+1) dimensions (Gliozzi–Pepe–Wiese 2010) Physics .

06

The Nambu–Goto levels and short tubes

The levels of a closed string with no longitudinal momentum can be written in closed form in Nambu–Goto (with the sums of the left- and right-moving excitations both equal to n; ABT formula (21)). It is a formula that comes from naive light-cone quantisation (Aharony–Komargodski cite, as the source of this method, Arvis 1983, which treated a string with fixed ends).

$$E_n(L)=\sqrt{\sigma^2L^2+8\pi\sigma\left(n-\frac{d-2}{24}\right)} \quad (10)$$

Expanding gives $\sigma L+\frac{4\pi}{L}\left(n-\frac{d-2}{24}\right)+\cdots$, and $n=0$ returns to (7). The ground state becomes imaginary at $\sigma L^2<\pi(d-2)/3$, which in $d=4$ is $L_H\sqrt{\sigma}=\sqrt{2\pi/3}=1.447$. If the periodic direction is regarded as time, $1/L$ is the temperature (4-02). The number of string states grows exponentially with energy, so above some temperature the sum of the partition function stops converging. This point is called the **Hagedorn transition** (for the growth of the states, see the detour of 17-05) Physics . ABT write that L_H can be regarded as 'the Hagedorn/deconfining transition in the Nambu-Goto model', and go on to say that pure gauge theories at large N do not reach this length. As the length shrinks, a first-order deconfinement transition occurs before it, at $l_c\sqrt{\sigma}=\sqrt{\sigma}/T_c\simeq 1.65$ (the l_c below Table 3).

Table 3 Energies of the closed string ($d=4$, in units of $\sqrt{\sigma}$)

L√σ	σL−π/(3L)	Ground state of (10)	Error of the approximation	n=1 of (10)	E_1-E_0
1.6	0.9455	0.6824	+38.6%	5.0595	4.377
3.0	2.6509	2.6279	+0.88%	5.6602	3.032
6.0	5.8255	5.8229	+0.04%	7.6836	1.861

Computed By formula (10). The length $l_c\sqrt{\sigma}$ at which the tube disappears on the lattice is the product of $a\sqrt{\sigma}$ and l_c/a in Table 3 of ABT: SU(3) 1.544, 1.559, SU(5) 1.636, SU(6) 1.676. All are slightly longer than L_H, and the two at larger N lie on either side of ABT's 1.65.

How far does universality go? According to Aharony–Komargodski 2013 'all the terms up to (and including) order 1/L^5 are universal': the ground state agrees with (10) up to $1/L^5$, the excitations for $d > 3$ deviate universally at $1/L^5$, and theory-specific terms enter from $1/L^7$. These are results derived within the effective theory of long tubes Physics . Of the quantisation in (10) they also write 'There is no rigorous derivation'. ABT measured about 20 levels of closed tubes on the lattice for four-dimensional SU(N), and reported that many are well described by (10) down to short lengths where the $1/L$ expansion does not converge. Some states with $J^P = 0^-$ (J is the angular momentum about the axis of the tube, P the sign under reflection) deviate strongly Physics .

In the last column of Table 3, the first excitation $3.03\sqrt{\sigma}$ of the string at $L\sqrt{\sigma} = 3$ is of the same order as the 0^{++} glueball of SU(3), $3.405\sqrt{\sigma}$ (17-01 Table 2) Computed (this does not imply a correspondence). Turning "there is a single scale" into a mathematical sentence is the subject of 17-12.

07

The lattice string is rigid, and roughens somewhere

On the strong-coupling side of the lattice, shifting part of the surface by one lattice step adds plaquettes, each carrying a small weight u. Transverse fluctuations are suppressed exponentially, and the surface stays **rigid** (flat). In the world of (9) the variance diverges logarithmically, and the surface is **rough**.

For crystal surfaces there is a **roughening transition** between the two. Itzykson–Peskin–Zuber 1980 argued that in four-dimensional lattice gauge theory too, roughening occurs at a finite coupling without any phase transition of the vacuum, and that $\sigma(\beta)$ has an essential singularity there. An essential singularity here is a singularity of the kind where the function and all its derivatives stay finite, but the Taylor expansion no longer represents the function at that point. An example is the function set to e^{-1/x^2} for $x > 0$ and 0 for $x \le 0$; its Taylor series at $x = 0$ has all coefficients 0. However far the series is extended, this point cannot be seen. IPZ write 'The roughening transition is a natural barrier to the extrapolation of strong-coupling series for the string tension'. For SU(2), β_r is about 2 (with a large error), and the authors note that their four-dimensional argument is 'for the most part, intuitive' Physics .

The three-dimensional Z(2) gauge theory is **dual** to the three-dimensional Ising model. Dual means that under a change of variables the partition functions of the two

models agree (up to simple factors) (the detour of 2-09; IPZ cite Wegner 1971). Under this correspondence the surface of a Wilson loop becomes an interface of the Ising model (IPZ formula (8)) Known . Measuring the gauge coupling by $t = \tanh\beta$ (in terms of the temperature of the dual Ising model, $t = e^{-2J/T}$; IPZ formula (7)), the phase transition is at $t_c = 0.6418$. The roughening point, by the series estimate IPZ cite, is $t_r \approx 0.46$, before the phase transition Physics . The effective string theory applies only on the rough side (Gliozzi–Pepe–Wiese 2010) Physics .

Key point

Physics The theorem on the area law at strong coupling (Osterwalder–Seiler 1978, 5-04) is about the rigid side; the Lüscher term and the logarithmic thickness are about the rough side. Physics holds that between them lies a singularity of $\sigma(\beta)$ beyond which the strong-coupling series cannot be extended.

08

The Wulff shape

On the rigid side of § 07, the tension also depends on direction. The edge of a Wilson loop at angle θ to the lattice axes becomes a staircase, and the minimal surface has $RT(|\cos\theta| + |\sin\theta|)$ plaquettes. At leading order in strong coupling this gives the following.

$$\sigma(\theta) = \sigma_0\big(|\cos\theta| + |\sin\theta|\big) \quad (11)$$

At 45° it is $\sqrt{2}$ times larger Computed . In the continuum limit rotation symmetry is expected to return and the tension to become constant Physics .

What shape does a region of fixed area (a two-dimensional droplet) take when it is enclosed by a line whose tension depends on direction? With the tension $\tau(\mathbf{n})$ for each normal $\mathbf{n}$, we minimise $\oint \tau\, ds$ with the enclosed area fixed. The answer is given by the construction of Wulff 1901 (bibliographic data for the proof that the construction gives the minimum is in the table of sources) Known .

$$\mathcal{W}_\tau = \bigcap_{|\mathbf{n}|=1} \{\mathbf{x} : \ \mathbf{x}\cdot\mathbf{n} \le \tau(\mathbf{n})\}, \qquad \min_{\text{area}=A} \oint \tau\, ds = 2\sqrt{|\mathcal{W}_\tau|\, A} \quad (12)$$

If the tension is constant the answer is a circle; for $\tau = |n1| + |n2|$ of (11) it is the square $\{\max(|x|,|y|) \le 1\}$. The total at area 1 is 4 for the square and $8/\sqrt{\pi} = 4.51$ for a circle Computed .

09

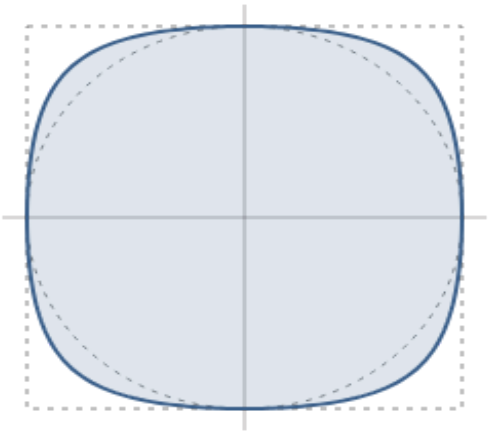
The droplet of the two-dimensional Ising model

An example where the direction-dependent tension is known exactly is the Ising model on the square lattice (coupling J, 2-09). The $\beta = 1/T$ of this section is the inverse temperature ($k_B = 1$; not the β of lattice gauge theory). The equilibrium shape of an island of the minority (a droplet) in the low-temperature phase is the following curve (the first exact analysis is attributed to Rottman–Wortis 1981) Known .

$$\cosh(\beta x) + \cosh(\beta y) = \sinh(2\beta J) + \frac{1}{\sinh(2\beta J)} \quad (13)$$

(13) is the boundary of $\mathcal{W}_\tau$ in (12) itself. Its size too is as in (12): the extent $\max \mathbf{x}\cdot\mathbf{n}$ in the direction $\mathbf{n}$ equals the tension $\tau(\mathbf{n})$ (in units of J). So x, y are not positions but coordinates measured in units of energy. The intercept with the x axis (the tension in the axis direction) agrees to 5 digits with $2J + \beta^{-1}\ln\tanh\beta J$ from Onsager's exact solution Computed . As $T \to 0$, (13) becomes a square, and the tension is the same staircase count as in (11). As $T \to T_c$ the right-hand side approaches 2, and for $u = \beta x$, $v = \beta y$ we have $\cosh u + \cosh v \approx 2 + (u^2 + v^2)/2$, so it becomes a circle.

Figure 2 Changing the temperature T/T_c with the knob changes the equilibrium shape of the two-dimensional Ising droplet (filled) from a square to a circle. The tension along the axis is normalised to 1; the dotted lines are the square and the circle



```
T/T_c = 0.30  τ(0°) = 1.9278 J (Onsager's formula 1.9278)  τ(45°)/τ(0°) = 1.1196 (square 1.4142, circle 1)
```

Table 4 Equilibrium shapes of the two-dimensional Ising model. The isoperimetric ratio is $4\pi\,\text{area}/\text{perimeter}^2$; the last column is the ratio of the total tension to that of a circular droplet of the same area

T/T_c	τ(45°)/τ(0°)	Isoperimetric ratio	Wulff/circle
→ 0	1.4142	0.7854	0.8862
0.2	1.1984	0.9406	0.9698
0.4	1.0690	0.9917	0.9958

T/T_c	τ(45°)/τ(0°)	Isoperimetric ratio	Wulff/circle
0.8	1.0035	1.0000	1.0000

Computed The numbers in this figure and table were computed on this machine ((13) solved by bisection for each angle).

At low temperature the diagonal tension falls first. A stepped edge has a number of arrangements (entropy), and the corners round off from there. Approaching T_c the correlation length becomes much longer than the lattice spacing, the orientation of the lattice is forgotten, and the shape becomes a circle. This is how the expectation that rotation symmetry returns in the continuum limit (5-05) looks in an exactly tractable model.

That the droplet takes the Wulff shape in the low-temperature two-dimensional Ising model has been shown from the microscopic model (Dobrushin–Kotecký–Shlosman 1992, 1993) Known .

The Wulff problem applies directly to droplets of a two-dimensional phase with fixed area like this one. A closed ring of flux in Yang–Mills has no constraint fixing its area, and left alone it shrinks. Still, a closed tube on the lattice must turn corners, so its energy depends on its shape through $\sigma(\theta)$ Physics . How it is used in the record of this book's study is in 23-02.

10

What the string picture hands to mathematics, and what it does not

What it hands over

Physics If a long tube is assumed to exist, the Lüscher coefficient, the coefficient of $\ln R$ in the thickness, and the form of the levels up to $1/L^5$ are fixed independently of the details.

Known For the low-temperature two-dimensional Ising model, the relation between the direction-dependent tension and the shape of the droplet is a theorem.

What it does not hand over

That a tube exists (the area law at large β) 'remains elusive' for the four-dimensional non-abelian theory (Chatterjee 2018). The lattice σ is extracted by subtracting the string correction, and Lucini–Teper, fitting only two points, note: 'we cannot claim to have evidence for both the 1/L functional form and for its coefficient'.

The string picture is about massless transverse fluctuations, and the glueball masses do not come out of it directly. ABT examine the possibility that the $J^P = 0^-$ levels that deviate from (10) represent 'the massive modes associated with the internal structure

of the flux tube' Physics . Since the only length scale is $1/\sqrt{\sigma}$, $m \sim \sqrt{\sigma}$ looks natural, but it is not derived (17-01 § 09, 17-12). The view as a sum over surfaces appears in the next chapter, 17-04, as the $1/N$ expansion.

What this chapter can say

What is said	What is not said
Physics The universality, derived within the effective theory that assumes a long tube exists, of the Lüscher coefficient, the coefficient of $\ln R$ in the thickness, and the levels up to $1/L^5$	That a long tube exists in four-dimensional non-abelian Yang–Mills (the area law at large β)
Computed That differences of regularisation enter only the term proportional to R (Table 1). The increase in thickness of the lattice free field (Table 2)	A theorem on the thickness of the Yang–Mills tube
Computed The difference between (10) and the Lüscher approximation, L_H, l_c (Table 3)	That (10) is exact for Yang–Mills
Known The Wulff construction and its justification for the low-temperature two-dimensional Ising model. Computed Table 4	That $\sigma(\theta)$ becomes constant in the four-dimensional continuum limit
Known The area law at strong coupling (Osterwalder–Seiler). The duality between the three-dimensional `Z(2)` gauge theory and the Ising model. $-1/12$ by Euler–Maclaurin	Extending the strong-coupling series beyond the roughening point
Physics The roughening transition of four-dimensional lattice gauge theory (the argument of IPZ), and the series estimate of t_r for three-dimensional `Z(2)`	A proof of the existence and location of the roughening transition

Sources and reproduction

Item	Kind	Source / tools
§ 01, (7)(10), l_c, the Hagedorn sentence and $l_c\sqrt{\sigma} \simeq 1.65$, $J^P = 0^-$	Primary	Athenodorou–Bringoltz–Teper, JHEP 1102 (2011) 030, arXiv:1007.4720 (text)
The order of universality, the caveat on (10), that (10) comes from naive light-cone quantisation	Primary	Aharony–Komargodski, JHEP 05 (2013) 118, arXiv:1302.6257 (text)

Item	Kind	Source / tools
The original sources of the Lüscher term and (10)	Primary	Lüscher–Symanzik–Weisz, NPB 173 (1980) 365 (bibliographic data only). Lüscher, NPB 180 (1981) 317 (bibliographic data only). Arvis, PLB 127 (1983) 106 "The Exact q Anti-q Potential In Nambu String Theory" (bibliographic data only; taken from reference [19] of Aharony–Komargodski). The original source of the name Nambu–Goto not checked
Formula (9), that effective string theory applies on the rough side	Primary	Lüscher–Münster–Weisz, NPB 180 (1981) 1 (bibliographic data only). Gliozzi–Pepe–Wiese, PRL 104 (2010) 232001, arXiv:1002.4888 (text)
Extraction of the lattice σ, $L\sqrt{\sigma} \geq 3$	Primary	Lucini–Teper, JHEP 0106 (2001) 050, hep-lat/0103027 § 4.1 (text). The original reads 'r $\geq 3/\sqrt{\sigma} \simeq 1.3$ fm'. The 1.22 fm in the text is the conversion with $\sqrt{\sigma} = 485$ MeV (the same as 17-01)
The roughening transition, $t = \tanh\beta$, t_c, t_r, duality	Primary	Itzykson–Peskin–Zuber, PLB 95 (1980) 259 (text). $t = \tanh\beta = e^{-2/T}$ is formula (7) of that paper, $t_c = 0.6418$ is in its text. $t_r \approx 0.46$ is the value $t_r \approx 0.4593$ from the series analysis of Weeks et al. that IPZ cite at formula (13) (an estimate as of 1980; the original sources of Weeks et al. are references [7][9] of IPZ, one an unpublished communication; not checked). In IPZ's own quantity, formula (19) gives $t_r \approx 0.47$. Duality: Wegner, J. Math. Phys. 12 (1971) 2259 (bibliographic data only; from the reference list of IPZ)
The status of the area law	Primary	Chatterjee arXiv:1803.01950 § 4 (text). Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 (bibliographic data only)
Formula (12), the microscopic justification of the Wulff shape	Primary	Wulff, Z. Kristallogr. 34 (1901) 449–530 (bibliographic data only; taken from the reference list of Fujimoto–Otsuka). The original source of the proof that the construction gives the minimum not checked. Explanation of the construction: Günther–Rikvold–Novotny, Physica A 212 (1994) 194, cond-mat/9406114 (text), and the review cited there, Rottman–Wortis, Phys. Rep. 103 (1984) 59 (bibliographic data only). Dobrushin–Kotecký–Shlosman, Wulff Construction: A Global Shape from Local Interaction, AMS Transl. Math. Monographs 104 (1992). The same authors' short paper A microscopic justification of the Wulff construction, J. Stat. Phys. 72 (1993) 1–14 (both bibliographic data only; checked on Crossref)
Formula (13)	Primary	Rottman–Wortis, PRB 24 (1981) 6274 (bibliographic data only; the attribution "the first exact analysis" is from the introduction of Fujimoto–Otsuka arXiv:1906.01925). Also Avron–van Beijeren–Schulman–Zia, J. Phys. A 15 (1982) L81, Zia–Avron, PRB 25 (1982) 2042 (bibliographic data only; listed by the same introduction as 'see also'). The shape checked against Onsager's tension
Tables 1–4, Figures 1 and 2, the Computed items in the text	Computed	python3 (standard library): regularised sums / the lattice free field / (10) and Table 3 of ABT / bisection for (13)

Next chapter to read: 17-04 Large N — a view that organises the sum over surfaces by $1/N$.

Large N — sending the number of colours to infinity sorts Feynman diagrams by surfaces

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — That as N grows, diagrams are sorted by genus into powers of N, and only planar diagrams survive ('t Hooft 1974). That the sum of diagrams becomes a sum over random surfaces, and fluctuations disappear. A phase transition that appears only at N=∞ on the two-dimensional lattice, and the N-dependence on the four-dimensional lattice.

Prerequisite chapters — 4-03 Perturbation theory and Feynman diagrams. 3-02, 7-02 and 17-01 are written on the spot as far as needed.

Said first — Large N is an expansion at the level of physics. The prize problem requires finite N, and no road back from N→∞ is known. Mathematically established results are limited to the strong-coupling lattice or to two dimensions. This book does not itself show either the mass gap or the construction of the continuum limit (the published construction of the continuum limit is read in Volume 2).

The order of this chapter

1. The idea of varying N — 1/N in place of a small number
2. The double-line notation — a line is two indices
3. 't Hooft's counting — the power of N is fixed by the genus
4. The genus expansion of a Gaussian matrix — actually counting (Detour: Catalan numbers and the semicircle)
5. Random surfaces — towards the string picture
6. The master field — fluctuations disappear as 1/N
7. The Gross–Witten transition — a kink only at N=∞
8. The four-dimensional lattice — ratios move slowly with N
9. What can be said, and what cannot — the shores of strong coupling and two dimensions

01

The idea of varying N

Four-dimensional pure Yang–Mills has no small number to expand in. The mass scale Λ has the form $e^{-1/(2b_0 g^2)}$ ($b_0 = 11N/(48\pi^2)$, formula (1) of 17-01) and shows up at no power of g (17-01 § 03). 't Hooft (1974) treated the size N of the group as a knob and considered the following limit (the group in the original paper is U(N)) Known .

$$N \to \infty, \qquad \lambda = g^2 N \ \text{ fixed} \quad (1)$$

According to the abstract, 'only planar diagrams with the quarks at the edges dominate', and 1/N corresponds to the coupling of the string models of the time. As a metaphor, the idea is to see $N = 3$ as "$N = \infty$ with a correction of order $1/9$"; more precisely, how small the correction is has to be checked with lattice numbers (§ 08). The problem statement calls the duality conjecture, that through the 1/N expansion a gauge theory might be equivalent to a string theory with coupling $1/N$, 'A prominent speculation', and writes that such a description could be 'perhaps a good starting point for a rigorous proof (for sufficiently large N)' Known . But the requirement is finite N (0-03).

02

The gauge field is a matrix: the double-line notation

The gauge field A_μ is an $N \times N$ Hermitian matrix at each point (3-02), and we regard the two indices of a component ${A_\mu}^i{}_j$ as the two ends of colour lines. The action is (Euclidean)

$$S = \frac{1}{2g^2}\int \operatorname{tr} F_{\mu\nu}F_{\mu\nu}\, d^4x = \frac{N}{2\lambda}\int \operatorname{tr} F_{\mu\nu}F_{\mu\nu}\, d^4x, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].$$

The trace of a product joins indices into a ring, as in $\operatorname{tr}(ABC) = A^i{}_j B^j{}_k C^k{}_i$, and the three-point and four-point vertices of (2) are all of this form. The colour part of the propagator (4-03) for U(N) is

$$\langle A^i{}_j\, A^k{}_l \rangle \ \propto\ \frac{\lambda}{N}\,\delta^i{}_l\,\delta^k{}_j. \quad (3)$$

One delta is one index line, so a gluon line is two lines of opposite orientation, a thin ribbon. This is the **double-line notation**. In the continuum action (at the level of the Lie algebra) the direction of the identity matrix drops out of the commutators, and its cross terms with the traceless part vanish too, so the U(N) of pure gauge theory is

SU(N) plus one **free U(1) field that does not interact** Known . As groups, U(N) does not split into SU(N)×U(1) (the identity matrix times $e^{2\pi ik/N}$ lies in both), and on the lattice U(N) (§ 07) this separation does not hold.

03

't Hooft's counting: the power of N is fixed by the genus of a surface

From (2)(3), each vertex carries N/λ, each propagator λ/N, and each closed index ring $\sum_i \delta^i{}_i = N$. A vacuum diagram with V vertices, E propagators and F index rings is proportional to

$$\left(\frac{N}{\lambda}\right)^{V}\left(\frac{\lambda}{N}\right)^{E} N^{F} = N^{\,V-E+F}\,\lambda^{\,E-V} \tag{4}$$

Gluing a disc to each index ring gives an oriented closed surface, and $V - E + F$ becomes the **Euler characteristic** $\chi = 2 - 2h$, fixed by the surface alone (h is the number of handles = the **genus**; for a tetrahedron on a sphere $4 - 6 + 4 = 2$). Therefore

$$\ln Z = \sum_{h\geq 0} N^{\,2-2h}\, f_h(\lambda) \tag{5}$$

where f_0 is the sum of diagrams that can be drawn in the plane (**planar diagrams**), and the corrections go in steps of $1/N^2$ Known . A quark line is a single index line, and opens a boundary in the surface, multiplying by $1/N$. (5) is a rearrangement of the perturbation series; convergence is a separate question (4-12). The reason for fixing λ lies in the β function (the formula describing how the coupling constant changes with the scale μ at which we look; 17-01 § 02). Writing it in terms of λ with the b_0 of § 01, N disappears Computed .

$$\mu\frac{d\lambda}{d\mu} = -\frac{11}{24\pi^2}\,\lambda^2, \qquad \Lambda = \mu\,\exp\!\left(-\frac{24\pi^2}{11\,\lambda(\mu)}\right) \tag{6}$$

The left-hand side is how λ changes when the scale μ is changed slightly, and since the right-hand side is negative, λ is smaller at shorter distances (larger μ). Fixing λ fixes Λ too, and on the lattice this choice gives a smooth limit (Lucini–Teper) Physics .

04

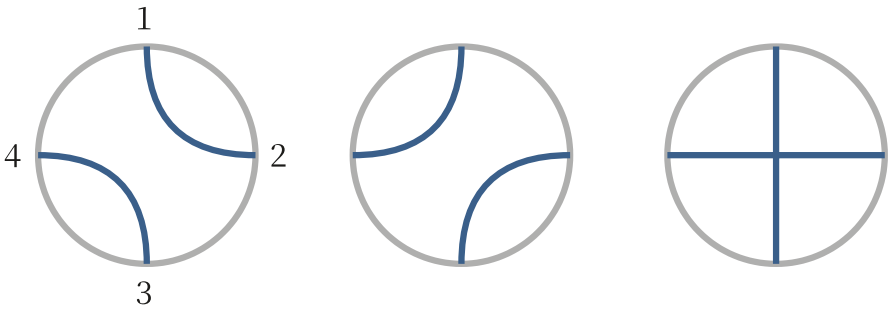
Counting: the genus expansion of a Gaussian matrix

The counting (4) can be checked in a model where spacetime is shrunk to a single point. Take one Hermitian matrix M distributed according to a Gaussian (1-08) (dM is the Lebesgue measure on the real components).

$$d\mu(M) \;\propto\; e^{-\frac{N}{2}\operatorname{tr} M^2}\,dM, \qquad \langle M_{ij}M_{kl}\rangle = \frac{1}{N}\,\delta_{il}\,\delta_{jk} \tag{7}$$

The expectation of a product of Gaussian variables is the sum over ways of pairing them (Wick's theorem, 2-07). The pairings of $\frac{1}{N}\operatorname{tr} M^4$ are the three of Figure 0: (a)(b) give $\frac{1}{N}\cdot N^3\cdot N^{-2}=1$ and (c) gives N^{-2}.

Figure 0 The three ways of joining legs 1–4 of $\operatorname{tr} M^4$ in pairs. (a)(b), where the strings do not cross, have 3 index rings; (c), where they cross, has 1 (it cannot be drawn on a sphere and needs a torus)



(a) 3 rings: 1 (b) 3 rings: 1 (c) 1 ring: $1/N^2$

In general

$$\left\langle \tfrac{1}{N}\operatorname{tr} M^4 \right\rangle = 2 + \frac{1}{N^2}, \qquad \left\langle \tfrac{1}{N}\operatorname{tr} M^{2k} \right\rangle = \sum_{g\ge 0} \varepsilon_g(k)\, N^{-2g} \tag{8}$$

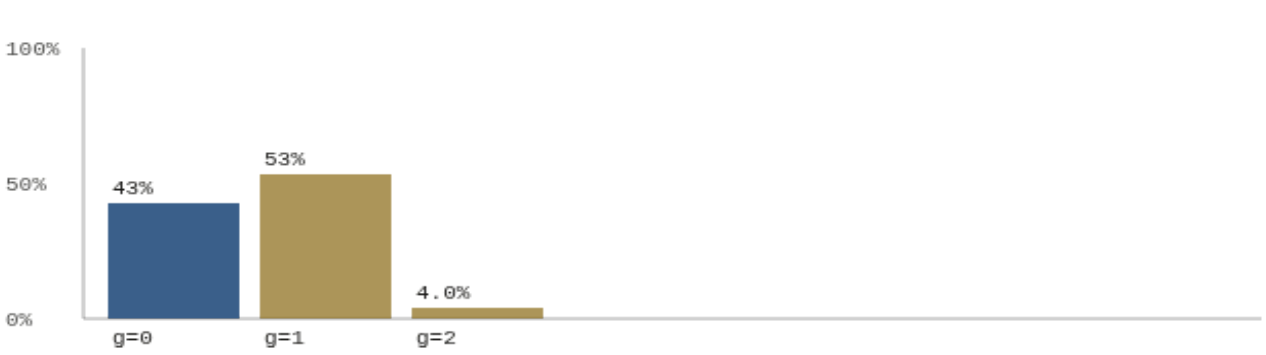
where $\varepsilon_g(k)$ is the number, among the $(2k-1)!!$ pairings, of those of genus g (with $k+1-2g$ index rings). The coefficients can be obtained from the recursion of Harer–Zagier (1986).

$$(k+1)\,\varepsilon_g(k) = (4k-2)\,\varepsilon_g(k-1) + (k-1)(2k-1)(2k-3)\,\varepsilon_{g-1}(k-2) \tag{9}$$

Table 1 $\varepsilon_g(k)$ (full enumeration of pairings; agrees with recursion (9))

k	g=0	g=1	g=2	g=3	Total
4	14	70	21		105
5	42	420	483		945
6	132	2310	6468	1485	10395
7	429	12012	66066	56628	135135

Figure 1 Varying N and k changes the share of genus-g diagrams in $\langle \frac{1}{N}\operatorname{tr} M^{2k}\rangle$. Raising N leaves only the planar diagrams (g=0)



k = 4 N = 2.00 ⟨(1/N)tr M^8⟩ = 32.81, planar part 14 (42.7%)

Computed The numbers of Table 1 and Figure 1 were computed on this machine (for $k \le 7$, full enumeration checked against the recursion). If JavaScript is disabled, see Table 1.

The total $(2k-1)!!$ grows much faster than the number of planar diagrams, and at $k = 9$, $N = 1$ planar diagrams are under 0.02%. The large-N limit is the limit that keeps only the few diagrams.

Detour: Catalan numbers and the semicircle

A subsection you may skip.

The $g = 0$ column of Table 1 is the Catalan numbers $\frac{1}{k+1}\binom{2k}{k}$. As $N \to \infty$ the distribution of the eigenvalues of M approaches the semicircle $\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$ ($|x| \le 2$) (Wigner's semicircle law) Known . Integrating numerically the $2k$-th moment $\int x^{2k}\rho(x)\,dx$ of this density agrees with the Catalan numbers to 10 digits for $k \le 6$ Computed . Counting planar diagrams is the same as finding the large-N limit, and Brézin–Itzykson–Parisi–Zuber (1978) used this as a way of counting Known .

05

The sum of diagrams becomes a sum over random surfaces

We read the ribbon diagrams inside out. Replace four-point vertices by squares, three-point vertices by triangles, and propagators by the gluing of two edges, and each diagram becomes one closed surface glued from polygons. (5) is

$$\ln Z = \sum_{\Sigma} N^{\chi(\Sigma)}\, w_{\lambda}(\Sigma) \qquad (10)$$

a sum over **random surfaces**, where N counts the topology and λ the number of polygons (w_λ is a weight that includes the momentum integrals). As a metaphor it is a sum over shapes of a membrane; more precisely, (10) is a rearrangement of a series, and convergence as a sum over surfaces has to be shown separately.

Physics reads this membrane as the world sheet of the string swept out by the tube of confinement (17-03). N^{2-2h} has the form of the perturbative expansion of a string with string coupling $1/N$; in the problem statement, 'may be equivalent to a string theory with 1/N as the string coupling constant' Physics . In two-dimensional QCD, the coefficients of the $1/N$ expansion are fixed by sums over maps of surfaces, and it can be read as a theory of closed strings (Gross–Taylor 1993) Known .

06

Fluctuations disappear: factorisation and the master field

One more thing follows from the counting (4): the covariance of quantities normalised with $\frac{1}{N}\operatorname{tr}$ is smaller by $1/N^2$ Known .

$$\left\langle \tfrac{1}{N}\operatorname{tr} A \cdot \tfrac{1}{N}\operatorname{tr} B \right\rangle = \left\langle \tfrac{1}{N}\operatorname{tr} A \right\rangle \left\langle \tfrac{1}{N}\operatorname{tr} B \right\rangle + O\left(N^{-2}\right) \quad (11)$$

This is **factorisation**, and for $A = B$ the standard deviation shrinks as $1/N$.

Table 2 $\frac{1}{N}\operatorname{tr} M^4$ from random samples of the matrix (7) Computed

N	Mean	$2 + 1/N^2$	N × standard deviation
2	2.2541(80)	2.2500	7.19
8	2.0152(38)	2.0156	6.09
128	2.0003(15)	2.0001	6.06

At $N = \infty$, quantities made from traces (for the matrix model (7), quantities unchanged under conjugation $M \to UMU^\dagger$; for gauge theory, gauge-invariant quantities) stop being random, and the measure behaves as though concentrated on a single configuration. By custom, where it concentrates is called the **master field**. As a metaphor it is the picture of a heaving sea freezing; more precisely, what freezes is the distribution of quantities made from traces, while the components of the matrix keep fluctuating.

The representative case in gauge theory where factorisation works is the **Wilson loop**. It is the $\frac{1}{N}\operatorname{tr}$ of the matrix (an $N \times N$ unitary matrix) obtained by parallel transport with the gauge field along a closed path C, defined in 17-01 § 04. The **Schwinger–Dyson equations** are relations among expectation values that come from the integral not changing when the integration variable is shifted. Putting factorisation into these equations for Wilson loops gives equations closed in the expectations of loops alone (Makeenko–Migdal 1979) Physics . In the plane $\mathbb{R}^2$, Lévy constructed the large-N limit of the Yang–Mills measure (7-02) and showed that each Wilson loop converges in probability to a deterministic limit, and that these equations hold both at finite N and in the limit Known .

07

Large N seen on the two-dimensional lattice: the Gross–Witten transition

We write as much lattice vocabulary as needed (for details see 5-01). Replace spacetime by a mesh of lattice points, and on each edge ℓ joining two neighbouring points place a group element $U_\ell \in U(N)$. Write U_p for the product of the elements on the four sides of the smallest square of the lattice (a **plaquette**) p, taken in order around it; the action is made from $\operatorname{tr} U_p$. The $\frac{1}{N}\operatorname{tr}$ of the product of edge elements along a closed path is the lattice **Wilson loop**. Each U_ℓ follows the **Haar measure** (the uniform probability measure on the group, 1-08 § 03) multiplied by the weight of the action.

On the two-dimensional lattice, the **tree gauge**, which sets some of the edge elements to 1, makes the plaquettes independent, and a Wilson loop enclosing n faces is exactly w^n (7-02 § 02, § 03). Large N can be studied with the integral over a single plaquette. Weighting the Haar measure of U(N), set

$$d\mu(U) \;\propto\; \exp\Big(\frac{N}{\lambda}\operatorname{tr}(U+U^\dagger)\Big)\,dU, \qquad w_N(\lambda) = \Big\langle \tfrac{1}{N}\operatorname{Re}\operatorname{tr} U \Big\rangle \tag{12}$$

(in terms of the β of 7-02, $\beta = 2N/\lambda$). At finite N, Weyl's integration formula (which turns the Haar integral of a function depending only on traces into an integral over the eigenvalues $e^{i\theta 1},\dots,e^{i\theta N}$) and Heine's identity (which turns that eigenvalue integral into a determinant) give a determinant whose entries are the modified Bessel functions

$$I_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\cos n\theta\, d\theta$$

Known .

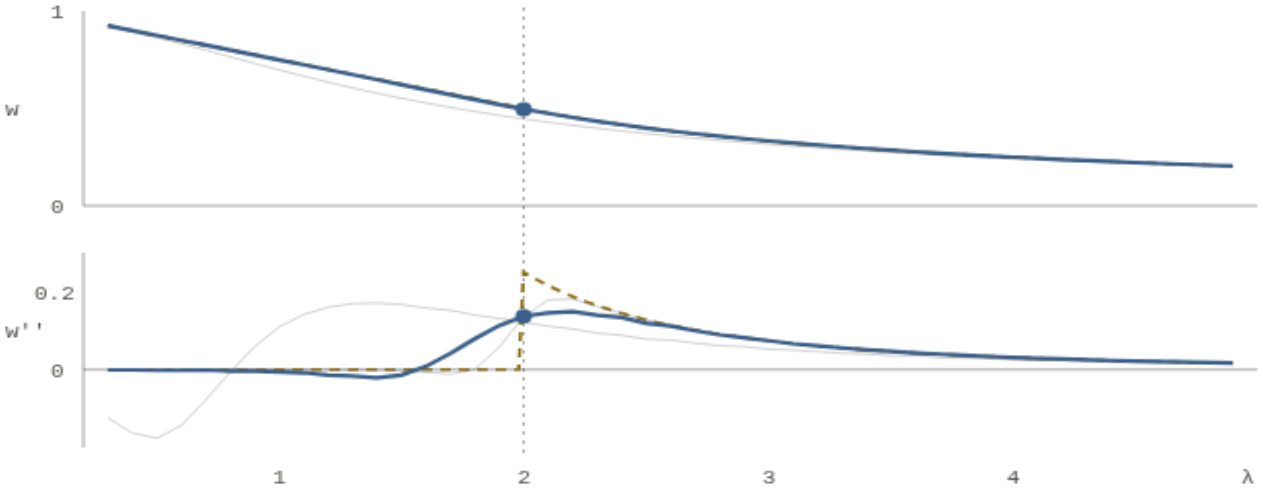
$$Z_N = \det\big[I_{j-k}(2N/\lambda)\big]_{j,k=1}^{N}, \qquad w_N = \frac{1}{2N}\frac{d\ln Z_N}{d(N/\lambda)} \tag{13}$$

Gross–Witten (1980) evaluated $N\to\infty$ by the **saddle-point method** (a method that approximates $\int e^{Nf(x)}dx$, containing a large parameter N, around the point where f is maximal) and found a third-order phase transition at $\lambda = 2$. The same transition appears in Wadia's 1979 preprint Known . In this normalisation the limit has the following form, and the finite-N values approach it as in Table 3 Computed .

$$w_\infty(\lambda) = \begin{cases} 1-\lambda/4 & (\lambda \le 2) \\ 1/\lambda & (\lambda \ge 2) \end{cases} \tag{14}$$

At $\lambda = 2$ both the value 1/2 and the slope −1/4 join up, and the second derivative jumps from 0 to 1/4 (the third derivative, for the free energy). The string tension is $\sigma = -\ln w$.

Figure 2 Choosing N draws $w_N(\lambda)$ in the upper panel and w''_N in the lower panel as thick lines (dashed is N=∞). Raising N brings the lower panel closer to the step at λ=2



N = 10 λ = 2.0 w = 0.49634 (N=∞ : 0.50000) σ = 0.700 w" ≈ 0.14 (N=∞ : step 0→0.25)

Table 3 $w_N(\lambda)$. The last column is formula (14)

λ	N=1	N=10	N=40	N=∞
1	0.69777	0.74937	0.74996	0.75000
2	0.44639	0.49634	0.49939	0.50000
4	0.24250	0.25000	0.25000	0.25000

Computed The numbers in this figure and table were computed on this machine ((13) evaluated to 300 digits; N=1,2 agree with direct integration to 10 digits; the second derivative in the figure is a difference with step 0.1). If JavaScript is disabled, see Table 3.

Three things can be read off Computed . (i) At finite N, Z_N is an entire function of N/λ and positive, so w_N is smooth, and the kink appears only at $N = \infty$. (ii) The manner of approach differs between the phases. The deviation is nearly $-1/(16N^2)$ at $\lambda = 1$; at $\lambda = 4$ it is exponentially small, 2×10^{-12} at $N = 20$; at $\lambda = 2$ doubling N only reduces it by about 1/2.5. (iii) Extending the strong-coupling formula $1/\lambda$ to $\lambda = 1$ gives 1, away from the correct 0.75. This is an example of the strong-coupling expansion not reaching beyond the transition (5-02).

Wadia's preprint (reprinted on arXiv) writes of this transition that it 'seems to be a pathology of the infinite N limit', and holds that in this model it is unrelated to the physical continuum limit (the $\lambda \to 0$ side). Still, $N \to \infty$ can create singularities that finite N does not have, so exchanging $N \to \infty$ and $a \to 0$ needs a reason.

08

The four-dimensional lattice: ratios move slowly with N

In four dimensions we rely on lattice numbers. Lucini–Teper (2001) fitted the continuum limits of $m_{0^{++}}/\sqrt{\sigma}$ (the lightest glueball and the string tension, 17-01) for SU(2) to SU(5) with a linear function of $1/N^2$, and summarised: 'all SU(N) gauge theories are close to SU(∞), at least as far as the low-lying mass spectrum is concerned' Physics .

$$\left.\frac{m_{0^{++}}}{\sqrt{\sigma}}\right|_N = a + \frac{b}{N^2} \quad (15)$$

Refitting (15) to the four points of the paper with error weights gives $a = 3.366 \pm 0.106$, $b = 1.93 \pm 0.54$, $\chi^2 = 0.26$ (2 degrees of freedom), with central values the same as the paper's $3.37(15) + 1.93(85)/N^2$. A constant alone gives $\chi^2 = 12.9$ (3 degrees of freedom) and does not fit Computed .

Table 4 Continuum-limit values of $m_{0^{++}}/\sqrt{\sigma}$ and the line (15) (in the Athenodorou–Teper column, SU(3) is from the 2020 paper, SU(2) and SU(∞) from the 2021 paper)

N	Lucini–Teper	Line (15)	Athenodorou–Teper
2	3.844(61)	3.848	3.781(23)
3	3.607(87)	3.580	3.405(21)
4	3.49(14)	3.487	
5	3.38(16)	3.443	
∞	3.37(15)	3.366	3.072(14)

Physics Extrapolated values from primary sources. Computed "Line (15)" is the fit on this machine (correlations and systematic errors ignored).

From SU(2) to SU(∞) the ratio changes by about 12.5% on the line (15), and by about 19% in the Athenodorou–Teper values Computed . The SU(∞) values of the two primary sources differ by about 2.0σ Computed . All are extrapolations presupposing that the continuum limit exists, and σ is extracted through the string picture. "N=3 is close to N=∞" is an observation within that premise.

09

What can be said as mathematics, and what cannot

Large-N results that have become theorems lie on the following shores Known .

- **Theories of planar diagrams**: 't Hooft (1983) treated a theory of planar diagrams alone in four-dimensional Euclidean space, and showed that 'if all particles involved have non-vanishing masses' and the couplings are small enough, an absolutely convergent procedure gives Green functions (expectations of products of fields; the

propagator is one of them) that uniquely solve the Dyson–Schwinger equations (the same as the Schwinger–Dyson equations of § 06) (abstract: 'an absolutely convergent procedure exists to obtain Green functions that uniquely solve the Dyson-Schwinger equations'). The gap is an assumption.

- **The strong-coupling lattice**: Chatterjee (2015) computed the expectations of large-N Wilson loops for SO(N) at strong coupling as absolutely convergent sums over trajectories of strings on the lattice, and Chatterjee–Jafarov, for the same strong-coupling SO(N), proved the $1/N$ expansion of Wilson loops as an asymptotic expansion (the first term is genus 0, higher orders surfaces of higher genus).
- **All N and β**: Cao–Park–Sheffield expressed the expectations of Wilson loops, for any matrix size $N \geq 1$, $\beta > 0$ and dimension $d \geq 2$, as sums over **embedded maps** (graphs drawn on a surface, mapped onto the lattice). For U(N) the weight of one map is proportional to the product of N^{χ} and the weights of the lattice edges (Weingarten functions, which themselves depend on N).
- **Two dimensions**: the planar master field (§ 06) and the area law at all couplings (7-02).

Seam

(1) The sum over surfaces holds for all β, but there is no estimate producing the area law or the gap on the continuum-limit side (large β). (2) Exchanging $N \to \infty$ and $a \to 0$ is not justified. (3) Even if something can be shown at $N = \infty$, the prize problem requires finite N, and the $1/N$ expansion has no guarantee of convergence. The problem statement too, about four dimensions, only goes as far as writing 'there might some day be an asymptotic solution in a large N limit'.

Large N is used again for strings (17-05) and for holography, which also takes $\lambda \to \infty$ (17-08). The distance to finite N is in 17-10.

What this chapter can say

What is said	What is not said
Known (4)(5) (as rearrangements of series), the difference between U(N) and SU(N).	Convergence of f_h. That the $1/N$ expansion approximates the theory in four dimensions.
Computed (6), Tables 1–3, Figures 1 and 2, the moments of the semicircle density, the check of (13) at N=1,2, the fit of Table 4.	Table 4 is a refit of extrapolated values from primary sources.

What is said	What is not said
Known Large N and the $1/N$ expansion for strong-coupling SO(N), sums over surfaces, the planar master field, the theory of planar diagrams assuming positive masses.	The area law and the gap on the weak-coupling side. Exchanging $N \to \infty$ and $a \to 0$. The return to finite N.
Physics The dominance of planar diagrams, the string coupling $1/N$, the loop equations, the smallness of the N-dependence of the ratios.	Not theorems of the four-dimensional continuum theory. This book does not itself show either the mass gap or the construction of the continuum limit (the published construction of the continuum limit: Volume 2).

Sources and reproduction

Item	Kind	Source / tools
(1), planar diagrams	Known	G. 't Hooft, Nucl. Phys. B72 (1974) 461. Abstract checked.
Quotations from the problem statement	Known	A. Jaffe, E. Witten, "Quantum Yang–Mills Theory" (Clay) § 6.1, § 6.6. Text checked.
Counting with matrix integrals	Known	E. Brézin, C. Itzykson, G. Parisi, J.-B. Zuber, CMP 59 (1978) 35. Abstract checked.
(9) / the semicircle law	Known	J. Harer, D. Zagier, Invent. Math. 85 (1986) 457 / E. P. Wigner, Ann. of Math. 62 (1955) 548. Bibliographic data only.
The factorisation (11), the difference between U(N) and SU(N)	Known	Treated as standard facts of large N (within the framework of 't Hooft 1974 and Brézin–Itzykson–Parisi–Zuber 1978). For these two points, the texts of both papers: bibliographic data only. (11) was also checked with the numbers of Table 2.
The determinant (13)	Known, Computed	The standard rewriting by Weyl's integration formula and Heine's identity (original sources: bibliographic data only, not checked). Checked on this machine against direct integration for N=1,2.
Two-dimensional QCD and strings	Known	D. J. Gross, W. Taylor, NPB 400 (1993) 181, hep-th/9301068. Abstract checked.
Loop equations	Physics	Yu. M. Makeenko, A. A. Migdal, PLB 88 (1979) 135. Abstract checked.
The planar master field	Known	T. Lévy, arXiv:1112.2452. Abstract checked.
The transition, the quotation from Wadia	Known	D. J. Gross, E. Witten, PRD 21 (1980) 446. Abstract checked. S. R. Wadia, arXiv:1212.2906 (a reprint of the 1979 preprint). Abstract checked. (14) not checked in the texts of either paper; confirmed as the limit of the finite-N values.
Table 4, quotations	Physics	B. Lucini, M. Teper, JHEP 0106 (2001) 050, hep-lat/0103027 (text checked). A. Athenodorou, M. Teper, arXiv:2007.06422 Table 17 (SU(3)) / arXiv:2106.00364 Table 23 (SU(2)), Table 31 (SU(∞)). Tables in the text checked.

Item	Kind	Source / tools
The theorems of § 09	Known	G. 't Hooft, "Rigorous construction of planar diagram field theories in four-dimensional Euclidean space", CMP 88 (1983) 1. S. Chatterjee, arXiv:1502.07719. S. Chatterjee, J. Jafarov, arXiv:1604.04777. S. Cao, M. Park, S. Sheffield, arXiv:2307.06790. Abstracts checked.
Tables, figures, (6), (13)	Computed	Python 3 on this machine (numpy, decimal at 300 digits).

Next chapter to read: 17-05 String theory and extra dimensions — what is needed to read the sum over surfaces as a theory of strings.

String theory and extra dimensions — 26, 10 and 11 are numbers that come from conditions of consistency

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — That quantising a string as a fundamental object takes away the free choice of the dimension of spacetime. How 26 for the bosonic string is derived in two ways (the first excited state / the central charge), and 10 for the superstring by the same balance. That 11 is a number that comes from supergravity.

Prerequisite chapters — 17-04 Large N. We use the zero-point energy and $-1/12$ of 17-03. Group representations (1-02), Lorentz transformations (2-04) and the harmonic oscillator (2-06) are written on the spot as far as needed.

Said first — The derivations are at the level of physics. The two-dimensional computations on the world sheet are exact, but string theory has no non-perturbative definition, and the existence of M-theory is a hypothesis. The story of extra dimensions shows nothing about the construction of four-dimensional Yang–Mills or about the mass gap. This book does not itself show either the mass gap or the construction of the continuum limit (the published construction of the continuum limit is read in Volume 2).

The order of this chapter

1. Making the string a fundamental object — how it differs from the string of the tube
2. Counting the transverse oscillations — the mass formula
3. Why 26 (I) — the first excited state is massless
4. Counting the levels — the tachyon and the growth (Detour: partition numbers and Hardy–Ramanujan, Detour: the growth of states and the Hagedorn temperature)
5. Why 26 (II) — the balance of the central charge

6. Why 10 — world-sheet supersymmetry (Detour: 26 − 10 = 16 and the heterotic string)
7. Why 11 — the number of supergravity
8. The "critical dimension" is a critical central charge — it need not be geometry
9. Back to Yang–Mills — the string of the tube, and holography

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01

Making the string a fundamental object

17-03 treated the tube of confinement as a single string, and in 17-04 § 05 $1/N$ played the role of the string coupling. String theory too is said to have begun from models of mesons (**dual resonance models**). The scattering amplitude of Veneziano (1968) is often cited as the starting point; for example, the abstract of Neveu–Schwarz (1971) opens with 'A new dual-resonance model of mesons is constructed ...' Known . For mesons, spin J and the square of the mass line up on a straight line (Regge trajectories). If a string of tension T and length $2R$ rotates with angular velocity $1/R$, its ends move at the speed of light, and integrating the energy and the angular momentum (2-04) gives $E = \pi T R$ and $J = \pi T R^2/2$, that is

$$J = \alpha' E^2, \qquad \alpha' = \frac{1}{2\pi T}. \ (1)$$

For the quantised open string, the maximum spin at level N is $J_{\max} = \alpha' M^2 + 1$ (Tong § 3.1.3). Putting the $\sqrt{\sigma} = 485$ MeV of 17-01 into T gives $\alpha' = 0.677\ \text{GeV}^{-2}$, $\sqrt{\alpha'} \approx 0.16$ fm Computed . The **fundamental string** of this chapter is not the effective theory of a tube; it is required to be quantisable without contradiction at however short a distance. Then the dimension D of spacetime stops being a free input. In this chapter, following the string literature, we write D for the dimension of spacetime. It is the same as the d of 17-03 and of the table of notation in the appendix, not the dimension of space alone.

02

Counting the transverse oscillations: the mass formula

A string sweeps out a surface (the **world sheet**) in spacetime $\mathbb{R}^{1,D-1}$. In **light-cone gauge** we use the light-cone coordinates $X^{\pm} = (X^0 \pm X^{D-1})/\sqrt{2}$, recombining time X^0 and one longitudinal direction X^{D-1}, and make X^+ proportional to the world-sheet time τ ($X^+ = x^+ + \alpha' p^+ \tau$; x^+ and p^+ are the $+$ components of the position and

momentum of the centre of mass of the string; Tong (2.13)). The oscillations of X^- do not vanish, but they are fixed by the transverse oscillations through the constraint equations. Only the $D-2$ transverse directions oscillate independently (Goddard–Goldstone–Rebbi–Thorn 1973) Known . This is the same counting as the transverse displacement $h \in \mathbb{R}^{d-2}$ of 17-03 ($d = D$). The n-th oscillation in direction i is a harmonic oscillator (2-06), and each application of the creation operator α^i_{-n} raises the **level** N by n. The mass of the open string is

$$\alpha' M^2 = N - a \tag{2}$$

where $-a$ is the zero-point energy (the sum of $\frac{1}{2}n$ over all oscillators). Its finite part is the $-1/12$ of 17-03 (17-03 § 04), so

$$a = -\frac{D-2}{2}\sum_{n\geq 1} n \;\longrightarrow\; -\frac{D-2}{2}\cdot\Big(-\frac{1}{12}\Big) = \frac{D-2}{24}. \tag{3}$$

The Lüscher term $-\pi(D-2)/(24R)$ of 17-03 and the $(D-2)/24$ of the Nambu–Goto levels (17-03 § 06) are this same number Known . Tong admits that this treatment of the sum is 'famously unconvincing', and derives it again by the central-charge method, the same as § 05 of this chapter (Tong § 4.4.1). The closed string has right-moving oscillations α and left-moving oscillations $\tilde{\alpha}$, and its levels satisfy $N = \tilde{N}$ (Tong (2.26)).

03

Why 26 (I): the first excited state must be massless

The internal states of a particle form a representation of the rotations that leave its momentum unchanged (the **little group**) (1-02; Wigner 1939). For mass $M > 0$ we can go to the rest frame and the little group is $SO(D-1)$; for zero mass it is the $SO(D-2)$ that preserves the momentum $(p, 0, \ldots, 0, p)$ (precisely, it is the group $ISO(D-2)$ of rotations and translations in $D-2$ dimensions, but what acts on finitely many states is the rotation part $SO(D-2)$). A photon in four dimensions has 2 polarisations, a massive spin-1 particle 3 Known . The level-1 states $\alpha^i_{-1}|0\rangle$ of the open string are a transverse vector with $D-2$ states, one short of the $D-1$ components a massive vector needs. To keep Lorentz symmetry it must be massless, and

$$\alpha' M^2 = 1 - a = 0 \implies a = 1 \overset{(3)}{\implies} \frac{D-2}{24} = 1 \implies D = 26. \tag{4}$$

Tong makes the same argument for the first excited state of the closed string ($(D-2)^2$ states) and writes 'these states will have to be massless. And this is only the case if the dimension of spacetime is D = 26' Physics .

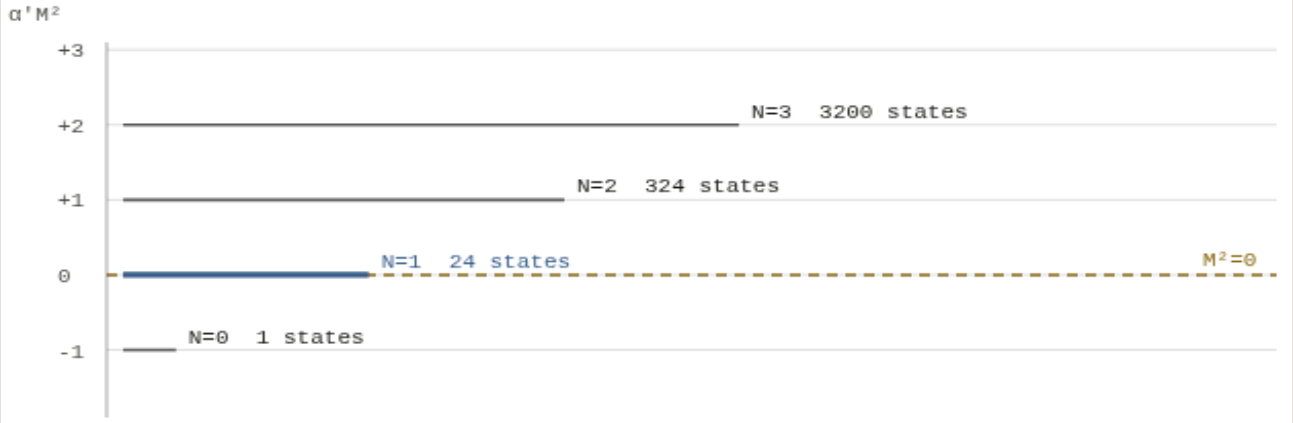
Checking with the Lorentz algebra

More carefully, we check whether the conserved quantities $M^{\mu\nu}$ that generate Lorentz transformations (rotations and boosts) satisfy the relations of the Lorentz algebra in light-cone gauge too. $M^{\mu\nu}$ includes the angular momentum carried by the oscillations of the string. The crux is $[M^{i-}, M^{j-}] = 0$, where i, j are transverse directions and $-$ is the light-cone direction X^- (Tong § 2.4). Since the oscillations of X^- are fixed by the transverse oscillations, this commutator can be written with the transverse oscillators.

$$[M^{i-}, M^{j-}] \propto \sum_{n>0} \left[\left(\frac{D-2}{24} - 1\right)n + \frac{1}{n}\left(a - \frac{D-2}{24}\right)\right](\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n) + (\alpha \leftrightarrow \tilde{\alpha}) \quad (5)$$

The proportionality factor $2/(p^+)^2$ is omitted. This is the formula for the closed string, and $(\alpha \leftrightarrow \tilde{\alpha})$ is a term of the same form for the left-moving oscillations. It vanishes for every n only when $D = 26$ and $a = 1$ Known . The original source is Goddard–Goldstone–Rebbi–Thorn (1973), whose abstract also states that the angular momentum operators generate the Lorentz group $O(D-1,1)$ 'only when the first excited state is a photon, i.e., a spin-one massless state, and when D = 26'. The first excited state of the closed string has $24^2 = 576 = 299 + 276 + 1$ states Computed , and Tong assigns these to the symmetric traceless, antisymmetric and trace fields $G_{\mu\nu}, B_{\mu\nu}, \Phi$ of $SO(24)$, reading the first as a massless spin 2, that is, the graviton Physics .

Figure 1 Varying D moves the levels $\alpha' M^2 = N - a$ up and down (the length of a line is the logarithm of the number of states). The thick level 1 (1/2 for the superstring) sits on $M^2 = 0$ only for D=26 for the bosonic string and D=10 for the superstring



D = 26, a = (D−2)/24 = 1, level 1: 24 states, α'M² = 0 → consistent as a massless vector (SO(24))

Computed The numbers in this figure were computed on this machine. If JavaScript is disabled, see (4) and Table 2.

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04
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Counting the levels: the tachyon and the growth of the number of states

$N = 0$ of the open string at $D = 26$ has $\alpha' M^2 = -1$: a **tachyon**, with negative squared mass (Tong § 3.1.1). It is a field whose mass term has the opposite sign (2-02), meaning that we are expanding around a maximum of the potential Physics . The number of states d_N at level N is the number of ways of splitting N into a sum of "positive integers in 24 colours".

$$\sum_{N \geq 0} d_N \, q^N = \prod_{n \geq 1} (1 - q^n)^{-24} \tag{6}$$

The 324 at level 2 is the dimension of the symmetric traceless rank-2 tensor of $SO(25)$, and the 3200 at level 3 is $2900 + 300$ (symmetric traceless rank 3 and antisymmetric rank 2) Computed . According to Tong all excitations fit into representations of $SO(D-1)$, and the condition is on level 1 alone. That the numbers at levels 2 and 3 match sums of dimensions of representations of $SO(D-1)$ we checked for $D = 4$–39 Computed . Both are polynomials in D of degree at most three, so this makes them identities valid for any D.

Table 2 Levels of the open bosonic string ($D = 26$). The ratio is to the Bessel form of the detour ($n = N - 1$)

N	$\alpha' M^2$	d_N	Ratio
0	−1	1	
1	0	24	
2	1	324	1.000009
3	2	3200	0.999998
10	9	6.392×10^{8}	1.00000000
400	399	1.446×10^{91}	1.00000000

Computed The numbers in this table were computed on this machine.

Detour: partition numbers and Hardy–Ramanujan

A subsection you may skip.

With one colour the coefficients of (6) are the partition numbers $p(N)$ ($p(4) = 5$ from $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$), and Hardy and Ramanujan (1918) gave $p(N) \sim \frac{1}{4\sqrt{3}\,N} e^{\pi\sqrt{2N/3}}$ Known . For (6) with 24 colours, setting $n = N - 1$ gives $d_N \approx \frac{1}{\sqrt{2}} n^{-27/4} e^{4\pi\sqrt{n}}$, but the approach is slow, and the ratio is still 0.71 at $N = 400$. What fits well is the form $2\pi n^{-13/2} I_{13}(4\pi\sqrt{n})$ using the modified Bessel function I_{13}, which already agrees to 5 digits at $N = 2$ (Table 2) Computed . $I_\nu(x)$ is a function that behaves like $e^x/\sqrt{2\pi x}$ when x is large. Putting this

in gives $2\pi n^{-13/2}\, e^{4\pi\sqrt{n}}/\sqrt{8\pi^2\sqrt{n}} = \frac{1}{\sqrt{2}} n^{-27/4} e^{4\pi\sqrt{n}}$, and the two formulas agree at large n. The ratio 0.71 is the error from replacing I_{13} by this approximation.

Detour: the growth of states and the Hagedorn temperature

A subsection you may skip.

An open string at level N has $\alpha' M^2 = N - 1$, so setting $n = N - 1$ as in the detour above gives $\alpha' M^2 = n$, that is $M = \sqrt{n/\alpha'}$. From the asymptotic form of the detour above, the number of states grows at the rate $e^{4\pi\sqrt{n}} = e^{4\pi\sqrt{\alpha'}M}$ (the power of n in front changes slowly compared with the exponential). Write Θ for the temperature (to distinguish it from the tension T; in units with Boltzmann's constant 1). The weight of a state of mass M is the Boltzmann factor $e^{-M/\Theta}$ (2-08), so the product of number and weight has the form $e^{(4\pi\sqrt{\alpha'} - 1/\Theta)M}$, and for $\Theta > \Theta_H = 1/(4\pi\sqrt{\alpha'})$ it grows with M and the sum diverges. This Θ_H is the **Hagedorn temperature** Physics . Putting $d = D = 26$ and $\sigma = 1/(2\pi\alpha')$ into the Hagedorn length L_H of 17-03 § 06 ($\sigma L_H^2 = \pi(d-2)/3$, where σ is the tension, the T of this chapter) gives $L_H = 4\pi\sqrt{\alpha'}$, and its reciprocal $1/L_H$ coincides with Θ_H Computed . It is the same reading as in 17-03, where $1/L$ became the temperature when the periodic direction was regarded as time.

05

Why 26 (II): the balance of the central charge

The second derivation looks at the D fields X^μ on the world sheet as two-dimensional free scalar fields (2-02). A two-dimensional theory whose form does not change under a change of scale (a **conformal field theory**) has a **central charge** c, and one free scalar has $c = 1$. More precisely, a conformal field theory is a theory invariant under angle-preserving maps (**conformal transformations**), of which changes of scale are a part. In two dimensions there are infinitely many such maps, and this constrains the theory strongly. c appears in the ground-state energy on a cylinder of circumference L (when the left and right c are equal).

$$E_0(L) = -\frac{\pi c}{6L} \quad (7)$$

(Blöte–Cardy–Nightingale 1986, Affleck 1986; Tong § 4.4.1) Known . The $-\pi(D-2)/(6L)$ of the closed tube of 17-03 is the case $c = D - 2$, and about the $-\pi c/(24R)$ of a tube with ends Tong writes 'c counts the number of degrees of freedom of the QCD flux tube'. The Lüscher coefficient measures the central charge of the tube.

In Polyakov's formulation the metric of the world sheet is also a field to be integrated, and changes of coordinates and changes of scale of the metric (Weyl transformations) are gauge symmetries. Fixing the gauge brings in auxiliary fields (ghosts; the same

mechanism as Faddeev–Popov in 4-08), with central charge -26. A quantum breaking of a gauge symmetry (an anomaly) is not allowed, so the total must be 0 (Tong § 5.3) Physics .

$$c_{\text{matter}} + c_{\text{ghost}} = D \cdot 1 + (-26) = 0 \implies D = 26 \quad (8)$$

The $D - 2$ fields and $-1/12$ in light cone, and the D fields of the covariant form with the ghosts' -26, select the same 26. More precisely, (8) is a condition only as long as the scale of the metric (Weyl) is kept as a gauge symmetry. For $D \neq 26$ the scale degree of freedom does not disappear but remains, and behaves as a new field called the Liouville field (the non-critical string of Tong § 5.3.2). The abstract of Polyakov (1981) too writes 'At D = 26 the string dynamics is that of harmonic oscillators ..., otherwise it is described by the nonlinear integrable theory', treating the $D \neq 26$ side as a different theory.

06

Why 10: world-sheet supersymmetry

The bosonic string has no fermions (relatives of the electron and the quarks), and has a tachyon. Add to the world sheet fermions ψ^μ as partners of X^μ ($c = 1/2$ each), make the **world-sheet supersymmetry** that exchanges the two a gauge symmetry, and the ghosts add central charge $+11$.

$$D\left(1 + \frac{1}{2}\right) + (-26 + 11) = 0 \implies D = 10 \quad (9)$$

In Tong § 5.3.1, 'c = 26 − 11 = 15' and 'D(1 + 1/2) = 15' Physics . We check it in light cone too. Depending on the boundary condition, the fermion oscillations take half-integer labels $r = \frac{1}{2}, \frac{3}{2}, \ldots$ (Neveu–Schwarz, NS) or integer ones (Ramond, R), and their zero-point energy has the opposite sign, $-\frac{1}{2}\sum r$. The finite parts are in Table 1.

Table 1 The sum minus $1/\varepsilon^2$ (cutoff $e^{-\varepsilon r}$)

ε	Integer $\sum n e^{-\varepsilon n}$	Half-integer $\sum r e^{-\varepsilon r}$
0.1	-0.0832917	0.0416302
0.001	-0.0833333	0.0416667
→ 0	-1/12	+1/24

Computed The numbers in this table were computed on this machine (closed forms checked against direct sums).

The zero-point energy per transverse direction is $-\frac{1}{24} - \frac{1}{48} = -\frac{1}{16}$ for NS and $-\frac{1}{24} + \frac{1}{24} = 0$ for R Computed . The NS levels are $\alpha' M^2 = N - \frac{D-2}{16}$ (N also takes half-integers), and the first excited state $\psi^i_{-1/2}|0\rangle$ is again a vector with $D - 2$ components. The

massless condition $\frac{1}{2} - \frac{D-2}{16} = 0$ gives $D = 10$, agreeing with (9) ("Superstring" in Figure 1). The tachyon of the NS ground state, $\alpha' M^2 = -\frac{1}{2}$, is said to be removed by a rule that discards part of the states (attributed to Gliozzi–Scherk–Olive 1977) Physics .

Detour: 26 − 10 = 16 and the heterotic string

A subsection you may skip.

Green–Schwarz (1984) named $SO(32)$ and $E_8 \times E_8$ as the groups for which the gauge and gravitational anomalies in 10 dimensions cancel, and Gross–Harvey–Martinec–Rohm (1985) built the heterotic string as 'a chiral combination of the closed D=26 bosonic and D=10 fermionic strings' Known . The difference between 26 and 10 is 16, and $SO(32)$ and $E_8 \times E_8$ both have rank 16 and dimension 496 (E_8 has 240 roots and rank 8, giving 248; 1-04) Computed . How this 16 becomes a gauge group is in 17-07.

07

Why 11: a number of supergravity, not of strings

11 is not a critical dimension of strings. The abstract of Cremmer–Julia–Scherk (1978) reads 'We present the action and transformation laws of supergravity in 11 dimensions' Known . **Supergravity** is a field theory containing the graviton and its supersymmetric partner (the gravitino), and that 11 is its upper limit is said to follow from the classification of Nahm (1978).

The skeleton is this Physics . **Supercharges** are the generators of the symmetry that exchanges bosons and fermions (supersymmetry), and Q is the number of their real components. With Q supercharges, the range of helicities (the amount of rotation about the direction of motion) of a massless multiplet is $Q/8$, and for spin at most 2 (range at most 4) $Q \leq 32$. 11 dimensions is said to be the last dimension that fits within this.

The fields of 11-dimensional supergravity are the graviton, a **3-form** and the gravitino. A 3-form is an antisymmetric field with three indices, the electromagnetic potential A_μ (2-03) with three indices instead of one. Counting the degrees of freedom with the little group $SO(9)$ (the $SO(D-2)$ of $D = 11$), the 128 bosons, graviton $45 - 1 = 44$ and 3-form $\binom{9}{3} = 84$, balance the gravitino's $9 \cdot 16 - 16 = 128$ Computed . In counting the gravitino we used, only as a number, that a **spinor** of $SO(9)$ (the quantity carrying the internal degrees of freedom of a fermion) has 16 components.

Witten (1995) argued in the abstract 'It is argued ... that eleven-dimensional supergravity arises as a low energy limit of the ten-dimensional Type IIA superstring'.

The **IIA string** is a kind of closed superstring in 10 dimensions in which the supersymmetries carried by the right- and left-moving oscillations have opposite handedness (chirality). With λ the coupling of the IIA string, the radius of the 11th circle in 11-dimensional units is

$$r(\lambda) = \lambda^{2/3} \quad (10)$$

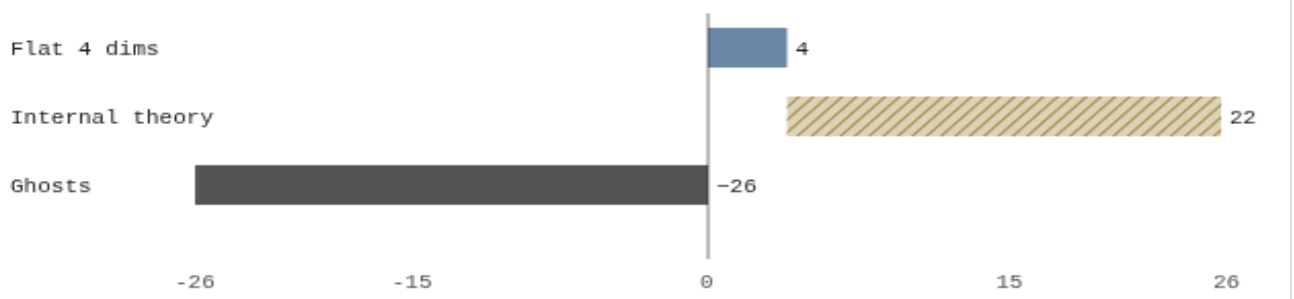
(formula (2.13) of that paper) Physics . At weak coupling the circle shrinks, and states with momentum n around the circle have mass $\sim |n|/\lambda$ in the string metric (formula (2.14)): heavy and not seen. As a metaphor, the 10 dimensions of the string are how a thin 11-dimensional tube looks from afar. More precisely, this is a conjecture from low-energy arguments, and the IIA string at strong coupling has no independent definition. Hořava–Witten (manuscript 1995, published 1996) **proposed** a link between the heterotic string and 11 dimensions in the form 'We propose'. The following paper (1996) argued that the gauge group is fixed to one E_8 for each boundary, wrote that this gives new material testing 'the hypothesis that there is a consistent quantum M-theory', and treated the existence of M-theory as a hypothesis. This 11-dimensional theory (M-theory) has no complete definition.

08

The "critical dimension" is a critical central charge: extra dimensions need not be geometry

What (8)(9) require is not a dimension but a central charge. Tong writes 'the "critical dimension" of string theory is something of a misnomer: it is really a "critical central charge"' Known . For flat four-dimensional spacetime, for the bosonic string it suffices to add **any conformal field theory with** $c = 22$ to the $c = 4$ of $X^0, \dots, X^3$, and Tong continues 'It is what we really mean when we talk about the "extra hidden dimensions of string theory"'. For the superstring flat four dimensions give $c = 6$ and the rest is $c = 9$ Computed . As a metaphor, only the total of the ledger is fixed and the items are free. More precisely, the internal theory must also satisfy other conditions (world-sheet supersymmetry and so on), and it can be read as geometry, as a manifold, only in special cases. How the extra dimensions then become invisible is in 17-06, and how gauge fields come out in 17-07.

Figure 2 Varying the number of flat dimensions d_{flat} changes the breakdown of the central-charge ledger. The internal theory (hatched) takes up what is removed, and the total stays 0



Bosonic string, flat 4 dims c = 4, internal c = 22, ghosts c = −26, total 0

Computed The arithmetic of this figure was checked on this machine. If JavaScript is disabled, read it as $d_{\text{flat}} + c_{\text{int}} = 26$ (for the superstring $\frac{3}{2}d_{\text{flat}} + c_{\text{int}} = 15$).

09

Back to Yang–Mills: the four-dimensional tube, and the dimensions of holography

Then is the string of the four-dimensional tube of 17-03, not being 26-dimensional, inconsistent? The abstract of Polchinski–Strominger (1991) states that the long-distance effective theory of four-dimensional strings such as 'QCD strings' is 'an interacting Poincaré-invariant conformal field theory with four bosons and c=26', and showed the compatibility to second order in perturbation Physics . The 4 bosons, which would have $c = 4$ if free, carry $c = 26$ through their interaction. What the critical dimension constrains is the fundamental string; the effective theory of a long tube is consistent as it is, in four dimensions. But the effective theory breaks down at short distances, and there is no mathematical derivation either (17-03 § 06).

The extra dimensions of the fundamental string return to Yang–Mills through holography (17-08), and the breakdown differs from model to model. Maldacena's $AdS_5 \times S^5$ (AdS_5 is a five-dimensional spacetime of constant negative curvature, with the four-dimensional directions and a radial direction; S^5 is the five-sphere) is 10 = 4 + radial + S^5; Witten 1998 § 4 is 11 = 4 + two circles + radial + S^4 (one circle fewer, 10, on reducing to IIA); the improved holographic QCD of Gürsoy–Kiritsis is 5 = 4 + radial. The dimensions that make the gap also differ by model: in Witten's type, a two-dimensional cigar formed by the radial direction and a shrinking circle; in the bottom-up type, a single radial direction and an infrared singularity (17-09) Physics . These extra dimensions are tools of a dual description, not claimed to be real dimensions, and are a separate matter from the experimental lower bounds (17-06).

Seam

(1) String theory has no non-perturbative definition, and the very existence of M-theory is a hypothesis. (2) Holographic computations work on the side of $N \to \infty$ and 't Hooft coupling $\to \infty$, the opposite direction to the limit that returns to four-dimensional pure Yang–Mills (17-10). (3) The problem statement (Jaffe–Witten,

2000) writes 'at present we do not know' of a non-trivial relativistic field theory in four dimensions satisfying axioms such as Wightman's (4-09). The four-dimensional theory was constructed by the construction published in 2026, as a limit of Wilson's lattice (the committee's assessment; Volume 2). The road of extra dimensions does not construct a four-dimensional theory in this sense.

What this chapter can say

What is said	What is not said
Known The counting of (3)(5) (as a two-dimensional world-sheet computation in light cone).	A non-perturbative definition of string theory. The existence of M-theory.
Physics The reading of (4) (reading the first excited state as massless) and the balance of (8)(9) (the central charge of the ghosts and the anomaly argument).	The treatment of the non-critical string (the Liouville field) for $D \neq 26$.
Computed Tables 1 and 2, Figures 1 and 2, 576, 496, 128, the Hagedorn temperature (the $1/L_H$ of 17-03), α'.	Why the Bessel form of Table 2 fits so well.
Physics The readings of the graviton and the tachyon, the relation between 11 dimensions and IIA, Hořava–Witten (a proposal).	Checking "11 is the upper limit" in the original source.
Physics That the effective theory of the four-dimensional tube is consistent with $c = 26$ (to second order in perturbation). The breakdown of the dimensions of the models.	Consistency of the effective string at all orders. That a tube exists in four-dimensional Yang–Mills. Constructing a four-dimensional theory from extra dimensions. This book does not itself show either the mass gap or the construction of the continuum limit (the published construction of the continuum limit: Volume 2).

Sources and reproduction

Item	Kind	Source / tools
$J_{\max}$, (2)–(9), each quotation	Known, Physics	D. Tong, "String Theory", arXiv:0908.0333 § 2.2–2.4, § 3.1, § 4.4.1, § 5.3 (including the non-critical string of § 5.3.2). Text checked.
Light-cone quantisation and $D = 26$ (the original source of (4)(5)) / 'At D = 26 …'	Known, Physics	P. Goddard, J. Goldstone, C. Rebbi, C. B. Thorn, Nucl. Phys. B56 (1973) 109 / A. M. Polyakov, Phys. Lett. B103 (1981) 207. Abstracts checked.

Item	Kind	Source / tools
Meson models, NS and R / the rule removing the tachyon	Known, Physics	G. Veneziano, Nuovo Cim. A57 (1968) 190 (bibliographic data only; its standing as the starting point of dual resonance models not checked) / A. Neveu, J. H. Schwarz, Nucl. Phys. B31 (1971) 86 / P. Ramond, Phys. Rev. D3 (1971) 2415 (abstracts checked) / F. Gliozzi, J. Scherk, D. Olive, Nucl. Phys. B122 (1977) 253 (abstract only; the attribution of the projection not checked).
The little group / formula (7)	Known	E. P. Wigner, Ann. of Math. 40 (1939) 149 / H. W. J. Blöte, J. L. Cardy, M. P. Nightingale, PRL 56 (1986) 742 / I. Affleck, PRL 56 (1986) 746. Bibliographic data only (content checked in Tong).
Anomaly cancellation / the heterotic string	Known	M. B. Green, J. H. Schwarz, Phys. Lett. B149 (1984) 117 / D. J. Gross, J. A. Harvey, E. J. Martinec, R. Rohm, PRL 54 (1985) 502. Abstracts checked.
11-dimensional supergravity / the upper limit	Known	E. Cremmer, B. Julia, J. Scherk, Phys. Lett. B76 (1978) 409 (abstract checked) / W. Nahm, Nucl. Phys. B135 (1978) 149 (abstract only; "11 is the upper limit" not checked).
(10), KK masses, Hořava–Witten	Physics	E. Witten, Nucl. Phys. B443 (1995) 85, hep-th/9503124 (text checked; formulas (2.13)(2.14)) / P. Hořava, E. Witten, Nucl. Phys. B460 (1996) 506, hep-th/9510209 (manuscript 1995-10) and Nucl. Phys. B475 (1996) 94, hep-th/9603142 (abstracts checked).
The four-dimensional tube and $c = 26$	Physics	J. Polchinski, A. Strominger, PRL 67 (1991) 1681. Abstract checked.
Asymptotic form of the partition numbers	Known	G. H. Hardy, S. Ramanujan, Proc. London Math. Soc. (2) 17 (1918) 75. Bibliographic data only.
Breakdown of the models / the problem statement	Physics, Known	J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200 / E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505, hep-th/9803131 § 4 / U. Gürsoy, E. Kiritsis, JHEP 02 (2008) 032, arXiv:0707.1324 (abstract) / A. Jaffe, E. Witten, "Quantum Yang–Mills Theory" (the Clay problem statement, 2000). Witten 1998 § 4 and the problem statement: text checked.
Tables, figures, arithmetic	Computed	Python 3 on this machine. zero_point.py (Table 1), levels.py (Table 2, $SO(D-1)$, Hagedorn, α'), counts.py (576, 128, 496, the balance).

Next chapter to read: 17-06 How extra dimensions become invisible — why extra dimensions that can be read as geometry do not show up in four-dimensional experiments.

How extra dimensions become invisible — waves on a small circle look like a tower of heavy particles

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — That extra dimensions being invisible means that fluctuations in those directions become a sequence of heavy particles (the KK tower), which cannot be produced at low energy. That the lowest rung of the tower is the first eigenvalue of the Laplacian, bounded by the diameter when the curvature is non-negative (Zhong–Yang). Two ways of hiding, Calabi–Yau and branes, and the experimental lower bounds.

Prerequisite chapters — 17-05 String theory and extra dimensions. We use 2-02 § 06, 2-05 § 08 and 1-10 § 07. Fourier series and the Rayleigh quotient are written on the spot, and the metric, the Laplacian, Ricci curvature and the Einstein equations are defined on the spot as far as they are used.

Said first — Whether the real world has extra dimensions is not known, and the experimental numbers are lower bounds. The gap of the KK tower comes from the internal space being small, and is a different thing from the mass gap of Yang–Mills (which arises from interaction on $\mathbb{R}^4$). This book does not itself show either the mass gap or the construction of the continuum limit (the published construction of the continuum limit is read in Volume 2).

The order of this chapter

1. What "seeing" means — the resolution is ħc/E
2. A field on a circle — one field becomes a tower
3. The lowest rung of the tower — the first eigenvalue and the diameter (the minimum of metric, Laplacian and Ricci curvature)
4. Kaluza and Klein — charge is momentum
5. The six dimensions of strings — Calabi–Yau
6. Confining to a brane — ADD and RS (Detour: extra dimensions without a gap)
7. Experimental lower bounds — Newton's law and colliders

8. Back to Yang–Mills — the KK gap is not the mass gap

01

What "seeing" means — the resolution is ħc/E

To see, in physics, is to collide particles and examine how they bounce off. To resolve a structure of size ℓ needs a probe with wavelength shorter than ℓ, that is, high energy (2-05).

$$\ell \approx \frac{\hbar c}{E}, \qquad \hbar c = 197.327\ \text{MeV}\cdot\text{fm} \quad (1)$$

1 TeV gives 2.0×10^{-19} m, the LHC collision energy of 13.6 TeV gives 1.5×10^{-20} m, and the Planck mass 1.22×10^{19} GeV gives 1.6×10^{-35} m Computed . From here on we set $\hbar = c = 1$ (2-04 § 08). As a metaphor, a garden hose seen from afar is a line, but up close the direction going around it becomes visible. More precisely, what this chapter says is that "fluctuations along a small circle need high energy".

02

A field on a circle — one field becomes a tower

Consider the five-dimensional spacetime $\mathbb{R}^{1,3} \times S^1$ with a circle of radius R attached to each point of space (the circle coordinate y identified with $y + 2\pi R$). A massless scalar field $\Phi(x, y)$ obeys the wave equation (2-02), and since it is periodic in y it can be expanded in a Fourier series (1-11).

$$(\partial_t^2 - \nabla^2 - \partial_y^2)\Phi = 0, \qquad \Phi(x,y) = \frac{1}{\sqrt{2\pi R}} \sum_{n\in\mathbb{Z}} \phi_n(x)\, e^{iny/R} \quad (2)$$

Substituting, ∂_y^2 is replaced by $-n^2/R^2$, and the equation splits coefficient by coefficient.

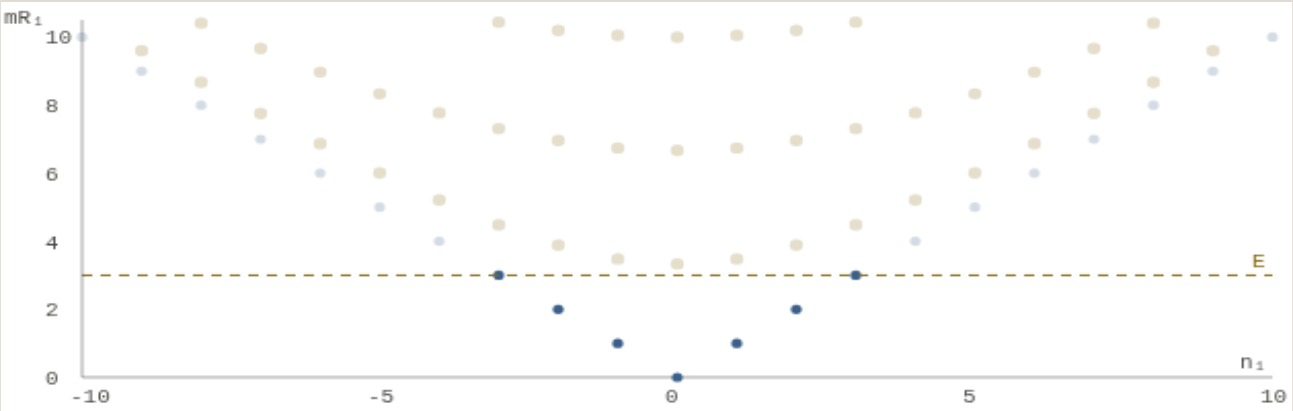
$$\left(\partial_t^2 - \nabla^2 + \frac{n^2}{R^2}\right)\phi_n = 0, \qquad m_n = \frac{|n|}{R} \quad (3)$$

This is the Klein–Gordon equation of mass m_n (2-02 § 06). One five-dimensional field looks, in four dimensions, like a massless **zero mode** ϕ_0 and an infinite sequence of fields of mass $1/R, 2/R, \ldots$. This is the **Kaluza–Klein tower** (KK tower) Known . The PDG writes 'At energies smaller than 1/R, the KK modes can be neglected, leaving the zero-mode action'. The couplings of the zero mode are diluted; for a quartic interaction

$\lambda_4 = \lambda_5/(2\pi R M_5)$ (PDG formula 84.6), weaker the larger the circle. M_5 is the fundamental mass scale of the five-dimensional theory, put in to make λ_5 dimensionless.

For a product of δ circles (a torus T^δ with radii R_i), each tuple of integers n_i gives a field with $m^2 = \sum_i n_i^2/R_i^2$. Figure 1 is the case of two circles.

Figure 1 The KK tower of two circles ($R_1 \geq R_2$). Each point is a four-dimensional field (horizontal n_1, vertical the mass). Varying E (dashed) and the ratio of radii changes the fields below E



E·R₁ = 3.00, R₂/R₁ = 0.30, fields below E: 7 (Weyl estimate 8.5), the thin direction is not yet visible

Table 1 The number of four-dimensional fields of mass at most E (including the zero mode). In brackets the Weyl estimate $\pi R_1 R_2 E^2$ (the area $4\pi^2 R_1 R_2$ times $E^2/4\pi$; the detour of 1-10)

R_2/R_1	$ER_1 = 2$	**5**	**10**	**30**
1	13 (12.6)	81 (78.5)	317 (314.2)	2821 (2827.4)
0.1	5 (1.3)	11 (7.9)	23 (31.4)	267 (282.7)
One circle	5	11	21	61

Computed The numbers of Table 1 and Figure 1 were computed on this machine (the figure is drawn on the spot from formula (3)). If JavaScript is disabled, see Table 1.

The row $R_2/R_1 = 0.1$ is the same as one circle for $ER_1 \leq 5$, and the thin direction first appears at $E \gtrsim 1/R_2$. The count grows as E to the power of (the number of visible extra dimensions).

03 The lowest rung of the tower — the first eigenvalue and the diameter

The lowest rung $1/R$ can be restated as an inequality. For a periodic function with mean 0, $f = \sum_{n\neq 0} c_n e^{iny/R}$, Parseval's identity gives (Wirtinger's inequality) Known

$$\int_0^{2\pi R} |f'|^2\,dy = 2\pi R \sum_{n\neq 0} \frac{n^2}{R^2}|c_n|^2 \;\geq\; \frac{1}{R^2}\int_0^{2\pi R} |f|^2\,dy \qquad (4)$$

This is the simplest example of the Poincaré inequality of 2-05 § 08 and 1-10 § 07, and the best constant R^2 is the reciprocal of the gap $1/R^2$ (the square of the lowest rung of the tower).

Measuring lengths — metric, diameter, Laplacian

To replace the circle by a general compact manifold K (1-05) we need a way of measuring lengths. An inner product given on each tangent space (1-05 § 06), varying smoothly from point to point, is called a **metric**. It is a way of measuring lengths and angles. With a metric we can measure the lengths of curves, and the length of the shortest path joining two points is their distance. The maximum of that distance over all pairs of points is the **diameter** d (for a circle of circumference $2\pi R$, $d = \pi R$; the d of this chapter is the diameter, not the spacetime dimension d of other chapters). The size of the gradient of a function, $|\nabla f|$, is also fixed by the metric, and the operator $-\Delta_K$ with $\int_K |\nabla f|^2 = \int_K \bar{f}\,(-\Delta_K f)$ is called the **Laplacian** (the Laplace–Beltrami operator). On a flat torus, in coordinates y_i measured by length, $-\Delta_K = -\sum_i \partial^2/\partial y_i^2$.

If spacetime is the unwarped product $\mathbb{R}^{1,3} \times K$ and the field is a scalar, expanding in the eigenfunctions of $-\Delta_K$ (eigenvalues $0 = \lambda_0 < \lambda_1 \le \cdots$) gives $m_j^2 = \lambda_j$. The first KK mass is the minimum of the Rayleigh quotient (1-10 § 06).

$$m_1^2 = \lambda_1(K) = \inf\left\{ \frac{\int_K |\nabla f|^2}{\int_K |f|^2} \;:\; \int_K f = 0,\ f \not= 0 \right\} \tag{5}$$

For form or spinor fields, or in warped backgrounds, it becomes the eigenvalue of another operator (the Hodge Laplacian, the Dirac operator and so on). In this chapter we restate "extra dimensions are invisible" as the Poincaré inequality on the part without the zero mode, $\int |\nabla f|^2 \ge \lambda_1 \int |f|^2$.

When the diameter suffices — Ricci curvature

Multiplying the metric by s^2 (multiplying all lengths by s) multiplies λ_1 by s^{-2}, so shrinking raises the tower. That "small" may be measured by the diameter d alone is a theorem when the Ricci curvature is non-negative.

Ricci curvature, as a metaphor, is the average of how much a bundle of geodesics (locally shortest paths) leaving a point around one direction gathers together as it goes. On a sphere it is positive (meridians gather at the poles), on a flat torus 0, on a saddle-shaped surface negative (they spread). More precisely, for a direction v of length 1, it is the sum $\mathrm{Ric}(v,v) = \sum_i K(v, e_i)$ of the bending (sectional curvature) of the $n-1$ sections spanned by v and the directions e_i orthogonal to v, which is $n-1$ times the average. In two dimensions there is a single section, so it is the same as the Gauss curvature. The bending itself is the curvature in the sense of 1-07 § 05 (the

discrepancy after going round a small loop) when the bundle of tangent spaces is given the connection fixed by the metric. $\mathrm{Ric}_K \geq 0$ is the condition that this sum is at least 0 at every point in every direction Known .

$$\mathrm{Ric}_K \geq 0 \implies \lambda_1(K) \geq \frac{\pi^2}{d^2} \qquad \text{(Zhong–Yang; equality only for the circle: Hang–Wang)} \,($$

Table 2 The ratio $\lambda_1 d^2/\pi^2$. (6) says it is at least 1 Computed

K	Ratio	K	Ratio
Circle	1	Square torus	2
Rectangular torus 1×0.2	1.04	Hexagonal torus	1.7778
Rectangular torus 1×0.05	1.0025	Spheres S^2, S^3, S^6	2, 3, 6

The tori were computed numerically and agree to 5 digits with the closed forms (rectangle $1+\rho^2$, hexagon $16/9$). The spheres are known values.

For thin tori the ratio approaches 1 but does not reach it. The curvature condition cannot be dropped. Join two spheres by a thin neck and put into (5) a function that is $+1$ on one and -1 on the other: the numerator comes only from the neck, so making the neck thinner leaves the diameter nearly unchanged while the Rayleigh quotient becomes as small as we like. At the base of the neck the curvature becomes strongly negative, and $\mathrm{Ric} \geq 0$ fails. This is a well-known example Known .

04

Kaluza and Klein — charge is momentum around the circle

The idea of extra dimensions goes back to Kaluza and Klein in the 1920s (the introduction of the PDG). Five-dimensional spacetime also has a metric in the sense of § 03 (in spacetime, what corresponds to the inner product at each point has, like the $\eta = \mathrm{diag}(-1,1,1,1)$ of 2-04, exactly one negative sign, and it may vary from point to point). Its 15 components split into 10 for the four-dimensional metric $g_{\mu\nu}$, 4 for a vector A_μ and 1 for a scalar ϕ. Rewriting formulas 5 and 8 of the review by Overduin–Wesson with signature $(-,+,+,+,+)$ gives[Note 1]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \phi^2\left(dy + \kappa A_\mu dx^\mu\right)^2, \qquad \kappa = \sqrt{16\pi G} \quad (7)$$

The **Einstein equations** are the equations of gravity that tie the bending (curvature) of the metric to the energy of matter. Here we do not go into their content and use only the counting of components. If no component depends on y, the five-dimensional Einstein equations split into 10+4+1=15 four-dimensional equations (ibid. formula 6). Setting ϕ constant, among the 15 the equations for the metric and vector components become the four-dimensional Einstein equations and Maxwell's equations (ibid.

formula 7) Known . But the remaining scalar equation is compatible with $\phi =$ constant only when $F_{\mu\nu}F^{\mu\nu} = 0$ (pointed out by Jordan and Thiry, ibid. § 3.4). The change of coordinates $y \mapsto y + \kappa\lambda(x)$ induces $A_\mu \mapsto A_\mu - \partial_\mu\lambda$, and the gauge transformation (2-03) appears as a re-choice of the coordinate in the direction of the circle.

Klein (1926) connected Kaluza's theory with wave mechanics (abstract) and made the circle small. The momentum n/R of the n-th component in (2) becomes, under (7), a charge (ibid. formula 25, $\phi = 1$).

$$q_n = \frac{n\sqrt{16\pi G}}{R}, \qquad n \in \mathbb{Z} \quad (8)$$

Charge is restricted to integer multiples of q_1 (the title of Klein's short communication of the same year is 'The Atomicity of Electricity as a Quantum Theory Law'). As a metaphor, charge is momentum around an invisible circle; more precisely, it is the momentum n/R times a constant κ fixed by convention.

Setting $q_1 = e$ ($e^2/4\pi = \alpha \approx 1/137.036$) gives $R = 2\ell_P/\sqrt{\alpha} \approx 3.8 \times 10^{-34}$ m (about 23 times the Planck length), and the first rung of the tower is $1/R \approx 5.2 \times 10^{17}$ GeV, about 10^{21} times the electron mass Computed . Overduin–Wesson, taking the radius as the Planck length, estimate the discrepancy at about 22 orders of magnitude, and write that it 'played a large role in the abandonment of five-dimensional Kaluza-Klein theory'. If the internal space has a non-abelian symmetry, non-abelian gauge fields can come out (17-07) Physics .

05

The six dimensions of strings — curling up into Calabi–Yau

Spacetime in superstring theory is ten-dimensional (17-05 § 06), so to make it look four-dimensional a six-dimensional internal space K is needed. Candelas–Horowitz–Strominger–Witten (1985) studied vacua leaving one supersymmetry in four dimensions, and wrote in the abstract 'This condition permits only a few possibilities'. The K satisfying the condition is said to be a Kähler manifold with Ricci curvature 0 (a complex manifold whose metric and complex structure mesh) whose holonomy (1-07) lies in SU(3): a **Calabi–Yau manifold** Physics .

That such metrics exist is a mathematical theorem. Yau announced in 1977 'We announce a proof of Calabi's conjectures on the Ricci curvature of a compact Kähler manifold' (the paper is from 1978). As a consequence, a compact Kähler manifold with vanishing first Chern class has a Ricci-flat Kähler metric Known . If it is Ricci-flat, (6) applies, so the scalar KK tower on a Calabi–Yau of diameter d starts above π/d.

On the size, the introduction of the PDG writes 'The extra dimensions were supposed to be compactified at a scale close to the Planck scale, and thus not testable experimentally in the near future'. The shape affects the particle content (the model of CHSW has 'an E6 gauge group with four standard generations of fermions').

06

Confining to a brane — ADD and RS

In 1998, Arkani-Hamed–Dimopoulos–Dvali (ADD) proposed a way of hiding in which the fields of the Standard Model are confined to a three-dimensional membrane (a 3-brane) and only gravity spreads into $4+\delta$ dimensions. Since they can be probed only by gravity, the extra dimensions may be large. The abstract reads 'The observed weakness of gravity on distances $\gtrsim 1$ mm is due to the existence of n ≥ 2 new compact spatial dimensions large compared to the weak scale' Physics . The weakness of gravity is the same dilution as the λ_4 of § 02, and the fundamental scale M_D and the reduced Planck mass $M_P \approx 2.4 \times 10^{18}$ GeV are related by (PDG formula 84.10)

$$M_P^2 = R^{\delta} M_D^{2+\delta} \quad (9)$$

Table 3 R fixed by (9) (reproducing the "10^9 km, 0.5 mm, ..., 0.1 MeV^{-1}" of PDG formula 84.11) Computed

δ	$M_D = 1$ TeV	$M_D = 4$ TeV
1	1.2×10^{12} m	1.8×10^{10} m
2	4.8×10^{-4} m	3.0×10^{-5} m
6	2.6×10^{-14} m	4.2×10^{-15} m

$\delta = 1$ is the size of the solar system, 'clearly ruled out' (PDG), and $\delta = 2$ gives 0.5 mm. The next year, 1999, Randall–Sundrum (RS) used a slice of five-dimensional anti-de Sitter space (PDG formula 84.16).

$$ds^2 = e^{-2ky}\eta_{\mu\nu}dx^{\mu}dx^{\nu} + dy^2, \qquad 0 \leq y \leq \pi R \quad (10)$$

Anti-de Sitter space is a spacetime of constant negative curvature (17-08). The metric (10) is the slice $0 \leq y \leq \pi R$ of that spacetime with curvature radius $1/k$. Scales at the end $y = \pi R$ are redshifted by a factor $e^{-k\pi R}$. The PDG writes that $k \sim M_P$ and $R \simeq 11/k$ give the TeV; at $kR = 11$, $ke^{-k\pi R} \approx 2.4$ TeV, and changing kR from 10 to 12 moves it from 55 TeV to 0.1 TeV Computed . Making a large ratio with an exponential resembles dimensional transmutation (17-01). More precisely, in RS the exponent comes from geometry (kR), in dimensional transmutation from the running of the coupling ($1/g^2$). The view that connects warped extra dimensions with strongly coupled four-dimensional theories is AdS/CFT (17-08), at the level of a conjecture Physics .

Detour: extra dimensions without a gap

A subsection you may skip.

The paper of RS from the same year argued that four-dimensional gravity can be reproduced even if the extra dimension is infinitely long: 'even without a gap in the Kaluza-Klein spectrum, four-dimensional Newtonian and general relativistic gravity is reproduced to more than adequate precision' Physics . A gap in the tower is not essential for invisibility.

07

Experimental lower bounds — Newton's law and colliders

Deviations from Newton's law are by custom expressed in Yukawa form (PDG formula 84.12).

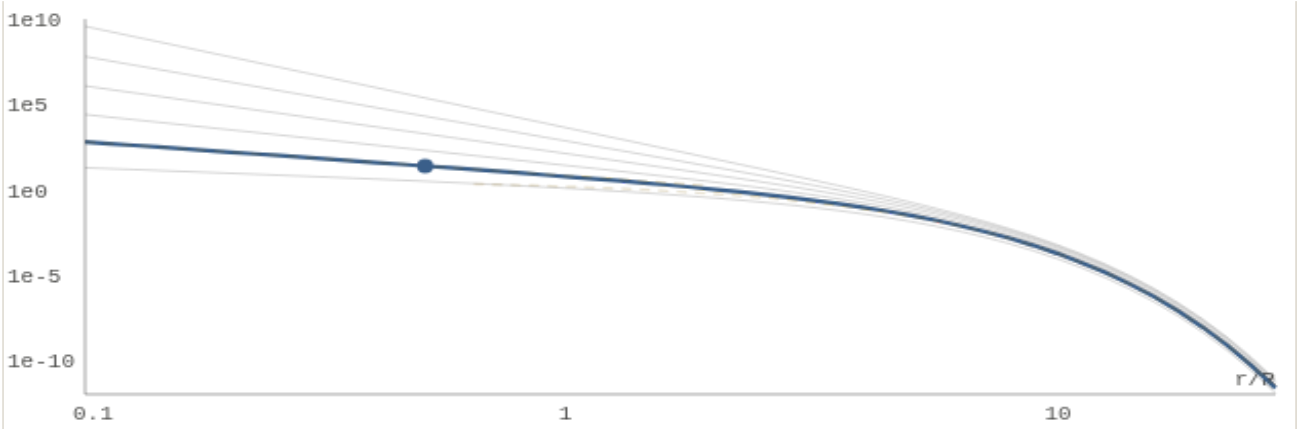
$$V(r) = -G_N \frac{m_1 m_2}{r}\left(1 + \alpha\, e^{-r/\lambda}\right) \quad (11)$$

The exchange of a field of mass m gives a Yukawa potential e^{-mr}/r, and one term is added for each KK mode. In a scalar (Newtonian) model, summing the whole tower of T^δ (radius R) gives

$$\frac{V(r)}{V_{\mathrm{N}}(r)} = \sum_{k \in \mathbb{Z}^\delta} e^{-|k|\, r/R} \xrightarrow{\;r \gg R\;} 1 + 2\delta\, e^{-r/R} \quad (12)$$

2δ is the number of fields on the first rung; for $\delta = 1$ the sum is $\coth(r/2R)$. For $r \ll R$, $V \propto r^{-1-\delta}$: the law of a space of $3+\delta$ dimensions. For the graviton the coefficient changes. The KK modes of the graviton are massive spin-2 fields, and the contribution of their polarisations is said to multiply the coefficient of the first rung by 4/3; the PDG takes, for a two-dimensional torus, $\alpha = 16/3$ (4/3 of the 4 in (12)) and $\lambda = R$ Physics . (12) is not the formula for gravity itself but a model for seeing how summing the tower produces a Yukawa-type deviation.

Figure 2 $V/V_{\mathrm{N}} - 1$ of (12) (log scale). Varying δ and r/R changes the deviation and the dimension of space that can be read off. The dashed lines are two approximations




```
δ = 2  r/R = 0.501  V/V_N = 25.13, readable dimension of space ≈ 5.0 (3 far away, 5)
```

Table 4 The ratio V/V_{N} of (12)

δ	$r/R = 0.1$	**1**	**10**
1	20.02	2.164	1.000091
2	628.3	6.507	1.000184
6	3.72×10^{9}	3721	1.000593

Computed The numbers of Figure 2 and Table 4 were computed on this machine (sums over $|k| \le 420$; $\delta = 1$ agrees with $\coth$ to 12 digits, and $r \to 0$ with the integral approximation to 5 digits). If JavaScript is disabled, see Table 4.

That the deviation vanishes as $e^{-r/R}$ for $r \gg R$ is the quantitative content of "invisible".

Table 5 Lower bounds from PDG 2025 (lengths converted with (1) Computed)

What was searched for	Lower bound
Inverse-square law (torsion balance)	For $\delta = 2$, $R < 30$ μm, $M_D > 4.0$ TeV
Supernova SN1987A / neutron stars	For $\delta = 2$, $M_D > 27$ / 1700 TeV
Graviton emission at the LHC (ATLAS / CMS)	$M_D > 5.9$–11.2 / 5.5–10.7 TeV ($\delta = 2$–6)
The lightest KK graviton of RS	ATLAS 4.5 TeV ($k/M_P = 0.1$), CMS 2.2–5.6 TeV
Gauge fields propagating in flat extra dimensions	$1/R > 4.16$ TeV (a reinterpretation by ATLAS; 4.7×10^{-20} m), $\gtrsim 6$ TeV from LEP2
All fields propagate (UED)	$1/R > 1.4$–1.5 TeV. The minimal model is excluded together with dark matter

One of the inverse-square-law experiments (Tan et al. 2016) confirms $|\alpha| \le 1$ down to $\lambda = 59$ μm (abstract). The observed gauge fields are four-dimensional down to about 5×10^{-20} m, and gravity follows the inverse-square law down to some tens of μm. Extra dimensions have not been found, but neither has it been shown that there are none.

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08
```

Back to Yang–Mills — the KK gap is not the mass gap

The KK gap is not the mass gap. The lowest rung of the tower is a gap that comes only from the internal space being compact; free fields have it too, and the zero mode has no gap. It is the same computation as the gap of a free field in a box (2-02 § 06, 2-05 § 06). What the prize problem asks for is a gap arising from interaction on $\mathbb{R}^4$, which has no compact directions (4-11), and Jaffe–Witten write 'no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume'.

Do not mix the two kinds of extra dimensions. The lower bounds of § 07 concern real extra dimensions. The extra dimensions of holographic models (17-08, 17-09) (the radial direction of anti-de Sitter space, S^5, the auxiliary circle of Witten's model) are tools of a dual description or of regularisation, and are not claimed to be real dimensions.

Within the models the tower is visible. In models that read the gap from geometry, the KK tower of the extra dimensions comes at the same height as the glueballs (in the Maldacena–Núñez model of $\mathcal{N} = 1$ supersymmetric Yang–Mills, if the gravity approximation is valid, 'comparable to the masses of Kaluza Klein states'; the authors also write that quantising the string on this background might allow the KK states to be separated) Physics . Removing the tower requires shrinking the extra dimensions, and then the coupling becomes weak (in four dimensions, because of asymptotic freedom), which is the opposite side from the region where the computation works (Csáki et al., of Witten's three-dimensional model: 'This is the opposite of the limit that is required for the supergravity description to be valid'). The order of limits is treated in 17-10.

Seam

What can be said as a theorem about "invisible" is only about linear fluctuations on a fixed internal space ((4)–(6)). Within the range we searched we found no result justifying that the tower separates in an interacting theory, in particular in the limit that returns to four dimensions.

Note 1 Overduin–Wesson take the four-dimensional signature $(+,-,-,-)$ and set $\kappa \equiv 4\sqrt{\pi G}$ (formula 8). (7) is aligned with this book's signature $(-,+,+,+)$, and $\kappa = \sqrt{16\pi G}$ is the same value. Back to text

What this chapter can say

What is said	What is not said
Known (2)–(8), Yau's theorem.	General lower bounds for towers in warped backgrounds or of forms and spinors. The consistency of keeping ϕ constant (the scalar equation requires $F_{\mu\nu}F^{\mu\nu} = 0$).
Computed Tables 1–5, Figures 1 and 2, Klein's radius, reproducing the numbers of the PDG with (9).	(12) is a scalar model. The 16/3 for the graviton is quoted from the PDG; the origin of the factor 4/3 is not derived in this chapter.
Physics The Calabi–Yau condition, ADD and RS, non-abelian gauge fields, the height of the towers in the models.	Whether real extra dimensions exist. That the towers of the models separate in the limit returning to four dimensions. This book does not itself show either the mass gap or the construction of the continuum limit (the published construction of the continuum limit: Volume 2).

Sources and reproduction

Item	Kind	Source / tools
(2)(3), (9)–(11), 16/3, Table 5, quotations	Known, Physics	Z. Demiragli, A. Pomarol, "84. Extra Dimensions" (PDG, PRD 110 (2024) 030001, revised August 2025). Text checked.
(7)(8), Kaluza, Klein	Known	J. M. Overduin, P. S. Wesson, Phys. Rep. 283 (1997) 303, gr-qc/9805018 § 3, § 4 (text checked; formulas 5–8 and 25, the condition for constant ϕ in § 3.4). Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. (1921) 966 (bibliographic data only). O. Klein, Z. Phys. 37 (1926) 895 (abstract checked); Nature 118 (1926) 516 (bibliographic data only).
(6)	Known	J. Q. Zhong, H. C. Yang, Sci. Sinica Ser. A 27 (1984) 1265 (bibliographic data only; the claim follows the survey by Ling–Lu, whose bibliographic data is not checked). F. Hang, X. Wang, IMRN 2007, rnm064, doi:10.1093/imrn/rnm064 (bibliographic data only).
The Calabi–Yau condition	Physics	P. Candelas, G. T. Horowitz, A. Strominger, E. Witten, NPB 258 (1985) 46. Abstract checked (the term Calabi–Yau and SU(3) holonomy not checked in the text).
Yau's theorem	Known	S.-T. Yau, PNAS 74 (1977) 1798 (abstract checked); CPAM 31 (1978) 339 (bibliographic data only).
ADD, RS, the inverse-square law	Physics	N. Arkani-Hamed, S. Dimopoulos, G. Dvali, PLB 429 (1998) 263. L. Randall, R. Sundrum, PRL 83 (1999) 3370; ibid. 4690. W.-H. Tan et al., PRL 116 (2016) 131101. Abstracts checked.
Quotations in § 08	Known, Physics	A. Jaffe, E. Witten, "Quantum Yang–Mills Theory" (the Clay problem description; the PDF published by Clay; text checked). J. Maldacena, C. Núñez, PRL 86 (2001) 588, arXiv:hep-th/0008001 (abstract checked). C. Csáki, H. Ooguri, Y. Oz, J. Terning, JHEP 9901 (1999) 017, arXiv:hep-th/9806021 (text checked).

Item	Kind	Source / tools
Tables, figures, numbers	Computed	Python 3 (numpy) on this machine: kk_count.py, zhong_yang.py, units_klein.py, add_rs.py, newton_kk.py (the values embedded in Figure 2 are the same computation output to 4 decimal places).

Next chapter to read: 17-07 Where gauge fields come from — extracting gauge groups from branes and geometry.

Where gauge fields come from — the group is read off from the number of branes, a lattice, the shape of a singularity

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — Four mechanisms by which, on the side of extra dimensions, the gauge group becomes an output (symmetries of the internal space, stacked branes, an internal lattice of the string, collapsing spheres). That the positions of branes become eigenvalues of matrices, and that ADE comes out of the linear algebra of "negative definite". The road down to pure Yang–Mills, and the expectations it relies on.

Prerequisite chapters — 17-06 How extra dimensions become invisible. We use roots (1-04), holonomy (1-07) and the central charge (17-05). Intersection numbers and lattices are defined on the spot.

Said first — All the mechanisms are at the level of physics, and neither string theory nor M-theory has a mathematical definition. That a group comes out of geometry and that the four-dimensional quantum Yang–Mills theory exists and has a mass gap are different questions. This book does not itself show either the mass gap or the construction of the continuum limit (the published construction of the continuum limit is read in Volume 2).

The order of this chapter

1. Is the group an input or an output? — four sources
2. Symmetries of the internal space — non-abelian Kaluza–Klein
3. Strings on branes — N branes give U(N)
4. Positions become matrices — eigenvalues are brane positions
5. The heterotic string — a 16-dimensional lattice (Detour: the same tower, different groups)
6. Compactifying on Calabi–Yau — the commuting part

7. Collapsing spheres and ADE — negative definiteness
8. Folding and geometric engineering
9. From eleven dimensions to pure Yang–Mills — two circles and an expectation
10. Back to Yang–Mills — the group is not where the difficulty lies

01

Is the group an input or an output? — four sources

In four-dimensional Yang–Mills theory the gauge group G is an input chosen by hand. The text of the prize problem (Jaffe–Witten) too says 'for any compact simple gauge group G', allowing any of the list in 1-04 (0-03). In theories with extra dimensions the group becomes an output, read off from something else Physics .

Symmetry	Number of branes	Lattice	Singularity
Isometries of the internal space § 02	N D-branes give U(N) § 03–04	The 16 left-over dimensions of the heterotic string § 05–06	The arrangement of collapsing spheres gives ADE § 07–08

What any of these roads fixes is the set of fields and the group seen at low energy, not the existence of a four-dimensional quantum theory. Of the four, three (branes, lattice, singularity) are stories within supersymmetric settings (theories with a symmetry exchanging bosons and fermions), and what that means for pure Yang–Mills we look back on in § 10.

02

Symmetries of the internal space — non-abelian Kaluza–Klein

In 17-06 § 04 the rotation of the circle became U(1). If the group $\mathrm{Isom}(K)$ of transformations of the internal space K that do not change distances (isometries) is large, each of its infinitesimal generators (Killing vector fields) gives one gauge field, and the commutators of the generators become the structure constants (1-03) Physics . For the generators of rotations of a sphere, $L_a = \varepsilon_{abc} x_b \partial_c$,

$$[L_a, L_b] = -\varepsilon_{abc}\, L_c, \qquad \dim SO(n+1) = \frac{n(n+1)}{2} \quad (1)$$

(the left formula was checked numerically Computed); S^n gives $n(n+1)/2$ gauge fields, and $\mathbb{CP}^2$ (the set of all complex lines through the origin of $\mathbb{C}^3$; real dimension 4) gives the 8 gauge fields of SU(3) Known . An example of a seven-dimensional space whose isometry group is locally (as a Lie algebra) SU(3)×SU(2)×U(1) (3-07) is $\mathbb{CP}^2 \times S^2 \times S^1$ ($4+2+1=7$ dimensions), and together with four dimensions the whole is $4+7=11$-dimensional.

Witten (1981) studied the road of compactifying eleven-dimensional supergravity on seven-dimensional spaces, and wrote in the abstract 'It is possible to obtain an SU(3) × SU(2) × U(1) gauge group, but the proper fermion quantum numbers are difficult to achieve'. The seven-dimensional spaces Witten studied are not necessarily the same as the example above.

03

Strings on branes — stacking N gives U(N)

Impose on open strings a boundary condition attaching their ends to a subspace, and that subspace is a **D-brane** (Polchinski 1995: 'extended objects defined by mixed Dirichlet-Neumann boundary conditions in string theory') Physics . One with p spatial dimensions is called a Dp-brane. Labelling the two ends of a string with brane numbers $1,\dots,N$, the states have $N\times N$ matrices as coefficients, and the massless vectors form the adjoint representation of U(N) (1-04) (Polchinski's lectures of 1996: 'this global symmetry of the world-sheet theory is elevated to a gauge symmetry in spacetime'). There are N^2 kinds of strings, and $\dim U(N) = N^2$.

Separating the branes makes the strings between them heavy (Witten 1995b § 3; T is the string tension, a_i the positions of the branes).

$$m_{ij} = T\,|a_i - a_j| \quad (2)$$

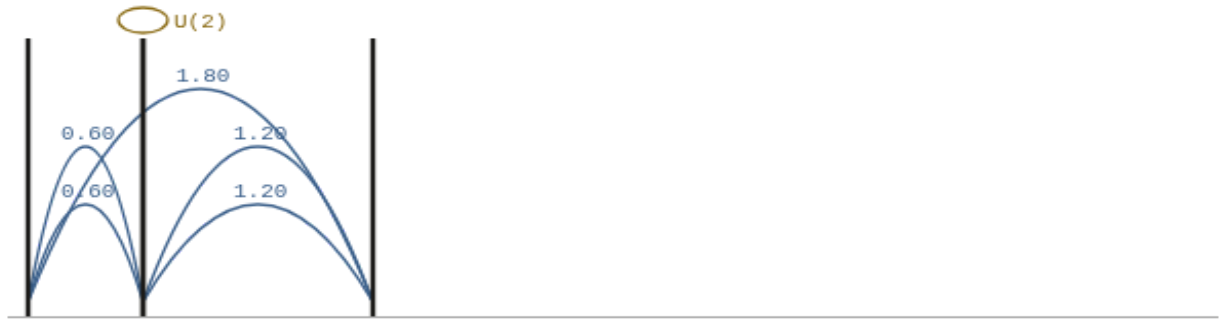
If they are stacked $n_1, n_2, \dots$ at a time, the remaining group is $U(n_1)\times U(n_2)\times\cdots$, and

$$\#\{\text{massless vectors}\} = \sum_k n_k^2, \qquad \#\{\text{heavy W}\} = N^2 - \sum_k n_k^2 \quad (3)$$

(the same count appears in Polchinski's lectures). Separating branes is the Higgs mechanism seen from the side of geometry. More precisely, when the adjoint scalars X^i of § 04 take diagonal values $\mathrm{diag}(a_1,\dots,a_N)$, the off-diagonal gauge fields acquire the masses (2). This mechanism, in which the values of scalars make gauge fields heavy, is called the Higgs mechanism (3-07) Physics .

Figure 1 Varying the spacings d_1, d_2, d_3 of four branes (vertical lines) changes the remaining group and the W masses. Arcs are heavy strings (numbers are mass/T); the loops at the top are strings between stacked branes. A

heavy string has two orientations ($i \to j$ and $j \to i$), so each arc, each mass, counts as 2

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Group U(2)×U(1)², massless 6, heavy 10, W masses/T (2 each): 0.60, 0.60, 1.20, 1.20, 1.80
```

Table 1 How four branes are stacked, and the numbers in (3) (a heavy string has two orientations, so 2 for each W mass)

Stacking	4	3+1	2+2	2+1+1	1+1+1+1
Remaining group	U(4)	U(3)×U(1)	U(2)×U(2)	U(2)×U(1)²	U(1)⁴
Massless / heavy	16 / 0	10 / 6	8 / 8	6 / 10	4 / 12

Computed Table 1 was counted on this machine, and Figure 1 is drawn on the spot from (2)(3). If JavaScript is disabled, see Table 1.

The W masses of four separated branes are d_1, d_2, d_3 and their consecutive sums, 6 in all: the relation between the 3 simple roots and the 6 positive roots of SU(4) (A_3). In § 07 the same structure appears as the areas of spheres.

04

Positions become matrices — the theory on the branes

The theory on stacked branes is 'the dimensional reduction of ten-dimensional supersymmetric Yang-Mills theory with gauge group U(n) from ten to p+1 dimensions' (Witten 1995b § 3) Physics . Splitting the components of the ten-dimensional gauge field into directions along the branes and perpendicular to them gives

$$A_M = (A_0, \dots, A_p;\, X^{p+1}, \dots, X^9), \qquad V = -\frac{T^2}{2} \sum_{i,j} \operatorname{tr}\left([X^i, X^j]^2\right) \;\geq 0 \quad (4)$$

where X^i are the $9-p$ scalars in the adjoint representation ($N \times N$ Hermitian matrices), and V is their potential. Since X is Hermitian, $[X^i, X^j]$ is anti-Hermitian and $\operatorname{tr}[X^i, X^j]^2 \leq 0$. Therefore $V \geq 0$. If $V = 0$, the matrices can be diagonalised simultaneously, and Witten reads the diagonal entries as the positions of the branes. The off-diagonal entries are the strings between branes, and the eigenvalues of the map $\delta \mapsto \sum_k [X^k, [X^k, \delta]]$ are $|a_\lambda - a_\mu|^2$ (times T^2, the square of (2)). We checked agreement within 4×10^{-15} on an example of two directions conjugated by a unitary matrix Computed .

On D3-branes the theory is four-dimensional $\mathcal{N} = 4$ U(N) supersymmetric Yang–Mills with 6 scalars and fermions (the starting point of 17-08), not pure Yang–Mills. The matrix indices have the same structure as the double lines of 17-04.

05

The heterotic string — a 16-dimensional lattice becomes a group

The heterotic string is a closed string whose right-movers follow the ledger of the superstring (central charge 15) and whose left-movers follow that of the bosonic string (26) (17-05 § 06, Gross–Harvey–Martinec–Rohm 1985). The left-movers are 16 short of the 10 of flat ten dimensions; Polchinski's lectures (1994), on the supersymmetric heterotic string, write 'an additional (0, 16) is needed from another unitary CFT', and go on to say that the only solutions are $E_8 \times E_8$ or SO(32) Physics .

A 16-dimensional lattice

In the lattice picture this 16 is seen as a torus $\mathbb{R}^{16}/\Gamma$ on which 16 left-moving bosons are wound Physics . The momentum P lies on the lattice Γ, and the states with $P \cdot P = 2$ become massless vectors. On the heterotic string wound on a torus, Witten (1995 § 4.6) writes 'The gauge symmetry is extended precisely when $V^\perp$ contains some P of $d_P = 1$'. Witten's $V^\perp$ is the space of left-moving charges. Witten takes the sign of the inner product opposite to this chapter and writes $d_P = -(P,P)/2$, so $d_P = 1$ corresponds to $P \cdot P = 2$ in this chapter's sign. Γ must be **even** (every $P \cdot P$ even) and **self-dual** (the points having integer inner product with all points are exactly Γ). This condition is said to come from the requirement that the one-loop amplitude, whose world sheet is a torus, not depend on how the shape of the torus is chosen (modular invariance), and Polchinski's lectures also write 'The constraints of modular invariance are strong'.

$$\dim G = 16 + \#\{P \in \Gamma : \ P \cdot P = 2\} \quad (5)$$

There are only two even self-dual lattices in 16 dimensions Known . Both can be written as "$x \in \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n$ with $\sum x_i$ even": two copies of $n = 8$ side by side give $E_8 \oplus E_8$, and $n = 16$ gives D_{16}^+. Both have 480 roots, and (5) is 496, but joining roots with non-zero inner product, the former splits into two sets of 240 ($E_8 \times E_8$) and the latter is connected ($\mathfrak{so}(32)$) Computed . The ten-dimensional group is written precisely as $\mathrm{Spin}(32)/\mathbb{Z}_2$. $\mathrm{Spin}(32)$ is the double cover of SO(32), the same relation as SU(2)→SO(3) in 1-02 (the $\mathbb{Z}_2$ divided out is a different element from the one that returns to SO(32)). The lattice spanned by the roots of D_{16}^+ is a sublattice of index 2 in D_{16}^+, and this $/\mathbb{Z}_2$ is said to correspond to that difference.

Seen from eleven dimensions, Hořava–Witten **proposed** that the two E_8 sit one on each of the walls at the two ends of the segment $S^1/\mathbb{Z}_2$ (the 1995 abstract says 'We propose'; the 1996 abstract writes that the group is fixed to 'a copy of E8 for each boundary component', explicitly presupposing 'the hypothesis that there is a consistent quantum M-theory') Physics .

Detour: the same tower, different groups

A subsection you may skip.

The eigenvalues of the Laplacian on a flat torus $\mathbb{R}^n/\Gamma$ are $4\pi^2\, v\cdot v$ for each point v of the dual lattice (the n-dimensional version of 17-06 § 02) Known ; if the lattice is self-dual, the number on each rung of the KK tower is the number of points of squared length $2k$. Counting, $E_8 \oplus E_8$ and D_{16}^{+} both give 480, 61920 and 1050240 for $k = 1, 2, 3$, agree on every rung with $k \le 6$, and also agree with $480\,\sigma_7(k)$ (σ_7 is the sum of the 7th powers of the divisors) Computed .

It is known that they agree on every rung, and Milnor (1964) is said to have given these two tori as an example of being non-isometric although their eigenvalues agree including multiplicities Known . The tower is the same, but the groups carried by the heterotic string differ.

06

Compactifying on Calabi–Yau — holonomy and the commuting part

To bring $E_8 \times E_8$ down to four dimensions, compactify on a six-dimensional Calabi–Yau manifold K (17-06 § 05). The holonomy of K (1-07) lies in SU(3), and in the standard construction the holonomy of the gauge field is said to be aligned with an SU(3) inside E_8 too Physics . A gauge transformation visible in four dimensions is a g constant over all of K that is unchanged by parallel transport along every closed path, that is, it must commute with every element of the holonomy Known .

$$G_4 = \{\, g \in E_8 :\ gh = hg \ \text{ for all } h \in SU(3) \,\} \quad (6)$$

Of the 240 roots of E_8, 72 are orthogonal to the roots of SU(3), giving E_6 (dimension 78) Computed . The abstract of Candelas–Horowitz–Strominger–Witten (1985) writes 'a model that in four dimensions has an E6 gauge group with four standard generations of fermions'.

Table 3 The part of E_8 that commutes with SU(n) (248 = remaining group + SU(n) + remainder)

Holonomy	SU(2) (K3)	SU(3) (Calabi–Yau)	SU(4)	SU(5)
Remaining group	E_7	E_6	SO(10)	SU(5)
Breakdown	133+3+112	78+8+162	45+15+188	24+24+200

Computed Table 3 was counted by building the 240 roots of E_8 in coordinates; the types were fixed by rank, number of roots and the determinant of the Cartan matrix.

The remaining 162 are $\mathbf{2}\times\mathbf{27}\times\mathbf{3}$ (pairs of the 27-dimensional representation of E_6 and the 3-dimensional one of SU(3), and their conjugates), and the four-dimensional fermions are said to come from here.

07

Collapsing spheres and ADE — the linear algebra of negative definiteness

A **K3 surface** is a Calabi–Yau manifold of complex dimension two (real dimension four), with holonomy SU(2) (the left end of Table 3) Known . Witten (1995 § 4.6), on the correspondence between the heterotic string wound on a torus and the IIA string wound on a K3 surface, writes 'Whenever the heterotic string on a four-torus gets an enhanced gauge group G, the corresponding K3 gets an orbifold singularity of type G' Physics . Then 'a configuration of r two-spheres S_i (with an intersection matrix given by the Dynkin diagram of G) collapses to a point' (Dynkin diagrams are defined later in this section), and each sphere gives one U(1). The W bosons are two-dimensional membranes wrapped on the collapsing spheres (as described in Bershadsky–Sadov–Vafa 1996 § 1). Their mass is said to be the tension of the membrane times the area of the sphere, the same form as the $T|a|$ of § 03.

Intersection numbers and negative definiteness

Why only ADE? Look at the **intersection numbers** $M_{ij}=S_i\cdot S_j$ of the spheres. For two spheres this is the sum over intersection points counted with orientation signs ±1. The intersection number of a sphere with itself (the self-intersection number) counts in the same way the intersection points with a slightly shifted copy, and for spheres in K3 it is -2 Known . Defining the "inner product" of classes of spheres by this intersection number, on K3 the inner product is positive in three directions and negative in nineteen (signature $(3,19)$) Known . According to Witten, in the collapsing limit the classes of the spheres lie in the subspace $V^{\perp}$ on which this inner product is negative definite, so M is negative definite ($x^{\mathsf T}Mx<0$ for every $x\neq0$). The collapsing spheres S_i can be taken to be **holomorphic curves** (curves of complex dimension one, locally the zero sets of holomorphic functions of complex coordinates), and intersection points of distinct holomorphic curves all count as positive because of the complex structure, so $M_{ij}\geq0$ Known . With $x=e_i+e_j$, $-4+2M_{ij}<0$ gives $M_{ij}\leq1$. Hence $M=-(2I-A)$, where A is the adjacency matrix of the graph whose vertices are the spheres and whose edges are the intersections. This graph is called the **Dynkin diagram** (those that appear here are of "simply laced" type, all edges single). We need the following.

$$x^{\mathsf{T}}(2I - A)\,x = 2\sum_i x_i^2 - 2\sum_{\text{edges } ij} x_i x_j \;>\; 0 \qquad (x \;/= 0) \quad (7)$$

If the graph is not connected we consider each component (the group is a product over components). If a connected graph has a cycle, the vector equal to 1 on the cycle makes the left-hand side at most 0, so it must be a tree. Examining the 201 trees with at most 10 vertices, only types A (a straight line), D (forked at one end) and E_6, E_7, E_8 remained (Table 4) Computed . That these are the only ones for any number of vertices is known as part of the classification of simple Lie algebras (1-04) Known .

Figure 2 Choosing a type changes the arrangement of collapsing spheres (circles are spheres, lines are intersections), the smallest eigenvalue of (7) and the dimension of the group. "Fold" shows by colour the orbits of the permutations of § 08



E₈: 8 spheres, 240 roots, group dimension 248, smallest eigenvalue 0.010956 (positive definite)

Table 4 The number of trees with n vertices, and the types satisfying (7) (for $n \le 3$ only the straight line)

n	4	5	6	7	8	9	10
Number of trees	2	3	6	11	23	47	106
Types satisfying it	A D	A D	A D E	A D E	A D E	A D	A D

Computed The numbers of Table 4 and Figure 2 were computed on this machine. The smallest eigenvalue agrees to 6 digits with $2 - 2\cos(\pi/h)$ (h is the number of roots ÷ the rank), and the figure is drawn with this formula. One beyond E_8 there is an integer vector making (7) 0 (centre 6, arms 3 / 4, 2 / 5, 4, 3, 2, 1). If JavaScript is disabled, see Tables 4 and 5.

08

Folding and geometric engineering — A to G are all there

ADE are only the types whose roots have equal lengths. B, C, F and G come out when the singularities are further bundled in another direction. Bershadsky et al. (1996 § 3) write that going once around over another parameter can permute the collapsing spheres by an automorphism of the Dynkin diagram (a permutation of vertices preserving edges), and that then 'the actual gauge group will be smaller', listing the following (the last is the cyclic permutation of three legs) Physics .

$$A_{2n-1} \to C_n, \qquad D_n \to B_{n-1}, \qquad E_6 \to F_4, \qquad D_4 \to G_2 \quad (8)$$

With this every type from A to G comes out of geometry. The number of orbits into which the permutation bundles the positive roots agrees with the number of positive

roots of the folded algebra (Table 5) Computed .

Table 5 Folding: number of orbits of positive roots = number of positive roots of the algebra after

Before → after	$A_5 \to C_3$	$D_5 \to B_4$	$E_6 \to F_4$	$D_4 \to G_2$
Dimension	35 → 21	45 → 36	78 → 52	28 → 14
Positive roots (fixed) → orbits	15 (3) → 9	20 (12) → 16	36 (12) → 24	12 (3) → 6

Computed Table 5 was counted by generating roots from the Cartan matrices ($A_3 \to C_2$, $A_7 \to C_4$ and $D_6 \to B_5$ also agree).

But the same paper notes that 'a given A-D-E singularity may *not* correspond to an enhanced gauge symmetry'. Computing field theories by this mechanism is **geometric engineering**; Katz–Klemm–Vafa (1997) reduced the mathematically established results of four-dimensional $\mathcal{N} = 2$ (17-11) to the T-duality of strings (strings wound on a circle of radius R and on a circle of radius α'/R give the same theory; α' is from (1) of 17-05), and write 'This construction is not based on any duality conjecture' Physics . What it can handle is supersymmetric theories with at least 8 supercharges (17-05 § 07) (corresponding to four-dimensional $\mathcal{N} = 2$ or more); within the range we searched we found no example handling the mass gap of non-supersymmetric pure Yang–Mills by geometric engineering.

09

From eleven dimensions to pure Yang–Mills — two circles

Witten (1998 § 4) describes a road from eleven dimensions down to four-dimensional **pure** Yang–Mills Physics . The starting point is the six-dimensional (2,0) theory on M5-branes (membranes of M-theory extending in five spatial dimensions) ((2,0) is the name of a type of six-dimensional supersymmetry; Witten calls it 'exotic'), which 'can apparently be interpreted' as M-theory on $AdS7 \times S^4$ ($AdS7$ is a seven-dimensional spacetime of constant negative curvature, treated in 17-08) (according to Maldacena's conjecture). No description by a Lagrangian is said to be known, and there is no mathematical definition either. This is wrapped on two circles.

$$(2,0) \xrightarrow{C_1(R_1)} \text{5-dim. } SU(N),\ g_5^2 = R_1 \xrightarrow{C_2(R_2)} \text{4-dim. } SU(N),\ g_4^2 = \frac{R_1}{R_2} \quad (9)$$

Making the fermions antiperiodic (changing sign once around) on C_2, the fermions acquire masses of order $1/R_2$, the scalars too 'very plausibly' become heavy at one loop, and 'If this is so, the low energy theory will be the pure SU(N) theory without supersymmetry'. In terms of the IIA string this is N D4-branes wrapped on $\mathbb{R}^4 \times S^1$, and the theory on the branes of § 03–04 comes down to four dimensions.

Witten writes that he would like to keep $\eta = g_4^2 N$ small, so that the coupling is weak at the scale of the circles and strong at much longer distances. A one-loop rough guide:

putting $b_0 = 11N/(48\pi^2)$ (formula (1) of 17-01), $\mu = 1/R_2$ and $g^2(\mu) = g_4^2 = \eta/N$ into (3) of 17-01 gives

$$\Lambda R_2 \;\sim\; \exp\!\Big(-\frac{24\pi^2}{11\,\eta}\Big) \quad (10)$$

which is 4×10^{-10} for $\eta = 1$ and 0.12 for $\eta = 10$ Computed . The coefficient changes with the convention for the coupling and the choice of $\mu = 1/R_2$, so this is a guide to the order of magnitude. But supergravity can compute only 'at least if η is large', and Witten writes of connecting to small η 'This looks like a tall order'. The order of limits is in 17-10, and how the gap appears as geometry in 17-09.

10

Back to Yang–Mills — the group is not where the difficulty lies

On every road the group comes with extra fields or supersymmetry (the $9-p$ scalars on the branes, the fields of mass $1/R_2$ from the two circles, and so on) Physics . And for the prize problem, the group is not where the difficulty lies. The problem statement allows any compact simple group; what is asked is that its quantum Yang–Mills theory exists on $\mathbb{R}^4$ satisfying the axioms (4-10) and has a mass gap (4-11). Geometry can produce all the names from A to G, but within supersymmetric settings, and the road to pure Yang–Mills (§ 09) relies on expectations and on the order of limits.

Seam

(1) Every construction stands on string theory or M-theory, neither of which has a non-perturbative mathematical definition. (2) Trying to define a four-dimensional theory from the side of extra dimensions makes the definition of the starting point ((2,0) theory and so on) the harder part. (3) Within the range we searched we found no result connecting the group coming out of geometry with the existence of a four-dimensional quantum theory with a gap.

What this chapter can say

What is said	What is not said
Known (1), the commutation of (6), that the diagrams satisfying (7) are limited to ADE, the signature and positivity of the intersection numbers of K3, the classification of 16-dimensional lattices and Milnor's example (the last two: bibliographic data only).	Proofs of the classification and of the properties of intersection numbers are not written.
Computed Tables 1–5, Figures 1 and 2, the eigenvalues around (4), the numbers of (10).	Lattice point counts up to $k \leq 6$. (10) is a one-loop rough guide, and its coefficient may change with convention.
Physics The U(N) of D-branes, the groups of the heterotic string, E_6, the proposal of Hořava–Witten, ADE and folding, geometric engineering, the pure SU(N) of (9).	None is a mathematical theorem. The existence of string theory, M-theory and the (2,0) theory. That the scalars become heavy at one loop. This book does not itself show either the mass gap or the construction of the continuum limit (the published construction of the continuum limit: Volume 2).

Sources and reproduction

Item	Kind	Source / tools (text = the text of the manuscript checked, abstract = the abstract che
§ 01, § 10	Known	A. Jaffe, E. Witten, "Quantum Yang–Mills Theory", the official problem descripti the Clay Mathematics Institute (text).
§ 02	Physics	E. Witten, NPB 186 (1981) 412 (abstract; the choice of seven-dimensional spaces checked).
§ 03, § 04	Physics	J. Polchinski, PRL 75 (1995) 4724, hep-th/9510017 (abstract); TASI Lectures on D-Branes, hep-th/9611050 (text). E. Witten, NPB 460 (1996) 335, hep-th/9510135 § 3 (text).
§ 05	Physics	D. J. Gross and 3 others, PRL 54 (1985) 502; NPB 256 (1985) 253 (abstracts). J. Polchinski, "What is String Theory?" (lectures of 1994), hep-th/9411028 § 2.6 (tex Hořava, E. Witten, NPB 460 (1996) 506; NPB 475 (1996) 94 (abstracts).
§ 05, § 07	Physics	E. Witten, NPB 443 (1995) 85, hep-th/9503124 § 4.6 (text). M. Bershadsky, V. Sadc Vafa, NPB 463 (1996) 420, hep-th/9511222 § 1 (text; the formula mass = tension × area for the membrane not checked).
Intersection numbers in § 07	Known	E. Witten, hep-th/9503124 § 4.6 (text; the squared length of the class of a sphere -2, the collapsing spheres are holomorphic). The general theory of the signatur the intersection form of K3 and the positivity of intersections of holomorphic cu W. Barth and 3 others, *Compact Complex Surfaces*, Springer (bibliographic data only).
Lattices, detour	Known	J. H. Conway, N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. J. Milnor, PN 51 (1964) 542. Both bibliographic data only.

Item	Kind	Source / tools (text = the text of the manuscript checked, abstract = the abstract che
§ 06	Physics	P. Candelas and 3 others, NPB 258 (1985) 46 (abstract; the construction aligning holonomies not checked).
§ 08	Physics	M. Bershadsky and 5 others, NPB 481 (1996) 215, hep-th/9605200 § 3 (text). S. Ka Klemm, C. Vafa, NPB 497 (1997) 173, hep-th/9609239 (abstract).
§ 09	Physics	E. Witten, ATMP 2 (1998) 505, hep-th/9803131 § 4 (text). (10) is the arithmetic of putting $g^2 = \eta/N$ into (1)(3) of 17-01 ($b_0 = 11N/(48\pi^2)$).
Tables, figures, numbers	Computed	Python 3 (numpy) on this machine: branes.py, ade.py, e8_commutant.py, lattice scales.py.

Next chapter to read: 17-08 Holography — the $\mathcal{N} = 4$ theory on branes, and the correspondence in which the radial direction becomes scale.

Holography — reading a four-dimensional field theory through the geometry of gravity in one more dimension

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — AdS/CFT is the conjecture that a four-dimensional field theory and a theory of gravity in five dimensions (times a sphere) are the same. That depth from the boundary measures scale, the dictionary linking mass and dimension, Wilson loops computed with a hanging string, and where the line falls for the correspondences that can be written exactly.

Prerequisite chapters — 17-05. We also use the 't Hooft coupling λ of 17-04 and the Nambu–Goto action of 17-03. For D-branes see 17-07. Tools not in the prerequisite chapters, such as the metric, geodesics and sources, are defined on the spot, one line each.

Said first — AdS/CFT is a conjecture, not a theorem. The representative example, $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, is conformal and has no mass gap; it is not pure Yang–Mills theory. There is no example in which holography has shown the mass gap as a mathematical theorem (models with a gap at the level of physics are in 17-09), and this book itself shows nothing about the mass gap or the construction of the continuum limit (the published construction of the continuum limit is read in Volume 2).

The order of this chapter

1. What is claimed (Detour: why "hologram")
2. The two faces of D3-branes
3. Anti-de Sitter space
4. Radial direction = scale — depth measures size (Figure 1)
5. The dictionary — mass and dimension
6. Hanging a string — Wilson loops (Figure 2)

7. Putting in a single scale — a model of the area law
8. Correspondences that can be written exactly — theorems and proposals
9. The distance to Yang–Mills

01

What is claimed — the boundary theory and gravity in the interior

The claim of holography is this. **A field theory without gravity in d dimensions describes, in other words, the same physics as a theory with gravity in $d+1$ dimensions.** The extra dimension is "depth" from the boundary; the former is called the boundary theory and the latter the interior (bulk) theory. The representative example is Maldacena (1997, ATMP 1998; the first year is when it appeared on arXiv, the second the year of the journal).

$$\mathcal{N}=4 \text{ SYM } \left(U(N),\, g_{\mathrm{YM}}\right) \quad \overset{\text{conjecture}}{\Longleftrightarrow} \quad \text{type IIB string theory on } AdS_5\times S^5 \ \left(N,\, g_s\right) \ (1)$$

On the left is four-dimensional $\mathcal{N}=4$ supersymmetric Yang–Mills theory (SYM below), on the right ten-dimensional string theory. **Type IIB** is one of the ten-dimensional superstring theories, differing from the type IIA of 17-05 in the combination of handedness (chirality) of its supersymmetries. The dimensions are 10 = 4 (boundary) + 1 (depth) + 5 (S^5) (17-05 § 09). The original paper writes 'We are led to the conjecture', and (1) is still a conjecture Physics . This chapter deals with the entrance; models in which the mass gap turns into geometry are left to 17-09.

Detour: why "hologram" — counting by area

A subsection you may skip.

The entropy of a black hole is proportional not to its volume but to the area A of its horizon (Bekenstein 1973, Hawking 1975) Known .

$$S/k_B = A/(4\ell_P^2), \qquad \ell_P^2 = G\hbar/c^3 \ (2)$$

For a solar mass the radius is 2953 m and $S/k_B = 1.05\times 10^{77}$, far less than the 2.6×10^{115} obtained by dividing the volume by the cube of the Planck length Computed . From here 't Hooft (1993) and Susskind (1994, J. Math. Phys. 1995) set up the idea of counting degrees of freedom by area, and Susskind wrote 'much like a holographic image' Physics . As a metaphor, it is a hologram recording a three-dimensional image on a flat plate. More precisely, (1) is a claim that two quantum theories are equivalent, and the "plate" is at the infinity of spacetime.

02

The two faces of D3-branes — where the conjecture came from

Open strings ending on N stacked D3-branes (membranes with three spatial dimensions on which the ends of open strings sit) become $U(N)$ gauge fields at low energy (17-07). Seen from gravity, the same branes curve spacetime. Maldacena read that in the low-energy limit the theory on the branes and the spacetime just near the branes describe the same thing Physics . The **metric** (line element) ds^2 is the formula giving the squared distance when the coordinates shift slightly, and the length of a curve is $\int ds$ (for flat space $ds^2 = dx_1^2 + dx_2^2 + \cdots$). The metric made by the branes is

$$ds^2 = f^{-1/2} dx_{\parallel}^2 + f^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \qquad f = 1 + \frac{4\pi g_s N \alpha'^2}{r^4} \quad (3)$$

Here $x_{\parallel}$ are the four coordinates along the branes (in $dx_{\parallel}^2$ only the time direction has the opposite sign; 2-04), r is the distance from the branes, $\alpha' = \ell_s^2$ is the square of the string length, and g_s is the string coupling. As $r \to 0$, $f \approx R^4/r^4$ ($R^4 = 4\pi g_s N \alpha'^2$), and setting $z = R^2/r$ gives

$$ds^2 = \frac{R^2}{z^2}\left(dx_{\parallel}^2 + dz^2\right) + R^2 d\Omega_5^2 \quad (4)$$

the product of AdS_5 of radius R and S^5 Known . There are two ways of writing the correspondence of couplings. In the convention widely used today $g_{\mathrm{YM}}^2 = 4\pi g_s$ and $R^4/\alpha'^2 = 4\pi g_s N = \lambda$ ($\lambda = g_{\mathrm{YM}}^2 N$); in the convention of the original paper, $g_{\mathrm{YM}}^2 = 2\pi g_s$, $R^4/\alpha'^2 = 2\lambda$ Known . The difference is only in the normalisation of the action; today's convention is the normalisation in which the instanton action is $8\pi^2/g_{\mathrm{YM}}^2$ (the review of Aharony–Gubser–Maldacena–Ooguri–Oz (3.8), § 4.2). (2) of 17-04 ($\frac{1}{2g^2}\int \mathrm{tr}\, F^2$ with matrix traces, $\lambda = g^2 N$) has the same normalisation, so with the λ of 17-04, $R^4/\alpha'^2 = \lambda$. In this chapter we do not use λ directly and write $R^2/\alpha' = \sqrt{4\pi g_s N}$.

Supergravity (the field theory with gravity that remains at low energy in string theory) is reliable when $R \gg \ell_s$, that is $g_s N \gg 1$, and quantum effects in the interior correspond to the $1/N$ corrections of the gauge theory (original paper) Physics . Making this compatible with $g_s \ll 1$ requires $N \gg g_s N \gg 1$.

Table 1 The string coupling $g_s = (R/\ell_s)^4/(4\pi N)$ needed for radius R

R/ℓ_s	$N = 3$	$N = 100$
2	0.424	0.0127
3	2.15	0.0645
5	16.6	0.497

R/ℓ_s	$N=3$	$N=100$
10	265	7.96

Example: taking the curvature radius to be 5 times the string length, $N=3$ gives $g_s=16.6$, a strong string coupling, and only at $N=100$ does it come down to 0.5 Computed .

03

Anti-de Sitter space — the boundary is infinitely far away

Anti-de Sitter space AdS_{d+1} is, in a flat space whose metric (§ 02) has signature $(-,-,+,\dots,+)$, the hyperboloid

$$-X_{-1}^2-X_0^2+X_1^2+\cdots+X_d^2=-R^2 \quad (5)$$

and the group $SO(2,d)$ preserving (5) is its symmetry as it stands (appendix of Maldacena). Choosing coordinates again gives the form of the first half of (4) (Poincaré coordinates). Below we consider the Euclidean version, moved to imaginary time (2-04). Writing the boundary coordinates as $x=(x_1,\dots,x_d)$,

$$ds^2=R^2\,\frac{\sum_{i=1}^{d}dx_i^2+dz^2}{z^2}\qquad (z>0)\quad (6)$$

For $d=1$ it is the upper half-plane, and Figure 1 draws this case. Example: the length going up along $x=0$ from $z=1$ to $z=\varepsilon$ is $\int_\varepsilon^1 R\,dz/z=R\ln(1/\varepsilon)$, infinite as $\varepsilon\to 0$. The boundary $z=0$ is infinitely far away Known . The dimension of the symmetry is $6\cdot 5/2=15$ for $SO(2,4)$ (in the Euclidean version $SO(1,5)$, of the same dimension 15; 1-03), and the four-dimensional conformal transformations number 15 (translations 4, Lorentz 6, dilatation 1, special conformal 4); in the words of Witten (1998a), 'SO(2, d) ... acts on M_d as the conformal group' Known .

04

Radial direction = scale — depth measures size

In (6), under $(x,z)\mapsto(sx,sz)$ both numerator and denominator are multiplied by s^2, and ds^2 does not change. This **isometry** of the interior (a transformation preserving length) is, on the boundary, the dilatation $x\mapsto sx$: "looking at something s times larger on the boundary" and "looking s times deeper" become the same operation. In the words of Susskind–Witten (1998), 'scale size gets transmuted into a spatial dimension' Known .

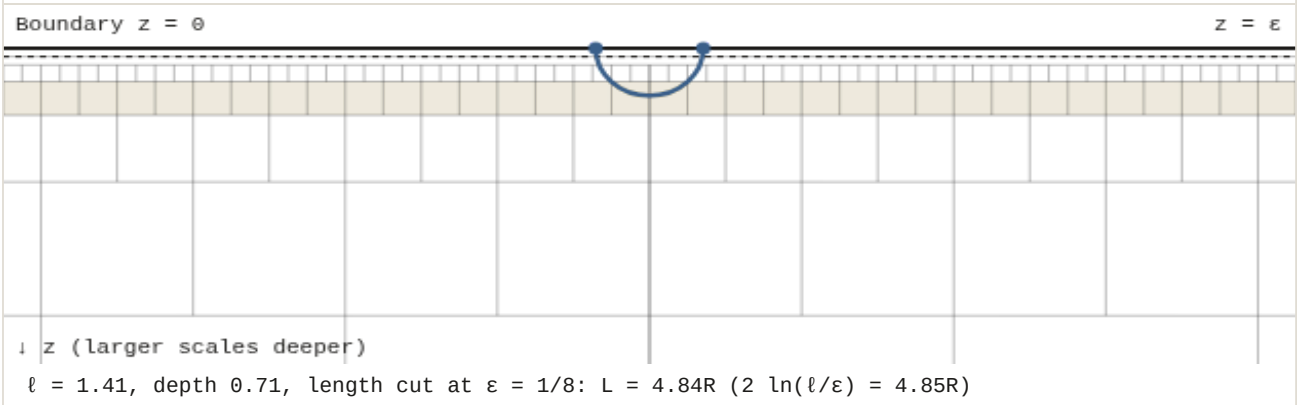
The **geodesics** of (6) (the shortest curves joining two points) are semicircles orthogonal to the boundary, and the semicircle hanging from the boundary interval

$[-\ell/2, \ell/2]$ goes down to depth $\ell/2$. The length cut off at depth ε is logarithmic in ℓ.

$$L(\ell) = 2R \ln \frac{\ell + \sqrt{\ell^2 - 4\varepsilon^2}}{2\varepsilon} \approx 2R \ln \frac{\ell}{\varepsilon}, \qquad e^{-mL} \approx \left(\frac{\varepsilon}{\ell}\right)^{2mR} \tag{7}$$

The amplitude for a particle of mass m to propagate through the interior is roughly e^{-mL}, so exponential decay turns into a power law on the boundary. Susskind–Witten (2.5) follow the same idea, writing it on a model of the sphere in a form without coefficients, $\delta^m / |X_1 - X_2|^m$. The exponent $2mR$ is an approximation for large mR; exactly it is (9) Physics .

Figure 1 The upper half-plane (6) and a dyadic tiling (every tile has area $R^2/2$). Changing the interval ℓ with the knob changes the depth of the semicircle and the L of (7). The coloured band is the level where the bottom lies



Computed The numbers in this figure were computed on this machine.

Table 2 L of (7) ($R = 1$; agrees with numerical integration along the semicircle). Each doubling of ℓ increases L by $2\ln 2 = 1.386$

ℓ/ε	L	$2\ln(\ell/\varepsilon)$
10	4.584863	4.605170
100	9.210140	9.210340
1000	13.815509	13.815511

Cutting near the boundary at $z = \varepsilon$ is an ultraviolet cutoff of the boundary theory. Susskind–Witten write 'the infrared regulator δ in the bulk theory also plays the role of an ultraviolet regulator in the boundary super Yang Mills theory' (δ corresponds to the ε of this chapter). They then divided the cut sphere into a lattice of size δ, assumed one degree of freedom per field per lattice cell, and counted the degrees of freedom as N^2/δ^3 (ibid. (3.1)) Physics . In Figure 1 the width of the tiles doubles at each level down, which recalls one step of the renormalisation group, grouping blocks in pairs (7-05). More precisely, it stands at the level where de Boer–Verlinde–Verlinde (1999, JHEP 2000) derived the form of the Callan–Symanzik equation (the formula describing how correlation functions change when the scale is changed; 4-04) from the Hamilton–Jacobi equation describing radial evolution in classical supergravity (an equation set

up by regarding the action as a function of the values at the end), and proposed 'We propose a direct correspondence' Physics .

05

The dictionary — fields in the interior and operators on the boundary

In the prescription of Gubser–Klebanov–Polyakov (1998) and Witten (1998a), the boundary value ϕ_0 of an interior field ϕ is read as the **source** of a boundary operator $\mathcal{O}$ (an external field added to the action in the form $\int \phi_0 \mathcal{O}\, d^d x$), and the following are set equal. The right end is the action I of the classical solution in the limit where supergravity can be used ($N, \lambda \to \infty$). The abstract of GKP begins 'We suggest'; it is a proposal Physics .

$$\left\langle \exp \int \phi_0\, \mathcal{O}\, d^d x \right\rangle_{\text{boundary}} = Z_{\text{interior}}\left[\phi \to \phi_0\right] \approx e^{-I[\phi_0]} \tag{8}$$

One consequence is the relation between mass and dimension. For solutions independent of x, the equation of a field of mass m is $z^{d+1}(z^{1-d}\phi')' = m^2 R^2 \phi$ (Witten 1998a (2.37)), and putting in $\phi = z^\alpha$ gives $\alpha(\alpha - d) = m^2 R^2$. We write the larger root as Δ (the letter customary in the literature, distinct from the Δ of the mass gap).

$$\Delta(\Delta - d) = m^2 R^2, \qquad \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2} \tag{9}$$

A solution with $\phi \approx \phi_0 z^{d-\Delta}$ near the boundary extends smoothly into the interior. Under the dilatation $(x, z) \mapsto (sx, sz)$, ϕ_0 is multiplied by $s^{\Delta - d}$, so for $\int \phi_0 \mathcal{O}$ to be invariant the **scaling dimension** of $\mathcal{O}$ (the Δ such that the operator is multiplied by $s^{-\Delta}$ under the dilatation $x \mapsto sx$) must be Δ, and the two-point function is proportional to $|x|^{-2\Delta}$ Physics . The condition for real roots, $m^2 R^2 \geq -d^2/4$, is the Breitenlohner–Freedman bound, which allows negative squared masses down to this point Known .

Table 3 Δ of (9) ($d = 4$). The right end is the power read off by solving the equation numerically from $z = 1$ to e^{-25} (agrees with $d - \Delta$)

$m^2 R^2$	−3	0	5	12
Δ	3	4	5	6
Numerical power	1.000000	0.000000	−1.000000	−2.000000

Example: if massless, $\Delta = 4 = d$, and the ϕ_0 in $\int \phi_0 \mathcal{O}\, d^4 x$ is a dimensionless number, that is, it plays the role of a coupling. The dilaton (the scalar whose expectation value fixes the string coupling) corresponds to $\operatorname{tr} F^2$. For the $\mathcal{N} = 4$ theory, Witten (1998a § 2.6) states that the operator obtained by differentiating the Lagrangian density with respect to the coupling (which includes $\operatorname{tr} F^2$) corresponds to the dilaton with $m^2 = 0$, and Csáki–Ooguri–Oz–Terning, citing earlier literature, write 'the operator tr F^2 in four

dimensions corresponds to the dilaton field ...' Physics . $\mathrm{tr}\, F^2$ creates the 0^{++} glueball (17-01), so in 17-09 we read glueball masses from the dilaton equation.

06

Hanging a string — computing the Wilson loop

The energy $E(L)$ of a heavy quark pair at distance L is read from the Wilson loop (5-01). In the prescription put forward at about the same time by Maldacena (1998) and Rey–Yee (1998, EPJC 2001), the quarks are the endpoints of a string ending on the boundary, and the area (Nambu–Goto) of the string hanging into the interior gives E Physics . In (4) the tension $1/(2\pi\alpha')$ is multiplied by R^2/z^2.

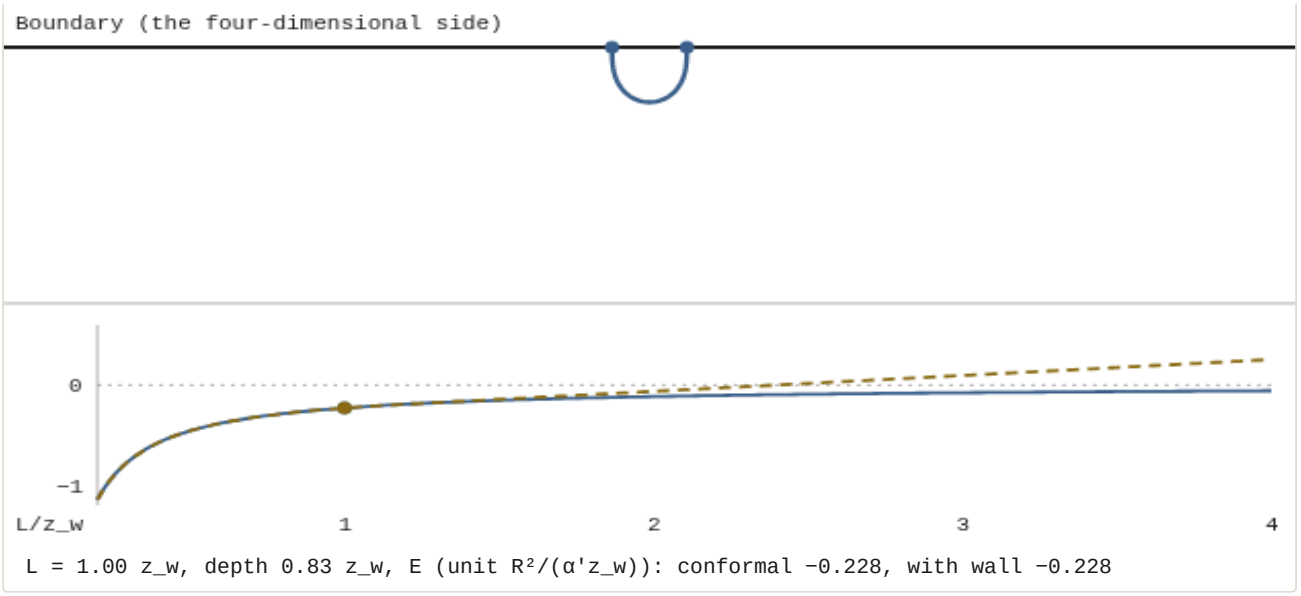
$$E = \frac{1}{2\pi\alpha'}\int_{-L/2}^{L/2} \frac{R^2}{z^2}\sqrt{1+z'^2}\,dx, \qquad L = 2z_0\int_0^1 \frac{y^2\,dy}{\sqrt{1-y^4}} = \frac{2\sqrt{2}\,\pi^{3/2}}{\Gamma(1/4)^2}z_0 = 1.19814\,z_0 \quad (10$$

The integrand does not contain x, so $z^2\sqrt{1+z'^2} = z_0^2$ (z_0 is the depth of the bottom) is conserved, giving the right-hand formula. The string goes down to a depth $z_0 = 0.8346L$ proportional to L. Subtracting the divergence at the boundary (the self-energy of point charges) by the amount for two straight strings gives

$$E(L) = -\frac{4\pi^2}{\Gamma(1/4)^4}\,\frac{R^2}{\alpha'}\,\frac{1}{L} = -0.22847\,\frac{\sqrt{4\pi g_s N}}{L} \quad (11)$$

where the constant agrees between numerical integration and the closed form to 8 digits, and is the same as (4.6)(4.8) of the original paper Computed . The original paper writes 'the energy goes as 1/L, a fact which is determined by conformal invariance. Notice that the energy goes as $(\mathrm{gN})^{1/2}$ as opposed to gN which is the perturbative result' Physics .

Figure 2 Top: a string with both ends on the boundary. Bottom: $E(L)$ (solid line (11); dashed line (12), with a wall). Changing L changes the depth and E, and with "Put a wall" the string lies down along the wall



Computed The numbers in this figure were computed on this machine (the shape of the string from a table of the integral (10), E from (11)(12)).

Table 4 E (unit $R^2/(\alpha' z_w)$) and the depth without a wall. $L_c = 1.19814\, z_w$

L/z_w	Conformal (11)	With wall (12)	$z0/z_w$
0.5	−0.45695	−0.45695	0.4173
1.19814	−0.19069	−0.19069	1.0000
2	−0.11424	−0.06307	1.6693
4	−0.05712	0.25524	3.3385
8	−0.02856	0.89186	6.6770

07

Putting in a single scale — cutting the geometry gives an area law

The $1/L$ of (11) is a consequence of conformality, so confinement ($E \approx \sigma L$, 17-01) needs a scale. The crudest model is to cut the interior at depth z_w. Polchinski–Strassler (2001, PRL 2002) stated that if there is a gap 'the gravitational redshift has a nonzero lower bound' and used the approximation of cutting the geometry Physics . Minimising (10) in this model, for $L \le L_c = 1.19814 z_w$ the conformal solution (bottom depth $0.8346 L \le z_w$) can be used as it is. For $L > L_c$ the bottom of the string reaches the wall. Where it touches the wall the string is tangent to it (slope 0), so the conserved quantity $z^2\sqrt{1+z'^2}$ becomes z_w^2, and the descending part is half of the conformal solution with $z_0 = z_w$. The halves on the two sides use up a width L_c, and the remaining $L - L_c$ is joined by a straight line along the wall.

$$E(L) = -0.19069\,\frac{R^2}{\alpha' z_w} + \sigma(L - L_c), \qquad \sigma = \frac{\sqrt{4\pi g_s N}}{2\pi z_w^2} \qquad (L > L_c) \quad (12)$$

A part proportional to L (an area law) appears, and at $L = L_c$ it joins (11) in both value and slope Computed . Example: at $L = 8z_w$ in Table 4 it is $+0.892$ against the conformal -0.029. As in (12) it has the form $\sigma(L - L_c)$ plus a constant, and for large L it lies on a straight line of slope σ. The string tension carries a factor $\sqrt{4\pi g_s N}$, while the glueball mass fixed by the box up to the wall ($\sim 1/z_w$) does not, so the larger the coupling, the further apart the two scales (the λ-dependence of the ratio and its sources are in 17-09 § 08). This is arithmetic within the model; the actual geometry is treated in 17-09 and 17-10.

08

Correspondences that can be written exactly — what is a theorem and what is a proposal

"Exact" has two meanings: being correct as mathematics closed on one side alone (hyperbolic geometry, Lie groups, ordinary differential equations), and the definitions of both sides agreeing within a model. Neither is a theorem of four-dimensional Yang–Mills theory, and (1) is not proved either.

Table 5 Correspondences and their level

Boundary side	Interior side	Level
Conformal group, dilatation	$SO(2,4)$, $(x, z) \mapsto (sx, sz)$	Known
$\ln(\ell/\varepsilon)$	Length of a geodesic (7)	The formula for the length Known (numbers Computed), its identification with the logarithm on the boundary Physics
Dimension Δ	The power of (9)	The power Known , the identification Physics
The form of Callan–Symanzik	Radial Hamilton–Jacobi	Physics (proposal)
$E(L)$	Minimal area of a string	The prescription Physics , (10)(11) Computed
—	Connectedness of the boundary	Known Mathematical theorem

The last row is a theorem of Witten–Yau (1999). We prepare three terms. An **Einstein manifold** is a manifold whose Ricci curvature (curvature averaged over directions) is a constant multiple of the metric; here the constant is negative (the space (6) is an example). A **conformal boundary** (in Penrose's sense) is a boundary at infinity, attached when multiplying the metric by the square of a function vanishing on the boundary makes it extend smoothly to the boundary (for (6), $z^2 ds^2$ extends to $z = 0$). $H_n(M; \mathbb{Z}) = 0$ means, roughly, that M has no n-dimensional "holes". If M is $n+1$-dimensional, N is its n-dimensional conformal boundary, M is complete (not cut off

partway in any direction), and the scalar curvature of N (curvature further condensed into a single number) is positive, then

$$H_n(M;\mathbb{Z}) = 0, \qquad \text{in particular } N \text{ is connected} \ (13)$$

(abstract: 'Let M be a complete Einstein manifold of negative curvature ... N must be connected') Known . What this chapter uses is only the conclusion that "the boundary N is in one piece (connected)". AdS/CFT is the motivation, but it is not a theorem about the duality itself. Example: the Physics rows of Table 5 all use (1) as a premise.

09

The distance to Yang–Mills — conformality, limits, where things stand

The boundary theory of (1) is not pure Yang–Mills theory. It has six scalar fields and fermions, is conformal so its coupling does not run (in contrast with 4-06), and no scale Λ is born (4-07). The $N, \lambda \to \infty$ where the gravity computation works is the opposite side from what the problem statement (0-03) asks for: any compact simple group, and a theory that agrees with asymptotic freedom ($\lambda \to 0$) at short distances. A conformal theory has no gap: if there is a state of mass $m > 0$, dilatation gives states of mass sm (for any $s > 0$) too.

$$m^2 \in \mathrm{spec}(P^2) \ \Rightarrow \ s^2 m^2 \in \mathrm{spec}(P^2) \ (s > 0) \ \Rightarrow \ \inf\left(\mathrm{spec}(P^2) \setminus \{0\}\right) = 0 \ (14)$$

P^2 is the square of the four-momentum (2-04), and the $1/L$ of (11) is another face of this circumstance Known . Example: the Clay page on the problem still writes of the mass gap 'no proof of this property is known'. There is no example in which holography has shown the mass gap as a mathematical theorem (models with a gap at the level of physics are in 17-09); string theory has no non-perturbative definition, (1) is a conjecture, and the limits lie outside the problem statement, all at once (16-05). The road back to four dimensions is in 17-10, and an example where a gap is visible from a duality under supersymmetry in 17-11.

What this chapter can say

What is said	What is not said
Known The geometry of (4)–(6), $SO(2, 4)$ and the conformal group, the power and the bound of (9), (13)(14).	A proof of (1). A mathematical definition of (8) (the quantum theory in the interior is undefined).

What is said	What is not said
Computed Tables 1–4, Figures 1 and 2, the constants of (10)(11), the numbers of (2).	The wall of (12) is a model, not the actual geometry of confinement.
Physics (1)(8), the prescription for Wilson loops, the radial direction and the renormalisation group, the wall model.	SYM has no gap. A bridge to pure Yang–Mills, finite N, asymptotic freedom. This book itself shows neither the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is in Volume 2).

Sources and reproduction

Item	Kind	Source / tools
(1)(3)–(5), the quotations in § 02, the convention of the original paper	Physics, Known	J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200. Text checked.
Today's convention $g_{\mathrm{YM}}^2 = 4\pi g_s$ in § 02	Known	O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Phys. Rept. 323 (2000) 183, hep-th/9905111. Text of (3.8) and § 4.2 checked.
The quotation in § 03, (2.37), the bound $m^2R^2 \geq -d^2/4$, the dilaton and the coupling operator in § 05	Known, Physics	E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150. Text checked for (2.35)–(2.37), the bound and the real-root condition (just after (2.35) and § 2.6), and the description of the dilaton (§ 2.6).
(8)	Physics	S. S. Gubser, I. R. Klebanov, A. M. Polyakov, Phys. Lett. B 428 (1998) 105, hep-th/9802109. Abstract checked.
The name of the bound	Known	P. Breitenlohner, D. Z. Freedman, Phys. Lett. B 115 (1982) 197; Ann. Phys. 144 (1982) 249. Bibliographic data only.
The idea of (7), the quotations in § 04	Physics	L. Susskind, E. Witten, hep-th/9805114. Text of (2.5), (3.1) and § 2– § 3 checked. J. de Boer, E. Verlinde, H. Verlinde, JHEP 0008 (2000) 003, hep-th/9912012. Text checked.
$\mathrm{tr}\, F^2$ and the dilaton	Physics	C. Csáki, H. Ooguri, Y. Oz, J. Terning, JHEP 9901 (1999) 017, hep-th/9806021. Text checked (the paper attributes this correspondence to earlier literature).
(10)(11), the quotations in § 06	Physics, Computed	J. M. Maldacena, Phys. Rev. Lett. 80 (1998) 4859, hep-th/9803002. Text checked.
The prescription of § 06 (the other one of the same period)	Physics	S.-J. Rey, J.-T. Yee, Eur. Phys. J. C 22 (2001) 379, hep-th/9803001. Abstract checked.
The quotation in § 07	Physics	J. Polchinski, M. J. Strassler, Phys. Rev. Lett. 88 (2002) 031601, hep-th/0109174. Text checked.
(13)	Known	E. Witten, S.-T. Yau, Adv. Theor. Math. Phys. 3 (1999) 1635, hep-th/9910245. Abstract and the opening of the text (M is $n+1$-dimensional, N is n-dimensional) checked.

Item	Kind	Source / tools
Detour	Known, Physics	J. D. Bekenstein, Phys. Rev. D 7 (1973) 2333; S. W. Hawking, Commun. Math. Phys. 43 (1975) 199 (bibliographic data only). G. 't Hooft, gr-qc/9310026; L. Susskind, J. Math. Phys. 36 (1995) 6377, hep-th/9409089 (abstracts checked).
The quotation in § 09	Known	Clay Mathematics Institute, “Yang–Mills & the Mass Gap” (web page). Text checked.
The requirements of the problem statement in § 09	Known	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official Clay problem description, PDF). For checking against the text, see the table of sources of 0-03.
Tables, figures, numbers	Computed	Python 3 (scipy) on this machine: params.py, geodesic.py, delta.py, wilson.py, bh_entropy.py.

Next chapter to read: 17-09 The mass gap becomes geometry — reading glueball masses as eigenvalues in the geometry of a shrinking circle.

The mass gap becomes geometry — when a circle shrinks and spacetime ends, a gap opens in the spectrum

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — The argument by which, when spacetime ends in a "cigar" where a circle shrinks, an area law and positive, discrete glueball masses (radial eigenvalues) come out. The numbers retraced on this machine, and the record against the lattice. That the ratios move a lot depending on which scalar is taken as the reference.

Prerequisite chapters — 17-08 (AdS, the dictionary, the string computation). The variational principle is in 2-05 § 08 and 1-10, glueballs in 17-01 § 06.

Said first — The gap here is in the supergravity approximation, $N \to \infty$, $\lambda \to \infty$, and is a linear eigenvalue problem on a fixed background. AdS/CFT is a conjecture, and the models are not pure Yang–Mills theory. This chapter does not solve the mass gap problem, and this book itself shows nothing about it.

The order of this chapter

1. What to translate into geometry
2. Putting in a scale with temperature
3. The circle shrinks (Figure 1, Detour: the temperature of a black hole)
4. The string lies down at the bottom
5. Glueballs are eigenvalues (Figure 2)
6. Why positive, why discrete
7. Comparing with the lattice (Later analysis: which scalar is lightest)
8. Two scales
9. Another way of closing off

01

What to translate into geometry — the gap and the area law

The boundary theory of AdS is conformal and has no gap (17-08 (14)), and in 17-08 § 07 a scale was put in with a wall placed by hand. In this chapter we look at models in which **the geometry itself ends at a finite depth**. Witten (1998) read two things from this Physics . The area law, from the coefficient of the boundary directions dx_i^2 being at least a positive constant in the interior. The gap, from the wave equation in the interior having only positive, discrete eigenvalues $m_1^2 < m_2^2 < \cdots$. Said on the boundary, the latter means that two-point functions such as that of $\mathrm{tr}\, F^2$ become sums of free propagators.

$$\langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle = \sum_{n \geq 1} |c_n|^2 \, \frac{e^{-m_n |x|}}{4\pi |x|} \qquad \text{(the boundary is three-dimensional)} \quad (1)$$

The fraction on the right is the propagator of mass m in three dimensions Known ; at long distance the first term dominates, and the correlation length is $1/m_1$ (2-05 § 05, 4-11). Example: looking at the two-point function of $\mathrm{tr}\, F^2$ in the model of § 05 (the lowest dilaton mode), in the normalisation of § 03 where the temperature is $T = 1/\pi$, $m_1 = 3.404$ and the correlation length is 0.294 Computed . But among all the supergravity fields, another scalar coming from fluctuations of the metric is lighter, $m^2 = 5.457$ (the dilaton is 11.588) (Brower–Mathur–Tan 2000; § 07) Physics . (1) is the leading term in $1/N$, and Witten writes that interactions destroy the discreteness but the gap remains.

02

Putting in a scale with temperature — the circle of imaginary time and antiperiodicity

Thermal equilibrium at temperature T is the path integral with imaginary time wound on a circle of period $\beta = 1/T$ (the inverse temperature; the β of this chapter is not the lattice inverse coupling) (2-07 § 06). Bosons return to themselves after one turn, and fermions change sign (antiperiodic), so the momenta in the direction of the circle are

$$\omega_n = \frac{2\pi n}{\beta} \ \text{(bosons)}, \qquad \omega_n = \frac{(2n+1)\pi}{\beta} \ \text{(fermions)}, \qquad n \in \mathbb{Z} \quad (2)$$

Known . Fermions have no $\omega = 0$ and drop out at long distances. Witten put the $\mathcal{N} = 4$ theory on $\mathbb{R}^3 \times S^1$ to break supersymmetry, and since the scalars also become heavy at one loop, held that a three-dimensional gauge field remains at long distances Physics . Csáki–Ooguri–Oz–Terning (1998; COOT below) call this QCD_3. As a metaphor, the

circle is a mould. More precisely, the radius R of the circle is an ultraviolet cutoff of the three-dimensional theory (17-06 § 08). Example: the lowest fermion momentum is πT, and the lowest non-zero boson momentum $2\pi T$.

03

The circle shrinks — the geometry of the cigar

If there is a circle on the boundary, the interior becomes an AdS black hole. Witten scaled that solution to obtain the following metric (the overall R^2 omitted) Physics .

$$ds^2 = g\,d\tau^2 + \frac{d\rho^2}{g} + \rho^2 \sum_{i=1}^{3} dx_i^2, \qquad g = \frac{\rho^2}{b^2} - \frac{b^2}{\rho^2}, \qquad \rho \geq b \quad (3)$$

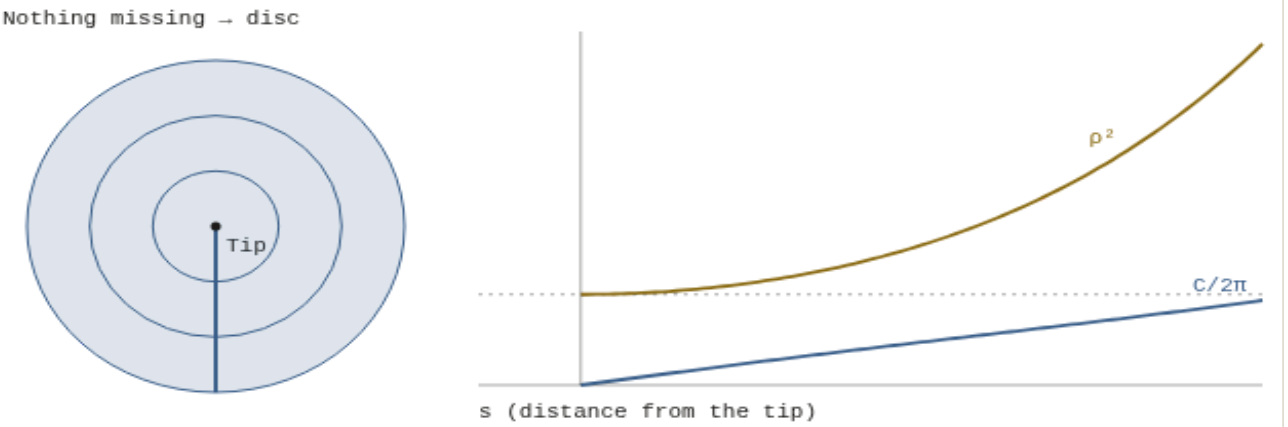
b is the curvature radius, and $\rho \to \infty$ is the boundary, where it returns to AdS (17-08 § 03). At $\rho = b$, $g = 0$ and the circle shrinks. Written with the distance r from the tip ($\rho \approx b + r^2/b$),

$$g\,d\tau^2 + \frac{d\rho^2}{g} \;\approx\; dr^2 + r^2\,d\theta^2, \qquad \theta = \frac{2\tau}{b} \quad (4)$$

these are polar coordinates in the plane. But if the period of τ deviates from πb, the tip is the vertex of a cone (a singularity). Smoothness fixes the period to πb. Near the boundary, dividing the metric by ρ^2 gives $d\tau^2/b^2 + \sum dx_i^2$, so the circumference measured on the boundary is $\pi b/b = \pi$, and the temperature is fixed to $T = 1/\pi$ independently of b Known . The numbers of this chapter are written in this normalisation (circumference π, $T = 1/\pi$). "$b = 1$" in the tables means that b was set to 1; the eigenvalues of (7) do not depend on b, since b disappears on setting $\rho = bu$.

(ρ, τ) is a cigar-shaped disc. As a metaphor, spacetime "ends there". More precisely, the tip is the centre of the disc. For $b = 1$ it can be written $\rho^2 = \cosh 2r$, agreeing with numerical integration to 8 digits Computed .

Figure 1 Left: near the tip (the edges of a sector of angle $2\pi\beta/\beta_1$ identified). Right: the radius of the circle $C/2\pi$ (blue) and the coefficient ρ^2 of dx^2 (yellow). Moving the period away from 1 makes the tip a cone



β/β₁ = 1.00, circumference/(2π×distance) → 1.00, smooth (centre of a disc), minimum of the dx² coefficient b² = 1

Values of the curves of Figure 1 ($\beta/\beta_1 = 1$, $b = 1$; s is the distance from the tip)

ρ	s	$C/(2\pi s)$	ρ^2
1.001	0.0316	0.9997	1.002
1.1	0.3186	0.9719	1.210
2	1.0317	0.9385	4.000
4	1.7324	1.1522	16.00

Computed The numbers in this figure were computed on this machine (the curves are $\rho^2 = \cosh 2s$; the table gives values from numerical integration, and closer to the tip $C/(2\pi s)$ comes closer to 1).

Detour: the temperature of a black hole is the same computation

A subsection you may skip.

If the imaginary-time Schwarzschild metric $f\,d\tau^2 + dr^2/f + \cdots$ ($f = 1 - r_s/r$) is smooth at $r = r_s$, the period is $4\pi r_s$. Reading this as $\hbar c/(k_B T)$ gives

$$T = \frac{\hbar c^3}{8\pi G M k_B} \quad (5)$$

and for a solar mass $T = 6.17 \times 10^{-8}$ K Computed . This derivation is attributed to the method of Gibbons–Hawking (1977) Known .

04

The string lies down at the bottom — the area law

The Wilson loop is fixed as $e^{-\alpha(D)}$ by the regularised area $\alpha(D)$ of a surface in the interior (17-08 § 06) Physics . In (3) the surface can only enter $\rho \geq b$, and the coefficient ρ^2 of dx_i^2 is at least b^2, so with A the enclosed area,

$$\alpha(D) \;\geq\; b^2 A - (\text{boundary and regularisation terms}), \qquad \sigma = b^2 \times (\text{string tension}) \quad (6)$$

and the area law (17-01 § 04) comes out (Witten). We computed the area per unit length $E(L)$ of a string joining two points at distance L by the same method as 17-08 (10).

Table 1 The string in the cigar (unit $R^2/(2\pi\alpha')$, $b = 1$). Conformal is 17-08 (11)

L	$E(L)$	**Conformal** $-1.4355/L$	$E - L$
0.1	−14.355	−14.355	–
1	−1.307	−1.436	−2.307
2	−0.032	−0.718	−2.032
6	4.000	−0.239	−2.000

At short distances it is the conformal $1/L$, at long distances $E \approx L - 2.000$, and the slope 1 is $\sigma = b^2$ Computed . The string goes down to the bottom $\rho = b$ and lies flat. No wall (17-08 § 07) is needed. Example: rescaling the boundary scale by s makes the temperature T/s and multiplies the tension by $1/s^2$, so $\sigma \propto T^2$ (Witten).

05

Glueballs are eigenvalues — the radial equation of the dilaton

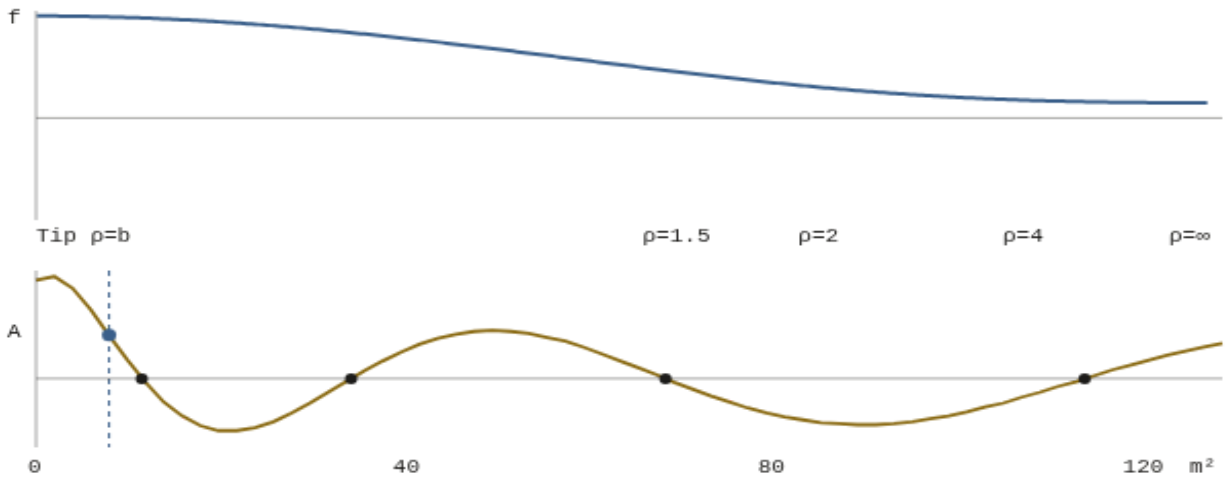
$\mathrm{tr}\, F^2$ corresponds to the dilaton (17-08 § 05). Witten studied waves $\phi = f(\rho) e^{ik\cdot x}$ on (3) that do not depend on τ (waves with momentum in the τ direction are heavier; a comparison within the same field; § 08). The equation is

$$-\frac{1}{\rho}\,\frac{d}{d\rho}\Big(\rho^3 g\,\frac{df}{d\rho}\Big) = m^2 f, \qquad m^2 = -k^2 \quad (7)$$

(for $b = 1$, $\rho^3 g = \rho^5 - \rho$) Physics , where m is the mass of a three-dimensional particle. There are two conditions. **Smooth at the tip**: one solution diverges as $\log(\rho - b)$, and the smooth solution is unique up to a constant multiple. **Normalisable at infinity**: of the solutions $f \sim 1$ and $f \sim \rho^{-4}$, the latter. The constant component is the source on the boundary (17-08 (8)).

Smoothness at the tip is the condition that the slope measured with the radial coordinate r of polar coordinates in the plane ((4)) is 0. Since $\rho - b \propto r^2$, the slope measured with ρ is not 0; from (7), $f'(1) = -m^2 f(1)/4$ ($b = 1$), which for the first eigenfunction is $-2.90 f(1)$ Computed . The abstract of de Mello Koch et al. (1998) also states this condition: there is no smooth solution whose slope in ρ vanishes at the horizon. The method is shooting: the zeros of the constant component $A(m^2)$ that the smooth solution from the tip leaves at infinity are the eigenvalues.

Figure 2 Top: the solution f shot from the tip (left) (horizontal axis θ, $\rho = \sec\theta$). Bottom: the constant component A left on the boundary (times $1 + m^2/4$). Varying m^2, at $A = 0$ the solution lands on the boundary at 0



m² = 8.0, constant component left on the boundary A = +0.148 (f=1 at the tip), nodes: 0

Computed The numbers in this figure were computed on this machine (the bottom is a table of a precise integration; the top is a Runge–Kutta integration of the same formula in the page, agreeing with the table to 4 digits).

Table 2 The eigenvalues m_n^2 of (7) ($b = 1$). There are $n - 1$ nodes. T_L is from (9)

n	This machine	COOT	$6n(n+1)$	$(\pi/T_L)^2(n+\frac{1}{2})^2$
1	11.5877	11.59	12	12.92
2	34.5270	34.53	36	35.89
3	68.9750	68.98	72	70.34
6	241.237	241.2	252	242.61

Shooting and the Chebyshev collocation method (moved to $u = 1/\rho^2$) agree to 7 digits and reproduce the 6 rows of Table 1 of COOT Computed . The ratio to the temperature is $m_1/T = 10.69$.

06

Why positive, why discrete — the variational principle and the length of the box

Positive. Multiplying (7) by ρf and integrating by parts, the boundary terms at the tip ($\rho^3 g = 0$) and at infinity vanish and

$$m^2 = \frac{\int_b^\infty \rho^3 g\,(f')^2\,d\rho}{\int_b^\infty \rho\, f^2\,d\rho} \tag{8}$$

If the numerator is 0 then f is constant, but a constant is not normalisable ($\int^\infty \rho\,d\rho = \infty$), so $m^2 > 0$ Known . The minimum of the right-hand side is m_1^2, and the gap is the

best constant of a Poincaré inequality (1-10, 2-05 § 08).

Discrete. Changing the variable to $t = \int_1^\rho d\rho'/\sqrt{\rho'^4 - 1}$ and the unknown function to $\psi = (\rho^6 - \rho^2)^{1/4} f$ (a Liouville transformation; $b = 1$), (7) becomes the Schrödinger equation on the interval $[0, T_L]$

$$-\frac{d^2\psi}{dt^2} + V(t)\,\psi = m^2 \psi, \qquad T_L = \int_1^\infty \frac{d\rho}{\sqrt{\rho^4 - 1}} = \int_0^1 \frac{ds}{\sqrt{1 - s^4}} = 1.31103 \quad (9)$$

Known . $V = (\rho^6 - \rho^2)^{-1/4} \frac{d^2}{dt^2} (\rho^6 - \rho^2)^{1/4}$ diverges at both ends: $V \approx -1/(4t^2)$ at the tip, $V \approx (15/4)/(T_L - t)^2$ at the end at infinity Computed . Even so, the length of the box T_L is finite, so the eigenvalues are discrete, just as for a particle in a box.

The type of each end also fixes the phase of the approximation. At an end where the potential is $(\nu^2 - \frac{1}{4})/u^2$ (u the distance from the end), the allowed solution is $\sqrt{u}\, J_\nu(mu)$ (J_ν the Bessel function of order ν), which away from the end oscillates like $\cos(mu - \nu\pi/2 - \pi/4)$. The tip has $\nu = 0$ and brings in a phase $\pi/4$, the end at infinity has $\nu = 2$ and brings in $5\pi/4$, so the n-th eigenvalue is $m_n T_L \approx \pi(n-1) + \pi/4 + 5\pi/4 = \pi(n + \frac{1}{2})$, that is $m_n \approx (\pi/T_L)(n + \frac{1}{2})$. At $n = 6$ the difference shrinks to 0.6% in m^2 (0.3% in m) Computed . The $6n(n+1)$ of Table 2 is an approximate formula from exact WKB analysis, which COOT cite as a private communication from Takei–Koike. Exact WKB analysis is a theory that handles mathematically divergent series expanded in a large parameter by summing them with Borel's method (4-12), and this formula assumes their summability (COOT note that there is no proof in this case).

In AdS the length $\int_0 d\rho/\rho^2$ down to the lower end $\rho = 0$ is infinite, solutions oscillate deep inside, and the spectrum is continuous from 0 (Witten). "Spacetime ends" here means "the Liouville length is finite". In the five-dimensional models of Gürsoy–Kiritsis–Nitti (2007) too, it is stated, by analysis within the model, that for the exponent α of the growth of the potential deep inside, confinement and a gap hold for $\alpha \geq 1$, and a purely discrete spectrum for $\alpha > 1$ (at $\alpha = 1$, a continuous spectrum above the gap). The same paper also has the sentence 'confinement and discrete spectra imply each other' Physics .

07

Comparing with the lattice — three and four dimensions

What supergravity produces are numbers in units of a single scale, so COOT matched the lowest dilaton state to the 0^{++} of the lattice (5-08) (in three dimensions, the lattice $N \to \infty$ value 4.065, printed as 4.07 in the table of COOT) and compared the rest. 0^{--} comes from the equation of a two-form field.

Table 3 Three dimensions (unit $\sqrt{\sigma}$). Lattice: Teper (values cited by COOT)

State	Lattice $N=3$	Lattice $N\to\infty$	COOT	This machine
0^{++*}	6.52(9)	6.18(13)	7.02	7.02
0^{++**}	8.23(17)	7.99(22)	9.92	9.92
0^{--}	6.48(9)	5.91(25)	6.10	6.10

Taking the dilaton as the reference, the ratio $0^{--}/0^{++}=1.50$ agrees with the lattice 1.45 ± 0.03 to 3%, and the excited states come out 14% and 24% higher than the $N\to\infty$ lattice Computed . Which scalar to take as the reference is treated in "Later analysis" below.

Four dimensions is the D4-brane background of 17-07 § 09 (radius R_2 of the antiperiodic circle). The equation of the dilaton-type scalar of COOT, with $y^2=\rho$ (the λ of the original paper) and in units placing the tip at $y=1$ (then $M_{\mathrm{KK}}=1/R_2=3$), is the following (the solutions at infinity are 1 and y^{-9}) Physics .

$$(y^7-y)\,f''+(10y^6-4)\,f'+m^2y^3f=0 \quad (10)$$

Table 4 Four dimensions (GeV, 0^{++} matched to 1.61). The right end is in units of $M_{\mathrm{KK}}=1/R_2$; 0^{++} and 0^{++*} are values from formula (10) of COOT, 0^{-+} the value from the three-form equation of COOT

State	Lattice (values cited by COOT)	COOT	This machine	m/M_{KK}
0^{++}	1.61(15)	Input	Input	2.03
0^{++*}	2.8 (no error)	2.38	2.38	3.01
0^{-+}	2.19(32)	1.83	1.93	2.44

$0^{-+}/0^{++}$ is 1.14 in COOT and 1.36 ± 0.32 on the lattice of the time. COOT obtain 0^{-+} from a component of the 11-dimensional three-form field. Solving that equation on this machine, the lowest value is $m^2=53.376$ (1.93 GeV if the 0^{++} of (10) is 1.61 GeV, ratio 1.20), agreeing with the value for the same equation (N4) in Table 2 of Brower–Mathur–Tan (2000; BMT below) Computed . The 1.83 GeV in the table of COOT cannot be reproduced from this equation. For the lattice 0^{++*}, a note says it may be a bound state of two glueballs.

Later analysis — which scalar is lightest

BMT turned all the supergravity fields (metric, three-form, size of the sphere) into radial eigenvalue problems. We list the four-dimensional scalar-type modes. "Assignment" is the quantum numbers J^{PC} read from the operator to which the mode couples.

Table 5 The lowest four-dimensional modes (m^2 in units placing the tip at 1; the right end in units of M_{KK})

Equation	Field of origin	Assignment	This machine	BMT Table 2	m/M_{KK}

Equation	Field of origin	Assignment	This machine	BMT Table 2	m/M_{KK}
S4	The $\tau\tau$ component of the metric	0^{++} (the lightest)	7.3083	7.308	0.901
T4	Metric, dilaton	2^{++}, 0^{++} and others (0^{++} couples to $\mathrm{tr}\, F^2$)	22.0966	22.097	1.567
V4	The τi components of the metric and others	1^{-+}, 0^{-+}	31.9853	31.985	1.885
(10) of COOT	Dilaton	0^{++} (COOT)	37.1699	–	2.032
N4	Three-form	1^{+-}, 0^{+-} (BMT), 0^{-+} (COOT)	53.3758	53.376	2.435

The T4 equation differs from (10) only in the first-order term.

$$(y^7 - y)\, f'' + (7y^6 - 1)\, f' + m^2 y^3 f = 0 \quad \text{(10')}$$

In the later analysis, the dilaton-type 0^{++} coupling to $\mathrm{tr}\, F^2$ is degenerate with 2^{++} at $1.567\, M_{\mathrm{KK}}$, and the lightest 0^{++} is another mode, S4, at $0.901\, M_{\mathrm{KK}}$ Physics . The 2.03 in Table 4 is the value from formula (10) of COOT. In three dimensions too there is a scalar (S3 of BMT, 5.457) lighter than the dilaton (11.588) Computed . BMT write 'the lowest mass scalar comes from the gravitational multiplet, not the dilaton', and call the agreement of ratios to a few per cent 'a misconception'. At the same time they write that the overall pattern of spins and masses corresponds well with the lattice results Physics .

So the ratios move depending on which modes are taken as numerator and denominator. The three-dimensional $0^{--}/0^{++}$ is 2.19 with S3 as the reference. In four dimensions, the three-form value of COOT divided by T4 is 1.55, and with the assignment of BMT (0^{-+} is V4) V4/T4 is 1.20 and V4/S4 is 2.09 Computed . For $SU(3)$ on today's lattice, $0^{-+}/0^{++} = 1.549(16)$ and $0^{++*}/0^{++} = 1.720(16)$ (the table of 17-01). The holographic ratios with the reference of COOT (1.14, 1.20, 1.48) come out 26%, 23% and 14% smaller respectively Computed .

08

Two scales — the tension is not predicted, and the scale of the circle does not go away

Glueball masses are fixed by an eigenvalue problem and do not depend on the coupling $\lambda = g_{\mathrm{YM}}^2 N$, but the string tension is proportional to λ. In the D4 model, from formula (2.7) of Kruczenski–Mateos–Myers–Winters (2003) Physics

$$\sigma = \frac{2}{27\pi}\,\lambda\, M_{\mathrm{KK}}^2, \qquad \frac{m0^{++}}{\sqrt{\sigma}} = \frac{m0^{++}}{M_{\mathrm{KK}}}\sqrt{\frac{27\pi}{2\lambda}} \quad (11)$$

(it may be off by a factor of 2 depending on the convention for λ). The value changes depending on which is put into $m0^{++}/M_{\mathrm{KK}}$ on the right: 2.03 from formula (10) of COOT, 1.567 from T4 of BMT (dilaton type), or 0.901 from S4 (the lightest).

Table 6 $m0^{++}/\sqrt{\sigma}$ from (11). The lattice $SU(\infty)$ is 3.072(14)

$m0^{++}/M_{\mathrm{KK}}$	$\lambda = 1$	10	30	100	1000	λ matching 3.072
2.03 (formula (10) of COOT)	13.2	4.19	2.42	1.32	0.42	19
1.567 (T4 of BMT)	10.2	3.23	1.86	1.02	0.32	11
0.901 (S4 of BMT)	5.87	1.86	1.07	0.59	0.19	3.7

At $\lambda \gg 1$, where supergravity is reliable, the ratio goes to 0 as $\lambda^{-1/2}$ Computed . The λ matching 3.072 is 3.7 to 19, moving by a factor of about 5 depending on which mode is taken, and in every case it is a fit, not a prediction. Klebanov–Strassler too acknowledge a 'large separation of scales' between the tension and the glueballs. On the lattice it is 3 to 4 (17-03).

In models cut off by a wall, the λ-dependence of the ratio is $\lambda^{-1/4}$. With a hard wall ((12) of 17-08) the tension is $\sigma = \sqrt{\lambda}/(2\pi z_w^2)$ and the glueball mass $\sim 1/z_w$ (Boschi-Filho–Braga 2003, imposing a Dirichlet condition at the wall). With a soft wall the tension is $\frac{R^2}{\alpha'}\frac{e}{4\pi}c$, from (19) of Andreev–Zakharov (2006) and $g = R^2/\alpha'$, and the squared mass is proportional to c ((15) of Karch–Katz–Son–Stephanov 2006; the case of mesons). In both the tension is proportional to $\sqrt{\lambda}$, so the ratio goes as $\lambda^{-1/4}$; the difference from the $\lambda^{-1/2}$ of the D4 model comes from the tension in (11) being proportional to λ Physics .

The other is the scale of the circle. In three dimensions, in the normalisation of § 03 the circumference on the boundary is π, so $R = 1/2$ and $m_1 = 3.404 = 1.70/R$ (COOT use another normalisation, placing the horizon at $b = 1/(2R)$). In four dimensions 0^{++} is $0.90/R_2$ to $2.03/R_2$, and whichever mode is taken, the glueballs are of the same order as the circle Computed . The KK modes on the sphere (17-06 § 02) are also 'of the same order' (COOT), and the states regarded as glueballs are chosen by hand. Removing them needs $\lambda \to 0$, which is the 'opposite' of the direction in which supergravity works. Witten himself called the computation at small $\eta = g_4^2 N$ 'a tall order' Physics . On the order of limits, see 17-10.

09

Another way of closing off — Klebanov–Strassler and the record

Klebanov–Strassler (2000) placed D3-branes (17-08 § 02) on a six-dimensional cone (the conifold) and built a background in which the singularity at the tip is deformed in the IR, leaving a finite sphere.

$$\sum_{i=1}^{4} z_i^2 = 0 \;\; \text{(conifold)} \;\longrightarrow\; \sum_{i=1}^{4} z_i^2 = \epsilon^2 \;\; \text{(deformed)}, \qquad z_i \in \mathbb{C} \tag{12}$$

Example: restricting z_i to real numbers, the right is an S^3 of radius ϵ, the sphere left at the tip Known . The flux of the five-form field through a cross-section of the cone (a quantity corresponding to the number of D3-branes) changes logarithmically with the radius r: $\mathcal{K}(r) = N + a\, g_s M^2 \ln(r/r_0)$. g_s is the string coupling (17-08 (3)), M the number of "fractional branes" wrapped on the sphere in the cone, and a a constant of order 1 (original paper). The authors read this decrease as a sequence (a cascade) in which the gauge group $SU(N+M) \times SU(N)$ shrinks step by step as $N \to N - M$ Physics . The deformation of the tip is held to be the breaking $\mathbb{Z}_{2M} \to \mathbb{Z}_2$ of the R-symmetry (the chiral symmetry of this theory) of the $\mathcal{N} = 1$ supersymmetric Yang–Mills remaining in the IR, made visible as geometry Physics . At the tip the coefficient of dx^2 is finite, so an area law comes out as in § 04. But the theory is $\mathcal{N} = 1$ supersymmetric, the UV cascade 'goes on forever', and the KK modes of S^3 and the glueballs 'both scale as $m/(g_s M)$'.

Table 7 The record (years are arXiv years)

Model	What fits	Input	What misses / is not predicted
Witten, QCD_3	$0^{--}/0^{++}$ 1.50 against 1.45(3) (with the dilaton as reference; 2.19 with S3 as reference)	0^{++}	Excited states 14–24%, tension, circle
Witten, QCD_4	$0^{-+}/0^{++}$ 1.14 against 1.36(32) (1998; with the dilaton of formula (10) of COOT as reference)	0^{++}	14–26% from today's lattice, ratios 1.14–2.09 depending on the choice of modes, $m/\sqrt{\sigma}$
Klebanov–Strassler (2000), Maldacena–Núñez (2000)	Area law, breaking of R-symmetry (chiral symmetry) (geometric)	—	$\mathcal{N} = 1$, same scale as KK
Gürsoy–Kiritsis–Nitti (2007)	$2^{++}/0^{++}$ 1.40 against 1.46(5)	0^{++*} $/0^{++}$	The potential is a fit, $N = \infty$

In every model the gap is shown as "a linear eigenvalue problem on a fixed background", and the part saying that the background describes pure Yang–Mills theory on $\mathbb{R}^4$ remains a conjecture. The Clay page too writes 'no proof of this property

is known'. The distance from the problem statement (0-03) is in 17-10, the seams in 16-05.

What this chapter can say

What is said	What is not said
Known (4)(8)(9)(12). Mathematics on a fixed background.	That the background describes pure Yang–Mills theory.
Computed Tables 1–6, Figures 1 and 2, (5), the behaviour of V of (9) at both ends. Reproducing the 0^{++} tables of COOT for $\mathbf{QCD_3}$ and $\mathbf{QCD_4}$. Reproducing the lowest values of S4, T4, V4, N4 in Table 2 of BMT and of S3 in three dimensions.	Reproducing the 0^{-+} (1.83 GeV) of $\mathbf{QCD_4}$. The convention for λ in (11). Settling which mode is to be regarded as which state.
Physics The readings of (1)(3)(6)(7)(10)(10′)(11), glueball = eigenvalue, the assignment of quantum numbers, the comparisons of Tables 3, 4, 5 and 7.	Outside $N, \lambda \to \infty$. This book itself shows neither the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is in Volume 2).

Sources and reproduction

Item	Kind	Source / tools
Witten's formulas and quotations	Physics, Known	E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505, hep-th/9803131. Text checked.
(10), Tables 2–4, quotations from COOT, $6n(n+1)$	Physics	C. Csáki, H. Ooguri, Y. Oz, J. Terning, JHEP 9901 (1999) 017, hep-th/9806021. Text checked ($6n(n+1)$ is cited by that paper as a private communication from Y. Takei, T. Koike).
Table 5, (10′), the later analysis of § 07	Physics	R. C. Brower, S. D. Mathur, C.-I. Tan, Nucl. Phys. B 587 (2000) 249, hep-th/0003115. Text checked (Tables 1 and 2, the table for $\mathbf{QCD_3}$, the quantum numbers of § 4, § 5.1).
Lattice	Physics	M. J. Teper, hep-lat/9804008, hep-lat/9711011; C. Morningstar, M. Peardon, PRD 56 (1997) 4043 (via COOT; bibliographic data only). A. Athenodorou, M. Teper, JHEP 11 (2020) 172; JHEP 12 (2021) 082 (from the checked survey records).
The condition at the tip, (11)	Physics	R. de Mello Koch et al., PRD 58 (1998) 105009, hep-th/9806125 (abstract, from the checked survey records); M. Kruczenski, D. Mateos, R. C. Myers, D. J. Winters, JHEP 0405 (2004) 041, hep-th/0311270, formula (2.7) (text checked: $\sigma = \frac{2}{27\pi} g^2_{\mathrm{YM}} N_c M^2_{\mathrm{KK}}$).

Item	Kind	Source / tools
The $\lambda^{-1/4}$ of the wall models in § 08	Physics	H. Boschi-Filho, N. R. F. Braga, JHEP 0305 (2003) 009, hep-th/0212207 (abstract checked); O. Andreev, V. I. Zakharov, Phys. Rev. D 74 (2006) 025023, hep-ph/0604204, (19) and $g = R^2/\alpha'$ (text checked); A. Karch, E. Katz, D. T. Son, M. A. Stephanov, Phys. Rev. D 74 (2006) 015005, hep-ph/0602229, (15) (text checked). The tension with a hard wall is (12) of 17-08 (computed on this machine; that a linear term comes out with a wall was also checked in (12)–(16) of H. Boschi-Filho, N. R. F. Braga, C. N. Ferreira, Phys. Rev. D 73 (2006) 106006, hep-th/0512295). The exponent $-1/4$ is arithmetic on this machine combining these.
(12), § 09, Table 7	Physics, Known	I. R. Klebanov, M. J. Strassler, JHEP 0008 (2000) 052, hep-th/0007191 (text checked); J. M. Maldacena, C. Núñez, PRL 86 (2001) 588 (abstract).
The exponent α of § 06, Table 7	Physics	U. Gürsoy, E. Kiritsis, F. Nitti, JHEP 0802 (2008) 019, arXiv:0707.1349 (text checked).
Detour, Clay	Known	G. W. Gibbons, S. W. Hawking, 'Action integrals and partition functions in quantum gravity', Phys. Rev. D 15 (1977) 2752 (bibliographic data only). The Clay page on the problem (from the checked survey records).
Tables, figures, numbers	Computed	Python 3 (scipy) on this machine: glueball.py, cigar_wilson.py, fig_shoot.py, rev_bmt.py (Tables 5 and 6), rev_liouville.py (the V of (9)) and 5 others.

Next chapter to read: 17-10 Back to four dimensions — the order of limits and the distance from the problem statement.

Back to four dimensions — the direction that removes the extra dimensions is opposite to the direction in which geometry can compute

> This chapter organises established knowledge. § 03 is this book's own investigation.

What this chapter shows — That $R \to 0$, which removes the extra dimensions, is because of dimensional transmutation the same direction as $\lambda \to 0$. That geometry can compute in the opposite direction, $\lambda \to \infty$, where the KK tower remains at the same height as the glueballs. That the uniformity needed to exchange the limits is missing.

Prerequisite chapters — 17-09. We also use dimensional transmutation in 17-01, 't Hooft's counting in 17-04 and the Curie–Weiss model in 2-08.

Said first — (2)(3) and the arithmetic of the tables are physical estimates, not theorems. Within the range we searched we found no result justifying the limit from the models to the four-dimensional theory, and this book itself shows nothing about the mass gap or the construction of the continuum limit (the published construction of the continuum limit is read in Volume 2).

The order of this chapter

1. Three knobs — N, λ, R and the problem statement
2. Dimensional transmutation fixes the direction — $R \to 0$ is $\lambda \to 0$
3. Supergravity works in the opposite direction — the window and the floor (Figure 1)
4. Extra things remain — the KK tower and supersymmetry
5. $N \to \infty$ — when the group is fixed to one
6. The order of limits — exchanging them needs uniformity
7. Detour: seeing non-uniform convergence (Figure 2)
8. The distance from the problem statement — five requirements
9. What remains — the problem moves

01

Three knobs — N, λ, R and the problem statement

The question of this chapter is **whether the models of 17-09 can be brought back to the four-dimensional theory of the prize problem**. The knobs are three: the N of $SU(N)$, the 't Hooft coupling $\lambda = g^2 N$ at the scale M_{KK} (17-04), and the radius of the circle of the extra dimension $R = 1/M_{KK}$ (the spacing of the KK tower, 17-06). With Λ the glueball mass, the ratio ΛR measures how visible the extra dimensions are. The requirements of the problem statement (Jaffe–Witten § 4, 0-03) are

G fixed one at a time (e.g. $SU(3)$), space is $\mathbb{R}^4$ ($R = 0$), agrees with asymptotic free

and the third is that the correlation functions agree at short distances with the predictions of asymptotic freedom and perturbative renormalisation theory (§ 4; quotation in 0-03 § 07) Known . The models, conversely, can compute where $N = \infty$, $\lambda \gg 1$ and ΛR is of order 1 (§ 03, § 05).

Example: the Sakai–Sugimoto model fixes $M_{KK} \simeq 949$ MeV and $\kappa = \lambda N_c/(216\pi^3) \simeq 7.45 \times 10^{-3}$ from experimental values. For $N_c = 3$, $\lambda \approx 16.6$ Computed . This is the value in the coupling convention of Sakai–Sugimoto, and can differ from the λ of (2) in the next section by a factor of about 2 because of the difference in convention. The numbers put into (2) below are rough guides. The 776 MeV of the ρ meson is 0.82 times M_{KK}, at the same height as the tower.

02

Dimensional transmutation fixes the direction — $R \to 0$ is $\lambda \to 0$

Removing the extra dimensions means taking $R \to 0$ with Λ fixed. Λ is born from the running of the coupling (17-01, 4-07). Rewriting the one-loop formula $\mu\, dg/d\mu = -b_0 g^3$ ($b_0 = 11N/(48\pi^2)$) of (1) of 17-01 in terms of $\lambda = g^2 N$ gives

$$\mu \frac{d\lambda}{d\mu} = 2gN\,\mu \frac{dg}{d\mu} = -2b_0 N\, g^4 = -\frac{2b_0}{N}\,\lambda^2 = -\frac{11}{24\pi^2}\,\lambda^2$$

Dividing both sides by λ^2 gives $d(1/\lambda)/d\ln\mu = 11/(24\pi^2)$, so $1/\lambda$ is linear in $\ln\mu$. Below, λ without an argument is the value at $\mu = M_{KK}$.

$$\frac{1}{\lambda(\mu)} = \frac{1}{\lambda} + \frac{11}{24\pi^2}\ln\frac{\mu}{M_{KK}}$$

Lowering μ decreases the right-hand side, which becomes 0 at $\mu = M_{KK}\exp\big(-24\pi^2/(11\lambda)\big)$ (the one-loop coupling diverges here). This scale where the coupling becomes strong is a rough guide to Λ:

$$\Lambda R \;\sim\; \exp\!\Big(-\frac{24\pi^2}{11\,\lambda}\Big) \quad (2)$$

Physics . $\sim$ means "equal up to a constant multiple", and the constant changes with the definition of Λ (the Λ parameter, or the glueball mass) and with conventions. Solving the other way gives

$$\lambda(M_{KK}) \simeq \frac{24\pi^2/11}{\ln\big(1/(\Lambda R)\big)} \;\xrightarrow[R\to 0]{}\; 0 \quad (3)$$

(a constant multiple only adds a constant to the denominator, so it does not affect the direction of $R \to 0$), and **removing the extra dimensions is the same direction as $\lambda \to 0$**. Witten (1998 § 4) writes 'Asymptotic freedom should say that as $\eta \to 0$, the particle masses become exponentially small' (η is the λ of this chapter), and Csáki–Ooguri–Oz–Terning, putting the masses as $M^2 = f(\lambda)\Lambda_{UV}^2$, state 'To take the continuum limit $\Lambda_{\mathrm{UV}} \to \infty$, we have to simultaneously take $\lambda \to 0$' Physics ((2) is (10) of 17-07). As a metaphor, R plays the role of the lattice spacing; more precisely, it is the same logic as sending the bare coupling to 0 as the spacing $a \to 0$ (5-05).

Table 1 Estimates of ΛR (with the constant multiple in (2) set to 1) and the rough guide $\lambda^{-3/2}$ to the string correction

λ	One loop (2)	Two loops	$\lambda^{-3/2}$
1	4.45×10^{-10}	2.17×10^{-9}	1
5	0.0135	0.0334	0.0894
10	0.116	0.215	0.0316
16.6	0.274	0.409	0.0147
100	0.806	0.565	0.0010

Computed Two loops: from $\mu\, dg/d\mu = -b_0 g^3 - b_1 g^5$ ($b_1 = \frac{34N^2}{3(16\pi^2)^2}$), the formula of 17-01 with the next term added, the right-hand side of (2) multiplied by $(11\lambda/48\pi^2)^{-51/121}$ ($51/121 = b_1/(2b_0^2)$). The coefficients can change with convention, and the rows with large λ are rough guides outside the range where (2) can be used (that is also why, at $\lambda = 100$, the two-loop column is smaller than the one-loop one; § 03). Example: $\Lambda R = 10^{-3}$ needs $\lambda \le 3.12$ (two loops 2.66), and 10^{-6} needs $\lambda \le 1.56$ (1.41).

03

Supergravity works in the opposite direction — the window and the floor

Gravitational geometry can compute when $\lambda \gg 1$ (17-08). Csáki et al. evaluated the leading string correction to the f in the $M^2 = f(\lambda)\Lambda_{UV}^2$ of § 02, and for the 0^{++} glueball of **three-dimensional** pure Yang–Mills (QCD_3) obtained

$$f(\lambda) = c_0 + c_1 \lambda^{-3/2} + \cdots, \qquad c_1 < 0 \quad (4)$$

(the lowest 0^{++} state and the first five excited levels) Physics . $c_1 < 0$ means that with the ultraviolet cutoff Λ_{UV} fixed, decreasing λ lowers the mass, the direction consistent with the expectation of a continuum limit. The power in four dimensions is not in this paper. Below we borrow $p = 3/2$ as a rough guide for four dimensions.

The same paper writes, for the three-dimensional case, of taking $g_4^2 N \to 0$ as $R \to 0$ (to keep the QCD_3 coupling $g_3^2 N = g_4^2 N / R$ constant): 'This is the opposite of the limit that is required for the supergravity description to be valid'. The grounds for four dimensions are Witten's sentence in § 02 and the sentence 'To take the continuum limit ...' in the introduction of Csáki et al. Witten too writes of showing asymptotic freedom and computing masses at small η: 'This looks like a tall order, given our limited knowledge of worldsheet field theory with Ramond-Ramond fields in the Lagrangian' (world-sheet field theory is explained in § 09).

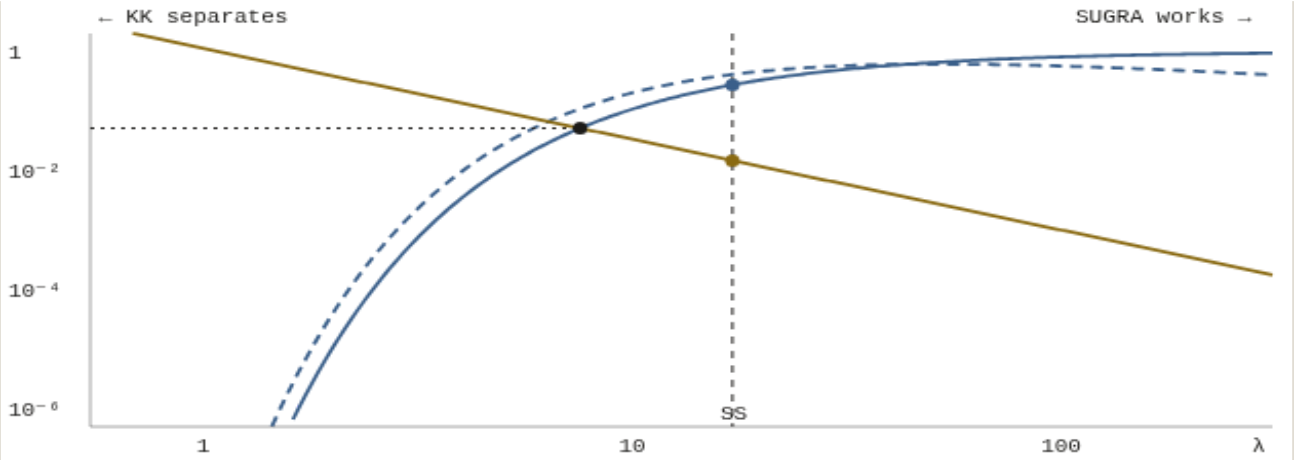
We look at the tug of war in numbers, with $e_{KK} = \Lambda R$ (the constant multiple in (2) set to 1) for how visible the tower is, and $e_{\text{str}} = \lambda^{-p}$ with coefficient 1 (borrowing (4), $p = 3/2$) as the rough guide to the string correction. We call the value at the crossing point λ^* of the two curves the **floor**.

$$F_p = e_{KK}(\lambda^*) = e_{\text{str}}(\lambda^*) \quad (5)$$

$\log e_{KK} + p \log \lambda$ is increasing in λ (at one loop and at two loops, if $p \geq 1$), so there is exactly one crossing point. The one-loop e_{KK} increases towards 1 and e_{str} decreases, so to the left of the crossing e_{str}, and to the right e_{KK}, exceeds F_p. That is, in the one-loop estimate $F_p = \min_\lambda \max(e_{KK}, e_{\text{str}})$, and **for any λ the larger of the two guides does not fall below F_p** Computed .

With the two-loop formula things differ. e_{KK} reaches its maximum 0.61 at $\lambda \approx 51$, and beyond that decreases towards 0 as $\lambda \to \infty$. So if the upper limit on λ is removed, the minimum of the max approaches 0 (for $p = 3/2$: 0.067 up to $\lambda \leq 10^4$, 0.038 for $\lambda \leq 10^5$, 0.014 for $\lambda \leq 10^6$) Computed . This is the behaviour of the perturbative formula for the running of the coupling extended to the large-λ side, outside the range where (2) can be used. The two-loop rows of Table 2 are crossing values, and also agree with the minimum of the max restricted to $\lambda \leq 316$ (the range of Figure 1).

Figure 1 Varying λ lets you read the values there of ΛR (blue; solid one loop, dashed two loops) and λ^{-p} (yellow). The horizontal dotted line is the one-loop floor F_p. SS is the λ of Sakai–Sugimoto (it can be off by a factor of about 2 by convention). The dashed line falling to the right of $\lambda \approx 51$ is outside the range where (2) can be used



λ = 16.63, ΛR: one loop 0.274, two loops 0.409, λ^−3/2 = 0.015, floor F = 0.051 (two-loop crossing 0.067)

Computed The numbers in this figure were computed on this machine (the curves are computed on the spot from the formulas; the floor agrees with Table 2).

Table 2 The floor F_p (the value at the crossing) and the crossing point λ^*

Formula for ΛR	p	λ^*	F_p
One loop	1	9.55	0.105
One loop	3/2	7.25	0.051
Two loops	1	7.77	0.129
Two loops	3/2	6.09	0.067

There is a **window** where both are around 5–13%, which does not contradict ratios agreeing with the lattice to a few per cent (when the three-dimensional $0^{--}/0^{++}$ is compared with the dilaton as reference; changing the reference mode or the states gives deviations of 14–26%: 17-09). But what a theorem needs is a **limit** sending the errors to 0, and in the one-loop estimate the floor blocks it. Both guides are crude, with coefficient 1; the point is not the value of the floor itself but that there is no direction making both small at once. Example: putting the $\lambda \approx 16.6$ of Sakai–Sugimoto straight into (2) gives $e_{KK} = 0.27$ (two loops 0.41) and $e_{\mathrm{str}} = 0.015$. Allowing for the difference of convention, halving λ (8.3) or doubling it (33) gives e_{KK} of 0.075–0.52 and e_{str} of 0.042–0.005, and their order does not swap Computed . This is a rough guide suggesting that the tower side remains large.

04

Extra things remain — the KK tower and supersymmetry

At $\lambda \gg 1$ the KK tower comes at the same height as the glueballs. The squared masses of fluctuations on a fixed background are eigenvalues of Laplace-type operators, and changing the metric as $g \mapsto s^2 g$ multiplies all eigenvalues by s^{-2}.

$$m_n^2 = \frac{c_n}{R^2} \qquad (c_n \text{ numbers fixed by the shape of the background alone})$$ (6)

If the background has a single length R, glueballs and KK modes alike are some multiple of $1/R$ (the counting of powers is Known ; the part saying that the background describes Yang–Mills is Physics). But the string tension involves λ, and Klebanov–Strassler write $T_s \sim g_s M\, m_{\text{glueball}}^2$ (the same circumstance as 17-08 § 07). That the tower does not separate is acknowledged by the authors in Table 3 (for Witten's background, by Csáki et al. and Klebanov–Strassler) Physics .

Table 3 What remains, and the authors' words

Model	What remains	The authors' words
Csáki et al. (Witten's background)	$SO(6)$ non-singlets	'their masses are of the same order as the SO(6) singlet states ... Presumably λ^{-1} corrections make them heavy'. Some fields removed by hand
Maldacena–Núñez	KK on the fivebranes	'the masses of glueballs are comparable to the masses of Kaluza Klein states on the fivebrane'
Klebanov–Strassler	KK on S^3	'in the SUGRA limit both scale as m/(g_sM)'. States explicitly that Witten's background also has extra modes
Polchinski–Strassler	$\mathcal{N} = 4$ matter	'the N=4 matter cannot be decoupled within the supergravity regime'
Sakai–Sugimoto	A tower of mass M_{KK}, $SO(5)$	'the same energy scale of the mass of the vector mesons'. A footnote says 'believed somewhat optimistically' (the latter in a footnote of hep-th/0507073)

How supersymmetry remains

In Witten's type (1998 § 4; fermions made antiperiodic on the circle), the fermions acquire masses of order $1/R$ at tree level, the scalars 'very plausibly get masses ... at one-loop level', and it continues 'If this is so, the low energy theory will be the pure SU(N) theory without supersymmetry' (§ 2.2, on the three-dimensional case, has a sentence to the same effect: at weak coupling the long-distance behaviour is 'very plausibly' only a three-dimensional gauge field). Klebanov–Strassler and Maldacena–Núñez preserve $\mathcal{N} = 1$ supersymmetry, so a further limit making the gluino (the fermion that is the supersymmetric partner of the gluon) heavy is needed. The five-dimensional model of Gürsoy–Kiritsis is of the kind (bottom-up) that puts a five-dimensional gravity model directly rather than deriving it from a string-theory background, and takes the β function as input ('using input ... as well as QCD') Physics . Example: within the range we examined, among the models built from string-theory backgrounds (top-down), only Witten's type yields non-supersymmetric pure Yang–Mills, and that too at the same scale as the tower.

05

$N \to \infty$ — when the group is fixed to one

Gravitational geometry becomes classical as $N \to \infty$. The quantities of the gauge theory are arranged by 't Hooft's counting (17-04) into a sum over genus h.

$$\log Z = \sum_{h \geq 0} N^{2-2h} F_h(\lambda) \quad (7)$$

What the gravity computation gives is the leading part of F_0 on the large-λ side, and for $SU(3)$ ($1/N^2 = 1/9$) all the remaining terms matter. That the ratios move slowly with N on the lattice (17-04) Physics does not give the existence of the theory at fixed N either. There is a gap in the kinds of group too: D-branes give $U(N)$ and $SU(N)$, and orientifolds (backgrounds divided by an operation that reverses the orientation of strings) give SO and Sp (17-07). Within the range we searched we found no holographic dual of four-dimensional pure Yang–Mills for the exceptional groups $G_2, F_4, E_{6,7,8}$, and geometric engineering can make types ADE, and types B, C, F, G by folding with outer automorphisms (automorphisms of Dynkin diagrams), but within supersymmetric theories (17-07). Example: in the problem statement, $SU(2)$ and G_2 are each a single theory.

06

The order of limits — exchanging them needs uniformity

The gravity computation takes the supergravity limit $\lambda \to \infty$ after the $N \to \infty$ that makes the string classical, and uses R as the unit.

$$N \to \infty \ \longrightarrow \ \lambda \to \infty, \qquad R \text{ fixed} \quad (8)$$

The problem statement's side fixes N and takes $R \to 0$, that is, by (3), $\lambda \to 0$. A standard theorem of analysis gives the condition under which two limits may be exchanged Known . A double sequence $a_{m,n}$ converges to b_m as $n \to \infty$ **uniformly in** m when

$$\sup_m \left| a_{m,n} - b_m \right| \xrightarrow[n \to \infty]{} 0$$

that is, when a single n makes the errors for every m small at once. Then the following holds.

$$a_{m,n} \xrightarrow[n \to \infty]{} b_m \ (m\text{-uniformly}), \quad a_{m,n} \xrightarrow[m \to \infty]{} c_n \implies \lim_{m \to \infty} b_m = \lim_{n \to \infty} c_n \quad (9)$$

Without uniformity the exchange is in general not allowed. Example: for $a_{m,n} = m/(m+n)$, $\lim_m \lim_n a_{m,n} = 0$ while $\lim_n \lim_m a_{m,n} = 1$ (since $\sup_m m/(m+n) = 1$, the

convergence to 0 as $n \to \infty$ is not uniform).

What is needed is that the $1/N$ corrections be uniformly small in λ, that the string corrections be uniformly small in N, and an unbroken road down to $\lambda \to 0$. (4) is the leading term of the correction in three dimensions, neither uniformity nor a limit. **Within the range we searched we found no result justifying this exchange.** As a metaphor, the supergravity region ('fixed ultraviolet cutoff … in the strong ultraviolet coupling regime') is in the position of a coarse lattice at strong coupling. More precisely, on the strong-coupling lattice the existence of the infinite-volume limit is proved (Osterwalder–Seiler 1978, 5-04), but this correspondence stays a metaphor.

07

Detour: seeing non-uniform convergence

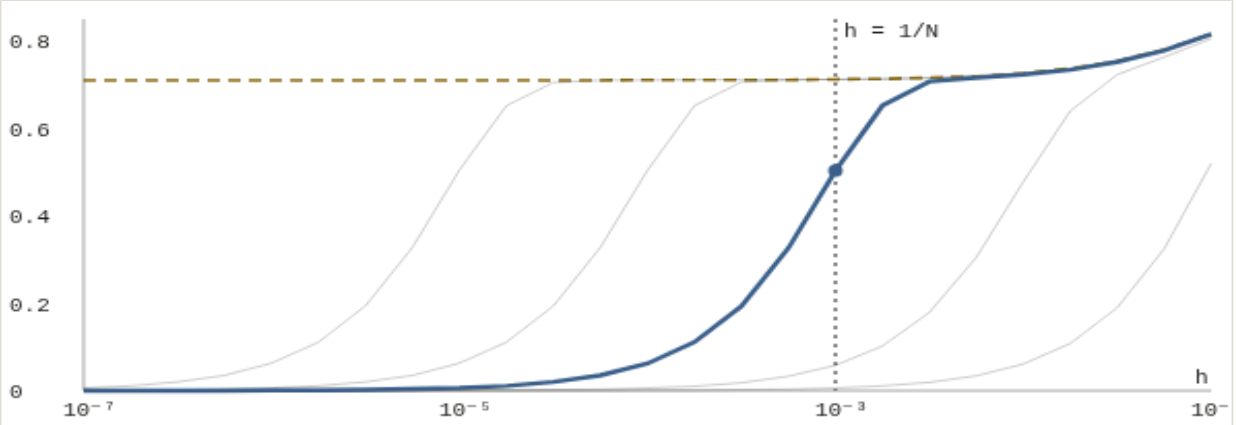
A section you may skip.

Spontaneous magnetisation changes value with the order of $h \to 0$ and $N \to \infty$ ((10) of 2-08). In the Curie–Weiss model ($J = 1$, inverse temperature $\beta = 1.25$) we count where it is non-uniform. The magnetisation per spin is written $\bar{m}$, as in 2-08, to distinguish it from mass. At finite N it can be written exactly as a sum of binomial coefficients.

$$\langle \bar{m} \rangle_{N,h} = \frac{\sum_k \binom{N}{k} \bar{m}_k \, e^{\beta N (\bar{m}_k^2/2 + h\bar{m}_k)}}{\sum_k \binom{N}{k} \, e^{\beta N (\bar{m}_k^2/2 + h\bar{m}_k)}}, \qquad \bar{m}_k = \frac{2k - N}{N} \tag{10}$$

By the symmetry of flipping all spins, $\langle \bar{m} \rangle_{N,0} = 0$, so $\lim_N \lim_{h \to 0} = 0$, while $\lim_{h \downarrow 0} \lim_N = \bar{m}^* = 0.71041$.

Figure 2 As N grows, the range over which $\langle \bar{m} \rangle_{N,h}$ (blue) clings to the $N = \infty$ curve (yellow dashed) extends to the left. The vertical dotted line is $h = 1/N$



N = 10³, ⟨m̄⟩ at h = 1/N = 0.503 ((11) at x = 1: 0.505)

Computed The numbers in this figure were computed on this machine ((10) evaluated exactly and embedded in steps of 0.25 in $\log_{10} h$).

Table 4 $\langle \bar{m} \rangle_{N,h}$ ($\beta = 1.25$). Bold is the $h = 1/N$ column

N	$h = 10^{-2}$	10^{-3}	10^{-4}	10^{-5}	$h = 0$
10^2	**0.4780**	0.0579	0.0058	0.0006	0
10^3	0.7231	**0.5028**	0.0625	0.0063	0
10^4	0.7255	0.7117	**0.5045**	0.0629	0
10^5	0.7257	0.7120	0.7105	**0.5047**	0
∞	0.7257	0.7120	0.7106	0.7104	$\bar{m}^*$ $(h \downarrow 0)$

Following a row to the right leads to 0, following a column down leads near $\bar{m}^*$, and the diagonal is about 0.5. For large N the distribution splits into two peaks at $\pm\bar{m}^*$ with weights in the ratio $e^{2\beta h N \bar{m}^*}$, so

$$\langle \bar{m} \rangle_{N,\, x/N} \longrightarrow \bar{m}^* \tanh(\beta \bar{m}^* x) \qquad (N \to \infty) \quad (11)$$

Against the value 0.50468 at $x = 1$, the exact value at $N = 10^5$ was 0.50467 Computed . **Non-uniformity sits in the band $h \sim 1/N$, which escapes to the left as N grows.** Here (10) can be written exactly, so the position of the band is known, but for the two knobs of (8) there is as yet no way of checking.

08

The distance from the problem statement — five requirements

The problem statement is 'Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on R^4 and has a mass gap $\Delta > 0$', and existence includes satisfying axiomatic properties (Jaffe–Witten § 4) Known .

Table 5 The requirements, and where the extra-dimensional models stand (every row at the level of Physics)

Requirement	Where things stand
Fix the group G	$N \to \infty$ for $SU(N)$. No duals for exceptional groups found (§ 05)
On $\mathbb{R}^4$	ΛR of order 1 on $\mathbb{R}^4 \times$ (extra dimensions) (§ 03)
Asymptotically free at short distances	The ultraviolet is a five-dimensional theory or the (2,0) theory (17-07). The cascade of Klebanov–Strassler 'goes on forever as the scale is increased'
Exists, satisfying the axioms (4-10)	AdS/CFT is a conjecture (17-08 § 08). The string-theory side has no non-perturbative definition
Mass gap	A supergravity eigenvalue problem (semiclassical). Witten: 'A mass gap likewise can be demonstrated ... by using the large λ behavior of the metric' (this λ is a radial coordinate of the metric)

None of the rows of Table 5 is satisfied yet. Jaffe–Witten § 6 wrote 'at present we do not know any non-trivial relativistic field theory that satisfies the Wightman (or any other reasonable) axioms in four dimensions' (a sentence from 2000; the existence of the four-dimensional theory is given by a construction published in 2026 as the limit of

Wilson's lattice, which the committee has assessed; Volume 2). On the mass gap, the problem page of the Clay Mathematics Institute still writes 'no proof of this property is known' Known (the definition of the gap is in 4-11). Example: the ratios the models hit to a few per cent (17-09; only the three-dimensional ratio with the dilaton as reference) are also numbers obtained while satisfying none of the rows of Table 5.

09

What remains — the problem moves

The problem does not vanish; it moves. Witten, noting that a singularity appears in the geometry as $\eta \to 0$, writes 'Apparently, in this approach, the mysteries of four-dimensional quantum gauge theory are encoded in the behavior of string theory near this singularity', and Csáki et al. write that under the conjecture the problem of QCD at large N reduces to 'that of controlling the α′ corrections to the two-dimensional sigma-model with the Ramond-Ramond background' Physics . We give the meanings of the words briefly. α' is the square of the string length (the string tension is $1/(2\pi\alpha')$, (1) of 17-05). The two-dimensional sigma model is the field theory on the two-dimensional surface swept out by a string (the **world sheet**), whose fields are maps from the world sheet to spacetime. Ramond–Ramond fields are fields sourced by D-branes. The α' corrections are corrections coming from the curvature radius of spacetime being finite compared with the string length, and (4) evaluates the leading term of this expansion. The 'worldsheet field theory with Ramond-Ramond fields' that Witten named in § 03 (a field theory on the world sheet whose Lagrangian contains Ramond–Ramond fields) is also this two-dimensional theory.

$$\text{4-dim. pure YM} \xleftarrow{\text{limit (3)}} \text{extra-dim. models} \xleftarrow{\text{conjecture}} \text{strings, M-theory} \ (12)$$

Example: neither arrow in (12) is a mathematical theorem (the left is unjustified, the right a conjecture), and the object at the right end has no non-perturbative definition either (17-13, 16-05).

An idea from this book's study (before verification)

Restricted to linear fluctuations of the supergravity fields, we see the "single length" of (6) as the reason the tower does not separate, and also as the reason that makes "a single scale" (17-12, 23-01) trivial on a fixed background. But the string tension involves λ (the formula of Klebanov–Strassler in § 04), and once strings are included the lengths are not exhausted by one. Going back to four dimensions is a change of vehicle from R to Λ, and the question remains after that. This is neither a theorem nor a claim of the literature (23-17).

What this chapter can say

What is said	What is not said
Known The powers in (6), (9), the quotations from the problem statement.	That the background describes four-dimensional pure Yang–Mills theory (a conjecture).
Computed Tables 1, 2 and 4, Figures 1 and 2, the floor of (5) (the value at the crossing) and that for the one-loop formula it is the minimum of the max, the maximum of the two-loop formula ($\lambda \approx 51$), $\lambda \approx 16.6$ and the range of a factor of 2 around it, the check of (11).	Coefficients independent of convention. e_{str} is a rough guide, not the actual string correction. The power in (4) is for the three-dimensional 0^{++}; the power in four dimensions is not given. The behaviour of the two-loop formula for $\lambda \gtrsim 51$ is outside the range of (2).
Physics That $R \to 0$ and $\lambda \to 0$ are the same direction, that the tower does not separate (the authors' words), the order in (8).	A justification of exchanging the limits (none found within the range searched). This book itself shows neither the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is in Volume 2).

Sources and reproduction

Item	Kind	Source / tools
(1), the problem statement in § 08 and the sentence of § 6	Known	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” § 4, § 6 (the Clay problem description). Text of the PDF checked.
The sentence on the Clay page in § 08	Known	Clay Mathematics Institute, “Yang-Mills & the Mass Gap” (the Millennium Prize Problems page). https://www.claymath.org/millennium/yang-mills-the-maths-gap/. Checked by reading the sentence.
The b_0 of (2) / the two-loop b_1	Known	The sources of 4-06 / W. E. Caswell, Phys. Rev. Lett. 33 (1974) 244 (bibliographic data only).
Witten's sentences, Witten's type	Physics	E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505, hep-th/9803131 § 2.2, § 4. arXiv manuscript checked.
(4), the sentences of Csáki et al.	Physics	C. Csáki, H. Ooguri, Y. Oz, J. Terning, JHEP 9901 (1999) 017, hep-th/9806021 (the abstract, (1.1)(1.2) of the introduction, the QCD_3 sentence of § 2, the concluding sentences). arXiv manuscript checked.
Klebanov–Strassler in Tables 3 and 5	Physics	I. R. Klebanov, M. J. Strassler, JHEP 0008 (2000) 052, hep-th/0007191. TeX manuscript checked.

Item	Kind	Source / tools
The other models in Table 3 and § 04	Physics	J. M. Maldacena, C. Núñez, Phys. Rev. Lett. 86 (2001) 588, hep-th/0008001; J. Polchinski, M. J. Strassler, hep-th/0003136; U. Gürsoy, E. Kiritsis, JHEP 0802 (2008) 032, arXiv:0707.1324. Abstracts checked for all.
Sakai–Sugimoto in § 01 and Table 3	Physics	T. Sakai, S. Sugimoto, Prog. Theor. Phys. 113 (2005) 843, hep-th/0412141; ibid. Prog. Theor. Phys. 114 (2005) 1083, hep-th/0507073. TeX manuscripts checked.
The strong-coupling lattice in § 06	Known	K. Osterwalder, E. Seiler, “Gauge Field Theories on the Lattice”, Ann. Phys. 110 (1978) 440. Abstract checked.
(9)	Known	Uniform convergence and the exchange of limits (for example W. Rudin, Principles of Mathematical Analysis, Chapter 7). Bibliographic data only.
Tables, figures, numbers	Computed	Python 3 (numpy, scipy) on this machine: scale_window.py (Tables 1 and 2), floor_crossing.py (uniqueness of the crossing in (5), the maximum of the two-loop formula, the minimum of the max for $\lambda \leq 316$, the range of a factor of 2 around $\lambda \approx 16.6$), floor_twoloop_check.py (the minimum of the max with the two-loop formula when the upper limit on λ is removed), curie_weiss_order.py (Table 4, Figure 2, (11)).

Next chapter to read: 17-11 Supersymmetry and Seiberg–Witten — an example where, under supersymmetry, a gap is visible from a duality.

2026-10-01 · chapter Yang–Mills and the mass gap Part 17 From the side of physics supersymmetry and electric–magnetic duality

Supersymmetry and Seiberg–Witten — switching to the dual, the strongly coupled vacuum becomes a weakly coupled theory of monopoles

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — That the low energies of the N=2 supersymmetric `SU(2)` theory are fixed by a single holomorphic function and its singularities and monodromies.
That at a strongly coupled point the mass of the monopole becomes 0, and that on breaking to N=1, confinement and a gap appear through monopole condensation as a weak-coupling calculation.
How far this is an argument of physics, and where theorems of mathematics begin.

Prerequisite chapters — 17-08. The tools are dual superconductivity in 17-02, dimensional transmutation in 17-01 and SU(2) in 1-04. Holomorphic functions, harmonic functions and the periods of a torus are defined on the spot.

Said first — The theory of this chapter is not pure Yang–Mills (it contains scalar fields and fermions). The central results are at the level of physics, and the N=2 theory itself has not been constructed mathematically. Nothing is shown about the mass gap of the problem statement.

The order of this chapter

1. What can be seen in a neighbouring theory — the one-loop coefficients
2. The minimum of supersymmetry — the set of fields, the flat directions, the coordinate u
3. A single holomorphic function — one loop and instantons
4. The contradiction with positivity — a harmonic function has no minimum in the interior
5. Exchanging electric and magnetic — $\tau \to -1/\tau$ and the BPS masses

6. Singularities and monodromy — the points where the monopole becomes light (Figure 1)
7. Writing with the periods of a curve — the periods of a torus and $\operatorname{Im}\tau > 0$ (Figure 2; Detour: at $u = 3$ we get $\tau = i$)
8. Reading the instanton coefficients from the exact solution — a series with rational coefficients, checked against one loop
9. The perturbation to N=1 — monopole condensation and the mass gap
10. The distance from pure Yang–Mills — the border between physics and mathematics

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01

What can be seen in a neighbouring theory — three theories and their one-loop coefficients

Pure Yang–Mills is hard because the coupling becomes strong at the low energies where the gap is decided, and perturbation theory does not work (17-01). Gauge theories with two sets of supersymmetry (N=2) are the typical **neighbouring theory** that can be computed into the strongly coupled side, and Seiberg and Witten (1994, SW below) proposed to fix the low-energy effective theory exactly in the case of `SU(2)`. The abstract says 'The "strongly coupled" vacuum turns out to be a weakly coupled theory of monopoles, and with a suitable perturbation confinement is described by monopole condensation' Physics .

The three theories differ in their set of fields and in the coefficient b of the one-loop running (in the convention where pure Yang–Mills has $11N/3$). One Weyl fermion in the adjoint representation (a two-component fermion; the Dirac fermion of the electron has four components) adds $-2N/3$, and one complex scalar adds $-N/3$ Known .

$$b_{\mathrm{YM}} = \tfrac{11}{3}N, \qquad b_{\mathcal{N}=1} = \tfrac{11}{3}N - \tfrac{2}{3}N = 3N, \qquad b_{\mathcal{N}=2} = \tfrac{11}{3}N - 2\cdot\tfrac{2}{3}N - \tfrac{1}{3}N = 2N \quad (1)$$

Example: for `SU(2)` these are, in order, $22/3,\ 6,\ 4$. The N=2 theory has no gap, since a massless photon remains; a gap appears only when it is broken to N=1 in § 09. Section 6.6 of the problem statement also cites Seiberg–Witten as an example of a route to understanding the gap in theories with matter ('for example, see [41]') Known .

02

The minimum of supersymmetry — the set of fields, the flat directions, the coordinate u

Supersymmetry is a symmetry that exchanges bosons and fermions, and N counts how many sets of its generators (supercharges) there are. A set of fields carried into one another by supersymmetry is called a **multiplet**. The smallest set for an N=2 gauge theory is the gauge field A_μ, two Weyl fermions λ, ψ and a complex scalar ϕ, all in the adjoint representation (1-04). The potential of the scalar is

$$V(\phi) = \frac{1}{g^2}\,\mathrm{tr}\,[\phi, \phi^\dagger]^2 \quad (2)$$

and it is 0 if ϕ and $\phi^\dagger$ commute. For `SU(2)` a gauge transformation brings it to $\phi = \frac{1}{2} a \sigma^3$ ($a \in \mathbb{C}$), and since a and $-a$ are carried into each other, the gauge-invariant coordinate is $u = \mathrm{tr}\,\phi^2 = \frac{1}{2}a^2$ Known . The vacua with $V = 0$ form a family numbered by the complex number u (a moduli space), and SW argue that this family is not lifted in the quantum theory either Physics .

For $a \neq 0$ the only elements commuting with ϕ are the diagonal `U(1)`, and the remaining two gauge fields (the W bosons) acquire mass by the **Higgs mechanism** (the mechanism by which a scalar field that takes a value gives mass to gauge fields; 3-07). The `U(1)` photon stays massless, so there is no gap. Example: if $\phi = \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $[\phi, \phi^\dagger] = \sigma^3$ and $V = 2/g^2 > 0$; if $\phi = \frac{1}{2} a \sigma^3$, then $V = 0$.

03

A single holomorphic function — the prepotential, one loop, instantons

A complex function is **holomorphic** if it can be differentiated as a function of a complex number. Under N=2 supersymmetry the low-energy effective theory can be written with a single holomorphic function $\mathcal{F}(a)$ (the **prepotential**) (Seiberg 1988, SW § 2) Physics . The complex number combining the coupling g and the θ angle,

$$\tau(a) = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} = \mathcal{F}''(a), \qquad ds^2 = \mathrm{Im}\,\tau(a)\,|da|^2 \quad (3)$$

gives the coupling of the photon and the distance (metric) on the family of vacua. Where $|a|$ is large, the theory is asymptotically free and weakly coupled (4-06). The original writes

$$\mathcal{F}(a) = \frac{i}{2\pi}\,a^2 \ln \frac{a^2}{\Lambda^2} + \sum_{k \geq 1} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k} a^2 \quad (4)$$

and continues 'Higher order perturbative corrections are absent. Instantons lead to new terms.' The first term is one loop, and the k-th term is the contribution of k instantons (3-05) Physics .

The power $4k$ comes from the $b = 4$ of (1). In the convention of (3) in 17-01, with $b_0 = b/16\pi^2 = 1/4\pi^2$,

$$\Lambda^4 = \mu^4 \exp\Big(- \frac{8\pi^2}{g^2(\mu)} \Big) \quad (5)$$

and the exponential is the weight of a single instanton. $(\Lambda/a)^4$ appears at no order of perturbation theory (4-12) Known . Example: in the units that put the singularities at $u = \pm 1$, one loop gives $\operatorname{Im}\tau \approx (2/\pi)\ln(2|a|)$ (from the numbers in § 07). The coefficient $2/\pi = 8\pi b_0$ agrees with the running of (5) Computed .

04

The contradiction with positivity — a harmonic function has no minimum in the interior

The metric of (3) must be positive. But the imaginary part of a holomorphic function is a **harmonic function**.

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Big) \operatorname{Im}\tau = 0 \qquad (a = x + iy) \quad (6)$$

The value of a harmonic function at the centre of a circle equals its average over the circle, and unless it is constant it cannot have a minimum in the interior of a region (the minimum principle) Known .

If τ is a single function on the whole a-plane, then $\operatorname{Im}\tau$ goes to $+\infty$ as $|a| \to \infty$, so it has a minimum somewhere, contradicting the principle. The original writes 'if Im τ(a) is globally defined it cannot be positive definite as the harmonic function Im τ cannot have a minimum', and concludes that the description by a is only locally correct Physics . Example: the one-loop $(2/\pi)\ln(2|a|)$ is negative for $|a| < 1/2$, and at $|a| = 1/4$ it is -0.44; that is, "$4\pi/g^2$ is negative" Computed . On the strongly coupled side another variable is needed.

05

Exchanging electric and magnetic — $\tau \to -1/\tau$ and the BPS masses

Maxwell's equations with magnetic charges added were symmetric under exchanging electric and magnetic ((6) in 17-02). Including the θ angle, the exchange is $\tau \to -1/\tau$,

and shifting θ by 2π gives $\tau \to \tau + 1$; the two generate the group $SL(2,\mathbb{Z})$ (integer entries, determinant 1) Known . SW set $a_D = \partial\mathcal{F}/\partial a$ and wrote the metric symmetrically in a and a_D.

$$ds^2 = \operatorname{Im}\left(da_D\, d\bar{a}\right), \qquad \tau_D = -\frac{1}{\tau} \quad (7)$$

With a_D as the coordinate the form is the same, and the coupling changes to $-1/\tau$. The original stresses that this is another description of the same theory Physics . The mass of a particle with electric charge n_e and magnetic charge n_m has a lower bound, and the mass m_{BPS} of particles equal to the bound (BPS states) is

$$m_{\mathrm{BPS}} = \sqrt{2}\,|n_e\, a + n_m\, a_D| \quad (8)$$

(Witten–Olive 1978, SW § 4) Physics . The W boson is $(n_m, n_e) = (0,1)$, the monopole $(1,0)$, and the dyons $(1,n)$. Example: classically $a_D = \tau_{\mathrm{cl}} a$, and the monopole is $|\tau_{\mathrm{cl}}| \approx 4\pi/g^2$ times heavier than the W, about 12.6 times for $g = 1$. On the strongly coupled side the ratio is reversed (Figure 2).

06

Singularities and monodromy — the points where the monopole becomes light

Differentiating the one-loop term of (4) gives $a_D \approx \frac{i}{\pi} a\left(\ln \frac{a^2}{\Lambda^2} + 1\right)$. For large u, $a \approx \sqrt{2u}$, so when u goes once round a large circle, $a \to -a$ and $\ln a^2 \to \ln a^2 + 2\pi i$, and $a_D \to -a_D + 2a$. Written as a matrix acting on the column vector $(a_D, a)^{\mathsf{T}}$ (the **monodromy**),

$$\mathsf{M}_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \quad (9)$$

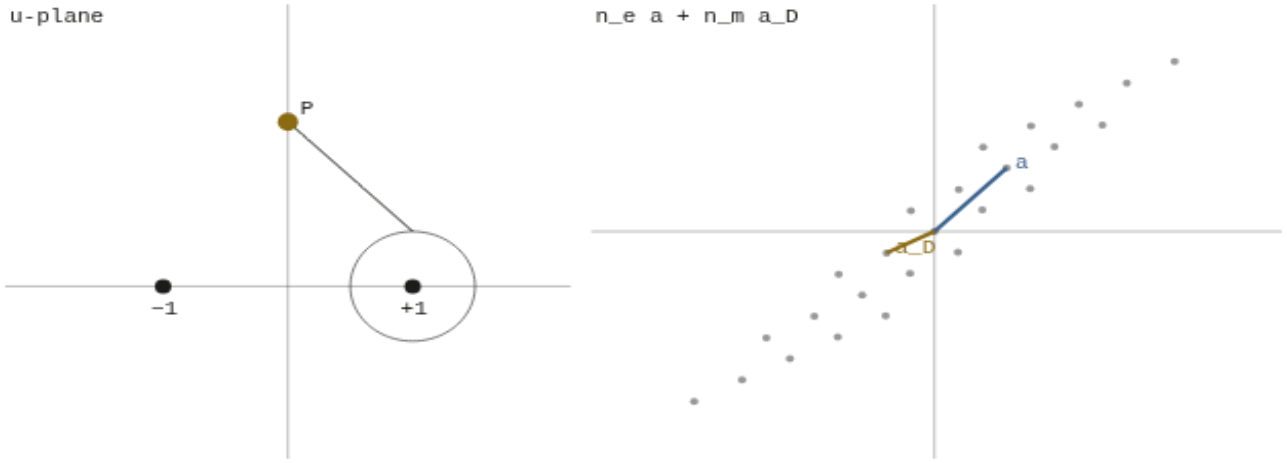
If every monodromy commuted with M_∞, then a^2 would be a global coordinate and we would be back at the contradiction of § 04. SW therefore proposed that there are two singularities at finite places (choosing units, $u = \pm 1$), where the mass of a monopole or a dyon becomes 0 Physics . Near $u = 1$ the effect of the light monopole gives $a \approx a_0 + (i/\pi) a_D \ln a_D$, and one turn gives $a \to a - 2a_D$.

$$\mathsf{M}_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad \mathsf{M}_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \qquad \mathsf{M}_1 \mathsf{M}_{-1} = \mathsf{M}_\infty \quad (10)$$

(The formulas of SW. In the same Figure 1, SW write the product in the order that puts the matrix of the loop traversed first on the right.) The charge $q = (n_m, n_e)$ of the

particle that becomes massless satisfies $q\mathsf{M} = q$; for M_1 it is $q = (1,0)$, that is, the monopole.

Figure 1 With "Play", u goes once round a singularity from $P = 1.5i$ (left), and a (blue) and a_D (yellow) move (right; the points are $n_e a + n_m a_D$). After one turn the pair is rewritten by a matrix



u = 0.000+1.500i, a = 1.256+1.256i, a_D = −0.832−0.424i

Computed The numbers in this figure were computed on this machine (the equation (PF) of § 07 solved along the path from P by the Runge–Kutta method; the matrices agree with Table 1).

Table 1 The matrices for one anticlockwise turn from the base point $P = 1.5i$ (deviation from integers at most 10^{-12})

Point encircled	Matrix	$q = (n_m, n_e)$ with $q\mathsf{M} = q$
$u = 1$: $\mathsf{M}(1)$	$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$	$(1,0)$ monopole
$u = -1$: $\mathsf{M}(-1)$	$\begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$	$(1,1)$ dyon
Both: $\mathsf{M}(\infty)$	$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$	None

In Table 1 the matrix for going round path A and then path B is written $\mathsf{M}_A\mathsf{M}_B$ (the order of analytic continuation). Seen from P, the large loop is "-1 and then $+1$", so $\mathsf{M}(-1)\mathsf{M}(1) = \mathsf{M}(\infty)$, and $\mathsf{M}(-1) = \mathsf{M}_\infty \mathsf{M}_1^{-1} = \mathsf{M}_\infty \mathsf{M}_{-1} \mathsf{M}_\infty^{-1}$ Computed . It is conjugate to the M_{-1} of SW, and the charge of the massless dyon changes from $(1,-1)$ to $(1,1)$. This is an instance of the original's 'complete democracy among dyons' (conjugating by M_∞ shifts the charge by 2).

07

Writing with the periods of a curve — the periods of a torus and $\operatorname{Im}\tau > 0$

SW § 6 assigns to each u the curve

$$E_u:\ y^2=(x-1)(x+1)(x-u)\quad (11)$$

Gluing the two sheets of $y=\pm\sqrt{\cdots}$ along two cuts joining the branch points $-1,1,u,\infty$ gives a surface with one hole (a **torus**). A torus is the plane divided by a lattice $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$, and with the orientation of $\tau=\omega_2/\omega_1$ chosen consistently, $\operatorname{Im}\tau>0$ Known . The branch points coincide only at $u=\pm1,\infty$, which agree with the singularities. Set the periods to be

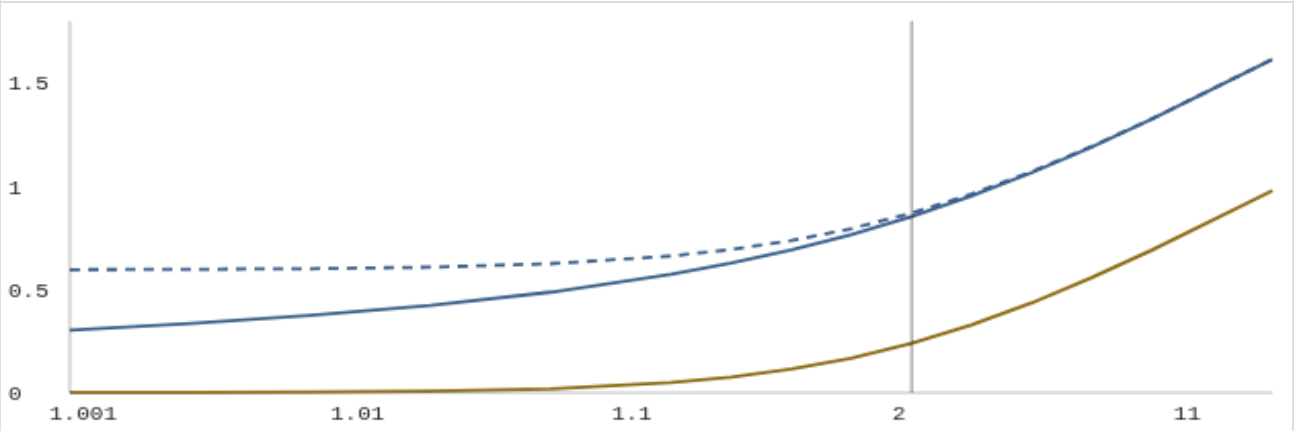
$$a=\frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,dx,\qquad a_D=\frac{\sqrt{2}}{\pi}\int_{1}^{u}\frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,dx,\qquad \tau=\frac{da_D/du}{da/du}\quad (12)$$

Differentiating in u, the integrand becomes a constant multiple of dx/y, and da/du and da_D/du are integrals along the two loops of the torus, that is, periods of the lattice (times the same constant). τ is their ratio, so $\operatorname{Im}\tau>0$ holds automatically (the properties of the curve are Known ; the correspondence with the theory is Physics). The two periods satisfy the same equation $4(u^2-1)\Pi''+\Pi=0$ (the Picard–Fuchs equation, (PF) below) (relative residual at most 10^{-15} on the real axis $u>1$ Computed). Figure 1 integrates this equation numerically along the path. Near $u=1$,

$$a_D\approx\frac{i(u-1)}{2},\qquad a\approx\frac{4}{\pi}-\frac{(u-1)\ln(u-1)}{2\pi}\quad (13)$$

and the $a\to a-2a_D$ of (10) comes out. Example: the numerical integration of (12) agreed with the closed form in hypergeometric functions to within 10^{-8}, and $a(1)=4/\pi$ Computed .

Figure 2 Choosing u (>1) on the real axis with the knob lets you read the mass ratio of monopole to W, $|a_D|/a$ (yellow), and the coupling $\operatorname{Im}\tau=4\pi/g^2$ (blue; the dashed line is one loop, $(2/\pi)\ln 2a$)



u = 1.989, W: √2a = 2.773, monopole: √2|a_D| = 0.662, ratio 0.239, Im τ = 0.853 (one loop 0.870)

Computed The numbers in this figure were computed on this machine (15 points of the numerical integration of (12) embedded).

Table 2 Periods and coupling on the real axis $u > 1$ (in units that put the singularities at $u = \pm 1$)

u	a	$\lvert a_D\rvert/a$	$\operatorname{Im}\tau$	One loop
1.001	1.2750	0.0004	0.303	0.596
1.5	1.6776	0.1448	0.735	0.771
3	2.4320	0.3727	1.000	1.007
100	14.1421	1.4912	2.128	2.128

The one-loop column is $(2/\pi)\ln 2a$. As $u \to 1$, $\operatorname{Im}\tau$ goes to 0 and the electric coupling diverges.

Detour: at $u = 3$ we get $\tau = i$

A subsection you may skip.

Under $x \mapsto (x+1)/2$ the branch points move to $0, 1, 2, \infty$ and the curve becomes $y^2 = x(x-1)(x-2)$. Under $x \mapsto 2 - x$ the right-hand side changes sign, so $(x, y) \mapsto (2 - x, iy)$ carries the curve to itself and returns to the start after 4 steps. This is a symmetry that turns the lattice by 90° onto itself, and such a lattice is square Known . If it is square, τ is a value carried to i by $SL(2, \mathbb{Z})$, and in the normalisation of (12) it was exactly $\tau(3) = i$ Computed . Since $-1/\tau = \tau$, it is the point where the electric and magnetic couplings are equal.

08

Reading the instanton coefficients from the exact solution — a series with rational coefficients, checked against one loop

Expanding (12) at large u and solving inversely for a gives a series with rational coefficients c_k Computed .

$$u = \frac{a^2}{2} + \sum_{k\geq 1} c_k\, a^{2-4k}, \qquad c1 = \tfrac{1}{4}, \quad c2 = \tfrac{5}{64}, \quad c3 = \tfrac{9}{128}, \quad c4 = \tfrac{1469}{16384} \tag{14}$$

Matone (1995) introduced $\mathcal{G} = \pi i\big(\mathcal{F} - \tfrac{1}{2} a\, a_D\big)$, and the abstract says it is modular invariant. Putting $\mathcal{G} = u$ into (4), the one-loop term gives $a^2/2$ and the k-th term gives $2\pi i k$ times the term in a^{2-4k}, so the coefficient of a^{2-4k} in (4) reads as $c_k/(2\pi i k)$. The equation of the abstract, $(1 - \mathcal{G}^2)\mathcal{G}'' + \tfrac{1}{4} a\, \mathcal{G}'^3 = 0$, also becomes the same equation as (PF) on setting $\mathcal{G} = u$ Computed .

We check this reading with a_D. Example: at $u = 2$ the numerical integration of a_D/i gives 0.473434. One loop alone gives 0.462621, up to $k = 1$ 0.473083, and up to $k = 3$ 0.473432; each added term shrinks the difference by more than one digit Computed .

Dorey–Khoze–Mattis (1996) computed one and two instantons directly and reported 'precise agreement' Physics .

09

The perturbation to N=1 — monopole condensation and the mass gap

In the language of N=1, the N=2 set splits into the set (A_μ, λ) (the vector multiplet) and the set Φ of ϕ, ψ. Adding a term $m\,\mathrm{tr}\,\Phi^2$ that gives mass to Φ, an N=1 theory remains at low energies. The original describes this theory as 'strongly believed to generate a mass gap', and lists confinement, the breaking $\mathbb{Z}_4 \to \mathbb{Z}_2$ and that there are exactly two vacua (citing Witten 1982) Physics . $\mathbb{Z}_4$ is the symmetry generated by the transformation that multiplies the phase of λ by i, and the remaining $\mathbb{Z}_2$ is multiplication by -1.

Near $u = 1$ we use the dual photon set A_D and the monopole fields $M, \tilde{M}$ (names of fields, not masses). The holomorphic function that fixes the interactions (the superpotential) and the conditions for the vacuum are

$$\widehat{W} = \sqrt{2}\,A_D M\tilde{M} + m\,U(A_D); \qquad \sqrt{2}\,M\tilde{M} + m\,\frac{du}{da_D} = 0, \quad a_D M = a_D\tilde{M} = 0 \tag{15}$$

where $U(A_D)$ is the coordinate u written as a function of A_D for the whole set. If $m \neq 0$ and $du/da_D \neq 0$, then $M, \tilde{M} \neq 0$ is required, and $a_D M = 0$ gives $a_D = 0$, that is, $u = 1$. From (13), $du/da_D = -2i$, so the solution with $|M| = |\tilde{M}|$ has $|M|^2 = \sqrt{2}\,|m|$ (in the units of § 06) Computed .

The monopoles condense. The dual photon acquires mass by the Higgs mechanism, and the original writes 'Expanding around this vacuum, it is easy to see that there is a mass gap'. What condenses is magnetic charge, so the electric field is squeezed into tubes and electric charges are confined (exactly the picture of 17-02) Physics . At $u = -1$ a dyon condenses, and there are two vacua. Example: for $m = 0.01$, $|M| = 0.119$ Computed . There are two scales, Λ and m, so the situation differs from "a single scale" of pure Yang–Mills (17-12).

10

The distance from pure Yang–Mills — the border between arguments of physics and theorems of mathematics

To reach pure Yang–Mills, two limits are needed: decoupling Φ with $m \to \infty$, and sending the mass of the fermion λ to infinity as well. The weak-coupling description of § 09 is for small m, and that the gap and confinement are not interrupted along the

limits remains an expectation; there is no means to control it (17-10) Physics . The whole argument also presupposes that the N=2 theory exists. The example of SW is the typical case where, in four dimensions, confinement was derived from a duality at the level of physics (semiclassical calculations also exist, such as Polyakov's in three dimensions; 17-02), but what remained on the side of mathematics took another form.

Table 3 Claims and levels

Claim	Level
The integral over the space of instantons with $\varepsilon 1, \varepsilon 2$ (the two parameters of rotations in four dimensions) put in has, as $\varepsilon 1, \varepsilon 2 \to 0$, a limit that agrees with the SW prepotential (Nekrasov's conjecture)	Known Theorem: Nakajima–Yoshioka, Nekrasov–Okounkov ($SU(N)$), Braverman–Etingof (general simple groups)
Invariants of four-manifolds made by counting the solutions of the Seiberg–Witten equations (a mathematical definition)	Known (3-06, the four-dimensions article)
That these invariants are equivalent to the Donaldson invariants (the abstract of Witten 1994: 'Recent developments ... suggest a new point of view', 'one can define equivalent four-manifold invariants')	Physics (the current state of proofs on the side of mathematics was not investigated in this chapter)
(11) and (12) describe the low energies of the N=2 theory, and for $m \neq 0$ condensation, confinement and a gap appear	Physics
The gap of pure Yang–Mills	This chapter says nothing

The mathematics in Table 3 consists of theorems and definitions about integrals over the space of instantons and about solutions of equations, not claims about the spectrum (4-11). Example: the coefficients of (14), once the conventions are aligned, are numbers guaranteed by a theorem as integrals over the space of instantons. That the same numbers are "properties of the N=2 quantum field theory" cannot be said, since the construction of the theory (7-01) is missing. It is accurate to read this as an instance, at the level of physics, of the "duality transformations" of § 6.6 of the problem statement (17-13, 16-05).

What this chapter can say

What is said	What is not said
Known (1), the curve (11) and the properties of the torus, the proof of Nekrasov's conjecture, the definition of the SW invariants, § 6.6 of the problem statement.	That the N=2 and N=1 theories exist as quantum field theories (they have not been constructed).

What is said	What is not said
Computed Tables 1 and 2, Figures 1 and 2, (PF), (14), the one-loop constant, $\tau(3) = i$, the $\lvert M\rvert^2 = \sqrt{2}\lvert m\rvert$ of the monopole field. All start from (12).	That (11) and (12) are derived from the N=2 theory (the determination by holomorphy and duality is an argument of physics). How far the equivalence of the SW invariants and the Donaldson invariants has been shown as mathematics (not investigated in this chapter).
Physics The points where the mass of the monopole becomes 0, condensation, confinement and the gap for $m \neq 0$, agreement with the direct instanton calculations.	Control of the two limits back to pure Yang–Mills, and what they imply for the gap. This book itself shows neither the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is in Volume 2).

Sources and reproduction

Item	Kind	Source / tools
The formulas and quotations of SW ((2)–(4), (7)–(13), (15))	Physics	N. Seiberg, E. Witten, Nucl. Phys. B426 (1994) 19–52 (Erratum B430 (1994) 485), hep-th/9407087. Text of the TeX manuscript checked. From the reference list of the same paper (bibliographic data only): Seiberg, Phys. Lett. 206B (1988) 75; Witten–Olive, Phys. Lett. 78B (1978) 97; Witten, Nucl. Phys. B202 (1982) 253.
§ 6.6 of the problem statement	Known	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the Clay problem description) § 6.6. Text of the PDF checked.
(1) and (5)	Known	The source in 17-01. $b = 4$ checked against the $(\Lambda/a)^{4k}$ and $\mathbb{Z}_{4N_c}$ of SW.
Matone in § 08	Physics	M. Matone, Phys. Lett. B357 (1995) 342–348, hep-th/9506102. Abstract only (the definition of $\mathcal{G}$, modular invariance, the equation for $\mathcal{G}$, 'we find u=u(a)'). $\mathcal{G} = u$ is from the check in § 08; the main text was not read.
Dorey–Khoze–Mattis in § 08	Physics	N. Dorey, V. V. Khoze, M. P. Mattis, Phys. Rev. D 54 (1996) 2921, hep-th/9603136. Abstract checked.
Nekrasov's conjecture and its proofs in Table 3	Known	N. Nekrasov, Adv. Theor. Math. Phys. 7 (2004) 831, hep-th/0206161; H. Nakajima, K. Yoshioka, Invent. Math. 162 (2005) 313–355, doi:10.1007/s00222-005-0444-1, math/0306198; N. Nekrasov, A. Okounkov, in *The Unity of Mathematics* (Progress in Mathematics, Birkhäuser) 525–596, hep-th/0306238; A. Braverman, P. Etingof, math/0409441. Abstracts checked.
The Seiberg–Witten invariants in Table 3	Known, Physics	E. Witten, “Monopoles and Four-Manifolds”, Math. Res. Lett. 1 (1994) 769–796, hep-th/9411102. Abstract checked. Mathematical textbook: J. W. Morgan, *The Seiberg–Witten Equations and Applications to the Topology of Smooth Four-Manifolds*, Princeton University Press, doi:10.1515/9781400865161 (bibliographic data only; year of publication not checked).

Item	Kind	Source / tools
Tables 1 and 2, Figures 1 and 2, (PF), (14), the numbers in §07–§08	Computed	Python 3 on this machine (numpy, scipy): `sw_periods.py` (the check of (12), the one-loop constant, Table 2, Figure 2), `sw_monodromy.py` (Table 1, the starting values of Figure 1, $\tau(3) = i$), `sw_pf_check.py` (the residual of (PF), agreement with Matone's equation), `sw_instanton.py` ((14) and the check of a_D).

Next chapter to read: 17-12 The sentence that plays the role of an equivalence principle — cutting out "a single scale" as an assumption in one sentence.

The sentence that plays the role of an equivalence principle — cutting out "a single scale" as one assumption

> This chapter is this book's own investigation. § 01, § 02, § 08 and § 09 are established knowledge; § 10 is a map of open questions.

What this chapter shows — We take out of the path of pure Yang–Mills one sentence that stands in the same place as the equivalence principle at the start of general relativity: "the tube of confinement has no length parameter other than the string tension". We follow in formulas how joining four lengths by three chains gives "the diffusive-relaxation-time assumption", and check it with the lattice tables and with examples where one scale breaks and where it holds.

Prerequisite chapters — 17-01 How physics sees it and 4-07 Dimensional transmutation and Λ. The Poincaré constant (1-10) and the correlation length (2-08) are defined on the spot as far as needed.

Said first — The correspondence with the equivalence principle is an analogy, not a mathematical equivalence. "A single scale" is not a theorem of the four-dimensional continuum theory, and neither the relaxation-time assumption nor any of the three chains has been proved. This chapter does not show the mass gap; it names the one sentence that remains unproved. The lattice numbers are values from primary sources.

The order of this chapter

1. General relativity began from one sentence — there is no second kind of mass
2. Detour: measuring a second kind of mass — the 10^{-15} of MICROSCOPE
3. A single scale — two kinds of length — the ball length and the string length line up
4. Looking at the lattice tables — the form of the extrapolation comes in from outside
5. Four lengths and three chains — splitting the relaxation-time assumption into a product of ratios

6. Which chain plays the role of the equivalence principle — a table of correspondences (an analogy)
7. Dropping candidates — the other three candidates
8. Examples where one scale breaks — U(1), supergravity, the leading strong-coupling term
9. Examples where one scale holds, and the input needed — three-dimensional $\mathbb{Z}_2$, and I and R
10. Where it differs from the equivalence principle — it cannot be put as an assumption

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01

General relativity began from one sentence

General relativity is sometimes explained as a combination of the **equivalence principle**, the **constancy of the speed of light**, **general covariance** (the form of the laws is the same in any coordinates) and **Riemannian geometry**. Riemannian geometry is a tool of mathematics, the constancy of the speed of light is a premise inherited from special relativity (setting $c = 1$ is a convention of units), and general covariance is a requirement on how laws are written. On top of these, the one sentence of physics newly added about gravity is the equivalence principle. We write falling in Newtonian mechanics.

$$m_{\mathrm{i}}\, a = m_{\mathrm{g}}\, g \quad \Longrightarrow \quad a = \frac{m_{\mathrm{g}}}{m_{\mathrm{i}}}\, g \tag{1}$$

m_{i} is the **inertial mass** (how hard a body is to move), and m_{g} is the **gravitational mass** (how strongly gravity pulls on it). They are defined by separate experiments, and the theory gives no reason for them to be equal. The equivalence principle says that the ratio $m_{\mathrm{g}}/m_{\mathrm{i}}$ is the same for every body; that is, **there is no second kind of mass**. Then gravity disappears locally inside a freely falling box, and gravity can be written as the geometry of spacetime (said to begin with Einstein 1907; the spacetime article) Physics. Example: if the ratio differs between materials by η, the difference in acceleration is about ηg (§ 02).

The question is: **does the path to the mass gap also contain "one sentence that is obvious physically and unproved mathematically"?** It does: "the tube of confinement has no length parameter other than the string tension". Its role, however, differs from that of the equivalence principle (§ 10).

02

Detour: measuring a second kind of mass

A section you may skip.

The difference in how two bodies fall is measured by the signed quantity $\eta = 2(a_1 - a_2)/(a_1 + a_2)$ (a_1, a_2 are the accelerations of the two bodies). Long ago there was the torsion balance of Eötvös and others (said to be a paper of 1922), and the satellite MICROSCOPE reported this final result with titanium and platinum Known .

$$\eta(\mathrm{Ti}, \mathrm{Pt}) = \left[-1.5 \pm 2.3\,(\mathrm{stat}) \pm 1.5\,(\mathrm{syst}) \right] \times 10^{-15} \quad (2)$$

Combining the errors in quadrature gives $\pm 2.75 \times 10^{-15}$, compatible with 0 Computed . More precisely, experiment only narrows the band and cannot show $\eta = 0$. Physics puts this one sentence as an axiom and keeps checking it.

03

A single scale — two kinds of length

Classical four-dimensional Yang–Mills has no constant with the dimension of mass, and in the quantum theory a single scale Λ appears (4-07). Physics takes a quantity whose mass dimension is k to be a constant multiple of Λ^k (for example the mass of the lightest glueball $m = c_1 \Lambda$ and the string tension $\sigma = c_2 \Lambda^2$; 17-01 § 03) Physics . Then the ratio $m/\sqrt{\sigma} = c_1/\sqrt{c_2}$ is a pure number fixed by the group alone. On the lattice (5-01) what is measured is am and $a\sqrt{\sigma}$ in lattice units; as $\beta \to \infty$ both go to 0, and in the ratio a cancels. Below, m, σ are numbers in lattice units. Making "a single scale" into an inequality uniform in β gives the following sentence (as the record of this book's study writes it).

$$c\,\sigma(\beta) \;\le\; m(\beta)^2 \;\le\; C\,\sigma(\beta) \qquad (\beta \ge \beta_0) \quad (3)$$

The constants $c, C > 0$ do not depend on β. It assumes neither $m > 0$ nor $\sigma > 0$; it only says that if they go to 0, they do so at the same rate. In terms of length, we call the length $\xi = 1/m$ of the lightest glueball (glue plus ball) the **ball length** below (the correlation length: the distance over which correlations decay as $e^{-|x|/\xi}$; 2-08 § 03). $\xi_\sigma = 1/\sqrt{\sigma}$ is the **string length**. $A \asymp B$ means that the ratio A/B is bounded above and below by positive constants independent of β. (3) is the same as $\xi \asymp \xi_\sigma$. It is a claim of the same form as "two kinds of mass", and in the language of physics it is the statement that **no second dimensionless parameter appears**. If there were a second parameter z, the ratio would be $F(z)$, and could go to 0 or to infinity.

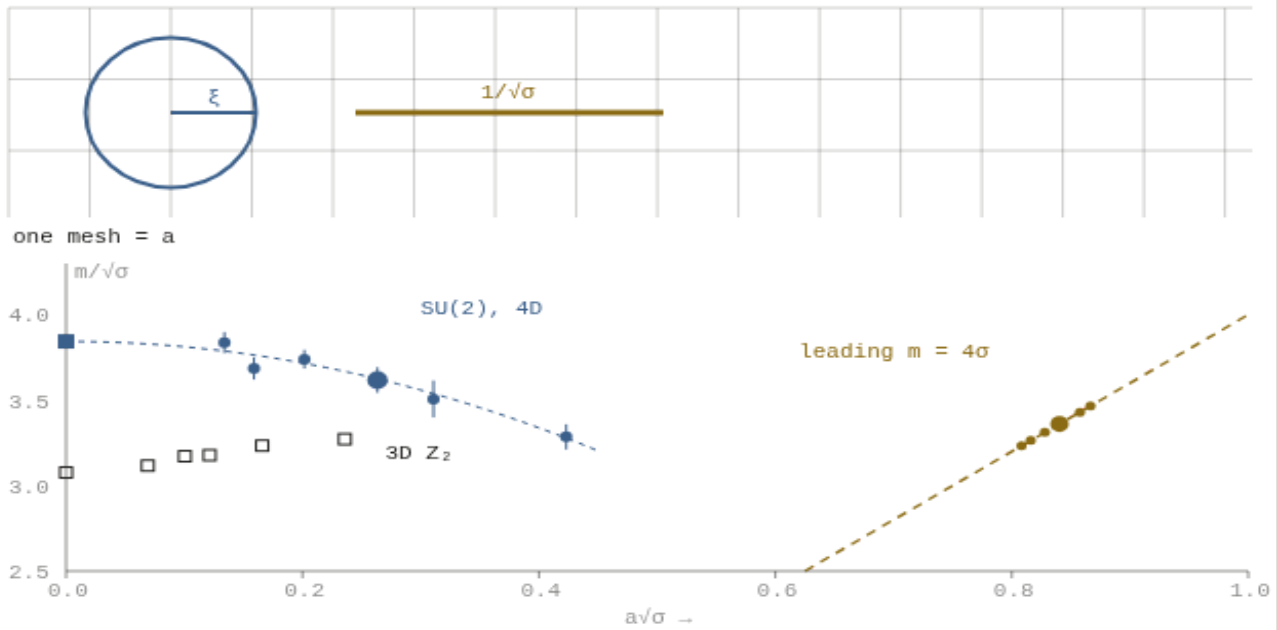
Example: for SU(2) at $\beta = 2.60$ (24^4), $a\sqrt{\sigma} = 0.13395$ and $am = 0.514$ (Lucini–Teper), so in lattice units $\xi_\sigma = 7.47$ and $\xi = 1.95$, and the ratio is 3.837 Computed . That this ratio stays bounded is the content of (3).

04

Looking at the lattice tables — the form of the extrapolation comes in from outside

These are the six SU(2) points of Lucini–Teper (2001). σ is the mass am_l of a tube of glue wound once round the periodic direction, plus the string-fluctuation correction $\pi/(3L)$, divided by the length L (in lattice steps) (equation (4) of the original paper: $am_l = a^2\sigma L - \pi/(3L)$). In using this formula the authors state 'we are assuming that we have linear confinement' (17-01 § 06).

Figure 1 The top is drawn in physical units (the string length $1/\sqrt{\sigma}$ fixed as the length of the bar). Raising β with the knob makes only the lattice mesh (spacing a) finer, and the ball length ξ (the radius of the circle) hardly changes. The bottom is the ratio $m/\sqrt{\sigma}$ (the square is three-dimensional $\mathbb{Z}_2$). The button overlays the leading strong-coupling term (dashed)



β = 2.4 (12⁴) a√σ = 0.2634 am = 0.953, in lattice units 1/√σ = 3.80、ξ = 1.05 m/√σ = 3.618, leading term 3.361

Table A $m/\sqrt{\sigma}$ for SU(2) (from Lucini–Teper Table 1. The master copy of the six values of am is 16-05 Table 3, and of $a\sqrt{\sigma}$ 5-09 Table 1)

β	Lattice	$a\sqrt{\sigma}$	am	$m/\sqrt{\sigma}$
2.25	8^4	0.4231(25)	1.390(30)	3.285(74)
2.30	10^4	0.3108(17)	1.090(33)	3.507(108)
2.40	12^4	0.2634(14)	0.953(19)	3.618(75)

β	Lattice	$a\sqrt{\sigma}$	am	$m/\sqrt{\sigma}$
2.475	16^4	0.2016(13)	0.754(10)	3.740(55)
2.55	20^4	0.15896(63)	0.586(10)	3.686(65)
2.60	24^4	0.13395(62)	0.514(8)	3.837(62)

Computed The numbers in this figure and table were computed on this machine (ratios of values from primary sources; the errors are a rough guide, treated as independent).

The string length in lattice units, $1/(a\sqrt{\sigma})$, grows from 2.36 to 7.47, but the ratio stays within 3.29–3.84, and broadly increases, approaching from below (it falls once from 2.475 to 2.55, within the errors). A fit with a constant independent of a is rejected with $\chi^2 = 38.6$ (5 degrees of freedom), so the ratio depends on a Computed . The continuum value is extrapolated after fixing the form of this dependence.

$$\frac{m}{\sqrt{\sigma}}(a) = r_0 + c\,(a\sqrt{\sigma})^p \quad (4)$$

Table B Fits of (4) (least squares weighted by the errors, all 6 points)

p	Continuum value r_0	χ^2 / dof
2	3.844(43)	2.39 / 4
1	4.053(73)	2.73 / 4
Primary source (Table 5)	3.844(61)	—

$p = 2$ agrees with the primary source in the central value, but $p = 1$ gives a χ^2 just as good, with a continuum value of 4.05 Computed . **The data alone cannot choose the form of the extrapolation.** The authors write 'the functional form of such corrections is known' and use $O(a^2)$, that is, $p = 2$ (5-05). 3.844 is a number resting on the premise that the limit exists and is approached in this form (17-01 § 08) Physics .

05

Four lengths and three chains — the diffusive-relaxation-time assumption

There are four lengths, each counted in lattice steps: the **bookkeeping length** ξ_{RG}, the string length ξ_σ, the ball length ξ, and the **length over which fluctuations are smoothed out** $\sqrt{CP}$.

The bookkeeping length is the length fixed by writing down, for each step that coarsens the lattice by a factor b (one step of the renormalisation group, 7-05), how much the coupling β falls, and piling up the steps. After k steps lengths are multiplied by b^k, so the number of steps needed to fall to a coupling of a fixed strength decides "how many lattice steps make one physical unit". We call that the bookkeeping length, and in this unit set the lattice spacing to $a := 1/\xi_{\mathrm{RG}}$ (the definition in the record of this

book's study; 23-02). Counting by the one-loop running alone (4-06), for SU(2), up to a constant factor, $\xi_{\mathrm{RG}} \approx e^{3\pi^2\beta/11}$ ($\beta = 4/g^2 = 2N/g^2$ of the table of notation; the coefficient in the exponent is a rewriting of the one-loop coefficient b_0) Known .

C_P is, for the probability distribution μ of the lattice gauge field (SU(2), Wilson action, infinite volume, no gauge fixing),

$$\mathrm{Var}_\mu(f) \;\le\; C_P\,\mathcal{E}(f), \qquad \mathcal{E}(f) = \mathbb{E}_\mu\Big[\sum_\ell |\nabla_\ell f|^2\Big] \quad (5)$$

the best constant for which this holds for every f (the Poincaré constant). Var_μ is the variance under μ, and ∇_ℓ is the derivative in the SU(2) directions of the link ℓ. Depending on whether the derivative is taken in the angle ϑ of the link ($U = e^{i\vartheta\cdot\sigma}$) or in the field $A = \sqrt{\beta}\,\vartheta$, $\mathcal{E}$ changes as $\mathcal{E}_A = \mathcal{E}/\beta$, and C_P differs by a factor β. In the free approximation $C_P = (g^2/4)\xi^2$ for the former and $C_P = \xi^2$ for the latter. Whether the band (6) below holds depends on this choice (with the angle normalisation the lower side breaks logarithmically even in the free approximation: 18-05 § 03). The best constant is the inverse of the gap (1-10 § 07). The dynamics at work is **Langevin dynamics**, a stochastic time evolution that moves each link little by little in the direction that lowers the action while adding a small random shake. Run for long, the distribution settles to μ. $\lambda_1 = 1/C_P$ is the gap of this dynamics, and C_P is the time of the slowest return. More precisely, C_P equals the return time only for reversible dynamics in continuous time (Langevin); for discrete-time updates or HMC it is a different quantity (18-03 § 06). The assumption the record of this book's study makes is that the length of fluctuations is on the same scale as the bookkeeping length, which in band form is (6) below. The names follow A-06 Index of names. **The relaxation-time assumption** is the statement at the level of powers $\lim \ln C_P / \ln \xi = 2$, and (6) is the band statement H_{band} (naming the normalisation, $H_{\mathrm{band}}[\mathcal{E}_\vartheta]$ or $H_{\mathrm{band}}[\mathcal{E}_A]$) combined with the chains (ii)(iii).

$$a^2 C_P \asymp 1 \qquad (\text{i.e. } \sqrt{C_P} \asymp \xi_{\mathrm{RG}}) \quad (6)$$

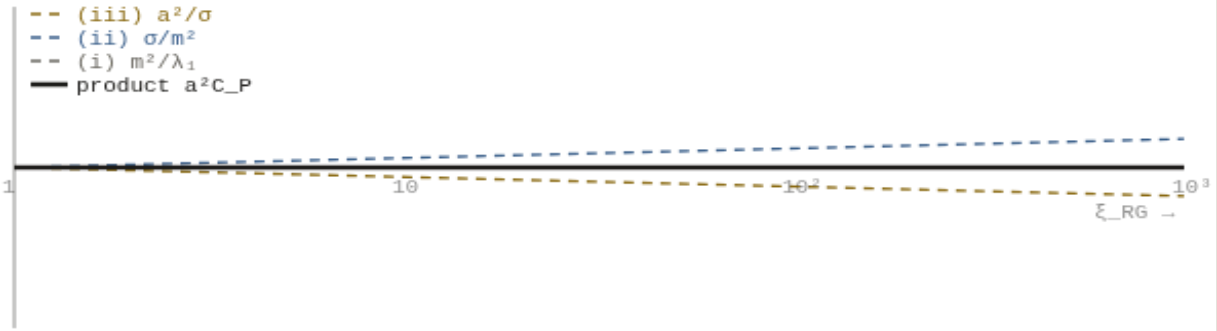
We split this into a product of three ratios.

$$a^2 C_P \;=\; \underbrace{\frac{a^2}{\sigma}}_{\text{(iii)}} \cdot \underbrace{\frac{\sigma}{m^2}}_{\text{(ii)}} \cdot \underbrace{\frac{m^2}{\lambda_1}}_{\text{(i)}} \quad (7)$$

That each factor is $\asymp 1$ we call one **chain**. (i) $\lambda_1 \asymp m^2$: the fluctuation length and the ball length line up (the relaxation chain). Langevin dynamics is diffusive, so smoothing out fluctuations of length ξ takes a time of order ξ^2 (dynamical exponent $z = 2$) Physics . That is why λ_1 is compared not with m but with m^2. (ii) $m^2 \asymp \sigma$: the ball length and

the string length line up (one scale, (3)). (iii) $\sigma \asymp a^2$: the string length and the bookkeeping length line up (confinement happens at the bookkeeping scale). If all three hold, (6) follows. Under (ii)(iii), (6) is the same as the band statement $C_P \asymp \xi^2$, and the double-well relaxation-time power statement follows from it (A-06). The converse does not follow: if the factors cancel each other, the product is constant even when chains are broken (Figure 2). The algebra in the direction from (7) to (6) is machine-checked Lean (`chain_band`, `sq_times_poincare_band`, `LatticeGaugeOneScale.lean`. The hypotheses are three band inequalities, and the conclusion is that $a^2 C_P$ lies in a positive band. The list is in 23-13), but none of the three chains themselves has been proved.

Figure 2 Changing the powers p of the three factors (factor $\propto \xi_{\mathrm{RG}}^{p}$) with the knobs tilts the three dotted lines and the solid line of the product $a^2 C_P$. If the product is horizontal, (6) holds. In the initial position ((iii) −0.3, (ii) +0.3) two chains are broken, yet the product is horizontal



`Product a²C_P ∝ ξ_RG^(+0.00)  (6) : holds, chains: (i) ○  (ii) ×  (iii) ×`

Computed A schematic drawing of the sum of the logarithms in (7), drawn on the spot (not a measurement).

06

Which chain plays the role of the equivalence principle

(i) is a relation between the dynamics chosen as a tool and the gap of the theory (the gap of the transfer matrix, 5-03); change the dynamics and the statement changes too. (iii) depends on how the steps are booked. If we call the small error dropped at each coarsening step the **shortfall**, (iii) is governed by the one-loop coefficient and by the shortfalls summed over the steps (23-02). In the record's arrangement, only (ii) is a sentence of the theory that depends neither on the tool nor on the bookkeeping.

The one sentence on this path

The tube of confinement has no length parameter other than the string tension ($m^2 \asymp \sigma$, equation (3)).

In the same pattern as "there is no second kind of mass", the record made the following table of correspondences.

General relativity	This path (analogy)
Riemannian geometry (the stage)	Lattice gauge theory (a probability measure in finite dimensions)
General covariance (a requirement on how laws are written)	One step of the renormalisation group (coarsening the lattice leaves the content the same)
The convention of units coming from the constancy of the speed of light ($c = 1$)	The convention of setting the scale (the convention of units that sets $\sqrt{\sigma} = 1$; 17-01 § 07)
The equivalence principle (one sentence of physics)	(ii) One scale (there is no second length parameter)

As a metaphor, just as general relativity made "weight" one kind, this sentence makes "length" one kind. More precisely, the table is an analogy and no theorem joins its left and right; the equivalence principle is an equality at each point, while (3) is a uniform inequality.

07

Dropping candidates — dual superconductivity, centre vortices, asymptotic scaling

(a) Dual superconductivity and **(b) centre vortices** (17-02) are statements about configurations after fixing a gauge and projecting. In raw form they depend on how the gauge is fixed (Gribov copies, 17-02 § 04), and do not become a sentence about the probability measure. The core that can be written gauge-invariantly ('t Hooft's vortex free energy) is a statement about σ, and merges into chain (iii). **(c) Non-perturbative asymptotic scaling** is the sentence that lengths in lattice units grow as perturbation theory predicts. The two-loop prediction is the following f_2 (SU(N), $g^2 = 2N/\beta$) Known .

$$f_2(\beta) = (b_0 g^2)^{-b_1/(2b_0^2)} \exp\Big(-\frac{1}{2b_0 g^2}\Big), \qquad b_0 = \frac{11N}{48\pi^2}, \quad b_1 = \frac{34N^2}{3(16\pi^2)^2} \tag{8}$$

The coefficients agree with the values in the PDG review (without quarks) (checked at $N = 3$ Computed). (c) says "the inverse of every length in lattice units, for example am/f_2 or $a\sqrt{\sigma}/f_2$, converges to a positive constant". If f_2 is read as the scale marking of physical units, the convergence of am/f_2 is itself the statement that the mass gap in physical units is positive and finite. Since it contains the conclusion we want to show, the record set it aside as circular.

It differs numerically too. For SU(2) with the bare coupling $g^2 = 4/\beta$, as $\beta = 2.40 \to 2.60$, $a\sqrt{\sigma}/f_2$ goes 57.26→48.23 (−15.8 %), while the ratio goes 3.618→3.837 (+6.1 %) Computed . The ratio moves less, and does not use perturbation theory either.

The bare coupling β carries large corrections peculiar to the lattice. Lucini–Teper use the mean-field improved coupling $\beta_I = \beta \cdot \frac{1}{N}\langle \mathrm{Re}\,\mathrm{Tr}\,U_p\rangle$ (U_p is the matrix of the plaquette) and write 'Transforming this to β_I is known to remove a large part of the lattice corrections at small a.' (§ 5 of the original paper, the sentence after equation (11); as the origin of this coupling the original paper cites earlier literature) Physics . The values obtained by finding the two-loop Λ with this coupling and extrapolating to the continuum limit are in Table 6 of the original paper, but for the SU(2) extrapolation it says 'the best continuum fit is very poor'.

08

Examples where one scale breaks

That (3) is not obvious is seen from examples where it breaks. **Three-dimensional compact U(1)**: Göpfert–Mack proved $\sigma > 0$ at all couplings Known , and also state that in the continuum limit, with the mass fixed in physical units, the string tension goes to infinity ($\sigma/m^2 \to \infty$) (from the abstract of the same paper; Villain action; the main text not read). If that description is right, there is confinement and yet the ratio $m/\sqrt{\sigma}$ goes to 0 (17-02 § 03). **The supergravity regime of holography**: the ratio goes to 0 as $\lambda^{-1/2}$, and $\lambda = g^2 N$ is the second parameter (17-09 § 08) Physics .

The leading term of the strong-coupling expansion is not a counterexample but an approximation with the opposite slope. With the action $S = \beta \sum_p \left(1 - \frac{1}{2}\,\mathrm{Tr}\,U_p\right)$ (the sum over all plaquettes, 5-01), the leading term for SU(2) gives, with $u = I_2(\beta)/I_1(\beta)$ (a ratio of modified Bessel functions), $\sigma = -\ln u$, and the lightest state is $m = -4\ln u = 4\sigma$ (5-02) Known , so

$$\frac{m}{\sqrt{\sigma}} = 4\,a\sqrt{\sigma} \quad (9)$$

which is the dashed line in Figure 1 (a straight line that would pass through the origin if extended). Extended to $a\sqrt{\sigma} \to 0$ the ratio falls to 0, the opposite direction to the lattice numbers. The leading term at $\beta = 2.40$ gives $a\sqrt{\sigma} = 0.840$ (measured 0.2634) Computed , outside the region where the expansion converges.

09

Examples where one scale holds, and the input needed

An example where it holds: the three-dimensional $\mathbb{Z}_2$ gauge model. The group is abelian, but by a duality transformation it maps to the three-dimensional Ising model; the glueball mass becomes the inverse of the correlation length, and the string tension becomes the interface tension. Agostini–Carlino–Caselle–Hasenbusch (1997), taking it

that $\sigma\xi^2$ should be constant in the scaling limit, found the continuum value $0.1056(19)$, checked that it agrees with the prediction $0.1024(88)$ of ϕ^4 theory, and obtained $m/\sqrt{\sigma} = 3.08(3)$ Physics .

The ratio $1/\sqrt{\sigma\xi^2}$ is 3.272(11) at $a\sqrt{\sigma} = 0.236$, 3.117(52) at 0.069, and 3.077(28) in the continuum, approaching from above (from Table 6 of the original paper Computed). The physical explanation is this. A **relevant direction** is a direction of couplings whose effect grows as the lattice is coarsened. Within the range that keeps the $\mathbb{Z}_2$ symmetry (in the Ising model, with no magnetic field), there is only one relevant direction near the critical point, so m and $\sqrt{\sigma}$ are fixed by one and the same ξ (2-08 § 07) Physics . It is not a theorem.

If the description of the continuum limit of three-dimensional U(1) is right, one scale breaks for the abelian U(1) and holds for the abelian $\mathbb{Z}_2$, so **it is not decided by whether the group is non-abelian**. What is needed is "an input that forbids a second small parameter". The record takes this input in four-dimensional SU(2) to be the positivity of the one-loop coefficient (asymptotic freedom, 4-06), and rewrote it in two forms. I: there is one relevant direction, and m and $\sqrt{\sigma}$ are fixed by the same ξ. R: the rotational symmetry broken by the lattice comes back at long distances (the shape of the tube matters; 17-03 § 08). Whether these two are enough is not known.

10

Where it differs from the equivalence principle — a sentence that cannot be put as an assumption

In general relativity the equivalence principle is an axiom. The prize problem asks us to *prove* existence and the gap from the definitions, so (ii) cannot be added as an axiom; it has to be made a theorem or bypassed.

The problem statement defines the gap Δ as there being no spectrum in $(0, \Delta)$, and continues 'The supremum of such Δ is the mass m, and we require m < ∞.' Known . Splitting the form $C_P \asymp \xi^2$ of (6) under (ii)(iii) (the band statement H_{band}) into upper and lower halves, in the record's reading $C_P \lesssim \xi^2$ (the relaxation-time upper-bound assumption) corresponds to the side of $\Delta > 0$, and $C_P \gtrsim \xi^2$ (the relaxation-time lower-bound assumption) to the side where the mass does not collapse to infinity (the form $m < \infty$). The precise treatment that splits the relaxation-time assumption itself into upper and lower halves is in 23-01. The lower-bound assumption is a different thing from "non-trivial" in the Clay sense (that there are interactions).

The lower half of (ii), $m^2 \gtrsim \sigma$, reduces to an operator inequality saying that the deficit of the tube is at least c/ℓ for perimeter ℓ (called the tube transfer-operator inequality in

the record) Lean (`mass_sq_ge_of_tube`, `LatticeGaugeTubeEntropy.lean`. Hypothesis: the norm of the transfer operator is at most $e^{-2\sqrt{c\sigma}}$, that is, a consequence of the tube inequality. Conclusion: $m^2 \geq 4c\sigma$. The list is in 23-13). There is no pattern of proof for the tube inequality itself (23-02). From finitely many measurements only an upper bound on m follows (a restatement of a known relation; 23-02). The one sentence can be named, but there are not yet tools to make it a theorem.

What this chapter can say

What is said	What is not said
Physics That "a single scale" can be written as (3), and that (ii) is the only chain that depends neither on the tool nor on the bookkeeping (the record's arrangement).	That the table of correspondences is a mathematical equivalence (it is an analogy). That (ii) is a sufficient input for the relaxation-time assumption.
Computed Tables A and B, Figures 1 and 2, the numbers in § 02, § 07, § 08 and § 09.	These are ratios and fits of values from primary sources, not measurements. The data alone cannot choose the form of the extrapolation.
Lean The algebra of the product in the direction from (7) to (6) (`chain_band`, `sq_times_poincare_band`). The reduction of the lower half of (ii) to the tube inequality (`mass_sq_ge_of_tube`).	Proofs of the chains (i)(ii)(iii), the relaxation-time assumption, (3) and the tube inequality. This chapter shows nothing about one scale, the mass gap or the construction of the four-dimensional continuum theory.
Known The area law at all couplings for three-dimensional U(1), the form of the leading term.	The continuum limit of three-dimensional U(1) (main text not checked). One scale for $\mathbb{Z}_2$ and SU(2) is at the level of numbers and physics.

Sources and reproduction

Item	Kind	Source / tools
The definitions of the gap and of $m < \infty$	Known	A. Jaffe, E. Witten, "Quantum Yang–Mills Theory" (the Clay problem statement) § 4. Text checked.
The starting point of the equivalence principle, the torsion balance	Physics	A. Einstein, Jahrb. Radioakt. Elektron. 4 (1907) 411–462. R. v. Eötvös, D. Pekár, E. Fekete, Ann. Phys. (Leipzig) 68 (1922) 11. Both bibliographic data only (secondary sources).
Equation (2)	Known	P. Touboul et al. (MICROSCOPE), PRL 129 (2022) 121102, arXiv:2209.15487. Abstract checked.
The inputs of Tables A and B and § 07, quotations, the mean-field improved coupling	Physics	B. Lucini, M. Teper, JHEP 0106 (2001) 050, hep-lat/0103027 Tables 1, 5 and 6, equations (4)(5) of the original paper, the sentence after equation (11) in § 5. Text checked.

Item	Kind	Source / tools
The $\mathbb{Z}_2$ inputs of § 09, the duality transformation, the ϕ^4 prediction 0.1024(88), 3.08(3)	Physics	V. Agostini, G. Carlino, M. Caselle, M. Hasenbusch, NPB 484 (1997) 331, hep-lat/9607029, equations (4)(15)(16) and Table 6 of the original paper. Text checked.
The leading term of the strong-coupling expansion ($u = I_2/I_1$, $m = 4\sigma$)	Known	J.-M. Drouffe, J.-B. Zuber, Phys. Rep. 102 (1983) 1–119. Bibliographic data only. The derivation is in 5-02.
't Hooft's vortex free energy (§ 07)	Known	G. 't Hooft, NPB 138 (1978) 1–25. Tomboulis–Yaffe, CMP 100 (1985) 313. Both bibliographic data only. 17-02.
The one-loop coefficient	Known	D. J. Gross, F. Wilczek, PRL 30 (1973) 1343. H. D. Politzer, PRL 30 (1973) 1346. Bibliographic data only.
Check of the coefficients of (8)	Known	J. Huston, K. Rabbertz, G. Zanderighi, "Quantum Chromodynamics", PDG, PRD 110 (2024) 030001, equation (9.3). Text checked.
Three-dimensional compact U(1)	Known	M. Göpfert, G. Mack, CMP 82 (1982) 545–606. Abstract (INSPIRE). Main text not read. The year is Crossref's year of publication; INSPIRE gives 1981.
The supergravity regime	Physics	The table of sources in 17-09.
(3), the relaxation-time assumption, the three chains, the table of correspondences, the judgement of candidates, I and R	Analogy, assumption	The arrangement of the record of this book's study. The chains and the relaxation-time assumption are assumptions, and the table of correspondences is an analogy. Lean covers only the algebra of (7) (`chain_band`, `sq_times_poincare_band`, `LatticeGaugeOneScale.lean`) and the reduction of the lower half of (ii) (`mass_sq_ge_of_tube`, `LatticeGaugeTubeEntropy.lean`). The list of theorems is in 23-13.
Tables A and B, Figures 1 and 2, the numbers in § 07, § 08 and § 09, the checks	Computed	Python 3 on this machine (numpy, scipy): `onescale.py` (propagation of errors, the least squares of (4), (8), I_2/I_1, the ratio for $\mathbb{Z}_2$).

Next chapter to read: 17-13 Where the way physics digs parts from mathematics — picking up, from the whole of Part 17, the places where physics and mathematics part. For the detailed form of the relaxation-time assumption, go to 23-01.

Where the way physics digs parts from mathematics — translating a sentence of physics into a sentence of mathematics brings the assumptions into the open

> This chapter organises established knowledge. § 04 is this book's own investigation.

What this chapter shows — The premises that come attached when a sentence of physics is translated into a mathematical statement. The seven moves through which premises enter, and where the theorems stop in each. How much of the string picture is in the denominator of the lattice ratio $m/\sqrt{\sigma}$ (recomputed from the tables of the primary source).

Prerequisite chapters — 17-10 (together with 17-01–17-03, 5-01 and 4-10).

Said first — This is an arrangement to avoid mistaking the level of results of physics for theorems; it does not discuss whether the methods of physics are good or bad. This chapter shows neither the existence of the mass gap nor the construction of the continuum limit. The numbers are recomputations from the tables of the primary sources.

The order of this chapter

1. Translating a sentence of physics into a sentence of mathematics
2. Measuring after putting that it exists
   - Detour: why the gap in a finite box is easy
3. The string picture enters the denominator
4. How much the ratio moves when the string picture is removed (Figure 1)
5. Extrapolating from finite β
6. Looking from the side of a small parameter
7. Moving to a neighbouring theory and coming back with a knob (Figure 2)
8. Looking after choosing a gauge
9. What physics regards as one

10. A list of the places where they part

01

Translating a sentence of physics into a sentence of mathematics

Table 5 of Lucini–Teper 2001 gives 3.844(61) as the continuum value of the ratio of the mass $m_{0^{++}}$ of the lightest glueball of $SU(2)$ to the string tension σ Physics . Translated into a mathematical statement, this sentence is

$$\lim_{\beta\to\infty} \frac{m(\beta)}{\sqrt{\sigma(\beta)}} = 3.844\ldots \quad (1)$$

($\beta = 2N/g^2$, and m and σ are quantities in units of the lattice spacing a; 5-01). (1) brings in four premises.

a. For each β the infinite-volume limit exists and is unique. Subsequential limits always exist by compactness, so the question is uniqueness.

b. For each β, $m(\beta) > 0$ and $\sigma(\beta) > 0$.

c. The limit $\beta \to \infty$ exists.

d. The limit is a ratio of quantities of a continuum theory satisfying the axioms (4-10).

For four-dimensional $SU(2)$ at large β, none of (a)–(d) is a theorem (§ 05) Known . "(61)" is an estimated numerical error, not an error bound given by inequalities. What this chapter calls "parting" is a place where some stage of the procedure of physics has no corresponding theorem and can only be put as an assumption.

02

Measuring after putting that it exists

The calculations of physics presuppose that a four-dimensional pure Yang–Mills theory exists, and is one. In a finite box B (a finite cube of lattice points) and at finite β, the probability measure on the link variables $U_\ell \in G$,

$$d\mu_{B,\beta}(U) = \frac{1}{Z_{B,\beta}} \exp\Big(\frac{\beta}{N} \sum_{P\subset B} \operatorname{Re}\operatorname{tr} U_P\Big) \prod_{\ell\subset B} dU_\ell \quad (2)$$

is properly defined (U_P is the product of the link variables once round the plaquette P, and dU_ℓ is Haar measure) Known . The premises enter in the limits. The problem statement does not accept weak existence by compactness (convergence of

subsequences) alone as a solution, unless the properties of the limit (the gap and the axioms) are shown by other means (footnote 2 of § 6.5 of the problem statement, 17-01). Further, in § 5 the problem statement names a **uniform** gap: 'the existence of a uniform gap for finite-volume approximations may play a fundamental role in the proof of existence of the infinite-volume limit' Known . The numbers of physics are "predictions made after assuming that it exists". For the construction published on 2026-11-10, which builds a theory satisfying the Wightman axioms as a limit of Wilson's lattice gauge theory, the committee's assessment is that it is proved in Lean. However, the infinite-volume limit is taken along a subsequence, and neither its uniqueness nor the mass gap, on which the numbers of physics rest, is shown ((a) above is a statement at each β of the lattice and lies outside this construction) (6-04).

Detour: why the gap in a finite box is easy

A subsection you may skip.

The β of this detour is the inverse temperature of the Ising chain (not the β of lattice gauge theory). The transfer matrix of the one-dimensional Ising chain (weight $e^{\beta s_i s_{i+1}}$, $s_i = \pm 1$) is

$$T = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}, \qquad \lambda_0 = 2\cosh\beta, \;\; \lambda_1 = 2\sinh\beta, \;\; \ln\frac{\lambda_0}{\lambda_1} = \ln\coth\beta \tag{3}$$

and correlations decay as $(\tanh\beta)^{|i-j|}$. The largest eigenvalue of a matrix whose entries are all positive is simple and strictly larger than the others (Perron–Frobenius) Known . Example: at $\beta = 1, 2, 3$ the gaps are 0.272, 0.0366 and 0.00496 Computed . It is positive at every finite β.

In lattice gauge theory too, the gap in a finite box is positive. More precisely, the link variables take values in a continuous group, so the transfer matrix is not a matrix but an integral operator with a positive continuous kernel. In a finite box the kernel is continuous and the group is compact, so this operator is compact. Then by an infinite-dimensional version of Perron–Frobenius (a generalisation of Jentzsch's theorem; the source is in 5-03) the largest eigenvalue is simple and isolated (the construction of the transfer matrix is Lüscher 1977 and Osterwalder–Seiler 1978; 5-03) Known . The question is whether, as $a \to 0$, $(\text{gap})/a$ remains positive uniformly in the volume (for two dimensions, 2-09).

03

The string picture enters the denominator

The σ in the denominator of (1) cannot be measured directly. What is measured is the mass m_l of a flux tube closed once round a periodic box (side L lattice points), and Lucini–Teper regard the tube as a string and subtract the Lüscher term (17-03 § 03) Physics .

$$am_l = a^2\sigma L - \frac{\pi}{3L} \quad (4)$$

The authors write 'in using eqn(4) we are assuming that we have linear confinement ..., that the effective string theory ... is in the Nambu-Gotto universality class, and that our string is in fact long enough', and also note 'with only two values of L ... we cannot claim to have evidence for both the 1/L functional form and for its coefficient'. The full form of the Nambu–Goto ground state (equation (23) of Athenodorou–Bringoltz–Teper) is

$$E_0(l)^2 = (\sigma l)^2 - \frac{\pi\sigma(d-2)}{3}, \qquad d = 4 \quad (5)$$

($l = aL$ is the length of the tube and d the dimension of spacetime; the primary source writes D). Taking the square root and expanding in $1/l$, the first two terms are (4) Known . That is, (4) is a long-string approximation, and the shorter the tube, the more the later terms matter. On the lattices of the primary source the tube lengths are $L\sqrt{\sigma} \approx 3.1$ to 3.4 (lengths in units of $1/\sqrt{\sigma}$) Computed , and that (4) is enough at these lengths is, as in the quotation above, put as an assumption by the authors too.

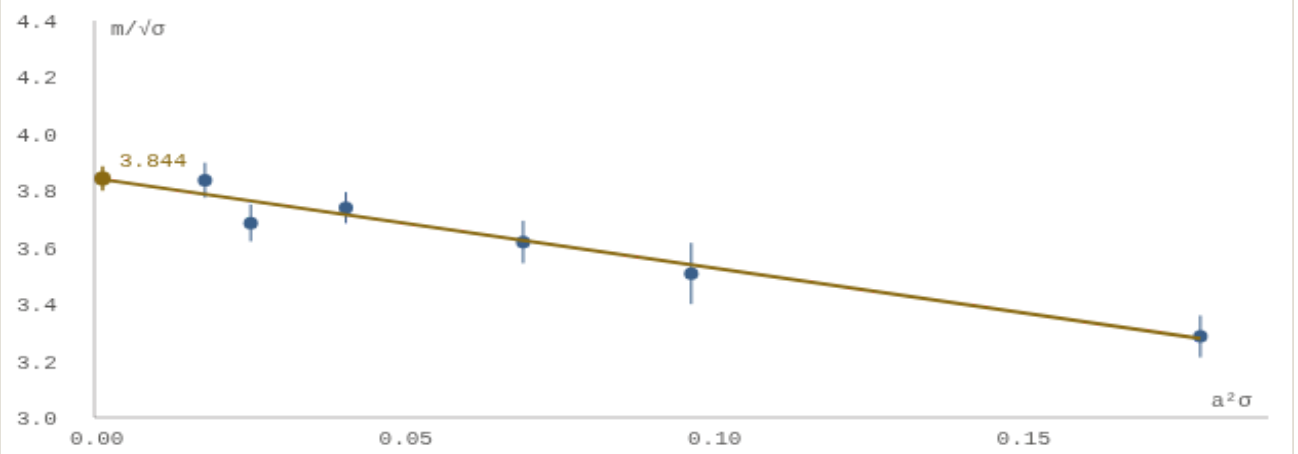
04

How much the ratio moves when the string picture is removed

From the tube masses in Table 1 of the primary source we extracted σ in three ways (no correction $a^2\sigma = am_l/L$, (4) and (5)) and recomputed the ratio Computed . No correction is the ruler that measures the contribution of the string picture.

How to read the fits — The ratio at each β is placed against the horizontal axis $a^2\sigma$ (the coarseness of the lattice), each point is weighted by the inverse square of its error, and a straight line is drawn (weighted least squares; the form of the line is (6) of § 05). The value at the left end $a^2\sigma = 0$ is the continuum value. χ^2 is the sum of the squares of the deviations between points and line, each divided by the error of the point; if it is of about the same size as the degrees of freedom (number of points − 2), the line fits the points within the errors. The errors of the ratios in Table 1 are the square root of the sum of the squared relative errors of $am_{0^{++}}$ and $a\sqrt{\sigma}$, treated as independent.

Figure 1 Changing how σ is taken and how many coarse points are dropped changes the fitted line and the continuum value (left end). The dashed line is equation (4), all points



Equation (4), points dropped: 0 → continuum value 3.844(43), χ²/dof 2.38/4, difference from equation (4), all points: 0.0 %

Computed The numbers in this figure were computed on this machine (the results of the same calculation as Table 1 embedded).

Table 1 $m_0{}^{++}/\sqrt{\sigma}$ for $SU(2)$, by how σ is taken (from Lucini–Teper Table 1. The master copy of the six values of am is 16-05 Table 3. How the errors are obtained is in "How to read the fits" above)

β	$L\sqrt{\sigma}$	No correction	(4)	(5)
2.25	3.38	3.447(78)	3.286(74)	3.278(73)
2.30	3.11	3.715(115)	3.508(108)	3.496(108)
2.40	3.16	3.825(80)	3.619(75)	3.608(74)
2.475	3.23	3.943(59)	3.740(55)	3.729(54)
2.55	3.18	3.894(69)	3.686(65)	3.676(64)
2.60	3.21	4.048(66)	3.837(62)	3.826(62)
Continuum value (all points)		4.059(45)	3.844(43)	3.833(42)

The central value 3.844 for (4) with all points agrees with Table 5 of the primary source. With or without the Lüscher term the continuum value moves by 5.6% (about 3.5 times the primary source's error 0.061), and the difference between (4) and (5) is 0.3% Computed . On the lattices of Table 1 the correction is 9–11% of the leading term σL. The precision of the ratio rests on **the leading correction of the string picture being right** (Athenodorou–Bringoltz–Teper report that many of the low states are well described by Nambu–Goto) Physics .

Translated into mathematics, the starting point of the string picture is a long tube of confinement, that is, confinement. Chatterjee 2018 writes that for four-dimensional non-abelian groups the area law at large β 'remains elusive', and that the area law at small β holds for all groups including $U(1)$, so it is no evidence Known .

> 05

Extrapolating from finite β

The primary source expects the lattice corrections of the plaquette action to start at a^2, and

$$\frac{m(a)}{\sqrt{\sigma(a)}} = \frac{m(0)}{\sqrt{\sigma(0)}} + c\,a^2\sigma \quad (6)$$

fits this (taking c as a constant), dropping the coarsest point if the fit is poor [Physics]. The knobs of Figure 1 reproduce this procedure; with (4) the continuum value stays within 3.84–3.86 [Computed] (for movement due to the form of the fit, see 17-12).

On the side of mathematics, the lattice correlation-length problem of Chatterjee 2018 (Chatterjee's Problem 5.1) asks for the following to be shown. Set $W_P = \operatorname{tr} U_P$, and let $f_\beta(x) = \langle W_{P_0} W_{P_x}\rangle - \langle W_{P_0}\rangle\langle W_{P_x}\rangle$ be the correlation (covariance) between the plaquette P_0 at the origin and the plaquette P_x nearest to the point $x \in \mathbb{R}^4$. For any infinite-volume limit of four-dimensional non-abelian lattice gauge theory, at every $\beta > 0$,

$$\lim_{|x|\to\infty} \frac{\log f_\beta(x)}{|x|} = -\frac{1}{\xi(\beta)}, \quad \xi(\beta) \in (0,\infty), \qquad \lim_{\beta\to\infty} \xi(\beta) = \infty \quad (7)$$

holds (ξ is the correlation length in lattice units; the continuum string-tension problem (Chatterjee's Problem 5.2) is the continuum limit of Wilson loops). In words: correlations decrease far away like $e^{-|x|/\xi}$, the length ξ is finite and positive at every β, and moreover grows without bound compared with the lattice spacing as the continuum limit is approached. The author writes 'Since none of the above has been proved', and states that at large β in dimensions higher than 2 even the uniqueness of the infinite-volume limit is unknown, and that $\beta_c = \infty$ in four dimensions is at the level of 'It is believed' [Known]. Example: even at the finest point ($\beta = 2.6$, 24^4), $\xi = 1/(am_{0^{++}}) = 1.95$ lattice spacings, and the ξ of the six points are 0.72–1.95 [Computed]. Extrapolation is the assumption that extends the slope over this range all the way to $\xi \to \infty$.

06

Looking from the side of a small parameter

Perturbation theory and semiclassics are both expansions around $g \to 0$, but the scale of dimensional transmutation (4-07) at one loop is

$$\Lambda = \mu\,\exp\Big(-\frac{1}{2b_0 g^2(\mu)}\Big), \qquad b_0 = \frac{11N}{48\pi^2} \quad (8)$$

$e^{-1/x}$ approaches 0 as $x \to 0^+$ faster than any power, and all its Taylor coefficients are 0 Known . If $\Delta = c\Lambda$, the gap appears at no order of the series in g^2 (an expectation) Physics (7-12, 4-12). Example: for $SU(3)$ at $g^2 = 0.5$, $\Lambda/\mu = 5.8 \times 10^{-7}$, while g^{20} is still 9.8×10^{-4} Computed .

The examples where semiclassics reaches the gap are none of them the theory of the problem statement. In three-dimensional compact $U(1)$, monopoles cause confinement (Polyakov 1977, semiclassical), and Göpfert–Mack proved the area law at all couplings on the lattice with the Villain action Known (the abstract of the same paper says that in the continuum limit the string tension in physical units goes to infinity; main text not read). On $\mathbb{R}^3 \times S^1$, Ünsal–Yaffe 2008 derived a gap and linear confinement for a deformed $SU(N)$ theory from a semiclassical analysis in the regime of small N where the circumference L is small enough (at the level of physics) Physics . That there is no phase transition on the way to $L \to \infty$ is a conjecture, based on the argument that there is no order parameter distinguishing small L from large L. What mathematics needs is an estimate non-perturbative in g, and the constructive methods too stop where they cross over to strong coupling (7-10).

07

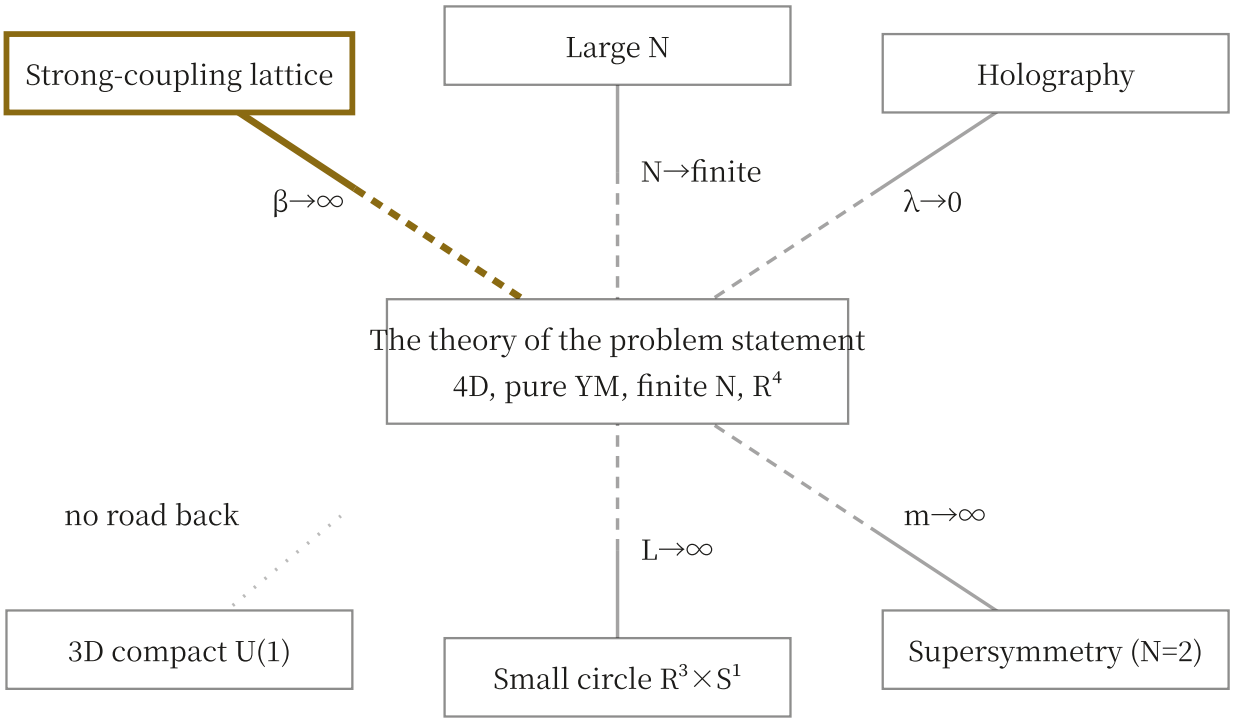
Moving to a neighbouring theory and coming back with a knob

In the move of going to a computable **neighbouring theory** and turning a knob to come back, the assumption is to put that nothing happens on the way. Translated into mathematics this is an exchange of limits, and uniformity is needed (17-10 § 06). For large N, for a dimensionless ratio F,

$$\lim_{a\to 0}\lim_{N\to\infty} F(N,a) \overset{?}{=} \lim_{N\to\infty}\lim_{a\to 0} F(N,a) \quad (9)$$

and what the problem statement needs is the limit at fixed N.

Figure 2 Choosing a shore (button, box, or row of Table 3) highlights the road back. Solid lines are the parts the calculation controls, dashed lines the roads back, and dotted lines shores with no road back



Strong-coupling lattice: The infinite-volume limit and the area law at small β (Osterwalder–Seiler). U(1) included Known , knob to come back: β→∞, needed on the way: The region of convergence does not reach weak coupling

Figure 2 is a picture of the content of Table 3, and has no numbers.

Table 3 Neighbouring theories and the knobs to come back

Shore	What can be said as mathematics	Knob to come back	What is needed on the way
Strong-coupling lattice	The infinite-volume limit and the area law at small β (Osterwalder–Seiler). U(1) included Known	β→∞	The region of convergence does not reach weak coupling
Large N	Sums over strings for strongly coupled SO(N) (Chatterjee). A representation as sums over surfaces at all β (Cao–Park–Sheffield) Known	N to finite	Uniformity of the 1/N corrections
Holography	The eigenvalue problem after fixing the background Known . The correspondence is a conjecture (17-08)	λ→0	Beyond supergravity; the definition of string theory
Supersymmetry (N=2)	Instanton integrals give the prepotential (17-11) Known	Mass→∞	That the gap is not interrupted
Small circle $R^3\times S^1$	Semiclassics at small N and small L (Ünsal–Yaffe) Physics	L→∞	That there is no phase transition
3D compact U(1)	The area law at all couplings (Göpfert–Mack, Villain action) Known	None	The scale does not become one (the abstract of the same paper)

For large N, 't Hooft 1983 is said to have constructed planar field theory under the condition "all particle masses non-zero and the coupling small enough" (bibliographic data and search summaries only), and the gap is an input. In § 6.6 the problem statement writes of 1/N as 'perhaps a good starting point for a rigorous proof (for sufficiently large N)' Known . Example: the lattice ratio is 3.405(21) for $SU(3)$ (Athenodorou–Teper 2020, 2021) and 3.072(14) for $SU(\infty)$ (same, 2021) Physics . Physics reads this as "large N is close", but what mathematics needs is a proof of the uniformity in (9).

08

Looking after choosing a gauge

Many mechanisms of confinement are told in terms of variables defined after choosing a gauge (monopoles, centre vortices, the Landau-gauge propagator) (17-02) Physics . On the side of mathematics, there is in general no globally continuous gauge fixing (Singer, 3-04), and even local conditions leave several solutions (Gribov, 4-08) Known . On the lattice, the expectation of a local quantity equals the expectation of that quantity averaged over gauge transformations (its gauge-invariant part), and components that change under gauge transformations have expectation 0 (Elitzur; for example $\langle U_\ell \rangle = 0$) Known .

The target of the problem statement is gauge-invariant local fields. For example, for $O = \operatorname{tr} F_{\mu\nu}F_{\mu\nu}$ (summed over μ, ν),

$$\left| \langle O(x)\, O(0) \rangle - \langle O \rangle^2 \right| \leq C\, e^{-\Delta |x|} \quad (10)$$

is one way the gap Δ shows itself (4-11). For the propagator, $D(0)$ is reported to look finite and non-zero on large lattices ('seems to') (Cucchieri–Mendes), and Zwanziger argues that copies inside the region do not affect expectations Physics . There is no theorem deriving (10) from $D(p)$.

There are gauge-invariant bridges too. The **centre** of a group G is the set of elements that commute with every element; for $SU(N)$ it is the identity matrix times an N-th root of 1, $e^{2\pi i k/N} I$. **Centre symmetry** is the symmetry under the transformation that multiplies all links of one layer by the same element of the centre (which does not change the values of the plaquettes), and we say it is "unbroken" when no measure changes under this transformation (precisely, on a slab of finite thickness, for every boundary condition; 17-02). Tomboulis–Yaffe 1985 are said to have bounded the Wilson loop from above by the free energy of centre vortices for $SU(2)$ (search summaries only). Chatterjee 2021 proved two things on the lattice for Wilson loops in representations acting non-trivially on the centre (representations in which some

element of the centre maps to something other than the identity matrix; for example the fundamental representation of $SU(N)$) Known : that if centre symmetry is unbroken there is confinement (in the sense $V(R) \to \infty$, weaker than the area law), and that if exponential decay of correlations for any boundary condition is assumed, centre symmetry is unbroken and the area law follows. What is missing is that this assumption holds at weak coupling in four dimensions (the claim of Tomboulis 2007 has been pointed out to have 'missing links'; Ito–Seiler).

09

What physics regards as one

Physics regards the scale after dimensional transmutation as one, and treats separate quantities as faces of the one scale (17-01 § 03). Translated into mathematics, the sentence splits.

Table 4 Sentences of physics and the translated statements

Sentence of physics	Mathematical statement and its current state
The mass and the string tension are constant multiples of the same Λ	That m^2/σ is bounded above and below uniformly in β (the diffusive-relaxation-time assumption of 17-12). Examples where it breaks: three-dimensional compact $U(1)$, the supergravity regime (17-09) Physics
Confinement and the gap are one phenomenon	Separate claims. The problem statement asks only for $\Delta > 0$; confinement is an extension problem ('natural extensions' in § 5 of the problem statement). There is no theorem deriving confinement from a gap in the sense of the problem statement (17-02 § 02). On the lattice, from the strong form, exponential decay for any boundary condition, the area law follows for groups with a non-trivial centre (Chatterjee 2021 Thm 2.4) Known . For finite groups, exponential decay of correlations of gauge-invariant quantities has been proved at weak coupling (Adhikari–Cao; a form on a box with no condition imposed at the boundary, different from the strong form) Known . That they do not confine is at the stage of belief (Chatterjee 2021 writes 'believed'; 5-10) Physics
The gap is the mass of the lightest glueball	The gap and an isolated one-particle state above it are different things. The latter is an extension problem (4-11) Known

10

A list of the places where they part

In every row, what is missing is not the line of argument of physics but the corresponding theorem.

Table 5 The seven moves, and where the translated statements stand

Move (section)	Translated into a mathematical statement	Current state

Move (section)	Translated into a mathematical statement	Current state
Placing it as existing (§ 02)	Construction of a limit satisfying the axioms	The construction published on 2026-11-10 (the committee's assessment is that it is proved in Lean; the infinite-volume limit is along a subsequence, with no uniqueness and no mass gap; 6-01)
Taking σ from the string picture (§ 03, § 04)	The area law at large β	'remains elusive' Known
Extrapolating (§ 05)	(7) and the continuum limit of Wilson loops	At the stage of being posed as problems (the lattice correlation-length problem, the continuum string-tension problem) Known
Expanding in a small parameter (§ 06)	An estimate non-perturbative in g	All Taylor coefficients of $e^{-1/x}$ are 0 Known . $\Delta = c\Lambda$ is an expectation of physics Physics
Coming back from a neighbouring theory (§ 07)	The uniformity in (9)	Within the range we searched, no justification found
Choosing a gauge (§ 08)	Gauge-invariant correlations (10)	Up to conditional lattice theorems Known
Regarding as one (§ 09)	Separate statements and a uniform comparison	At the stage of being cut out as an assumption Physics

Most of the places where they part lie not in divergences but in **limits and uniformity** (7-11), and for mathematics the numbers of physics are not steps of a proof but **targets and checks**. Where the translation succeeded, the result is **conditional theorems** on the lattice. The overall picture set beside the side of mathematics is in 16-05, and the parallel list of names is the parallel table in 16-05.

What this chapter can say

What is said	What is not said
Computed From Table 1, that the continuum value moves by 5.6% with or without the Lüscher term, and that the difference between (4) and (5) is 0.3%. That the central value for (4) with all points agrees with 3.844.	Whether the string picture is right. Confinement itself. The breakdown of the primary source's error 0.061 (0.043 in the fit on this machine).
Known For the seven moves, the translated statements and the current state as stated by the primary sources.	That the seven are exhaustive (the division is this chapter's arrangement). That the hypotheses of the conditional theorems hold.

What is said	What is not said
Physics The readings of semiclassics, large N, holography and supersymmetry in § 06 and § 07.	The existence of the limits taken by turning the knobs. The mass gap and the construction of the continuum limit (this book does not show them).

Sources and reproduction

Item	Kind	Source / tools
(1)(4)(6), the inputs of Table 1	Physics	Lucini–Teper, JHEP 0106 (2001) 050, hep-lat/0103027. Text (Table 1, Table 5, § 4) checked.
(5), the reports in § 03 and § 04	Physics	Athenodorou–Bringoltz–Teper, JHEP 1102 (2011) 030, arXiv:1007.4720. Text (equation (23), abstract) checked.
The problem statement in § 02, § 07 and Table 4	Known	Jaffe–Witten, "Quantum Yang–Mills Theory" (the Clay problem description) § 5 ('uniform gap', 'natural extensions'), § 6.5 (footnote 2), § 6.6 (1/N). Text checked.
(7), the area law, uniqueness	Known	Chatterjee, arXiv:1803.01950 § 4, § 5 (Problems 5.1 and 5.2). Text checked.
The transfer matrix of the detour, strong coupling in Table 3	Known	Lüscher, CMP 54 (1977) 283 (construction of a self-adjoint positive transfer matrix); Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 (both bibliographic data only). The theorems of Perron–Frobenius and Jentzsch: O. Perron, Math. Ann. 64 (1907) 248–263 / R. Jentzsch, J. reine angew. Math. 141 (1912) 235–244 (bibliographic data only). The generalisation: H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer (1974) p. 337 (bibliographic data only; the claim checked in secondary sources; details in the table of sources in 5-03).
Centre symmetry and finite groups in § 08 and Table 4	Known	Chatterjee, CMP 385 (2021) 1007–1039, arXiv:2006.16229. Text (Thms 2.2 and 2.4, the 'believed' paragraph on finite groups) checked. Adhikari–Cao, Ann. Probab. 53 (2025) 140–174, arXiv:2202.10375. Text (Theorem 1.1) checked.
Semiclassics in § 06	Known / Physics	Göpfert–Mack, CMP 82 (1982) 545–606 (Villain action; abstract (INSPIRE); main text not read; the year is Crossref's year of publication, INSPIRE gives 1981). Polyakov, NPB 120 (1977) 429 (bibliographic data only). Ünsal–Yaffe, PRD 78 (2008) 065035, arXiv:0803.0344 (abstract).
Large N in § 07	Known / Physics	't Hooft, CMP 88 (1983) 1 (bibliographic data and search summaries only). Chatterjee, arXiv:1502.07719; Cao–Park–Sheffield, arXiv:2307.06790 (abstracts). Athenodorou–Teper, arXiv:2007.06422 ($SU(3)$; Tables 14 and 17 of the text checked), arXiv:2106.00364 (Tables 24 and 34 [$SU(3)$] and Tables 31 and 38 [$SU(\infty)$] of the text checked).

Item	Kind	Source / tools
Gauges in § 08	Known / Physics	Singer, CMP 60 (1978) 7; Gribov, NPB 139 (1978) 1; Elitzur, PRD 12 (1975) 3978 (all bibliographic data only); Tomboulis–Yaffe, CMP 100 (1985) 313 (bibliographic data and search summaries only). Cucchieri–Mendes, arXiv:0710.0412; Zwanziger, hep-ph/0303028; Tomboulis, arXiv:0707.2179; Ito–Seiler, arXiv:0711.4930 (abstracts).
Table 1, Figure 1, (3), the numbers in § 03– § 06	Computed	Python 3 on this machine (standard library): string_sigma.py, extra.py. The inputs are only the values in the tables of the primary sources. (3) and Perron–Frobenius are standard facts.

Next chapter to read: 18-01 How to read the dictionary (the entrance to Part 18, which sets the words of physics and of mathematics side by side in two columns of standing, definition and reality).

How to read the dictionary — grading in two columns, and naming who judges "the same"

> This chapter is a guide (how to read the dictionary).

What this chapter shows — The two grades of each row of the dictionary: (a) the correspondence as definitions, and (b) existence in four-dimensional Yang–Mills. That the inconsistencies of a one-column grade are resolved by two columns (Figure 1).
Who judges "the same" (theorem, definition, convention, conjecture, analogy), and how to read the Conditions entry (Figure 2).

Prerequisite chapters — 0-03 Reading the problem statement word by word. § 03 and § 08 are easier to read after 4-10 and 4-11. Words not in 0-03 get a one-phrase explanation where they first appear, and the details are left to the linked chapters. Within Volume 3, this part is read after Part 17 (From the side of physics).

Said first — This chapter only fixes how to read. The grades in Table 2 are this book's reading, and the grounds are in 18-02–18-04. This book itself shows neither the existence of the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is read in Volume 2).

The order of this chapter

1. Two vocabularies pointing at the same thing
2. The entries of a row — nine entries (including the measurement column)
3. The definition column: the correspondence as definitions — Settled, Conditional, Unsettled
4. The existence column: existence in four dimensions — Constructed, Conditional, None (not yet constructed)
5. Two columns resolve the inconsistencies — Figure 1, Table 2
6. Which vocabulary to put first
   - Detour: false friends
7. Who judges "the same" — five judges
8. How to read the Conditions entry — Figure 2, Table 3

9. What comes along if the existence column moves

01

Two vocabularies pointing at the same thing — why a dictionary is needed

The "path integral", "vacuum" and "glueball" of physics are assigned in mathematics to "a probability measure", "a unit vector in the reconstructed Hilbert space (1-10)" and "an isolated eigenvalue of the mass operator $M = \sqrt{H^2 - P^2}$ (the operator that reads the rest mass from the energy H and the momentum P)". Whether words of different origins point at the same thing, and how certain the correspondence is, differs from row to row, and can only be checked row by row. That is the subject of this chapter.

There are three answers: it can be said by a theorem to be the same, it can be said only under assumptions, or it cannot be said. Moreover, in four dimensions the object on the side of mathematics has often not yet been made (0-03 § 04). So we give a row the following form.

$$\underbrace{\text{physical word}}_{\text{origin}} \longleftrightarrow \underbrace{\text{mathematical word}}_{\text{origin}} \qquad [\ (a)\ \text{correspondence as definitions}\ ;\ (b)\ \text{existen}$$

$\longleftrightarrow$ only places the two side by side as a pair; whether they are the same is stated by the two grades. Example: "instanton $\longleftrightarrow$ anti-self-dual connection" is (a) Settled, (b) Constructed (classical). For classical fields there is the precedent of Wu–Yang (1975) (abstract: 'identification with the mathematical concept of connections'; 1-07). An attempt to read the assumptions of this book's record in the words of physics is in 18-05.

02

The entries of a row — nine entries and a sample

There are nine entries: concept, the physical word and its origin, the mathematical word and its origin, the definition column, the existence column, the measurement column, judge, conditions (the assumptions needed for the identification and the gap that remains), and chapter.

The sample is the Feynman–Kac formula. The imaginary-time evolution of $H = -\frac{1}{2}\Delta + V$ (Δ is the Laplacian, not the Δ of the mass gap) can be written as an expectation over Brownian motion B_s started from x (a random path made by piling up independent small Gaussian displacements; 1-08) Known .

$$(e^{-tH}f)(x) = \mathbb{E}_x\Big[\exp\Big(-\int_0^t V(B_s)\,ds\Big) f(B_t)\Big] \quad (2)$$

Filling in the entries: the concept is "writing e^{-tH} as an integral over paths", the physical word is the path integral (Feynman 1948), the mathematical word is the Feynman–Kac formula (Kac 1949), (a) Settled, (b) Not applicable (a row of quantum mechanics with finitely many degrees of freedom, so not asked; § 04), the judge is a theorem, the conditions are "V bounded below, etc.; the real-time e^{iS} does not become a measure", and the chapter is 2-07. Two kinds of thing go into the Conditions entry. "V bounded below, etc." is a hypothesis of the theorem, and "the real-time e^{iS} does not become a measure" is a gap that remains even after the identification. Among the arrows, $\Longleftrightarrow$ is a theorem in both directions and $\Longrightarrow$ a theorem in one direction, while $\longleftrightarrow$, as in (1), only places the two side by side as a pair.

The measurement column: the direction reachable by finite measurement. Besides (a) and (b) we attach one more grade: for a quantity on the side of mathematics, which direction follows as a firm inequality from numbers measured with finitely many observables, finite time t and finitely many lattices. There are four values: **Upper bound only**, **Lower bound only**, **Both** and **Out of reach** (statements about limits, uniformity or all observables, which finite measurement settles in neither direction). Rows that name no quantity (rows that are only a correspondence of definitions or theorems) get "—". The grounds for the judgement are (8) and (9) of § 08. A sum of exponentials with weights $w_n \geq 0$ is log-convex, so the effective mass $m_{\mathrm{eff}}(t)$ decreases in t and approaches μ_O from above, and at every finite t, $m \leq \mu_O \leq m_{\mathrm{eff}}(t)$ Known ((3) in 4-11). So the gap and its relatives (positions of eigenvalues, decay rates) fall under "Upper bound only", and the lengths and times that are their inverses (the correlation length ξ, the Poincaré constant C_P) under "Lower bound only". This is the box "The asymmetry of upper and lower bounds" in 1-10 made into an entry of the dictionary. The row of the sample, the Feynman–Kac formula, gets "—".

03

The definition column: the correspondence as definitions — Settled, Conditional, Unsettled

The definition column asks: "**if both objects existed**, would they be the same?" Whether they exist is the job of the existence column.

Settled	Conditional	Unsettled
There is a theorem giving the same object, or the side of	The same only under assumptions that are in neither definition (the	They seem to point at the same thing, but no theorem says so.

mathematics defines it so.	assumptions go into the Conditions entry).	

Even when Settled, the arrow has a direction. The sample is the Osterwalder–Schrader (OS) reconstruction theorem (4-10). We call the axioms of real-time quantum field theory the Wightman axioms (4-09), and the five conditions imposed on imaginary-time correlations (Schwinger functions) the Osterwalder–Schrader axioms: regularity (E0), Euclidean covariance (E1), reflection positivity (E2; correlations with the quantity reflected in the mirror of time are not negative), permutation symmetry (E3) and clustering (E4). The **linear-growth** condition (E0′) is a strengthening of regularity, requiring the estimates of the n-point functions to stay within roughly factorial growth in n (4-10 § 03) Known .

$$\textbf{there: Wightman theory} \implies (\mathrm{E0}) \wedge \cdots \wedge (\mathrm{E4}), \qquad \textbf{back: } (\mathrm{E0}') \wedge (\mathrm{E1}) \wedge \cdots \wedge (\mathrm{E4}) \implies \mathbf{W}$$

On the way there, analytically continuing the correlations of a Wightman theory to imaginary time, using that the energy is positive, gives the five axioms. On the way back the linear-growth condition is needed, and whether this condition follows from a Wightman theory is not known (Summers § 3, 4-10 § 03). Since the conditions differ on the way there and the way back, even though (a) is Settled we write the arrow as two $\Longrightarrow$, not as $\Longleftrightarrow$. How the construction published on 2026-11-10 used this way back to reconstruct a Wightman theory is read in 12-08 Reconstruction and the Wightman axioms.

The sample of Conditional is the mass gap and the decay of correlations. § 5 of the problem statement writes in one direction: 'An important consequence of the existence of a mass gap is clustering'. The converse needs a dense family of observables to decay at a uniform rate (§ 08).

Unsettled comes in three types. (i) No definition on the side of mathematics: the four-dimensional Λ. (ii) Definitions on both sides, but within the range we searched no theorem saying they are the same: centre symmetry (Chatterjee writes that the mathematical physics literature has no mathematical definition, and then that his definition on the lattice was inspired by the physical definition through the Polyakov loop). (iii) Correspondences inside conjectures or analogies: the radial direction of holography and the scale of renormalisation (17-08).

Read as two groups, "Settled" are pairs that **are settled to point at the same thing**, and "Conditional" and "Unsettled" are pairs that **mathematical certainty has not been able to assert**. The dictionary of mass and length (18-03 § 01) adds a fourth grade, "Does not correspond", for pairs known to be different quantities. Row 9 of Table 2 (a false friend, the detour in § 06) is graded at this level in the entry "gap in physical units" of 18-03.

04

The existence column: existence in four dimensions — Constructed, Conditional, None

The existence column writes, with a **level**, whether the object the mathematical word points at has been made in four dimensions. There are four levels: classical (smooth connections), formal power series (each order of perturbation), lattice (spacing a fixed), and continuum quantum theory (a measure and a Hilbert space on $\mathbb{R}^4$); the problem statement asks for the last (0-03 § 03).

Constructed	Conditional	None
Made at the level the row names (the level is added).	Only versions of limited range (strong coupling, lattice fields satisfying an admissibility condition, and so on).	Not yet made at the named level. It has not been shown not to exist (versions at lower levels go into the Conditions entry).

These are the three grades. For rows that are not objects of four-dimensional Yang–Mills (quantum mechanics with finitely many degrees of freedom, extra dimensions, and so on) (b) is not asked, and we write "Not applicable" (the sample in § 02).

The last rule is the key. "None" as the name of a grade means "not yet constructed"; it has not been shown not to exist. The problem asks exactly about that existence. The measure on a finite lattice exists for every β (the coefficient of the coupling in the lattice action) (5-01), but the continuum measure is its limit, and for four-dimensional non-abelian groups it has not yet been constructed Known .

$$d\mu_a(U) = \frac{1}{Z}\, e^{-S_W(U)} \prod_{\ell} dU_\ell \quad \text{(exists)}, \qquad \mu = \lim_{a \to 0} \mu_a \quad \text{(not constructed)} \ (4)$$

U_ℓ is the link variable and S_W the Wilson action. Example: instantons are Constructed classically (ADHM 1978), and the transfer matrix (the operator that advances imaginary time by one step on the lattice, § 08) is Constructed on the lattice (Lüscher 1977). Rows 10 and 11 of Table 2 point at the same geometry, but 10 names the classical level and 11 the lattice level. The second Chern number of a smooth connection exists classically, so the (b) of 11 is Conditional because assigning an integer to a lattice field needs an admissibility condition (U_p close to 1 on each plaquette). On the strongly coupled lattice an isolated one-particle state has also been shown (Schor 1984), but at the lattice level, and on the opposite side from the continuum limit (5-09).

05

Two columns resolve the inconsistencies — fourteen samples

With a one-column grade, "are they the same as definitions?" and "do they exist in four dimensions?" are mixed into one grade, and which of the two decided the grade changes from row to row.

$$\text{one-column grade} = f\big((a),(b)\big), \qquad f \text{ chosen implicitly row by row } (5)$$

Example: rows 4 and 5 of Table 2 are the same pair, yet asking about definitions gives Conditional, and asking about a four-dimensional instance gives Unsettled. The same holds for 6 and 7, and 1 (the path integral) is Settled as a definition but Conditional in one column. In two columns these fall into the same cell. With a one-column grade that "takes the weaker", the rows of continuum quantum theory almost all line up at "None", and the correspondences of definitions become invisible.

Figure 1 The slider gives the one-column grade (left) and the two columns (right; vertical (a), horizontal (b)). The numbers are rows of Table 2; pressing a point or a row shows its grades. Dashed lines join the same pairs



The positions are the grades of Table 2 (this book's reading). With JavaScript disabled, the still figure and the table can be read.

Table 2 Fourteen samples (details in 18-02–18-04). "One column" is the grade given without separating the two questions. (c) is the measurement column of § 02 (the direction reachable by finite measurement).

#	Physical word ⟷ mathematical word	One column	(a)	(b)	(c)	Judge
1	Path integral ⟷ Euclidean probability measure	Conditional	Settled	None	—	Definiti
2	Wick rotation ⟷ OS reconstruction theorem	Settled	Settled (⇒)	None	—	Theorer
3	Existence of the lattice Hamiltonian ($T = e^{-aH}$ positive and self-adjoint) ⟷ reflection positivity of the Wilson action	Settled	Settled	Constructed (lattice)	—	Theorer
4	One-particle state ⟷ isolated eigenvalue of the mass operator	Conditional	Conditional	None	Upper bound only	Theorer conditic
5	Glueball ⟷ isolated eigenvalue of the mass operator	Unsettled	Conditional	None	Upper bound only	Theorer conditic convent
6	Spacetime and lattice ⟷ the $a \to 0$ limit of $a\mathbb{Z}^4$	Unsettled	Conditional	None	Out of reach	Definiti + conditic
7	Continuum limit ⟷ scaling limit	Conditional	Conditional	None	Out of reach	Definiti + conditic
8	Mass gap ⟷ spectral gap, exponential decay of correlations	Conditional	Conditional	None	Upper bound only	Theorer conditic
9	Gap in physical units ⟷ uniform gap in lattice units	Unsettled	Unsettled (false friend)	Conditional (strong coupling, etc.)	Out of reach	Convent
10	Instanton ⟷ anti-self-dual connection	Settled	Settled	Constructed (classical)	—	Theorer
11	Topological charge (lattice) ⟷ second Chern number of a lattice field	Conditional	Conditional	Conditional (lattice: fields with U_p close to 1 on each plaquette)	—	Theorer conditic
12	Renormalisability ⟷ BPHZ, Epstein–Glaser (for YM, BRS and flow equations)	Conditional	Conditional	Constructed (formal power series)	—	Theorer (for YM see the note below)

#	Physical word ⟷ mathematical word	One column	(a)	(b)	(c)	Judge
13	Λ, "a single scale" ⟷ the diffusive-relaxation-time assumption: $C_P \asymp \xi^2$	Unsettled	Unsettled	None	Lower bound only (C_P; the relaxation-time upper-bound assumption is out of reach)	Analogy
14	Radial direction of holography ⟷ Fefferman–Graham expansion	Unsettled	Unsettled	None	—	Conject

Note: column (b) of Table 2 is the reading from before the published construction (2026-11-10). Rows 2, 6 and 7 moved with the published construction (6-01, §09).

Notes on the words of Table 2. Row 11 names the level of the lattice (§ 04). The "theorem" in the judge of row 12 is Hepp 1966 for scalar fields, Becchi–Rouet–Stora 1976 for gauge theories (for a semisimple group, with a choice of gauge in which every field gets a mass, showing the identities of gauge symmetry to all orders of perturbation), and Efremov–Guida–Kopper 2017 for massless $SU(2)$ pure Yang–Mills (by the flow equations of the renormalisation group). The source for pure Yang–Mills with a general compact group has not been confirmed. In row 13, Λ is the mass scale the theory produces by itself (4-07), C_P is the Poincaré constant (how slowly configurations return to equilibrium under the stochastic gradient flow, that is, Langevin dynamics; 1-10, 18-03), ξ is the correlation length (the length over which correlations decrease as $e^{-r/\xi}$, the inverse of the gap in lattice units), and the diffusive-relaxation-time assumption is the assumption $C_P \asymp \xi^2$ made by the record of this book's study (that the two are proportional within constant factors; 23-01). The Fefferman–Graham expansion of row 14 is the formula expanding the interior metric in powers of the depth from the boundary (18-04).

06

Which vocabulary to put first — look up by the physical word, judge by the mathematical statement

The dictionary is ordered so that it can be looked up from the physical words that many readers know first, and the grades are judged after translating into mathematical statements (17-13 § 01). When a physical word has two meanings, the row is split. The criterion for splitting is whether the grade in the definition column differs between the two meanings. If it is the same, they stay in one row; if it differs, they are split. The sample is "unitarity" Known .

positive norm	$\Longleftrightarrow$ reflection positivity	(Settled; within tl
positive norm $\wedge$ asymptotic completeness	$\Longrightarrow$ unitarity of the S-matrix	(Conditional; und

Positive norm means that the inner product of states is positive definite (there are no states of negative norm). Asymptotic completeness means that the Hilbert space is exhausted by states of free particles in the far past and far future (asymptotic states) (18-02 § 05). The upper row is a statement within the correspondence between OS and Wightman: from reflection positivity come a positive-definite Hilbert space and $H \geq 0$ (reconstruction). The converse is the way-there direction of (3): the positivity of a Wightman theory becomes the reflection positivity of the Schwinger functions (4-10). The lower row is a statement under the Haag–Ruelle scattering theory, which builds asymptotic states by assuming an isolated one-particle mass shell.

Put into one row, the Settled grade would read as if it extended to the S-matrix. "Confinement" is another example of a split row. The three criteria are $V(R) \to \infty$ ($V(R)$ is the energy of a heavy quark pair placed at distance R, read from how Wilson loops decrease), the area law $\sigma > 0$ ($\sigma = \lim_{R\to\infty} V(R)/R$ is the string tension), and unbroken centre symmetry (invariance under transformations by the centre of the group; in physics, the expectation of the Polyakov loop at finite temperature is 0). In physics they mean almost the same thing, but the mathematical theorems are one-directional and on the lattice. Chatterjee's Theorem from Centre Symmetry to Confinement (Theorem 2.2 of Chatterjee 2021) shows, on the lattice $\mathbb{Z}^d$ at fixed coupling, "centre symmetry unbroken $\Rightarrow V(R) \to \infty$" (for Wilson loops in representations acting non-trivially on the centre; 18-03 § 03).

Detour: false friends

A subsection you may skip.

In learning foreign languages, words that look alike but differ in meaning are called "false friends". The typical one is "gap". A uniform gap in lattice units is, in physical units, a property of the side where the mass goes to infinity, the opposite side from the continuum limit, so we make the definition column Unsettled and write "false friend" in the Conditions entry (row 9 of Table 2). Precisely, the same word points in the two languages (physics and mathematics) at objects with different conditions, and one holding does not imply the other holding (it lies rather on the opposite side). "Continuum limit" is not quite a false friend, but it is an example where the conditions included differ. The scaling limit of probability theory does not include "the limit is not Gaussian", which the physical word implicitly includes (for the four-dimensional Ising type and ϕ^4 it is Gaussian: Aizenman–Duminil-Copin 2021, 7-08). This difference goes into the Conditions entry of row 7 of Table 2. For the difference between "the mass of a particle" and the gap, see 16-08 and 16-05.

07

Who judges "the same" — five judges

Theorem: there is a theorem in both directions or in one ((2), (3), (6)). **Definition**: the side of mathematics has fixed an object as the meaning of the physical word. Fixing the path integral as a countably additive probability measure is such a case. § 4 of Jaffe's review lists as a "problem" of the time that the path integral had no definition as 'a true integral, given by a countably-additive Borel measure'. This way of fixing it does not go as far as agreement with other calculations. **Convention**: the habit of calling things by the same name. Calling the lowest state measured in each channel on the lattice a "glueball", and calling uniform exponential decay in lattice units a 'strictly positive mass gap' (Shen–Zhu–Zhu). **Conjecture**: for the correspondence between the radial direction and scale, the side of physics too writes that 'what cutoff on the field theory corresponds to a radial cutoff in the bulk?' is 'left unanswered' (Heemskerk–Polchinski). **Analogy**: pointing out a similarity of form. "A single scale" and the relaxation-time assumption (17-12).

$(a) = \text{Settled} \;\Rightarrow\; \text{the judge includes a theorem or a definition,} \qquad \text{judge only convention, cc}$

Example: row 5 of Table 2 is "Theorem + conditions, convention". The correspondence with an isolated eigenvalue is a conditional theorem (1-10), and calling the lowest state a "particle" is a convention. When they coexist we write both, and grade the definition column by the stronger.

08

How to read the Conditions entry — a single correlation gives only an upper bound on the gap

The Conditions entry is where the hypotheses of theorems are kept. We read row 8 of Table 2 with a small model. The transfer matrix is the operator $T = e^{-H}$ that advances imaginary time by one step on the lattice (taking the step as 1; 4-10, 5-03). Let its levels be $0 = E_0 < E_1 < E_2 < \cdots$; the state Ω with $E_0 = 0$ is the vacuum. The correlation at imaginary-time separation t can be written with the levels and the overlaps $w_n \geq 0$ of the observable O with each level (if the levels are discrete) Known .

$$C_O(t) = \langle O(t)O(0)\rangle - \langle O\rangle^2 = \sum_{n\geq 1} w_n e^{-E_n t}, \qquad w_n = \left|\langle n|O|\Omega\rangle\right|^2 \quad (8)$$

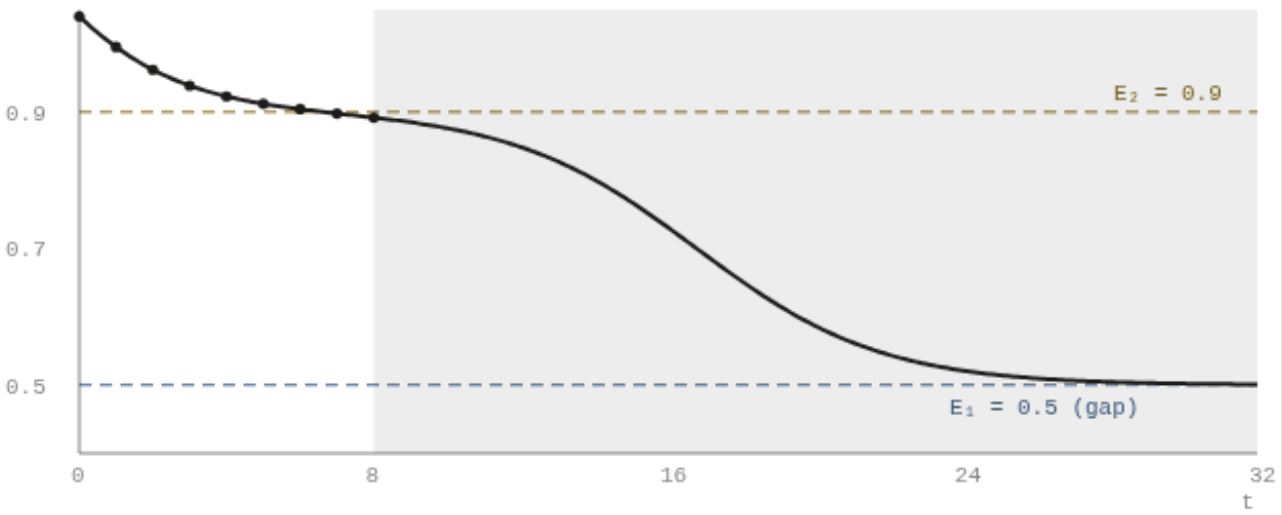
The gap is $m = E_1$. The rate at which the correlation decreases is fixed by the first level with $w_n > 0$.

$$\mu_O = \min\{E_n : w_n > 0\} \;\geq\; m, \qquad m_{\text{eff}}(t) = \log \frac{C_O(t)}{C_O(t+1)} \;\xrightarrow[t\to\infty]{}\; \mu_O \quad (9)$$

What a single correlation can say is only the upper bound $m \le \mu_O$. One observable suffices for the upper bound, and equality needs $w_1 > 0$. The lower bound $m \ge c$ needs every member of a family of observables dense in the part orthogonal to the vacuum to decrease at rate at least c (uniformity). Which observables overlap with E_1 is not known in advance, so a family is needed. This is the content of the conditions of row 8 (4-11 § 02, 18-03. Easy from above and hard from below is the same shape as the variational principle: the box "The asymmetry of upper and lower bounds" in 1-10). The measurement column of the dictionary (§ 02) attaches this asymmetry to each row.

Example: if $E = (0.5, 0.9, 1.4)$, $w_2 = 1$, $w_3 = 0.5$ and $w_1 = 10^{-3}$, then $m_{\mathrm{eff}}(8) = 0.892$, 1.78 times the gap, and it comes within 5% only from $t = 24$ Computed . For $t \le 8$ the difference between $w_1 = 10^{-3}$ and $w_1 = 0$ is at most 1.3% Computed ; at finite t a small overlap cannot be told apart from no overlap.

Figure 2 Changing the overlap w_1 and the measured range t_max changes the effective mass (line) and the values that can be read (points). Dashed lines are E_1 (the gap) and E_2; the shading is the unmeasured range



w₁ = 1.0e-3 m_eff(8) = 0.892 (1.78 times the gap), within 5%: t = 24 onwards

Computed The numbers in this figure were computed on this machine ((8) and (9) computed on the spot).

Computed **Table 3** $m_{\mathrm{eff}}(t)$ (with the same E, w_2 and w_3 as above). The right end is the first integer t with $m_{\mathrm{eff}} \le 1.05 E_1$.

w_1	$t = 1$	8	16	32	Within 5%
1	0.7013	0.5132	0.5005	0.5000	7
0.1	0.9322	0.6018	0.5054	0.5000	13
0.01	0.9888	0.8108	0.5481	0.5001	18
0.001	0.9953	0.8918	0.7305	0.5009	24

w_1	$t = 1$	8	16	32	Within 5%
0	0.9960	0.9036	0.9001	0.9000	—

09

What comes along if the existence column moves — reading the problem from the dictionary

What the problem asks for is to move the existence column from "None" to "Constructed" at the level of continuum quantum theory. "None" here means, as defined in § 04, not that it has been shown not to exist, but that it has not yet been made. What then comes along is shown by the definition column.

$$(b):\ \text{None} \to \text{Constructed} \ \text{ and } \ (a) = \text{Settled} \implies \text{the row can be used as it stands}$$ (10)

In Table 2 the existence column is "None" in 9 rows, and the "Constructed" rows 3, 10 and 12 are also at levels below continuum quantum theory. If a four-dimensional measure satisfying the conditions of the way back in (3), $(\mathrm{E0}')$–$(\mathrm{E4})$, were made, the Settled rows 1 and 2 could be used as they stand, and the reconstruction theorem would give the Hilbert space, the Hamiltonian and the vacuum. For the Conditional rows 4 and 5 the upper gap (§ 5 of the problem statement counts 'an upper gap, in addition to the mass gap' as an extension), and for row 8 a uniform decay rate, that is the gap itself, must be checked separately. The Unsettled rows 13 and 14 and the false friend of row 9 are not settled even when the measure is made. In this book's reading, this is also why existence and the gap stand side by side in one sentence of the problem statement (0-03 § 01).

The construction published on 2026-11-10 moved part of this to the level of continuum quantum theory (the committee's assessment is that it is proved in Lean): a Wightman theory for each component of the phases, with its Hilbert space, Hamiltonian and vacuum, and local operators corresponding to polynomials in the curvature (6-01). The probability measure (row 1 of Table 2), one-particle states (4, 5) and uniform decay (8) remain None (6-04). The existence columns of Table 2 and of 18-02 to 18-04 are left as the reading from before publication.

What this chapter can say

	What is said	What is not said
Known	(2), the way there and the way back of (3), (6) (under the premises attached), (8), (9). The constructions of ADHM, Lüscher 1977 and Schor 1984	That these hold in four-dimensional continuum Yang–Mills theory
This book's reading	How the two-column grades, levels and judges are fixed, how the measurement column is fixed, (7) and (10), the grades of Table 2	That the grades are unique (the grounds are in 18-02–18-04)
Computed	$m_{\mathrm{eff}}(t)$ of the three-level model (Table 3, Figure 2)	The actual spectrum of Yang–Mills
Overall	—	This book itself shows neither the existence of the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is in Volume 2)

Sources and reproduction

Item	Kind	Source / tools
The problem statement, the definitions in § 07	Text	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official problem statement of the Clay Mathematics Institute; the problem was announced in 2000). In: J. Carlson, A. Jaffe, A. Wiles (eds.), *The Millennium Prize Problems*, CMI / AMS, pp. 129–152 (year of publication not checked) / A. Jaffe, “Constructive Quantum Field Theory”, in *Mathematical Physics 2000* (Imperial College Press, 2000) 111–127, § 4 “Quantum Theory as Statistical Physics” (venue checked in Crossref)
(2), row 12 of Table 2	Bibliographic data only	Feynman, RMP 20 (1948) 367 / Kac, Trans. AMS 65 (1949) 1 / 't Hooft, NPB 33 (1971) 173 / Hepp, CMP 2 (1966) 301
Row 12 of Table 2 (proofs for YM)	Abstract	C. Becchi, A. Rouet, R. Stora, “Renormalization of gauge theories”, Ann. Phys. 98 (1976) 287–321 / A. N. Efremov, R. Guida, C. Kopper, “Renormalization of SU(2) Yang–Mills theory with flow equations”, J. Math. Phys. 58 (2017) 093503, arXiv:1704.06799
(3)	Abstract, text	Osterwalder–Schrader, CMP 31 (1973) 83, CMP 42 (1975) 281 (abstracts; the abstract of the second paper puts necessity in parentheses: '(necessary and) sufficient') / Glaser, CMP 37 (1974) 257 (abstract: points out and corrects an error in the first paper, and suggests that temperedness near Minkowski points needs an additional assumption) / S. J. Summers, “A Perspective on Constructive Quantum Field Theory”, arXiv:1203.3991, § 3 and footnote 15 (text) / A. Jaffe, “Reflection Positivity Then and Now”, arXiv:1802.07880 (text)

Item	Kind	Source / tools
(6)	Text, bibliographic data only	Osterwalder–Schrader 1973/75 (above) / Jaffe's review (above) § 4, § 7.1 (text: that reflection positivity gives a Hilbert space and a positive Hamiltonian, and that the Haag–Ruelle theory assumes an isolated one-particle mass shell) / R. Haag, Phys. Rev. 112 (1958) 669, D. Ruelle, Helv. Phys. Acta 35 (1962) (bibliographic data only)
Rows 3, 4 and 5 of Table 2	Abstract	Lüscher, CMP 54 (1977) 283 / Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 / Schor, NPB 231 (1984) 321
6, 7, the definition of centre symmetry, the Theorem from Centre Symmetry to Confinement (Theorem 2.2)	Text	Chatterjee, arXiv:1803.01950 § 5 / arXiv:2006.16229 (CMP 385 (2021) 1007)
9, 10, 11, 14, § 07	Abstract	Shen–Zhu–Zhu, arXiv:2204.12737 / ADHM, Phys. Lett. A 65 (1978) 185 / Lüscher, CMP 85 (1982) 39 / Heemskerk–Polchinski, arXiv:1010.1264
§ 01, detour	Abstract	Wu–Yang, PRD 12 (1975) 3845 / Aizenman–Duminil-Copin, Ann. Math. 194 (2021) 163
Table 2, Figure 1	This book's reading	The primary sources are in 18-02–18-04. The layout is `fig1_layout.py`
Table 3, Figure 2	Computed	Python on this machine (`meff.py`)

Next chapter to read: 18-02 A dictionary of the theory itself.

A dictionary of the theory itself — the Settled pairs line up in front of a single measure

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — Nineteen pairs of words for the space, the measure and the states on which the theory rests, with their two-column grades and the grounds for them.
That many of the Settled pairs lie inside a single premise: "if a measure satisfying the conditions exists".
A calculation that sees that local symmetry does not break (Elitzur's theorem) through the order in which the external field is removed.

Prerequisite chapters — 4-02 and 4-10. The two columns of grades are from 18-01. We also draw on tools from 4-11 (particles, spectrum), 5-03 (the transfer matrix), 3-04 (orbit space, Singer's theorem), 2-08 (the order of removing the external field) and 5-01 (Haar measure). Words not in the prerequisite chapters are defined where they first appear.

Said first — We only line up pairs and grade them; we make neither the measure nor the Hilbert space of four-dimensional Yang–Mills theory. "None" in the existence column means, as in 18-01 § 04, "not yet made"; it has not been shown not to exist. The exception is the entry "gauge fixing": that there is no global continuous gauge fixing (on the four-dimensional sphere) is Singer's theorem ("ruled out" in Table 0).

The order of this chapter

1. The overall picture — nineteen pairs and one premise (Table 0)
2. The container — the path integral, the measure, the space of states
3. Real time and imaginary time — three faces of a correlation
4. The contents — the vacuum, the Hamiltonian, particles
5. Scattering and positivity — two kinds of "unitarity"
6. Approximation — spacetime, regularisation, renormalisation
7. Fields have no values at points — operator-valued distributions

8. Gauge — slices and physical states
9. Local symmetry does not break — Elitzur's theorem (the entry "local symmetry does not break")
    - Detour: gauge theory at $\beta = \infty$ is the Ising model

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01

The overall picture — nineteen pairs and one premise

The grades are the two columns of 18-01. (a) is "if both existed, would they be the same?", and (b) is whether the mathematical object exists in four-dimensional Yang–Mills theory (level in parentheses). The judge is what supports "the same" (theorem, definition, convention, and so on; 18-01 § 07), and the chapter is the chapter that treats it in detail.

Table 0 A dictionary of the theory itself

#	Physical word	Mathematical word	(a) Definitions	(b) 4D YM	(c) Finite measurement	Jud
1	Path integral (quantum mechanics)	Feynman–Kac formula	Settled	Not applicable*	—	The
2	Functional integral	Euclidean probability measure	Settled	None (Constructed at the lattice level)	—	Def
3	Space of states	Reconstructed Hilbert space	Conditional	None (Constructed at the lattice level)	—	The cor
4	Wick rotation	OS reconstruction theorem	Settled (⇒)	None	—	The
5	Green functions	Wightman and Schwinger functions	Settled (with direction)	None	—	The
6	Källén–Lehmann representation	Spectral measure	Settled	None (Constructed at the lattice level)	Upper bound only	The

#	Physical word	Mathematical word	(a) Definitions	(b) 4D YM	(c) Finite measurement	Jud
7	Vacuum and uniqueness	Ω and clustering	Settled	None (Constructed at the lattice level)	Out of reach	The
8	Hamiltonian	Logarithm of the transfer matrix	Conditional	Constructed (lattice)	—	The cor
9	One-particle state	Isolated eigenvalue of the mass operator	Conditional	None	Upper bound only	The cor
10	S-matrix	Haag–Ruelle scattering theory	Conditional	None	—	The cor
11	Positive norm	Reflection positivity	Settled	Constructed (lattice)	—	The
12	Unitarity of the S-matrix	Positive norm + asymptotic completeness	Conditional	None	—	The cor
13	Spacetime and lattice	$a\mathbb{Z}^4 \to \mathbb{R}^4$	Conditional	None	Out of reach	Def + cor
14	Regularisation	Family of measures with lattice spacing a	Conditional	Constructed (lattice)	—	Def + cor
15	Renormalisability	BPHZ, Epstein–Glaser	Conditional	Constructed (formal power series)	—	The
16	Field $\phi(x)$	Operator-valued distribution	Conditional	None	—	Def + cor
17	Gauge fixing	Section of the orbit space	Settled	None (ruled out by Singer's theorem)	—	Def
18	Physical states	Reconstruction from gauge invariants	Unsettled	Constructed (lattice)	—	Cor (a t on latt
19	Local symmetry does not break	Elitzur's theorem	Settled	Constructed (lattice)	—	The

Note: column (b) of Table 0 is the reading from before the published construction (2026-11-10). As at the end of §01, rows 3, 4, 5 and 8 leave room to be regraded at the level of the published construction.

Notes on the words of Table 0. (c) is the measurement column of 18-01 § 02 (the direction reachable by finite measurement), given only for rows that name a quantity. The entries "Källén–Lehmann representation" and "one-particle state" are rows where a single correlation gives an upper bound on the lowest point (18-01 § 08), and the entries "vacuum and uniqueness" and "spacetime and lattice" are statements about limits. "None" in (b) means not yet constructed at the continuum level, and "None (Constructed at the lattice level)" means rows made only at the lattice level. Only "ruled out" in the entry "gauge fixing" has the opposite sense: it is a theorem that a global section does not exist. * "Not applicable" in the entry "path integral (quantum mechanics)" means that, being a pair from quantum mechanics with finitely many degrees of freedom, it has no four-dimensional entry (18-01 § 04).

Many of the Settled pairs lie inside a single premise. If on $\mathbb{R}^4$ there is a measure μ satisfying the Osterwalder–Schrader (OS) conditions, the reconstruction theorem gives the rest (4-10 § 05) Known .

$$\mu \text{ satisfies } (\mathrm{E0}')\text{–}(\mathrm{E4}) \implies (\mathcal{H},\ H \geq 0,\ \Omega), \quad \{S_n\} \to \{W_n\}, \quad d\rho_O \geq 0 \tag{1}$$

On the left are the OS conditions (growth, Euclidean covariance, reflection positivity, symmetry, clustering; 4-10 § 03), and on the right the Hilbert space, Hamiltonian and vacuum, the passage from the Schwinger functions S_n to the Wightman functions W_n, and the positivity of the spectral measure. Before the publication of 2026, the left was missing in four-dimensional Yang–Mills theory, so at the continuum level the existence column lined up almost entirely at "None" (Table 0 is that reading). The exceptions are rows for which the premise is not enough even if it holds (the entries "one-particle state", "S-matrix" and "unitarity of the S-matrix"), and rows decided outside the premise (Singer's theorem in the entry "gauge fixing", Elitzur's theorem in the entry "local symmetry does not break"). Example: for the four-dimensional free field the left of (1) exists, and the objects of every row except the gauge rows are present.

The committee's assessment is that the construction published on 2026-11-10 showed a variant of the OS axioms for each component of the phases, starting not from a measure but from the side of correlations (Schwinger functions), and reconstructed the Hilbert space, Hamiltonian and vacuum on the right (6-01, 12-08). The existence column of the entries "space of states", "Wick rotation", "Green functions" and "Hamiltonian" leaves room to be regraded at that level (18-01 § 09). The measure, one-particle states and the S-matrix remain None (6-04).

02

The container — the path integral, the measure, the space of states

"Container" is a metaphor. More precisely, it means the space on which the theory rests (the totality of field configurations), the measure on it, and the Hilbert space made from them.

Path integral (quantum mechanics) ⟷ Feynman–Kac formula

The physical word is "a weighted sum over all the paths a particle can take", and the mathematical word is an expectation over Brownian motion. They can be said to be the same after moving to imaginary time and putting conditions such as the potential being bounded below.

Writing the e^{-tH} of quantum mechanics in this form is the Feynman–Kac formula (Kac 1949; equation (2) of 18-01, 2-07 § 05) Known . Feynman's (1948) sum has the real-time weight $e^{iS/\hbar}$, and moving it to imaginary time gives this formula. The real-time weight does not become a measure.

Functional integral ⟷ Euclidean probability measure

The physical word is a sum over all field configurations with weight e^{-S}, and the mathematical word is a probability measure on the space of configurations. They can be said to be the same because the side of mathematics defined $\int \mathcal{D}A\, e^{-S}$ as a countably additive probability measure (the judge is a definition; § 4 of Jaffe's review).

$$\frac{1}{Z}\int \mathcal{D}\phi\; e^{-S(\phi)} F(\phi) \;\rightsquigarrow\; \int F\, d\mu, \qquad \mu:\ \mathcal{S}'(\mathbb{R}^4) \text{ etc.: a probability measure on it} \quad (2)$$

Example: the free field is the Gaussian measure with covariance $(-\Delta + m^2)^{-1}$ (Δ is the Laplacian, not the Δ of the mass gap) (4-10 § 02). Two-dimensional Yang–Mills theory has also been constructed (Driver 1989, 7-02). On the lattice, equation (9) of 4-02 is a genuine measure. The measure of four-dimensional Yang–Mills has not been constructed (the construction published on 2026-11-10 does not make the measure either; 6-04), and the space it rests on (a completion of the space of connections divided by gauge) has not yet been fixed either (3-04).

Space of states ⟷ reconstructed Hilbert space

The physical word is Fock space (the space that lines up states by number of particles, 2-06), and the mathematical word is the Hilbert space (3) made from the measure. They can be said to be the same for free fields, and, with interactions, when asymptotic completeness (§ 05) holds.

Let $\mathcal{E}_+$ be the totality of functions determined only by fields at positive times, and Θ time reversal (4-10 § 05).

$$\mathcal{H} = \mathcal{E}_+ / \mathcal{N}, \qquad \mathcal{N} = \{F : \mathbb{E}[\Theta F\, F] = 0\} \quad (3)$$

$\mathcal{N}$ is the totality of functions of norm 0, and dividing by it and completing gives $\mathcal{H}$. (i) For free fields it agrees with Fock space (4-10 § 07). (ii) With interactions, the representation of the canonical commutation relations at time 0 becomes inequivalent to the Fock representation (ϕ^4_3, Jaffe § 6). The Fock space of physics is the space of asymptotic particles, so agreement needs asymptotic completeness.

03

Real time and imaginary time — three faces of a correlation

"Three faces" is a metaphor. More precisely, it means the values of the same correlation read at different places (real time, imaginary time, time-ordered) of a single analytic function (the entry "Green functions").

Wick rotation ⟷ OS reconstruction theorem

The physical word is the computational procedure of setting $t = -i\tau$ and rotating the contour, and the mathematical word is the theorem that builds a Wightman theory from Schwinger functions. They can be said to be the same under the conditions (E0′)–(E4), in the direction from imaginary time to real time.

Wick rotation is Wick's (1954) procedure (4-02 § 02), and its mathematical partner is the reconstruction theorem, which has a direction Known .

$$\{S_n\} \text{ satisfy (E0}'\text{)–(E4)} \implies \text{a Wightman theory}, \qquad W_n(\ldots,(-i\tau_j, x_j),\ldots) = S_n(\ldots,(\tau$$

A condition on the ordering of times is needed because S_n does not change when its arguments are exchanged, while W_n changes with the order in which the operators stand (the example in the entry "Green functions"). Care is needed in the reverse direction. The Schwinger functions obtained from a Wightman theory need not satisfy (E0′). The first OS paper (1973) claimed an equivalence, but the proof had an error, which Glaser (1974) fixed. The second (1975) states it in the form of a sufficient condition with (E0′) added (the abstract has '(necessary and) sufficient', with parentheses).

Green functions ⟷ Wightman functions and Schwinger functions

The physical word "Green function" refers to any of the time-ordered product G_T, the Wightman function W and the Schwinger function S, while mathematics defines these

three as separate functions. They can be said to be the same in the sense of the theorem that the three are values of one analytic function. Example: for an oscillator of angular frequency ω ($\hbar = m = 1$),

$$F(z) = \frac{e^{-\omega z}}{2\omega} \ (\mathrm{Re}\, z \geq 0): \qquad S(\tau) = F(|\tau|), \qquad W(t) = F(it), \qquad G_T(t) = F(i|t|) \tag{5}$$

W is the boundary value on the imaginary axis, S is the restriction to the real axis folded by $|\tau|$, and G_T is glued together by sign (Wightman 1956, Symanzik 1966). For $\tau > 0$, $W(-i\tau) = F(\tau) = S(\tau)$, which is the two-point case of (4). For $\tau < 0$, extending the formula for W as it is grows as $e^{\omega|\tau|}$, disagreeing with $S(\tau)$. $W \to S$ follows from positive energy alone, but $S \to W$ needs (E0′).

Källén–Lehmann representation ⟷ spectral measure

The physical word is the formula that writes a two-point correlation as a superposition over energies (masses), and the mathematical word is the spectral measure of the Hamiltonian. They can be said to be the same when H exists by reconstruction (under the premise of (1)). The representation of Källén (1952) and Lehmann (1954) can be written, in the form of equation (6) of 4-11, as

$$\langle O\Omega,\ e^{-\tau H} O\Omega\rangle = \int_{[0,\infty)} e^{-E\tau}\, d\rho_O(E), \qquad d\rho_O(E) = d\langle O\Omega,\ P_H(E)\, O\Omega\rangle \geq 0 \tag{6}$$

$P_H(E)$ is the projection onto the part of H with energy at most E (the spectral theorem of 1-10 § 05). The positivity of the weights comes from reflection positivity (Table 2 of 4-10). Example: (5) is the case where ρ has weight at the single point $E = \omega$, and on the lattice $\rho_O = \sum_n |\langle n|O|\Omega\rangle|^2 \delta_{E_n}$ (equation (8) of 18-01). The breaking of positivity of the gauge-fixed gluon propagator is treated in 17-02.

04 The contents — the vacuum, the Hamiltonian, particles

"Contents" is also a metaphor. More precisely, it means the particular vector (the vacuum) and operators (the Hamiltonian, the mass operator) fixed on the Hilbert space of § 02.

Vacuum and uniqueness ⟷ Ω and clustering

The physical word is "the state with nothing in it", and the mathematical word is $\Omega = [1]$, obtained in the reconstruction as the class of the constant 1. They can be said to be the same when the vacuum of physics is read as "the state of lowest energy" (a theorem under the premise of (1)).

Uniqueness means that the eigenvalue 0 of H is simple (Jaffe § 2), and on the Euclidean side it is paired with clustering Known .

$$\Omega \text{ unique} \iff \lim_{|a|\to\infty} S_{n+m}\big(x_1,\dots,x_n,\,y_1+a,\dots,y_m+a\big) = S_n(x)\,S_m(y), \qquad a=(0,$$

Non-uniqueness of the infinite-volume limit, degeneracy of the ground state and phase transitions are the same phenomenon (Jaffe § 7.2). Example: on the low-temperature side of the two-dimensional Ising model, the difference between the lowest two levels closes with the width (2-09 § 08). The identification with the procedure of physics that builds the vacuum by switching on interactions slowly from the vacuum without interactions (attributed to Gell-Mann and Low) is not mathematically justified, since Haag's theorem (attributed to Haag 1955) denies in general the existence of the interaction picture (the way of writing that evolves in time by the interaction alone on the space of free fields).

Hamiltonian ⟷ logarithm of the transfer matrix

The physical word is the operator that generates the advance of time (the energy), and the mathematical word is the logarithm of the transfer matrix T that advances imaginary time by one step on the lattice. They can be said to be the same at the lattice level; agreement with the continuum H is conditional on the continuum limit.

$$H_a = -\frac{1}{a}\log T, \qquad T = T^{*} > 0 \quad (8)$$

a is the lattice spacing in the time direction. A positive self-adjoint T was made by Lüscher (1977) (5-03 § 05), and it connects to the H of Kogut–Susskind (1975) by sending the time step to 0 (Creutz 1977, 5-03 § 07) Known . It is a different thing from the generator L of stochastic quantisation (18-03 § 05).

One-particle state ⟷ isolated eigenvalue of the mass operator

The physical word is a particle classified as an irreducible representation of the Poincaré group (Wigner 1939), and the mathematical word is an isolated eigenvalue of the mass operator M. They can be said to be the same in the reconstructed theory, when that eigenvalue is isolated in the spectrum (4-11 § 05).

$$M = \sqrt{H^2 - \vec{P}^{\,2}}, \qquad m \text{ an eigenvalue of } M \text{ with } (m-\delta, m+\delta)\cap\sigma\big(M|_{\Omega^{\perp}}\big) = \{m\} \quad (\delta >$$

$\vec{P}$ is the momentum, $\sigma(\cdot)$ the spectrum, and $\Omega^{\perp}$ the part orthogonal to the vacuum. Jaffe writes that scattering theory assumes this and 'It entails both a lower gap and an upper gap' (§ 7.1; § 5 of the problem statement also counts the upper gap among the extensions) Known . It has been shown for weakly coupled $P(\phi)_2$ (Glimm–Jaffe–

Spencer 1974), and there is also a one-particle state at strong coupling on the lattice (Schor 1984). Example: for the free field the continuous spectrum above m starts at $2m$. The glueball row is in 18-03 § 04.

05

Scattering and positivity — two kinds of "unitarity"

"Unitarity" in physics is used in two senses, so we split it into the entry "positive norm" (no states of negative norm) and the entry "unitarity of the S-matrix" (probability conserved in scattering) (18-01 § 06). We put the tool of scattering (the entry "S-matrix") first.

S-matrix ⟷ Haag–Ruelle scattering theory

The physical word is the amplitude from a set of particles in the far past to a set of particles in the far future, and the mathematical word is an operator built from two isometries. They can be said to be the same when there is an isolated one-particle mass (the entry "one-particle state").

The mass hyperboloid is the surface $p^2 = m^2$ in momentum space (on the side $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$), and the one-particle states of mass m lie on it. An isometry is a map that preserves inner products, and Ran is the range (the totality of values of the map). $\mathcal{F}$ is the Fock space of asymptotic particles, the space that lines up, by number of particles, the states of particles flying freely in the far past (or future). The scattering theory of Haag (1958) and Ruelle (attributed to 1962) builds the isometries $W^{\mathrm{in}}, W^{\mathrm{out}} : \mathcal{F} \to \mathcal{H}$ from an isolated mass hyperboloid.

$$S = (W^{\mathrm{out}})^* W^{\mathrm{in}}, \qquad \mathrm{Ran}\, W^{\mathrm{in}} = \mathrm{Ran}\, W^{\mathrm{out}} \implies S^*S = SS^* = 1 \quad (10)$$

The rightward direction of (10) is elementary. For an isometric W, $W^*W = 1$, and WW^* is the projection onto the range; if the ranges are the same, $W^{\mathrm{out}}(W^{\mathrm{out}})^*$ does not change the values of W^{in}, so $S^*S = (W^{\mathrm{in}})^* W^{\mathrm{out}} (W^{\mathrm{out}})^* W^{\mathrm{in}} = (W^{\mathrm{in}})^* W^{\mathrm{in}} = 1$, and $SS^* = 1$ likewise Paper . Example: for the free field $W^{\mathrm{in}} = W^{\mathrm{out}}$ and $S = 1$. Asymptotic completeness is the claim that the two ranges are the whole of $\mathcal{H}$. Within the range we searched, we found no theorem deriving asymptotic completeness in the usual sense in general. Dybalski–Gérard (2014) showed a "generalised" form under lower and upper mass gaps Known . There are partial results for individual models (unitarity of the S-matrix for weakly coupled $P(\phi)_2$ on even states with energy below about $4m$: Spencer–Zirilli 1976).

Positive norm ⟷ reflection positivity

The physical word is that there are no states of negative norm (Wightman positivity), and the mathematical word is the reflection positivity of the Schwinger functions. They can be said to be the same within the correspondence between OS and Wightman, and this is a theorem.

Reflection positivity gives the Hilbert space and $H \geq 0$, and is preserved in limits (Jaffe § 4), so if a limit exists it can be carried over from the lattice. It holds for Wilson's lattice (Osterwalder–Seiler 1978, Lüscher 1977) Known . Example: negative weights of Pauli–Villars type break it (4-10 § 08).

Unitarity of the S-matrix ⟷ positive norm + asymptotic completeness

The physical word is that probability is conserved in scattering ($S^*S = 1$), and the side of mathematics is the entry "positive norm" with the agreement of the ranges in (10) added. They can be said to be the same when asymptotic completeness holds.

For gauge theories, the abstract of Kugo–Ojima (1979) writes that "unitarity of the physical S-matrix" is shown in general ('quite generally') on the basis of representations of the algebra of the BRS charge (the Q_B of § 08). This is a general theory assuming representations of the algebra and asymptotic fields, and its stage is a space with indefinite metric (an inner product that allows negative norms). Those assumptions have been checked within perturbation theory.

06

Approximation — spacetime, regularisation, renormalisation

Spacetime and lattice ⟷ the $a \to 0$ limit of $a\mathbb{Z}^4$

The physical word is four-dimensional spacetime with Minkowski signature ($\mathbb{R}^4$ after Euclideanisation), and the side of mathematics is the $a \to 0$ limit of measures on the lattice $a\mathbb{Z}^4$ of spacing a. They can be said to be the same when the limit exists and the rotational symmetry $O(4)$ comes back.

The rotational invariance of Euclideanised spacetime is the $O(4)$ invariance of Euclidean covariance (Osterwalder–Schrader axiom E1) (for the signature, the existing article One sign is different). What remains on the lattice is rotations by 90° and reflections, and the abstract of Wilson (1974) writes of the strong-coupling limit 'no Lorentz (or Euclidean) invariance' Known .

symmetries of the hypercube (order $2^4 \cdot 4! = 384$) $\subset O(4)$, $\qquad \mu_a \xrightarrow[a\to 0]{?} \mu$ ($O(4)$ -invaria

Example: for the free field on the lattice, the difference between correlations along an axis and along a diagonal vanishes as a^2 (Table 2 of 4-02). For pure gauge theories, the continuum limit of the lattice has been shown for non-abelian groups in two dimensions and for U(1) in three and four dimensions (Driver 1989, Gross 1983, Driver 1987, among others). The abstract for three dimensions writes that with the Villain action it converges to the free electromagnetic field. The free electromagnetic field is invariant under rotations, so rotational symmetry has come back too.

Regularisation ⟷ family of measures with lattice spacing a

The physical word covers all devices that temporarily stop infinities, and what the side of mathematics can treat as a family of measures is mainly the lattice. They can be said to be the same when the limit with the device removed does not depend on the choice of device (universality).

§ 6.5 of the problem statement writes that there are few ways of recovering the reflection positivity lost in dimensional regularisation, Pauli–Villars and the like. What Reisz (1989) shows is that the continuum limit of the lattice is finite at each order of perturbation.

$$\lim_{a\to 0} \mu_a^{\text{Wilson}} \stackrel{?}{=} \lim_{a\to 0} \mu_a^{\text{Villain}} \stackrel{?}{=} \cdots \quad (12)$$

The non-perturbative (12) has been shown in two dimensions. Chevyrev–Shen (2023, arXiv:2302.12160) showed on the two-dimensional torus, for connected compact groups, that lattice gauge theories with the Wilson, Villain and Manton actions give the same limit. Convergence of the two-dimensional lattice approximation is also in Driver (1989).

Renormalisability ⟷ BPHZ, Epstein–Glaser

The physical word is that infinities vanish at each order of perturbation after adjusting finitely many constants, and the mathematical word is the finiteness theorems of a subtraction rule (BPHZ) and of an inductive construction from causality (Epstein–Glaser). They can be said to be the same at the level of formal power series (4-05).

In physics, Dyson (1949), and for Yang–Mills 't Hooft (1971) and Becchi–Rouet–Stora (1976). In mathematics, Bogoliubov–Parasiuk (1957), Hepp (1966), Zimmermann (1969) and Epstein–Glaser (1973).

$$S_n^{\text{ren}}(x;g) = \sum_{k\geq 0} c_{n,k}(x)\, g^k \in \mathbb{C}[[g]] \qquad (\text{each } c_{n,k} \text{ finite}) \quad (13)$$

The abstract of Epstein–Glaser writes that the inductive construction gives 'at each order, finite terms' and is equivalent to BPHZ Known . It does not include convergence

of the sum (the series is thought to diverge, 4-12).

07

Fields have no values at points — operator-valued distributions

Field $\phi(x)$ ⟷ operator-valued distribution

The physical word is an operator $\phi(x)$ at each point x, and the mathematical word is an operator-valued distribution, in which only $\phi(f)$, smoothed with a test function f, is an operator. They can be said to be the same when $\phi(x)$ is read as "a symbol for the smoothed value, just before the smoothing width is sent to 0".

Distributions are in 1-11 § 04 (the reading of fields is Jaffe § 3) Known . For the free field of mass m ($\omega_p = \sqrt{p^2 + m^2}$),

$$\phi(f) = \int \phi(x) f(x)\, d^4x, \qquad \|\phi(f)\Omega\|^2 = \int \frac{d^3p}{(2\pi)^3} \frac{|\hat{f}(\omega_p, p)|^2}{2\omega_p} < \infty \quad (14)$$

Example: taking f to be a bump of integral 1 and width ε, $f_\varepsilon(x) = \varepsilon^{-4} g(x/\varepsilon)$, we have $\hat{f}_\varepsilon(p) = \hat{g}(\varepsilon p)$, so in four dimensions the right-hand side diverges as ε^{-2} (1-09). The point value $\phi(x)$ is the limit as f approaches δ, so as an operator it does not exist. The point fields of Yang–Mills are $\mathrm{Tr}\, F_{ij}F_{kl}(x)$, and § 4 of the problem statement requires those operators. The local-field footnote of the problem statement (footnote 1 of Jaffe–Witten) says that a natural one-to-one correspondence between classical differential polynomials and operators 'does not exist', and expects a correspondence between the spaces of dimension $\le d$ (0-03 § 08). Dimension here is the power under a change of the length scale (2 for F, 4 for $\mathrm{Tr}\, F^2$; 1-11 § 06). A Wilson loop is a quantity on a curve, outside the axioms of point fields (4-10 § 09).

08

Gauge — slices and physical states

Gauge fixing ⟷ section of the orbit space

The physical word is choosing one representative from each gauge orbit by a condition $F(A) = 0$, and the mathematical word is a section of the projection to the orbit space $\mathcal{A}/\mathcal{G}$ (a map that assigns continuously one point to each orbit); as definitions they are the same. On the four-dimensional sphere, however, it is a theorem that there is no global section.

The geometry of slices is in 3-04 § 05. The source of the Faddeev–Popov (1967) rule is

$$1 = \Delta_{\mathrm{FP}}(A) \int_{\mathcal{G}} dg\ \delta\big(F(A^g)\big), \qquad \Delta_{\mathrm{FP}}(A) = \Big|\det \frac{\delta F(A^g)}{\delta g}\Big|_{g=1} \quad (15)$$

(15) holds when the condition surface $F = 0$ meets each orbit exactly once. (15) evaluates the determinant at the single point $g = 1$ and takes it outside the integral, so if it meets many times the right-hand side becomes a sum of ratios of the determinants at each intersection, and is neither 1 nor the number of intersections. Putting the determinant inside the integral gives the number of intersections $N(A)$.

$$N(A) = \int_{\mathcal{G}} dg\ \delta\big(F(A^g)\big) \Big|\det \frac{\delta F(A^g)}{\delta g}\Big| \quad (15a)$$

The actual Faddeev–Popov rule uses $\det$ without absolute value, so it counts intersections with signs (the detour in 4-08 § 05). Gribov (1978) pointed out that "exactly once" breaks down in non-abelian theories, and the abstract of Singer (1978) writes, on S^4, 'no continuous choice of exactly one connection on each orbit can be made' Known . The lattice integrates with Haar measure (the uniform probability measure on the group, unchanged by multiplication), so no gauge fixing is needed and it avoids this row (5-01, 4-08).

Physical states ⟷ reconstruction from gauge invariants

The physical word is states satisfying the Gauss constraint, or BRST cohomology, and the mathematical word is the space made by carrying out (3) with gauge-invariant quantities only. They can be said to be the same up to the lattice level; in the continuum it is Unsettled.

The Gauss constraint is the non-abelian version of Gauss's law $\nabla \cdot \boldsymbol{E} = 0$, and on the side of states it is the condition "the state does not change under gauge transformations at fixed time" (Kogut–Susskind 1975, 4-01). On the lattice, $\psi(U^g) = \psi(U)$ for gauge transformations g at fixed time. The BRST Q_B is an operator satisfying $Q_B^2 = 0$ (the BRST charge), and $\mathbf{Ker}\, Q_B / \mathbf{Im}\, Q_B$ is "the states annihilated by Q_B, identified up to the range of Q_B". An indefinite metric is an inner product that allows negative norms.

$$\mathcal{H}_{\mathrm{phys}}^{\mathrm{BRST}} = \mathbf{Ker}\, Q_B / \mathbf{Im}\, Q_B \ \ \text{(indefinite metric)}, \qquad \mathcal{H}_{\mathrm{inv}}^{\mathrm{OS}} = \mathcal{E}_{+}^{\mathrm{inv}} / \mathcal{N} \ \ \text{(positive definite)} \quad (16$$

On the lattice, transition probabilities between gauge-invariant states are not negative (Lüscher 1977), and gauge fixing does not change the Green functions of gauge singlets (operators unchanged by gauge transformations) (Creutz 1977) Known . In the continuum it has not been shown that the two sides of (16) are the same, and non-perturbative BRST remains hard to define because of Gribov copies.

09

Local symmetry does not break — Elitzur's theorem

Local symmetry does not break ⟷ Elitzur's theorem

The physical word is the claim "gauge symmetry (local symmetry) does not break spontaneously", and the mathematical word is a theorem stating it as a limit in an external field. They can be said to be the same when compared in the order "put in an external field, make the volume infinite, then remove the field".

The abstract of Elitzur (1975) argues, in the form 'It is argued that', that spontaneous breaking of local symmetry is impossible in a symmetric gauge theory without gauge fixing, and says it is shown in a simple system of abelian gauge fields on a lattice Known . Mathematical successors include a proof, for an abelian Higgs model on the lattice, that the magnetisation becomes 0 when the external field is removed (De Angelis–de Falco–Guerra 1978) Known . If the external field is 0, the Haar average makes the expectation 0 (equation (9) of 17-02). Spontaneous breaking is a question of the order in which the external field is removed (2-08).

Comparing in the order of removing the external field

We write Elitzur's argument out for the weight $\exp(\beta \sum_p U_p + h \sum_\ell U_\ell)$ of $\mathbb{Z}_2$ lattice gauge theory in d dimensions (link variables $U_\ell = \pm 1$, with U_p the product of the four links round a plaquette). A gauge transformation at one point x flips the $2d$ links touching x, and does not change $\sum_p U_p$. So for ℓ touching x,

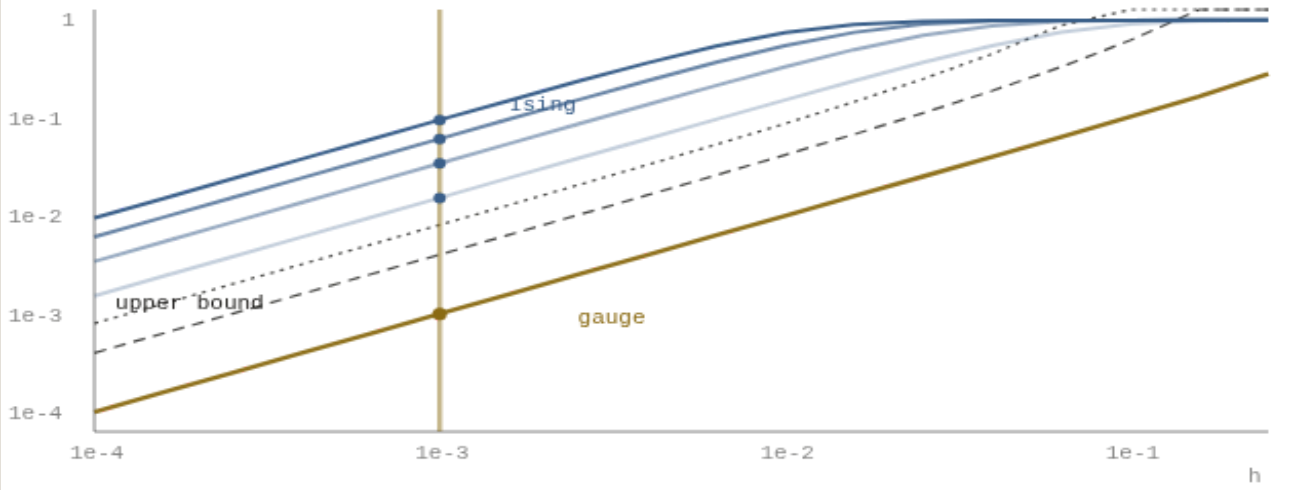
$$\langle U_\ell \rangle_h = -\langle U_\ell \, e^{-2hQ_x} \rangle_h, \quad Q_x = \sum_{\ell' \ni x} U_{\ell'} \quad \Longrightarrow \quad |\langle U_\ell \rangle_h| = \tfrac{1}{2} \big| \langle U_\ell (1 - e^{-2hQ_x}) \rangle_h \big| \le \tfrac{1}{2}$$

Paper . The right-hand side depends neither on the volume nor on β. In the Ising model (global $\mathbb{Z}_2$) all spins must be flipped, and the upper bound from the same argument is $\frac{1}{2}(e^{2h|\Lambda|} - 1)$, which loses its meaning as the volume grows. The only difference is whether one symmetry transformation touches finitely many variables or all of them. The slope at $h = 0$ is

$$\frac{d\langle U_\ell \rangle_h}{dh}\Big|_0 = \sum_{\ell'} \langle U_\ell U_{\ell'} \rangle_0 = 1, \qquad \frac{d\langle \sigma_0 \rangle_h}{dh}\Big|_0 = \sum_y \langle \sigma_0 \sigma_y \rangle_0 \approx M_0^2 \, |\Lambda| \quad (18)$$

On the left, the term with $\ell' = \ell$ is 1. The terms with $\ell' \neq \ell$ are 0 because, gauge-transforming at the endpoint of ℓ that does not touch ℓ', only U_ℓ of the two changes sign and $U_{\ell'}$ does not (in a box of side at least 3 such an endpoint always exists). The right is on the low-temperature side, where even distant spins keep the correlation M_0^2, so it is proportional to the volume.

Figure 1 Choosing the external field h (log scale) shows below ⟨σ⟩ (Ising, blue, darker in the order L = 4 to 10) and ⟨U⟩ ($\mathbb{Z}_2$ gauge, ochre) for the two-dimensional L×L periodic box. "Play" lowers h towards 0 (the direction of removing the external field). The dashed and dotted lines are the upper bound (17) (d = 2, 4)



h = 1.0e−3 ⟨σ⟩ : L=4 0.0152, L=6 0.0342, L=8 0.0607, L=10 0.0946 ⟨U⟩ 0.00100, upper bound 0.00402

Computed The numbers in this figure were computed on this machine (exact values by diagonalising the transfer matrix).

Computed Table 1 The two-dimensional $L \times L$ periodic box. Ising at $\beta = 0.6$ (low-temperature side), gauge at $\beta = 2$

h	$\langle\sigma\rangle$ $L=4$	**6**	**8**	**10**	$\langle U\rangle$ $L=4$–**10**	**(17)** $d=2$
10^{-4}	0.0015	0.0034	0.0061	0.0095	0.000100	0.00040
10^{-3}	0.0152	0.0342	0.0607	0.0946	0.001000	0.00402
10^{-2}	0.1512	0.3288	0.5396	0.7314	0.010002	0.04164
10^{-1}	0.8979	0.9792	0.9810	0.9810	0.1017–0.1016	0.61277

The Ising columns rise with L, and $\langle\sigma\rangle/(hL^2)$ at $h = 10^{-4}$ is 0.9529, 0.9505, 0.9494, 0.9489, approaching 0.9479, the square of the formula $M_0 = (1 - \sinh^{-4} 2\beta)^{1/8}$ cited for the spontaneous magnetisation (attributed to Yang 1952) Computed . In the gauge column, for $h \le 10^{-2}$, $L = 4$–10 agree to 6 decimal places, and the slope is the 1 of (18). The "breaking of gauge symmetry" of physics is said to be reread as, for example, the breaking of a global symmetry after gauge fixing Physics ; taken literally it is a false friend (18-01).

Detour: gauge theory at β = ∞ is the Ising model

A subsection you may skip.

As $\beta \to \infty$ all plaquettes line up at $U_p = +1$ (a flat field). On a simply connected region one can write $U_{xy} = g_x g_y$ ($g_x = \pm 1$), so the external-field term becomes

$$h \sum_{\langle xy \rangle} U_{xy} = h \sum_{\langle xy \rangle} g_x g_y \qquad (19)$$

the Ising model with coupling $K = h$. The expectation of a link is its nearest-neighbour correlation (about $\tanh h$ for small h), and does not become a magnetisation. The periodic box (torus) of Table 1 is not simply connected. The products of links once round the box (holonomies, 1-07) fall into four classes, ± 1 horizontally and vertically, and the $\beta \to \infty$ limit is the sum of four Ising models with periodic and antiperiodic boundary conditions. At the computed $h = 0.1$ and 0.3 (the high-temperature side of Ising), the difference from the periodic Ising model alone becomes exponentially small with L.

Computed Table 2 $\langle U \rangle$ in the $L \times L$ periodic box and the nearest-neighbour correlation of Ising ($K =$ difference between the periodic Ising model alone and the sum of the four classe

h	Gauge $\beta = 4$, 8	Sum of the four classes	Periodic Ising	Difference $L = 4$	
0.1	0.101689	0.101689	0.101689	1.2e−3	1.4e
0.3	0.352256, 0.352257	0.352257	0.356234	6.9e−2	1.7e

The first three columns are $L = 8$. At $h = 0.1$ they agree to 6 decimal places; at $h = 0.3$ a difference is visible between the gauge value 0.35226 and periodic Ising 0.35623, and the sum of the four classes agrees Computed .

What this chapter can say

	What is said	What is not said
Known	(4)(6)(7)(8)(15)(15a). The claims of the primary sources cited	That these hold in four-dimensional continuum Yang–Mills theory (the objects do not yet exist)
Paper The derivations of this chapter (elementary; (17)(18) follow the argument of Elitzur 1975)	The rightward direction of (10) (if the ranges are equal, S is unitary), the upper bound (17), the left of (18)	Estimates of the same form for non-abelian groups or continuum theories
This book's reading	The grades and judges of Table 0, the reading of (1)	That the grades are unique
Computed	Tables 1 and 2, Figure 1 (two dimensions, $L \leq 10$)	Four dimensions, non-abelian groups, $L \to \infty$
Overall	—	This book itself shows neither the existence of the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is in Volume 2)

Sources and reproduction

Item	Kind	Source / tools
Quotations from Jaffe and the problem statement	Text	Jaffe, *Constructive Quantum Field Theory* (2000, public PDF) § 2– § 7 / Jaffe–Witten, *Quantum Yang–Mills Theory* (Clay Mathematics Institute, 2000) § 3– § 6.5, the local-field footnote and the weak-existence footnote
Abstract checked	Known	Wick, PR 96 (1954) 1124 / Wightman, PR 101 (1956) 860 / Symanzik, JMP 7 (1966) 510 / Osterwalder–Schrader, CMP 31 (1973) 83, 42 (1975) 281 / Glaser, CMP 37 (1974) 257 / Lüscher, CMP 54 (1977) 283 / Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 / Creutz, PRD 15 (1977) 1128 / Kogut–Susskind, PRD 11 (1975) 395 / Schor, NPB 231 (1984) 321 / Dybalski–Gérard, CMP 332 (2014) 1167 / Spencer–Zirilli, CMP 49 (1976) 1 / Kugo–Ojima, PTPS 66 (1979) 1 / Wilson, PRD 10 (1974) 2445 / Gross, CMP 92 (1983) 137 / Driver, CMP 123 (1989) 575 / Reisz, NPB 318 (1989) 417 / Chevyrev–Shen, arXiv:2302.12160 (2023) / Epstein–Glaser, AIHP A 19 (1973) 211 / Faddeev–Popov, PLB 25 (1967) 29 / Gribov, NPB 139 (1978) 1 / Singer, CMP 60 (1978) 7 / Neuberger, PLB 183 (1987) 337 / Elitzur, PRD 12 (1975) 3978 / De Angelis–de Falco–Guerra, PRD 17 (1978) 1624 / Yang, PR 85 (1952) 808 (the formula for M_0: main text not checked)
Bibliographic data only	Known	Feynman, RMP 20 (1948) 367 / Kac, Trans. AMS 65 (1949) 1 / Driver, CMP 110 (1987) 479 / Gell-Mann–Low, PR 84 (1951) 350 (attribution from secondary sources) / Haag, Kong. Dan. Vid. Selsk. Mat.-Fys. Medd. 29 (12) (1955), PR 112 (1958) 669 / Ruelle, Helv. Phys. Acta 35 (1962) 147, as attributed (bibliographic data not checked) / Källén, Helv. Phys. Acta 25 (1952) 417 / Lehmann, Nuovo Cim. 11 (1954) 342 / Wigner, Ann. Math. 40 (1939) 149 / Glimm–Jaffe–Spencer, Ann. Math. 100 (1974) 585 / Dyson, PR 75 (1949) 486, 1736 / 't Hooft, NPB 33 (1971) 173 / Becchi–Rouet–Stora, Ann. Phys. 98 (1976) 287 / Bogoliubov–Parasiuk, Acta Math. 97 (1957) 227 / Hepp, CMP 2 (1966) 301 / Zimmermann, CMP 15 (1969) 208
Table 0	This book's reading	The primary sources are the rows above and 18-01
Tables 1 and 2, Figure 1, the slope of (18)	Computed	Python 3 on this machine (numpy): `elitzur.py`, `table.py`, `detour_beta_inf.py`, `holonomy_check.py` (the four classes), `holonomy_brute.py` (full enumeration of the 3×3 box)

Next chapter to read: 18-03 A dictionary of mass and length.

A dictionary of mass and length — pairs that overlap on the lattice and come apart on the side of the limit

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — We pair 20 physical words about "mass" and "length" with mathematical words, and see two layers: an inner layer where they overlap once the lattice is fixed, and an outer layer, on the side where the lattice spacing is sent to 0, where the conditions gather. We also fix here the distinction between "two times" and two η.

Prerequisite chapters — 4-11, 5-08 and 2-08. The Poincaré constant is in 1-10, and the two columns of grades in 18-01. Words not in the prerequisite chapters (asymptotic freedom, stochastic differential equations, log-Sobolev, mixing time, the Polyakov loop, central charge, and so on) are defined in a line each where they are used.

Said first — We only line up correspondences; we show neither the mass gap, nor confinement, nor the continuum limit. The assumptions of the record of this book's study (the diffusive-relaxation-time assumption and others) remain assumptions.

The order of this chapter

1. The overall picture — two layers, two-column grades, how to read Table 0
2. The mass gap and the correlation length — lattice units and physical units
3. Potential, string tension, confinement — only one direction is a theorem
4. Λ, one scale, glueballs — the distinction between a gap and a particle
5. Two times — H_T and L, the unit of time
6. Autocorrelation times and the Poincaré constant — times that split into three
7. Critical slowing down — z, the conventional theory, the variational lower bound, two η. Detour: for the free field the relaxation time equals the susceptibility
8. Fluctuations of the tube and topological freezing — effective strings, Wulff, bottlenecks. Detour: sectors of topological charge and narrow passages

01

The overall picture — two layers and two-column grades

In physics, mass and length are two sides of one coin. More precisely, in units with $\hbar = c = 1$ the inverse of a mass has the dimension of length; the force carried by a particle of mass m stops reaching beyond a distance of about $1/m$ (Yukawa type, 3-01), and correlations fade as $e^{-r/\xi}$ with $\xi = 1/m$. In mathematics this one number splits into separate quantities: the spectral gap, the decay rate of correlations, and the rate of relaxation of a stochastic dynamics.

The pairs of Table 0 fall into two layers. The inner layer is pairs that become the same thing by the spectral theorem alone (1-10) once the lattice spacing a and the dynamics are fixed (the gap of the transfer matrix and the decay rate of correlations, for example). The outer layer is pairs that appear where $\xi \to \infty$ and $a \to 0$ are joined, and the conditions and the Unsettled grades gather there.

The two columns of the table are the grades of 18-01. (a) is "if both objects existed, would they be the same?": to the three grades Settled, Conditional and Unsettled we add "Does not correspond" for pairs known to be different quantities (the "false friend" of 18-01). (b) is the existence column of 18-01 given at the level "lattice (a fixed)", read as: Constructed = exists at every β in finite volume; Conditional = only versions of limited range, such as strong coupling; None = nothing shown at this level (including things known only numerically). At the continuum four-dimensional level every row is "None" (the construction published on 2026-11-10 does not make the mass gap, Wilson loops or one-particle states either; 6-04). Notation: $\beta = 2N/g^2$, lengths in lattice units, $m_{\mathrm{lat}} = 1/\xi$, $\Delta = m_{\mathrm{lat}}/a$. $A \asymp B$ means that the ratio A/B is bounded above and below by positive constants (A-01).

Table 0 The list of the dictionary of mass and length (from top, in the order of § 02– § 08)

#	Physical word	Mathematical word	(a) Definitions	(b) Lattice (a fixed)	(c) Finite measurement
1	Mass gap	Spectral gap, uniform decay	Conditional	Conditional (strong coupling)	Upper bound only
2	Correlation length ξ	Inverse of the gap of the transfer matrix	Conditional	Conditional (strong coupling)	Lower bound only
3	Gap in physical units	Uniform lower bound in lattice units	Does not correspond (false friend; settled as a different quantity)	Conditional (strong coupling)	Out of reach

#	Physical word	Mathematical word	(a) Definitions	(b) Lattice (a fixed)	(c) Finite measurement
4	$V(R)$	Decay rate of Wilson loops	Conditional	Conditional	Upper bound only
5	String tension σ	Coefficient of the area law, the d of the continuum string-tension problem (Chatterjee's Problem 5.2)	Conditional	Conditional ($\sigma > 0$ at strong coupling)	—
6	Confinement	One-directional theorems	Conditional	Conditional	—
7	Centre symmetry	Centre invariance on a slab	Unsettled	None (whether it holds is unproved)	—
8	Λ, one scale	No counterpart	Unsettled	None	—
9	$m/\sqrt{\sigma}$	The limit of the pair of the lattice correlation-length problem and the continuum string-tension problem	Unsettled	None (numerical only)	Out of reach
10	Glueball	Isolated eigenvalue of the mass operator	Conditional	Conditional (strong coupling; see note)	Upper bound only
11	Langevin relaxation	Generator L	Settled	Constructed (finite volume)	Upper bound only ($\mathrm{gap}(L)$)
12	$z = 2$	$\mathrm{gap}(L) \asymp \mathrm{gap}(H_T)^2$	Unsettled	None (a statement about a family; see note)	Out of reach
13	Exponential autocorrelation time	Poincaré constant C_P	Settled in continuous time	Constructed (finite volume)	Lower bound only
14	Thermalisation time	Mixing time	Conditional	Constructed (finite volume)	Lower bound only
15	Dynamical exponent z	$C_P \asymp \xi^z$	Unsettled	None (a statement about a family; see note)	Out of reach
16	$z = 2 - \eta_s$	Variational lower bound $C_P \geq \chi$	Conditional	Constructed (lower bound only)	Lower bound only

#	Physical word	Mathematical word	(a) Definitions	(b) Lattice (a fixed)	(c) Finite measurement
17	No slowing down away from criticality	Uniform mixing ⟺ uniform log-Sobolev	Conditional	Conditional (strong coupling)	Out of reach
18	Lüscher term	Casimir energy of a strip	Unsettled	None (numerical only)	—
19	Roughening, shape of the tube	Wulff construction	Unsettled	None (a picture of physics)	—
20	Topological freezing	Bottleneck in state space	Conditional	None (freezing numerical only)	Lower bound only

Note: (c) is the measurement column of 18-01 §02 (the direction reachable by finite measurement). Relatives of the gap (the entries "mass gap", "V(R)", "glueball" and "Langevin relaxation") reach an upper bound from a single correlation or test function, and the lengths and times that are their inverses (the entries "correlation length ξ", "exponential autocorrelation time", "thermalisation time", "z=2-η_s" and "topological freezing") swap direction and reach a lower bound. Statements about uniformity, families or limits (the entries "gap in physical units", "m/√σ", "z=2", "dynamical exponent z" and "no slowing down away from criticality") are out of reach. The ground for the entry "topological freezing" is (14) (choosing one set gives a lower bound on C_P; §08), and for the entry "thermalisation time" the definition of mixing time (§06; if the total variation distance exceeds ε on a single event, that time is before the mixing time). The entry "string tension σ" (the string tension) gets "—", since the main text of this book does not treat a form that pins down one side from finite measurements. The (b) of the entry "glueball" has the form that for each momentum there is a dispersion curve isolated from the energy spectrum, since the lattice has no Lorentz symmetry and the mass operator M is not defined as it is (Schor 1984; §04). Both sides of the entries "z=2" and "dynamical exponent z" are defined on a fixed lattice, but $\asymp$ is a statement about a family with the lattice spacing sent to 0, so on a single lattice it has no content. "None" in (b) means that nothing has been shown about that family.

02

The mass gap and the correlation length — lattice units and physical units

The problem statement (Jaffe–Witten) defines the mass gap as the Hamiltonian H having no spectrum in $(0, \Delta)$ (§ 4), and lists equal-time clustering as a consequence (§ 5) Known . After OS reconstruction (4-10), if the vacuum Ω is unique, the spectral theorem gives

$$\langle O\Omega, e^{-tH} O\Omega\rangle - |\langle \Omega, O\Omega\rangle|^2 = \int_{[\Delta,\infty)} e^{-tE}\, d\rho_O(E) \;\le\; e^{-t\Delta}\, \|O\Omega\|^2 \quad (1)$$

(ρ_O is the spectral measure of $O\Omega$ with 0 removed). The converse of (1) holds only when a dense family of observables decreases at a *uniform* rate (the entry "mass gap"). On the lattice, reflection positivity lets one build a positive self-adjoint transfer matrix $T = e^{-HT}$ (Lüscher 1977, 5-03), and from its eigenvalues $\lambda_0 > \lambda_1 \ge \cdots$,

$$m_{\mathrm{lat}} = -\log \frac{\lambda_1}{\lambda_0}, \qquad \xi = \frac{1}{m_{\mathrm{lat}}} \quad (2)$$

The correlation length of the plaquette correlation of Chatterjee (2018) is in general $\xi_{\mathrm{plaq}} \le 1/m_{\mathrm{lat}}$ (equality needs overlap with the lowest state). Example: for the free field $\xi = 1/(2\,\mathrm{arsinh}(m/2))$ (Table 1).

Two quantities under one word

Many of the results called "the mass gap is positive" in mathematics are uniform lower bounds in lattice units, $m_{\mathrm{lat}} \ge c > 0$. An example that can be checked is Shen–Zhu–Zhu (2023), whose abstract states a 'strictly positive mass gap' for strongly coupled lattice Yang–Mills Known . Then $\Delta = m_{\mathrm{lat}}/a \ge c/a \to \infty$, which in physical units is a property of the side where the mass goes to infinity, the opposite side from the continuum limit. A gap in physical units needs $\xi \to \infty$ to cancel $a \to 0$, and the lattice correlation-length problem (Chatterjee's Problem 5.1) also asks for $\xi(\beta) \to \infty$ as $\beta \to \infty$ (5-09, 16-08). This is why the entry "gap in physical units" in Table 0 is "Does not correspond".

03

Potential, string tension, confinement — only arrows in one direction are theorems

The static potential $V(R)$ is the energy of an infinitely heavy quark pair placed at distance R, read, following Wilson (1974), from the $R \times T$ Wilson loop (5-01) as

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \log\langle W_{R,T}\rangle, \qquad \sigma = \lim_{R\to\infty} \frac{V(R)}{R} \quad (3)$$

Taking the left as the definition it is Settled, but identifying it with the ground-state energy of static charges needs the $T \to \infty$ limit and overlap with the ground state (the entry "V(R)"). From the monotonicity and concavity of V shown by Bachas (1986), $\sigma \in [0, \infty)$ exists Known . The area law holds at strong coupling for every group (Osterwalder–Seiler 1978, 5-02). The continuum string tension is a proposal to define it as the d of Chatterjee's continuum string-tension problem.

$$\log\langle W_{R/\varepsilon,\,T/\varepsilon}\rangle = -c\,(R+T) - d\,RT + o(1), \qquad \varepsilon(\beta)\to 0,\ c(\beta)\to\infty,\ d \neq 0 \quad (4)$$

He himself writes that it is not clear whether the perimeter term is needed. The physical rule of thumb is $\sqrt{\sigma} \approx 485(6)$ MeV. This is a conversion that tentatively applies to the $r_0\sqrt{\sigma} = 1.160(6)$ of SU(3) pure gauge theory the length scale $r_0 = 0.472$ fm fixed by lattice calculations of QCD, and the authors themselves note that it is a conversion that cannot be made exactly, since pure gauge theory and QCD are different theories (Athenodorou–Teper 2020) Physics .

Three ways of saying confinement

The three are (a) $V(R) \to \infty$, (b) $\sigma > 0$ and (c) unbroken centre symmetry. The physical definition of (c) is at finite temperature. On a lattice whose time direction is a ring of length $1/T$ (T is the temperature), the trace of the product of links once round the time direction from a point is called the Polyakov loop, and its expectation being 0 is taken as the sign of confinement (Polyakov 1978, Susskind 1979; the universality argument through an order parameter transforming under the centre is Svetitsky–Yaffe 1982) Physics . A centre transformation multiplies all the time-direction links crossing the surface at some time by an element of the centre of the gauge group (for $SU(N)$, $e^{2\pi ik/N}$ times the identity matrix). The action does not change and only the Polyakov loop is multiplied by $e^{2\pi ik/N}$, so if this symmetry is unbroken the expectation is 0.

What has been shown in mathematics is only the one-directional implications of Table A (Chatterjee 2021) Known . The definition of unbroken centre symmetry (Definition 2.1 of Chatterjee 2021) is that for some $N \geq 1$ the theory restricted to the slab $\{0, \dots, N\} \times \mathbb{Z}^{d-1}$ is invariant under centre transformations for every boundary condition, and the definition of strong decay (Definition 2.3 of Chatterjee 2021) is uniform exponential decay for every cube and boundary condition (strong decay). The Wilson loops of the Theorem from Centre Symmetry to Confinement (Theorem 2.2 of Chatterjee 2021) are those in representations acting non-trivially on the centre.

Table A Implications between the ways of saying confinement (lattice)

Direction, example	Status
Strong exponential decay (Definition 2.3 of Chatterjee 2021) $\Longrightarrow$ unbroken $\wedge$ area law	Theorem (the Theorem from Uniform Decay to the Area Law)
Unbroken centre symmetry (Definition 2.1 of Chatterjee 2021) $\Longrightarrow$ $V(R) \to \infty$	Theorem (the Theorem from Centre Symmetry to Confinement)
Area law $\Longrightarrow$ $V(R) \to \infty$	Trivial
$V \to \infty \Longrightarrow \sigma > 0$	Not shown
Weak decay (for particular boundary conditions only) $\Longrightarrow$ confinement	Not shown
Finite groups, weak coupling: decay of two-point correlations for particular boundary conditions / no confinement	Theorem (Adhikari–Cao 2025) / belief

The gap and confinement are separate claims (17-02, 16-08, 5-10). On centre symmetry, Chatterjee proposes the definition of unbroken centre symmetry on the grounds that there is "to my knowledge" no mathematical definition, and cites Borgs–Seiler (1983) for the finite-temperature analogue. Whether the two definitions separate the same phases is unproved (the entry "centre symmetry").

04

Λ, one scale, glueballs — the distinction between a gap and a particle

The value of the dimensionless coupling g changes with the scale μ at which it is measured. Asymptotic freedom means that g becomes smaller the larger μ is (the shorter the distance) (4-06), and the one-loop coefficient that fixes the rate is b_0. Solving the one-loop formula,

$$\Lambda = \mu \exp\!\Big(-\frac{1}{2b_0\, g^2(\mu)}\Big), \qquad b_0 = \frac{11N}{3}\cdot\frac{1}{16\pi^2} \quad (5)$$

becomes a mass scale independent of μ (dimensional transmutation, 4-07). In physics every mass is a constant multiple of Λ, and $m/\sqrt{\sigma}$ is read as a pure number Physics . The side of mathematics has no definition of the four-dimensional Λ (the lattice Λ_L is in 18-04). The record of this book's study writes "a single scale" as chain (ii) $m^2 \asymp \sigma$, and calls it, by *analogy*, the sentence that plays the role of an equivalence principle (17-12). There is no proof. If the continuum string-tension problem holds with $\varepsilon(\beta)$ and $\varepsilon\xi \to \ell \in (0,\infty)$, then

$$\frac{m}{\sqrt{\sigma}} = \lim_{\beta\to\infty}\frac{1/(\varepsilon\,\xi)}{\sqrt{d}} = \frac{1}{\ell\sqrt{d}} \quad (6)$$

and this pairing is a reading not in Chatterjee's text. The numbers are 3.781(23) for SU(2), 3.405(21) for SU(3) and 3.072(14) for SU(∞) (0^{++}, Athenodorou–Teper 2020, 2021) Physics . There are also opposite examples. In three-dimensional compact U(1) (Villain action) the string tension is positive at every coupling, the continuum limit is a free field of mass m_D, and σ/m_D^2 in physical units diverges to infinity (Göpfert–Mack, abstract) Known . It is an example where there is confinement but the scale does not become one (17-12).

Glueballs are measured as the lowest decay rate in each J^{PC} (spin, parity, charge conjugation) channel (5-08; the SU(3) 0^{++} is $r_0 m = 4.21(11)(4)$, Morningstar–Peardon 1999) Physics . The rotational symmetry of the lattice is only the 48 rotations and reflections that carry the cube onto itself (the group O_h), so the channels are divided by the irreducible representations of O_h. The correspondence with continuum spin presupposes the recovery of rotational symmetry. A one-particle state in mathematics is an isolated eigenvalue of the mass operator

$$M = \sqrt{H^2 - P^2} \;\geq 0 \quad (7)$$

(P is the momentum; § 6 of the problem statement), and in § 5 the problem statement counts "an isolated one-particle state (the upper gap)" among the extensions Known . A gap and a particle are separate claims. The lattice has no Lorentz symmetry and M is not defined as it is, so what has been shown at strong coupling on the lattice has the

form that for each momentum there is a dispersion curve isolated from the energy spectrum (Schor 1983, 1984) Known .

05

Two times — the H_T of the transfer matrix and the L of Langevin

There are two times. One is the imaginary time of the path integral: correlations are carried by $T = e^{-H_T}$, and the gap is $1/\xi$. The other is the "fictitious fifth time" s of stochastic quantisation (Parisi–Wu 1981, 7-09) (the abstract of Zwanziger 1981): the whole configuration moves by the following stochastic differential equation and settles to e^{-S}/Z (the time of Monte Carlo methods is of the same kind).

$$d\phi_x = -\frac{\partial S}{\partial \phi_x}\, ds + \sqrt{2}\, dW_x(s) \quad (8)$$

$W_x(s)$ is a Brownian motion independent at each point x (1-09), and dW_x is the Gaussian increment over time ds, with mean 0 and variance ds. The first term moves in the direction that lowers the action, and the second adds fluctuations. On a computer one advances with step h as $\phi_x \leftarrow \phi_x - h\,\partial S/\partial\phi_x + \sqrt{2h}\,Z_x$ (Z_x independent standard normals), which is called the Euler–Maruyama method. The operator that generates the time change of the expectation of a function f (the generator) is

$$Lf = -\Delta f + \nabla S \cdot \nabla f, \qquad \langle f, Lf\rangle_\mu = \mathcal{E}(f) = \int |\nabla f|^2 \, d\mu \quad (9)$$

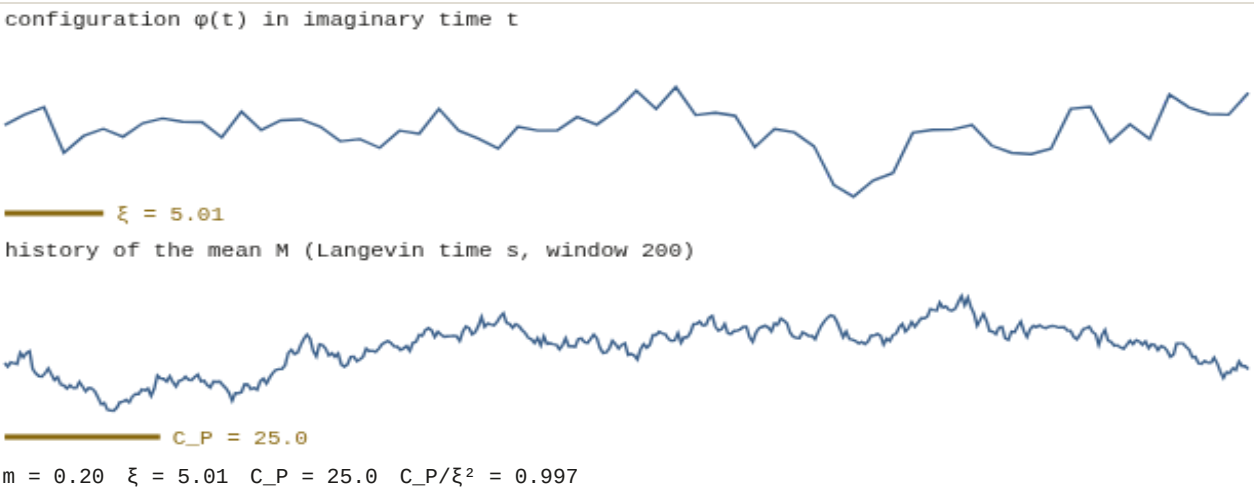
($\mu = e^{-S}/Z$, and $\mathcal{E}$ is the Dirichlet form; the Δ in (9) is the Laplacian, not the Δ of the mass gap), and the inverse of the gap above 0 is the Poincaré constant C_P (1-10). On a finite lattice both are defined (the entry "Langevin relaxation"), but H_T acts on the Hilbert space of a time slice, while L acts on the space $L^2(\mu)$ of functions of the whole configuration (functions square-integrable with respect to μ), and no theorem joins the two in general. For the free field $\mathrm{gap}(H_T) = 2\,\mathrm{arsinh}(m/2) \approx m$ and $\mathrm{gap}(L) = m^2$, so $C_P \approx \xi^2$ (1-10).

The diffusive-relaxation-time assumption and the unit of time

The relaxation-time assumption of the record of this book's study is the assumption that $\mathrm{gap}(L) \asymp \mathrm{gap}(H_T)^2$ holds uniformly in the lattice spacing (23-01, 18-05). However, the unit of time of L is fixed by the normalisation of S and of the noise. Replacing S by βS multiplies the gap of L, in the Gaussian case, by exactly β. In gauge theory with $S = \beta\times$ (Wilson action), in the free approximation $C_P \approx \xi^2/\beta$; that is, C_P is proportional to $g^2(a)\,\xi^2$, and this factor moves with the lattice spacing. Written at the level of a band (constant factors), the relaxation-time assumption is a statement made after

fixing this normalisation once (23-01). By the convention of the record, the relaxation-time assumption is put at the level of powers, and the band level is split off as a separate assumption that names the normalisation (18-05 § 03).

Figure 1 Changing m with the knob, the bar ξ at the top (the configuration φ(t) in imaginary time) grows as about 1/m, and the bar C_P at the bottom (the history of the mean M along the Langevin time s) as $1/m^2$. "Play" runs equation (8)



Computed The numbers in this figure were computed on this machine. The fluctuations are Euler–Maruyama with step $h = 0.05$. At this step the mean M is multiplied by $1 - hm^2$ each step and noise is added, a first-order recurrence whose integrated autocorrelation time $1/m^2 - h/2$ agrees with $1/m^2$ to within 2.5% over the whole range of the knob ($0.1 \le m \le 1$). Measured on the histories run, it was checked at $m = 0.3,\ 0.5,\ 1$ and agrees to within 10%. Without JavaScript it can be read from Table 1.

Computed Table 1 The two gaps of the free field on the lattice (zero-momentum channel)

m	gap(H_T)	ξ	$C_P = 1/m^2$	C_P/ξ^2
1.00	0.962424	1.039	1.00	0.9263
0.10	0.099958	10.004	100	0.9992
0.02	0.020000	50.001	2500	1.0000

The gap agrees to 6 digits with the slope of the correlation on a ring of length 8192, and C_P with the smallest eigenvalue of $-\Delta_{\text{lat}} + m^2$ on a ring of 256 points Computed . $C_P/\xi^2 \to 1$ is the $z = 2$ of the free field.

06

Autocorrelation times and the Poincaré constant — equality only in continuous time

When the dynamics is run from equilibrium, the correlation between values of an observable f separated by time t, divided by the variance, is called the autocorrelation $\rho_f(t)$. The time of its slowest decay is τ_{exp}, $\tau_{\text{int},f} = \frac{1}{2} + \sum_{t \ge 1} \rho_f(t)$ is the integrated autocorrelation time, and the variance of the sample mean is about $2\tau_{\text{int},f}\,\text{Var}(f)/n$

(Wolff 2004) Known . In continuous time the sum is replaced by an integral, $\tau_{\mathrm{int},f} = \int_0^\infty \rho_f(s)\,ds$. For reversible dynamics in continuous time $\rho_f(s) = \int e^{-s\lambda} d\nu_f(\lambda)$, with ν_f supported on $[\mathrm{gap}, \infty)$, so

$$\tau_{\exp} = \frac{1}{\mathrm{gap}(L)} = C_P, \qquad \tau_{\mathrm{int},f} = \int \frac{d\nu_f(\lambda)}{\lambda} \;\le\; C_P \quad (10)$$

For a reversible chain in discrete time (eigenvalues $1 > \lambda_2 \ge \cdots > -1$) they split into three (Levin–Peres–Wilmer § 12.2) Known .

$$C_P = \frac{1}{1-\lambda_2}, \qquad t_{\mathrm{rel}} = \frac{1}{1-\lambda_\star}, \qquad \tau_{\exp} = \frac{-1}{\log \lambda_\star}, \qquad \lambda_\star = \max_{\lambda \neq 1} |\lambda| \quad (11)$$

Example: for the AR(1) chain $X_{n+1} = \rho X_n + \sqrt{1-\rho^2}\,Z$ (eigenvalues ρ^k) with $\rho = -0.9$, $C_P = 5.263$, $t_{\mathrm{rel}} = 10$ and $\tau_{\exp} = 9.491$, all three different, and τ_{int} is 0.026 for x and 4.763 for x^2. In continuous time (Ornstein–Uhlenbeck) all three line up at 1 Computed . For a lazy chain (one that stays put with probability 1/2 each time, so all eigenvalues are at least 0), $\lambda_\star = \lambda_2$ and $C_P = t_{\mathrm{rel}}$ (LPW § 12.2, Exercise 12.3) Known .

Differences of dynamics, and mixing time

The heat-bath method (redrawing variables one at a time from their conditional distributions), Langevin and HMC (hybrid Monte Carlo, Duane et al. 1987: add fictitious momenta, move by classical mechanics for a fixed time, and correct by acceptance or rejection) have different generators, and regarding the time of HMC as the C_P of Langevin is an assumption (5-07). Thermalisation time and mixing time are also a Conditional pair. The mixing time $t_{\mathrm{mix}}(\varepsilon)$ is the number of steps until, from any initial state, the total variation distance (the largest difference between the probabilities two distributions give to the same event) between the distribution and the equilibrium distribution π is at most ε. The $\log(1/\pi_{\min})$ in the upper bound $t_{\mathrm{mix}}(\varepsilon) \le t_{\mathrm{rel}} \log(1/(\varepsilon\pi_{\min}))$ (LPW Theorem 12.4; $\pi_{\min}$ is the smallest probability of π) grows with the volume. This is a theorem for reversible chains with finitely many states; for gauge fields taking values in a continuous group $\pi_{\min}$ has no meaning, and other tools are needed, such as the log-Sobolev constant (§ 07).

07

Critical slowing down — z, the conventional theory, two η

As ξ grows, the relaxation of the dynamics also becomes slower (critical slowing down). Physics writes this with the dynamic scaling hypothesis $\tau \propto \xi^z$ and calls z the dynamical exponent (Halperin–Hohenberg 1969, Hohenberg–Halperin 1977). z changes with the dynamics and the observable. (8) is the relaxational dynamics

without conserved quantities called model A in the classification of Hohenberg–Halperin. In mathematics it is $C_P \asymp \xi^z$, but the existence of a power is itself a hypothesis, and there is no theorem fixing as a power the ξ-dependence of C_P in Yang–Mills (the entry "dynamical exponent z").

The conventional theory and RG

The static exponents are those of 2-08: $\xi \sim |t|^{-\nu}$, $\chi \sim |t|^{-\gamma}$ (t is the distance in temperature from the critical point), and η_s is the amount by which the critical correlation $G(x) \sim |x|^{-(d-2+\eta_s)}$ deviates from the power of the free field (the anomalous dimension). From the scaling relation $\gamma = \nu(2-\eta_s)$, $\chi \propto \xi^{\gamma/\nu} = \xi^{2-\eta_s}$ Physics . The conventional theory (called by van Hove's name) is summarised as making the relaxation time proportional to χ, giving $z = 2 - \eta_s$ Physics . The abstract of Halperin–Hohenberg–Ma (1972), who applied Wilson's expansion method to a continuum version of the kinetic Ising model, writes that the results 'disagree with the mode-mode coupling approach and the conventional (Van Hove) theory'. RG (the ε expansion) and numerics both suggest $z > 2 - \eta_s$ (Table 2) Physics .

Table 2 The dynamical exponent z and $2-\eta_s$ (η_s is the static anomalous dimension)

Model, method	$2-\eta_s$	z	$\eta_{\mathrm{dyn}} = 2 - z$
2D Ising, Glauber numerics	1.75	2.1665(12)	−0.1665
2D Ising, 5 loops (ε expansion)	1.75	2.14(2)	−0.14
3D Ising, Glauber numerics	1.9637	2.032(3)	−0.032
3D Ising, 5 loops (ε expansion)	1.9637	2.0235(8)	−0.0235
Free field, Langevin	2	2	0

Computed $2-\eta_s$ ($\eta_s = 1/4$ in two dimensions, $\eta_s = 2\Delta_\sigma - 1$ in three, with the scaling dimension of the spin $\Delta_\sigma = 0.5181489(10)$) and η_{dyn} computed on this machine. z and Δ_σ are values from primary sources Physics . Glauber is a dynamics that updates spins one at a time at random.

The variational lower bound

The counterpart on the side of mathematics is the variational lower bound. Putting $f = M = \sum_x \phi_x$ into (9), $|\nabla M|^2 = |\Lambda|$ (the number of points) and $\mathrm{Var}(M) = |\Lambda|\chi\Lambda$, so

$$C_P \;\geq\; \frac{\mathrm{Var}(M)}{\mathcal{E}(M)} = \chi\Lambda \quad (12)$$

which is a theorem holding on every lattice of finite volume Known . For the free field $\chi = 1/m^2 = C_P$, with equality (Table 1). Moving from here to a lower bound on the exponent needs two hypotheses: dynamic scaling $C_P \asymp \xi^z$ and static scaling $\chi \asymp \xi^{\gamma/\nu}$, under which $z \geq \gamma/\nu = 2 - \eta_s$ Physics (a lower bound regarded as standard in the

Monte Carlo literature). An example of a firm lower bound from an inequality of the same type is $z_{\mathrm{SW}} \geq \alpha/\nu$ for the Swendsen–Wang method (α is the specific-heat exponent; the abstract of Li–Sokal 1989) Known . The value of the conventional theory sits exactly at this lower bound, and the z of Table 2 all lie above it.

In gauge theory this form does not reach ξ^2 Physics . Measuring the dimension Δ of an operator by the power when correlations behave at short distances like $|x|^{-2\Delta}$, the lowest dimension of a gauge-invariant local operator is 4. The two-point correlation of a dimension-4 operator decreases as $|x|^{-8}$, so its sum χ in four dimensions is decided at short distances and does not grow with ξ (23-01).

Two η

We split η into two. The static η_s is the anomalous dimension above, the dynamic one is $\eta_{\mathrm{dyn}} := 2 - z$, and they are equal only within the conventional theory (in Table 2 they differ even in sign). This book always attaches the subscript, and which one the record's η is is fixed in 18-05.

No slowing down away from criticality

Conversely, the premise of physics is that if ξ is finite there is no slowing down (the entry "no slowing down away from criticality"). For the mathematical form we use two tools. The log-Sobolev inequality bounds the entropy, instead of the variance, by $\mathcal{E}$:

$$\int f^2 \log \frac{f^2}{\int f^2 \, d\mu} \, d\mu \;\leq\; 2C_{LS}\, \mathcal{E}(f)$$

It is stronger than the Poincaré inequality and gives $C_P \leq C_{LS}$ Known . The Dobrushin–Shlosman mixing condition is that, whatever boundary condition is imposed on a finite box, correlations inside the box decrease exponentially with distance at a uniform rate. The mathematical form of "no slowing down away from criticality" is, for discrete spin systems (finite spin space, finite range), the equivalence theorem "uniform mixing independent of boundary conditions (Dobrushin–Shlosman) $\Longleftrightarrow$ uniform log-Sobolev" (Stroock–Zegarlinski 1992; bibliographic data only) Known . A finite correlation length alone is not enough of a hypothesis. For gauge fields taking values in a continuous group the equivalence is not found within the material of this chapter. For lattice Yang–Mills, Shen–Zhu–Zhu (2023) show Poincaré, log-Sobolev and exponential decay at strong coupling. Neither result fixes the power on the side $\xi \to \infty$.

Detour: for the free field the relaxation time equals the susceptibility

A subsection you may skip.

For the free field $S = \frac{1}{2}\sum_x[(\phi_{x+1} - \phi_x)^2 + m^2\phi_x^2]$, summing (8) over points, the nearest-neighbour coupling terms cancel, and the mean M obeys

$$dM = -m^2 M\, ds + \sqrt{2/N}\, dW(s)$$

(N is the number of points and W a single Brownian motion). M is a single Ornstein–Uhlenbeck process, with relaxation time $1/m^2$, the same value as the susceptibility $\chi = \sum_x \langle \phi_x \phi_0 \rangle = 1/m^2$ (in the calculation on this machine too, χ and C_P agree for every m in Table 1) Computed . The "relaxation time $\propto \chi$" of the conventional theory has the same form as this relation, exact for the free field. With interactions the two are said to drift apart, and the $z > 2 - \eta_s$ of Table 2 corresponds to that drift Physics .

08

Fluctuations of the tube and topological freezing — effective strings, Wulff, bottlenecks

Effective strings and the Lüscher term

A long flux tube spontaneously breaks transverse translation, and Goldstone bosons live in 1+1 dimensions (Lüscher 1981). A Goldstone boson is a massless fluctuation that appears when a continuous symmetry is spontaneously broken; here it is the fluctuation of the transverse position of the tube. These fluctuations add a universal $1/R$ term to $V(R)$ (Lüscher–Symanzik–Weisz 1980, 17-03).

$$V(R) = \sigma R + \mu + \frac{\gamma}{R} + O(R^{-2}), \qquad \gamma = -\frac{\pi(d-2)}{24} \quad (13)$$

A two-dimensional conformal field theory (a two-dimensional field theory with a symmetry that includes scaling of distances) has a number called the central charge c, and one free boson has $c = 1$. The part of the ground-state energy of fluctuations confined to a strip of width R that depends on the width (the Casimir energy) is proportional to c (Blöte–Cardy–Nightingale 1986, Affleck 1986): $-\pi c/(24R)$ for an open strip and $-\pi c/(6L)$ for a periodic strip of circumference L. Setting $c = d - 2$, the former is γ and the latter at $d = 4$ is $-\pi/(3L)$, which agrees with the correction subtracted when extracting the string tension on the lattice (Lucini–Teper 2001) Computed . The lattice σ contains the physical assumption "the central charge of the tube is $d - 2$" (the entry "Lüscher term").

Rigid tubes and roughening

At strong coupling, where the area law has been shown, the tube is said to be rigid. More precisely, each shift of the surface by one lattice step costs a small weight, so transverse fluctuations are exponentially suppressed (17-03). The shape of the tube at strong coupling can be read by analogy with the Wulff construction from a direction-dependent tension. The Wulff construction is a drawing that gives, under a tension $\tau(\mathbf{n})$ depending on direction, the shape that minimises the total tension of the boundary for a fixed enclosed area (17-03). It is said to be separated from the region of fluctuating tubes by the roughening transition. This is a picture of physics found in the title (abstract) of Lüscher 1981, and no theorem showing its existence in lattice gauge theory is found in the material of this chapter Physics (the entry "roughening, shape of the tube").

Topological freezing and bottlenecks

In HMC for pure gauge theory, the effective dynamical exponent of the square of the topological charge is about 5, and Wilson loops decouple from this slow mode (the abstract of Schaefer–Sommer–Virotta 2011) Physics . The mathematical word is a bottleneck: putting into the variational principle a smoothed version f of the indicator function of a set A of configurations (configurations with topological charge close to 1, say),

$$C_P \;\geq\; \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} \;\approx\; \frac{\mu(A)\big(1-\mu(A)\big)}{\mathcal{E}(f)} \qquad (14)$$

and the narrower the passage joining A and its outside, the larger C_P Known (the two-valley model, 1-10). More precisely, the configurations near the boundary of A have small measure, so $\mathcal{E}(f)$ becomes small. (14) is a theorem, but that it is the cause of freezing, and carrying the phenomenon of HMC over to the C_P of Langevin, have not been checked (the entry "topological freezing", 23-02).

Detour: sectors of topological charge and narrow passages

A subsection you may skip.

For continuum fields the topological charge (the second Chern number, 3-05, 18-04) is an integer, and sectors of configurations with different values are not joined by smooth deformations. On the lattice the topological charge is an integer only approximately, and the sectors are joined through rough configurations at the scale of the lattice spacing. These passages are expected to become narrower as $a \to 0$ Physics . Taking the A of (14) to be "configurations with topological charge close to 1", this expectation translates into the lower

bound on CP becoming larger. There is no theorem showing how fast the passages become narrow.

What this chapter can say

What is said	What is not said
Known One direction of (1), (2), the existence of σ, the Theorem from Centre Symmetry to Confinement and the Theorem from Uniform Decay to the Area Law (Chatterjee 2021), (10)(11), $CP = t_{\mathrm{rel}}$ for lazy chains, the variational lower bounds (12)(14), "uniform mixing $\Longleftrightarrow$ uniform log-Sobolev" for discrete spin systems, the results for three-dimensional compact U(1) (abstract).	The converse theorems for every pair. That these quantities exist in the four-dimensional continuum theory. Which way the "Unsettled" pairs will move. The equivalence of mixing and log-Sobolev for gauge fields taking values in a continuous group.
Computed Table 1, the differences in Table 2, Figure 1 and the autocorrelation time of the mean M, the example in § 06, that for the free field the relaxation time agrees with χ, $\pi/(3L)$.	The z of Table 2 are not measurements on this machine.
Physics The numbers for $m/\sqrt{\sigma}$, $\sqrt{\sigma}$, glueballs and z, $z \geq \gamma/\nu$ under the two scaling hypotheses, the rigidity and roughening of the tube, the passages between topological sectors.	Not theorems. The form of extrapolation and $c = d - 2$ are assumptions. The conversion of $\sqrt{\sigma}$ into MeV is borrowed from another theory.
Which pairs the relaxation-time assumption and chain (ii) correspond to (the entries "Λ, one scale" and "z=2"), and that the relaxation-time assumption is a statement with the normalisation of time fixed.	Their proofs. Nothing is shown about the mass gap, confinement or the continuum limit.

Sources and reproduction

Item	Kind	Source / tools (level of checking)
Gap, clustering, one-particle states, transfer matrix	Known	Jaffe–Witten (the Clay problem statement) § 4– § 6 (text). Lüscher, CMP 54 (1977) 283 (abstract only); Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 (abstract: transfer matrix, infinite-volume limit and analyticity at strong coupling, Wilson's confinement estimate; exponential clustering not checked in the main text); Schor, NPB 222 (1983) 71, 231 (1984) 321 (abstracts only); Shen–Zhu–Zhu, CMP 400 (2023) 805 (abstract only; 'strictly positive mass gap').

Item	Kind	Source / tools (level of checking)
ξ, V, σ, confinement, centre symmetry	Known	Chatterjee, arXiv:1803.01950; CMP 385 (2021) 1007, arXiv:2006.16229 (text). Bachas, PRD 33 (1986) 2723; Wilson, PRD 10 (1974) 2445; Polyakov, PLB 72 (1978) 477; Susskind, PRD 20 (1979) 2610; Svetitsky–Yaffe, NPB 210 (1982) 423; Adhikari–Cao, Ann. Probab. 53 (2025) (abstract only). Borgs–Seiler, CMP 91 (1983) 329 (bibliographic data only). Göpfert–Mack, CMP 82 (1982) 545–606, doi:10.1007/BF01961240 (checked in the abstract; INSPIRE; main text not read; the year is Crossref's year of publication, INSPIRE gives 1981).
$m/\sqrt{\sigma}$, $\sqrt{\sigma}$, 0^{++}, $\pi/(3L)$	Physics	Athenodorou–Teper, JHEP 11 (2020) 172 (the conversion of $\sqrt{\sigma}$ and the caveat are in § 5.2), JHEP 12 (2021) 082; Lucini–Teper, JHEP 06 (2001) 050; Morningstar–Peardon, PRD 60 (1999) 034509 (text).
Two times, autocorrelation, mixing time, HMC	Known, Physics	Levin–Peres–Wilmer, *Markov Chains and Mixing Times* 2nd ed. § 12.2, Exercise 12.3, Theorem 12.4 (text). Zwanziger, NPB 192 (1981) 259; Wolff, CPC 156 (2004) 143; Duane et al., PLB 195 (1987) 216 (abstracts only). Parisi–Wu, Sci. Sin. 24 (1981) 483 (bibliographic data only).
Critical slowing down, Table 2, the variational lower bound, the equivalence away from criticality	Physics, Known	Halperin–Hohenberg–Ma, PRL 29 (1972) 1548 (abstract). Nightingale–Blöte, PRB 62 (2000) 1089 (bibliographic data only; the source of the two-dimensional 2.1665(12)); Liu et al., PRE 108 (2023) 034118 (abstract: reconfirmation of 2.1665(12) and the three-dimensional value); Adzhemyan et al., PLA 425 (2022) 127870, arXiv:2111.04719 (abstract: the 5-loop values); Kos et al., JHEP 08 (2016) 036 (abstract); Li–Sokal, PRL 63 (1989) 827 (abstract). Halperin–Hohenberg, Phys. Rev. 177 (1969) 952, RMP 49 (1977) 435 (abstracts only). Stroock–Zegarlinski, CMP 144 (1992) 303 (title: the equivalence of the log-Sobolev inequality and the Dobrushin–Shlosman mixing condition), CMP 149 (1992) 175 (title: the log-Sobolev inequality for discrete spin systems) (bibliographic data only). The authority for van Hove's form $z = 2 - \eta_S$ and for the lower bound $z \geq \gamma/\nu$ not checked.
Effective strings, central charge, topological freezing	Physics	Lüscher–Symanzik–Weisz, NPB 173 (1980) 365; Lüscher, NPB 180 (1981) 317 (roughening in the title); Blöte–Cardy–Nightingale, Affleck, PRL 56 (1986) 742, 746; Schaefer–Sommer–Virotta, NPB 845 (2011) 93 (abstract only).
Wulff construction, roughening	Physics	Sources not checked in this chapter (Dobrushin–Kotecký–Shlosman: bibliographic data not checked). The construction and examples are in 17-03 and its table of sources.
The relaxation-time assumption, chain (ii)	Assumption, analogy	The arrangement of the record of this book's study (23-01, 17-12).

Item	Kind	Source / tools (level of checking)
Table 1, the differences in Table 2, Figure 1, the detour in § 06 and § 07, the numbers in § 08	Computed	Python 3 on this machine (numpy): `two_times.py`, `zero_mode.py`, `ar1.py`, `exponents.py`.

Next chapter to read: 18-04 A dictionary of scales and limits. For the correspondence with the assumptions of the record, see 18-05.

A dictionary of scales and limits — words of flow translate into maps, words of limits into questions of existence

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — We pair 23 physical words about scale and limits with mathematical words, and grade them in the two columns of 18-01. We also see that a change of scheme is a one-off shift (Table 1), and the flow of the approximate Migdal–Kadanoff recursion (Figure 1).

Prerequisite chapters — 4-06, 7-05 and 17-08. The two columns are in 18-01, and mass and length in 18-03. The characters and Bessel functions of § 04 are in 7-02, and the minimum is supplied in § 04.

Said first — We only line up correspondences; we construct neither a flow nor a continuum limit. Figure 1 is a calculation of an approximate recursion, not a claim about the four-dimensional lattice theory.

The order of this chapter

1. The overall picture — Table 0
2. The renormalisation group and the β function — sequences of steps and Polchinski's equation
3. Flow on the lattice — Table 1
4. Migdal–Kadanoff and hierarchical lattices — Figure 1 / Detour: why the decrease per step is $(1-b^{-2})/3$
5. The continuum limit and a(β) — two kinds of scaling
6. Universality, Symanzik, marginality — the price of improvement
7. Large N, the master field, holography, the KK tower
8. Topology, instantons, the Wilson flow, the Gribov region
9. What had no word on the side of mathematics — the trace anomaly

01

The overall picture — flow, limits, geometric scales

The words fall into three groups. Words of **flow** (renormalisation group, β function, Λ) translate into properties of the maps of the steps. Example: "the β function is negative" becomes "a step that doubles the lattice spacing lowers β by an almost constant amount on the weak-coupling side" (§ 02, § 03).

Words of **limits** (continuum limit, $a(\beta)$, universality) translate into the question "does the limit exist, and is it unique?". Example: "universality" is the uniqueness claim "changing the details of the action leads to the same limiting measure" (§ 05, § 06).

Words of **geometric scale** (N, an extra direction, flow time, topology) look at the theory by adding another scale. Classically, many of them agree by theorems. Example: an "instanton" is exactly an anti-self-dual connection (§ 07, § 08).

Notation: $\beta = 2N/g^2$ is the coefficient of the lattice coupling (not the inverse temperature of statistical mechanics), b is the factor of one step (multiplying the lattice spacing a by b), and $\kappa = 4Nb_0$ (0.3715 for SU(2), A-01). Writing asymptotic freedom (4-06) in the lattice coupling gives the running per step. For SU(2), doubling a lowers β by about 0.2575, and halving a raises it by the same amount.

$$\beta(ba) = \beta(a) - \kappa \ln b + O(1/\beta) \qquad \text{(one loop; } O(1/\beta) \text{ is two loops and beyond)} \quad (1)$$

Table 0 The list of the dictionary of scales and limits ((a) correspondence as definitions, (b) existence in four dimensions, (c) direction reachable by finite measurement; levels and conditions in the text)

#	Physical word	Mathematical word	(a)	(b)	(c)
1	Renormalisation group	Sequence of steps, UV stability	Conditional	Conditional	—
2	β function	Recursion for the effective coupling	Conditional	Conditional	—
3	Polchinski's equation	Polchinski equation of a decomposition of the measure (heat flow)	Settled	None	—
4	Δβ of MCRG	The map of one step	Conditional	None	—
5	Change of scheme for Λ	Reparametrisation of the coupling	Settled	Constructed (formal power series)	—
6	Migdal–Kadanoff	Exact recursion on hierarchical lattices	Conditional	Constructed (hierarchical lattice)	—

#	Physical word	Mathematical word	(a)	(b)	(c)
7	Continuum limit	Scaling limit	Conditional	None	Out of reach
8	a(β), setting the scale	ε(β) (the continuum string-tension problem)	Unsettled	None	—
9	Scaling	Limit of dimensionless ratios	Unsettled	None	Out of reach
10	Asymptotic scaling	Asymptotic form of ε(β)	Unsettled	None	Out of reach
11	Universality	Uniqueness of the limit	Conditional	None	Out of reach
12	Symanzik's $O(a^2)$	The shortfall per step δ(b)	Unsettled	None	—
13	Improved action	Loss of reflection positivity (contributions with negative weight)	Settled (for actions with dimension-6 operators added)	Constructed (lattice)	—
14	Marginal coupling	Critical Sobolev exponent	Settled	Constructed (classical)	—
15	Large N, planar diagrams	Planar maps, sums over strings	Conditional	Conditional	—
16	Master field	Limit of non-commutative distributions	Conditional	None	—
17	Radial direction of holography	Fefferman–Graham expansion	Unsettled	None	—
18	KK tower	λ_1 of the internal Laplacian	Conditional	Not applicable	Upper bound only
19	Topological charge	Second Chern number	Conditional	Conditional (lattice)	—
20	Instanton	Anti-self-dual connection	Settled	Constructed (classical)	—
21	Wilson flow	Yang–Mills heat flow	Settled	Constructed (classical)	—
22	Gribov region	Region where the FP operator is positive	Settled	Constructed (classical)	—
23	Trace anomaly	No counterpart	Unsettled	None	—

Note: column (b) of Table 0 is the reading from before the published construction (2026-11-10). As in §02 and §09, the rows for the continuum limit (row 7) and the trace anomaly (row 23) are candidates to move with the published construction.

(c) is the measurement column of 18-01 §02 (the direction reachable by finite measurement). The rows about limits and uniqueness (the entries "continuum limit", "scaling", "asymptotic scaling" and "universality") are out of reach, and for the λ_1 of the entry "KK tower" a single test function gives an upper bound (1-10). "None" in (b) means, as in 18-01 §04, "not yet made".

The entries "Δβ of MCRG", "scaling" and "asymptotic scaling" have lattice numbers (at the level of physics), treated in §03 and §05.

02

The renormalisation group and the β function — translated into sequences of maps

The renormalisation group is the operation of integrating out fluctuations at short scales and moving to an effective action at long scales. Wilson (1971) rewrote Kadanoff's theory of scaling in the form of differential equations (abstract: 'The Kadanoff theory of scaling ... is cast in differential form') Known . The existence of the map and convergence to a fixed point are premises in physics. In mathematics this becomes a **sequence of steps** that estimates the compositions of the one-step map R_b uniformly (7-05).

$$\mu_a \xrightarrow{R_b} \mu_{ba} \xrightarrow{R_b} \mu_{b^2 a} \to \cdots, \qquad \text{control the effective action } S_k \text{ uniformly in } k \quad (2)$$

For four-dimensional pure gauge theory, Bałaban (1987, 1989) built a sequence of effective actions and showed UV stability in finite volume. Chatterjee (2018) § 6 calls the existence of convergent subsequences UV stability Known . In Bałaban's method, convergence of the orbits and limits of observables are unproved (7-07). The construction published on 2026-11-10 takes limits of correlations of local observables with a sequence of renormalisation steps of a different design (Parts 9–11). Agreement of the sign of the β function (the entry "β function") with an exact flow is a theorem for the two-dimensional Gross–Neveu model (Gawędzki–Kupiainen 1985). In four dimensions it goes as far as Bałaban's recursion in the small-field approximation.

Polchinski's equation

Polchinski (1984) wrote, for four-dimensional $\lambda\phi^4$, the change of the effective action as the momentum cutoff is moved continuously as an exact renormalisation group equation. On that basis he showed perturbative renormalisability Known . The same equation exists on the side of mathematics. Splitting the covariance of a Gaussian measure into a sum over scales and integrating one layer at a time from the fine scales, the remaining effective action V_t obeys this equation. Rewritten for e^{-V_t}, it is convolution with a Gaussian kernel, that is, the flow of the heat equation Known . The pair is equation to equation, so the (a) of the entry "Polchinski's equation" is Settled.

There are also criteria standing on this equation. The log-Sobolev inequality is an inequality of the form $\mathrm{Ent}_\mu(f^2) \le 2C \int |\nabla f|^2 d\mu$. It is a stronger version of the Poincaré inequality $\mathrm{Var}_\mu(f) \le C \int |\nabla f|^2 d\mu$ (1-10), with the variance replaced by the entropy $\mathrm{Ent}_\mu(g) = \int g \ln g \, d\mu - \int g \, d\mu \, \ln \int g \, d\mu$, and it guarantees how fast a stochastic dynamics

returns to equilibrium. Bauerschmidt–Bodineau (2021) extended the Bakry–Émery criterion (1-10) to multiple scales by controlling Polchinski's equation, and showed an asymptotically optimal log-Sobolev inequality for the massive continuum sine-Gordon model (a two-dimensional model, with coupling $\beta_{\mathrm{sG}} < 6\pi$) Known . The criterion is a tool standing on the pair, so it goes on the side of the conditions. There is no application to four dimensions, so (b) is "None".

03

Flow on the lattice — MCRG and a one-off shift

MCRG (Swendsen 1979) fixes $\Delta\beta$ so that observables blocked with $b = 2$ match a lattice with coupling $\beta - \Delta\beta$. The first results for SU(3) (Gupta et al. 1984) reported no asymptotic scaling for fundamental-representation coupling $K_F < 6.1$ Known . $\Delta\beta$ is the amount by which β is lowered when a is doubled, the same quantity, measured by another method, as "the step that halves a" in Table 4 of 4-06. The mathematical partner is the one-step map $\beta \mapsto \beta - \Delta(b)$ ($\Delta(b)$ is the decrease per step, not the Δ of the mass gap), and with the same projection (the way of matching to the Wilson form) chosen, they are the same quantity (the entry "Δβ of MCRG").

Consider one step within the same way of fixing things (scheme). If at leading order at weak coupling ($\beta \to \infty$) Δ does not depend on β, then two steps in succession and a single multiplication by $b_1 b_2$ are the same, so

$$\Delta(b_1 b_2) = \Delta(b_1) + \Delta(b_2) \quad \Longrightarrow \quad \Delta(b) = C \ln b \ \text{(3)}$$

(Cauchy's functional equation under monotonicity). The constant C is not fixed by additivity alone; comparing with (1) (one loop) gives $C = \kappa$. From two loops on, Δ keeps a β-dependence of order $O(1/\beta)$, and (3) is the leading-order formula that ignores it.

Integrating (1), the one-loop formula is $\beta = -\kappa \ln(a\Lambda)$ (the one-loop part of equation (8) of 4-06). A change of the way of fixing things, $\tilde{\beta} = \beta - c + O(1/\beta)$, only moves the map by conjugation and does not change the leading term of Δ. What changes is the reading of Λ: from $\tilde{\beta} = -\kappa \ln(a\Lambda e^{c/\kappa})$, $\tilde{\Lambda} = \Lambda e^{c/\kappa}$, that is, Λ is multiplied once only by $e^{c/\kappa}$ (the entry "change of scheme for Λ"). Dashen–Gross (1981) found this ratio to be $\Lambda_{\mathrm{MS}}/\Lambda_{\mathrm{L}} = 38.852704\, e^{-3\pi^2/(11N^2)}$ (19.823 for N = 2) (Wilson action) Known .

Computed Table 1 The Dashen–Gross ratio and the corresponding shift of β, $c_N = \kappa_N \ln(\text{ratio})$ ($\kappa_N = 11N^2/12\pi^2$). c_N is the amount by which β is lowered, $\beta_{\mathrm{MS}} = \beta_{\mathrm{L}} - c_N$ (the one-loop relation compared at $\mu = 1/a$). Number of steps $c_N/(\kappa_N \ln 2)$

N	Ratio	c_N	Steps

N	Ratio	c_N	Steps
2	19.823	1.1096	4.309
3	28.809	2.8092	4.848
∞	38.853	–	5.280

For SU(2), $c_2 = 1.1096$ is about 4.3 steps of the running $\kappa \ln 2 = 0.2575$ of one step with $b = 2$. Moving from the Wilson action to MS is the same as lowering β once only by about 4.3 steps, that is, taking once only about 4.3 extra steps that double a. A map whose constant shifts at every step, $\Delta(b) = \kappa(\ln b - \delta(b))$, breaks (3) and moves the reading of Λ step by step. The record of this book's study calls $\delta(b)$ in this form "the shortfall" (23-02, 18-05).

04

Migdal–Kadanoff and hierarchical lattices — reading an approximation as an exact recursion on another lattice

Kadanoff (1976) rederived Migdal's (1975) recursion formula as an approximation to decimation, and proposed the scheme of potential (bond) moving (abstract) Known . Here we write one step as a pair of two operations. **Bond moving** gathers b^{d-2} plaquettes in one place and turns the weight f into $f^{\,b^{d-2}}$. **Decimation** combines $b \times b$ plaquettes of a plane into one. Decimation is exact in two dimensions (7-02).

We set down the notation a line at a time. Let the eigenvalues of an element U of SU(2) be $e^{\pm i\theta}$, and write the weight of one plaquette of the Wilson action as $e^{\beta \cos\theta}$ ($\beta = 4/g^2$). χ_n is the character of the n-dimensional irreducible representation (spin $(n-1)/2$), $\chi_n(U) = \sin n\theta / \sin\theta$, with dimension $d_n = n$. I_n is the modified Bessel function $I_n(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos n\theta \, d\theta$. Expanding the weight in characters as $f = a_1 \sum_n d_n c_n \chi_n$, we call the coefficients c_n ($c_1 = 1$) the normalised coefficients, and decimation is $c_n \mapsto c_n^{\,b^2}$. 7-02 writes the weight as $e^{\beta \,\mathrm{Re\,tr}\, U}$, so the β of 7-02 is the $\beta/2$ of this chapter.

$$c_n \longmapsto \left[c_n\left(f^{\,b^{d-2}}\right)\right]^{b^2}, \qquad \text{for the SU(2) Wilson action } c_2 = \frac{I_2(\beta)}{I_1(\beta)} \quad (4)$$

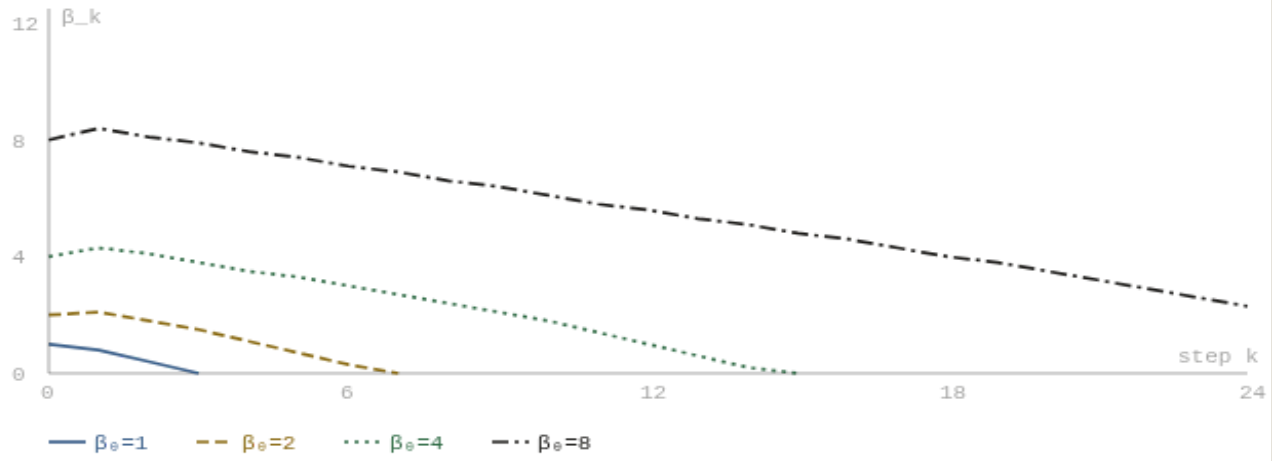
The approximation lies only in bond moving. On the lattice made by repeating the two operations in a nested way (a hierarchical lattice), (4) is exact by construction. For spin systems there are the hierarchical lattices of Griffiths–Kaufman (1982). For gauge theories Ito (1985) showed permanent confinement for groups such as SU(N) in a four-dimensional hierarchical lattice gauge theory on which the MK recursion holds exactly (abstract: 'hierarchical lattice gauge theories in which the Migdal-Kadanoff approximate renormalization recursion formulas hold exactly') Known . The

"approximation" of physics is, in mathematics, "an exact map on another lattice" (the entry "Migdal–Kadanoff"; (b) is at the level of this hierarchical lattice).

Example: at strong coupling $c_2 \approx \beta/4$, so one step with $d = 4$, $b = 2$ is $\beta' \approx 4\beta^4$, and with decimation alone $\beta^4/64$ (at $\beta = 0.1$, 3.895×10^{-4} and 1.560×10^{-6}) Computed . The latter is the strong-coupling formula of one step of bookkeeping in the record (23-02), and the missing factor b^{d-2} is known in the record too.

Where the flow goes, and the decrease per step

Figure 1 The Migdal–Kadanoff flow ($b = 2$). Changing d with the knob and the group with the buttons changes where β_k goes from four starting points. "Play" advances the steps. The colour and dash pattern of each line correspond to the starting point β_0 in the legend at the bottom



d = 4, SU(2): 1 → towards strong coupling, 2 → towards strong coupling, 4 → towards strong coupling, 8 → 24 steps, down to 2.3

Computed The numbers in this figure were computed on this machine (the weight function iterated with 8000 points and 400 characters; β is read from the c_2 of (4), and for U(1) from I_1/I_0). Without JavaScript it can be read from the next two paragraphs.

For $d < 4$ the flow goes towards strong coupling, for $d > 4$ towards weak coupling, and $d = 4$ is the border. For SU(2) at $d = 4$, all the starting points examined ($\beta_0 \le 20$) go towards strong coupling ($\beta_0 = 8$ reaches $\beta < 0.05$ in 31 steps, and 20 in 79 steps) Computed . For U(1), starting points with $\beta_0 \ge 1$ stop at values of about $\beta \approx 1.4$ or more and lie on a line of fixed points (5-06). The first step goes up, since it moves from the Wilson form to the settled form of the recursion; this too is a one-off shift.

The decrease per step after the form has settled at weak coupling (SU(2) at $d = 4$, the 10th step from $\beta_0 = 60$) is 0.2503, 0.2965 and 0.3126 for $b = 2, 3, 4$ Computed . The step given by one loop, $\kappa \ln b$, is 0.2575, 0.4081 and 0.5150. The MK values match the formula $(1 - b^{-2})/3$ of the detour below (0.2500, 0.2963, 0.3125) Computed . For $b = 2$ it is 97% of the one-loop value, but two steps with $b = 2$ (0.5006) and one step with $b = 4$ (0.3126) differ, so (3) is not satisfied. The $a(\beta)$ of MK depends on b.

Detour: why the decrease per step is (1−b⁻²)/3

A subsection you may skip.

There is a weight whose form is preserved by decimation $c_n \mapsto c_n^{b^2}$: the weight with coefficients $c_n = e^{-(n^2-1)/(2\beta)}$, called the SU(2) **heat kernel** (the Q_t of 7-02 § 04 with $t = 4/\beta$). Decimation only moves it as $\beta \mapsto \beta/b^2$. When θ is small, this weight can be written

$$f_\beta(\theta) \propto \frac{\theta}{\sin\theta}\, e^{-\beta\theta^2/2}, \qquad \frac{\theta}{\sin\theta} = 1 + \frac{\theta^2}{6} + \cdots \approx e^{\theta^2/6}$$

Known . It differs from the Wilson weight $e^{\beta\cos\theta} \approx e^{\beta} e^{-\beta\theta^2/2}$ only by the prefactor $\theta/\sin\theta$. Raising to the power b^2 by bond moving ($d = 4$), the prefactor is also multiplied b^2 times. The heat-kernel form needs only one, so the extra $b^2 - 1$ enter the exponent as $e^{(b^2-1)\theta^2/6}$.

$$f_\beta^{b^2} \propto \frac{\theta}{\sin\theta}\, e^{-\left(b^2\beta - \frac{b^2-1}{3}\right)\theta^2/2} : \qquad \beta \mapsto b^2\beta - \frac{b^2-1}{3} \xrightarrow{\text{decimation}} \beta - \frac{1-b^{-2}}{3}$$

The decrease per step is $(1 - b^{-2})/3$, independent of β. Unlike $\kappa \ln b$, it is not additive in b, because the decrease comes not from a logarithm but from the coefficient $1/6$ of θ^2 in $\theta/\sin\theta$. This calculation is only the leading term for small θ, so agreement with the numbers in the text (0.2500 and 0.2503, and so on) is within the range checked by computation Computed .

For how this book uses this, see 15-04 § 02 (the invariant curve and the contraction of the same single-step map).

05

The continuum limit and a(β) — two kinds of scaling

On the continuum limit (the entry "continuum limit"), Chatterjee (2018) § 5 summarises the usage of physics: if the correlation length diverges at a critical point β_c one says 'the model has a continuum limit' Known . The scaling limit is the operation of shrinking the lattice to $\varepsilon\mathbb{Z}^d$, normalising the field and letting it converge weakly as a distribution (2-08); once what is held fixed and the normalisation are specified, it is the same operation. The non-triviality that the physical word includes is not in the definition, and the limits of the four-dimensional Ising type and of ϕ^4 are Gaussian (Aizenman–Duminil-Copin 2021, 7-08).

There are two ways of choosing $a(\beta)$ (the entry "a(β), setting the scale"). The explanation in § 5 fixes the correlation length, and the continuum string-tension problem quantifies existentially with Wilson loops (the same formula as equation (4) of 18-03).

$$\log\langle W_{R/\varepsilon, T/\varepsilon}\rangle = -c\,(R+T) - d\,RT + o(1), \qquad \varepsilon(\beta) \to 0,\ c(\beta) \to \infty,\ d\ /= 0 \quad (5)$$

The d here is the coefficient of the area law (the number corresponding to the continuum string tension), a different symbol from the dimension d in (8). The continuum string-tension problem does not presuppose the lattice correlation-length problem (Problem 5.1), and the existence of either $\varepsilon(\beta)$ is unproved. Physics fixes the scale by the string tension (Creutz 1980) or r_0 (Sommer 1994) Physics .

Scaling (the entry "scaling") is that dimensionless ratios stop depending on a, and **asymptotic scaling** (the entry "asymptotic scaling") is that $a(\beta)$ follows the two-loop formula.

$$\frac{m_{\mathrm{lat}}(\beta)}{\sqrt{\sigma_{\mathrm{lat}}(\beta)}} \to \text{constant} \qquad \text{and} \qquad a\Lambda_{\mathrm{L}} = (b_0 g^2)^{-b_1/(2b_0^2)} e^{-1/(2b_0 g^2)}\bigl(1 + O(g^2)\bigr) \tag{6}$$

In the lattice numbers the two behave separately. The deviation on the right shows up in the report that MCRG for SU(3) sees no asymptotic scaling for $K_F < 6.1$ (Gupta et al. 1984). In Table 4 of 4-06 too, the data for the step that halves a (0.35–0.52) do not reach the two-loop value (about 0.61) Physics . The left is supported by dimensionless ratios lining up at one value in extrapolations that assume a^2 corrections (the $m/\sqrt{\sigma}$ of Lucini–Teper 2001, the $\sqrt{8t_0}/r_0$ of Lüscher 2010) Physics . Fits for SU(N) with $N > 4$ do not pin down $a(\beta)$ in units of $\Lambda_{\overline{\mathrm{MS}}}$ (Allton–Teper–Trivini 2008) Physics . The mathematical partners are, for the left, the limit of the pair of the lattice correlation-length problem and the continuum string-tension problem (the entry "m/√σ" of 18-03), and for the right, the asymptotic form of $\varepsilon(\beta)$; neither has a theorem (5-05, 5-09).

06

Universality, Symanzik, marginality — two errors in different variables

Universality (the entry "universality") is that the same limit comes out independently of the details of the action. Griffiths (1970) and others stated it as a hypothesis, and Wilson (1971) explained it by irrelevant variables. In mathematics it is uniqueness of the limit, and the theorems are in two dimensions: Ising on isoradial graphs (Chelkak–Smirnov), and the Wilson, Villain and Manton actions going to the same measure for the YM measure of the trivial bundle on the two-dimensional torus (Chevyrev–Shen) Known . In four dimensions, that the Wilson action and improved actions give the same theory is a premise.

Symanzik's (1983) improvement (the entry "Symanzik's O(a²)") is the idea of reducing the errors at finite a systematically by adding irrelevant terms. Under a single action, the error of a dimensionless ratio is expanded in a.

$$R(a) = R(0)\bigl(1 + s\,a^2 + \cdots\bigr) \ \ (a \to 0\text{: vanishes}) \qquad \text{versus} \qquad \sum_{j<k} \kappa\,\delta(b_j) \ \ (\text{accumulates ste}$$

For Yang–Mills the logarithmic corrections to a^2 are also said to be weak (Husung–Marquard–Sommer 2020) Known . The record's $\delta(b) = c_2/b^2 + c_3/b^3 + O(b^{-4}\ln b)$ is an expansion in the factor b, a different variable, so the pairing with Symanzik remains an analogy (Unsettled). Example: over 20 steps, the left is $4^{-20}s \approx 9 \times 10^{-13}s$, and the right, with b fixed, is $20\kappa\delta(2)$. In an example where the factor increases by one at each step, $b_j = j + 2$, the leading terms alone give $2.39\kappa\delta(2)$ (the limit with the steps continued without end is $4(\pi^2/6 - 1) \approx 2.58$ times) Computed .

The price of improvement (the entry "improved action")

Lüscher–Weisz (1984) wrote that adding dimension-6 operators to the Wilson action means 'physical positivity is lost in general', and showed that the transfer matrix can still be defined. It has two properties. Complex eigenvalues can appear ('may occur'). The spectral decomposition of two-point functions always has contributions with negative weight ('there are always contributions ... that come with a negative weight') Known . In the language of reflection positivity (5-03), the mathematical word of the entry "improved action" is "loss of reflection positivity (contributions with negative weight)", grounded on the 'always'. (a) is Settled for actions with dimension-6 operators added (the judge is a theorem). The kernel that sets $\delta(2) = 0$ in the record also has negative weights; whether it is of the same type is treated in 18-05.

Marginality (the entry "marginal coupling")

Shrinking as $A_\lambda(x) = \lambda A(\lambda x)$,

$$\int_{\mathbb{R}^d} \operatorname{tr} F_\lambda^2 \, d^d x = \lambda^{4-d} \int_{\mathbb{R}^d} \operatorname{tr} F^2 \, d^d x, \qquad g^2 \sim (\text{length})^{d-4} \tag{8}$$

so at $d = 4$ the action is scale invariant (the border in Figure 1). In mathematics this is the critical Sobolev exponent. The L^p norm $\|F\|_{L^p} = (\int |F|^p \, d^d x)^{1/p}$ gives stronger control the larger p is, and under shrinking $\|F_\lambda\|_{L^p}^p = \lambda^{2p-d}\|F\|_{L^p}^p$. The exponent that does not change under a change of scale, $p = d/2$, is the critical exponent (1-11, 1-12). Uhlenbeck's (1982) weak compactness (the theorem extracting a convergent subsequence from a sequence of connections) needs $p > d/2$ for boundedness of the L^p norm of the curvature. The four-dimensional action has $p = 2 = d/2$, exactly at the threshold Known . The problem statement writes the outlook that asymptotic freedom rescues the case where the Sobolev inequality is at the borderline (0-03). The agreement goes only as far as classical dimension counting (7-09, the four-dimensions article).

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07
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Large N, the master field, holography, the KK tower — words that add another scale

The four words of this section look at the theory by adding one scale outside four-dimensional spacetime: the size N of the group, the depth from the boundary into the interior, and the size of rolled-up extra dimensions.

Large N and planar diagrams (the entry "large N, planar diagrams")

't Hooft (1974) showed that with $\lambda = g^2 N$ fixed and $N \to \infty$, planar diagrams dominate (perturbation theory, 17-04). Two things agree as theorems: planar maps of matrix models (Guionnet–Maurel-Segala), and Wilson loops of the strongly coupled SO(N) lattice written as sums over trajectories of strings (Chatterjee, Basu–Ganguly) Known .

The master field

The idea that as $N \to \infty$ fluctuations disappear and gauge invariants are fixed by a single configuration. In mathematics it corresponds to a deterministic limit of non-commutative distributions. A non-commutative distribution regards the totality of expectations of traces of products, $\frac{1}{N}\mathbb{E}\,\mathrm{tr}(X_{i_1}\cdots X_{i_k})$, for a set $X_1, X_2, \ldots$ of random $N \times N$ matrices, as the "distribution" of the matrices (the language of free probability). For planar two-dimensional Yang–Mills, Lévy showed that Wilson loops approach a deterministic limit in this sense (17-04) Known .

The radial direction of holography

Since the infrared of the interior is mirrored in the ultraviolet of the boundary (Susskind–Witten 1998), the radial direction is read as the scale of the boundary (17-08). What is certain in mathematics is only the expansion of the metric near the boundary (Fefferman–Graham). It expands the metric of the interior in powers of the depth z from the boundary, in the form $ds^2 = (dz^2 + g_z)/z^2$, $g_z = g_{(0)} + z^2 g_{(2)} + \cdots$ ($g_{(0)}$ is the metric of the boundary) Known . The side of physics too says the correspondence with the cutoff is 'left unanswered' (Heemskerk–Polchinski 2010) Physics .

The KK tower

Rolling up extra dimensions into an internal space K makes a sequence of masses appear. For a massless scalar on an unwarped product,

$$m_1^2 = \lambda_1(\Delta_K) \qquad \text{(for a circle of radius } R,\ 1/R^2\text{)} \ (9)$$

(17-06) Known . $\lambda_1(\Delta_K)$ is the smallest non-zero eigenvalue of the Laplacian on K. No construction is known that separates the KK tower from Λ while keeping geometric calculation (17-10). Pure Yang–Mills has no extra dimensions, so (b) is Not applicable.

08

Topology, instantons, the Wilson flow, the Gribov region — words that are Settled classically

The four words of this section agree with mathematical words by theorems for classical fields. Where they are carried into quantum fields, they become conditional or at the level of physics.

Instantons

The conventions of this section are the same as A-01: g is put in front of the action and not included in F: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ ($A_\mu = A_\mu^a T^a$). The Hermitian generators are normalised as $\operatorname{tr} T^a T^b = \frac{1}{2}\delta^{ab}$, so the action is $S = \frac{1}{2g^2}\int \operatorname{tr} F_{\mu\nu}F_{\mu\nu} = \frac{1}{4g^2}\int F^a_{\mu\nu}F^a_{\mu\nu}$. Setting the dual to $\tilde F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ (ε totally antisymmetric with $\varepsilon_{1234} = 1$), $\int (F^a \mp \tilde F^a)^2 \geq 0$ bounds the action below by the topological charge.

$$S \geq \frac{8\pi^2}{g^2}|Q|, \qquad Q = \frac{1}{32\pi^2}\int F^a_{\mu\nu}\tilde F^a_{\mu\nu}\, d^4x \in \mathbb{Z} \quad (10)$$

Equality holds only for (anti-)self-dual fields (3-03). (12) of 3-03 is written in the canonical normalisation $D_\mu = \partial_\mu - igA_\mu$, and rewriting with $A = gA_{\text{3-03}}$ (and so $F = gF_{\text{3-03}}$) agrees with (10). With the solution of BPST (1975), the construction of all solutions by ADHM (1978), and Uhlenbeck's (1982) "finite-action fields on $\mathbb{R}^4$ come from S^4", they are Settled as classical objects Known . Approximating the path integral by sums over instantons is a separate, quantum claim.

Topological charge

The topological charge is the second Chern number. The second Chern number is an integer determined by $\frac{1}{8\pi^2}\int \operatorname{tr} F \wedge F$ for a connection on a closed four-dimensional manifold (for example S^4) (in the convention with A anti-Hermitian, A-01), unchanged when the connection is varied continuously (the four-dimensional version of the first Chern number of 1-07). It differs from the Q of (10) only by a sign convention (the detour in 3-05). On the lattice, from fields with each plaquette U_p close enough to the identity (action density below a threshold: admissible) a bundle can be built, and it is an integer (Lüscher 1982, 3-05) Known . Otherwise the sectors are 'a dynamical property of the theory' (Lüscher 2010, SU(3)) Physics .

The Wilson flow

The Wilson flow is the equation $\dot{B}_\mu = D_\nu G_{\nu\mu}$, $B_\mu|_{t=0} = A_\mu$, which smooths the field by adding a flow time t. $G_{\mu\nu}$ is the field strength of B, and D_ν the covariant derivative by B. It is steepest descent of the action, and the equation is exactly the Yang–Mills heat flow. The radius $\sqrt{8t}$ is the value at $d = 4$ of the spread $\langle |x|^2 \rangle = 2dt$ of the heat kernel $e^{t\Delta}$. Its action on quantum fields goes as far as the three-dimensional case under a GFF assumption (Cao–Chatterjee) and two-dimensional YM and three-dimensional YMH (a model including a Higgs field, CCHS) through stochastic quantisation Known . Four dimensions is at the stage of Lüscher's 'suggest' Physics .

The Gribov region

Gribov (1978), faced with the ambiguity of the Landau gauge $\partial_\mu A_\mu = 0$, restricted the integral to fields where the FP determinant is positive (3-04, 4-08). The FP operator $M(A) = -\partial_\mu D_\mu(A)$ is the operator expressing the change in the gauge condition when the field is moved a little by a gauge transformation, and its determinant is the Faddeev–Popov determinant that multiplies the path integral. $D_\mu(A)X = \partial_\mu X - i[A_\mu, X]$ is linear in A, so

$$\Omega = \{A : \ \partial_\mu A_\mu = 0, \ M(A) \geq 0\}, \qquad M\big(tA_1 + (1-t)A_2\big) = tM(A_1) + (1-t)M(A_2) \quad (11)$$

and Ω is convex ($M \geq 0$ in the sense of positive semi-definite as an operator). Zwanziger (1982) showed 'bounded in every direction', and Dell'Antonio–Zwanziger (1991) showed that every orbit enters this region Known . The picture that the measure on it produces confinement is at the level of physics (17-02).

09

What had no word on the side of mathematics — the trace anomaly

In classical four-dimensional Yang–Mills, the scale invariance of (8) makes the trace $T^\mu{}_\mu$ of the energy-momentum tensor vanish (3-03). The trace is a quantity that measures locally the change of the action when the field is shrunk by $x \mapsto \lambda x$, and it is 0 if the action does not change under a change of scale. In the quantum theory, integrating the fluctuations needs a cutoff (a lattice spacing, say), and a length enters there. The coupling $g(\mu)$ runs with respect to that length, so scale invariance breaks. Writing the way it breaks locally is the trace anomaly.

Here we write F in the canonical normalisation ($D_\mu = \partial_\mu - igA_\mu$, the convention of 3-03). It is related to the field of the convention of (10) by $F_{(10)} = gF$, and writing (12) in the convention of (10), the coefficient becomes $\frac{\mu\, dg/d\mu}{2g^3}$.

$$T^{\mu}{}_{\mu} = \frac{\mu\, dg/d\mu}{2g}\, F^{a}_{\mu\nu} F^{a\,\mu\nu}, \qquad \mu\frac{dg}{d\mu} = -b_0 g^3 + O(g^5) \quad (12)$$

This is a formula cited in standard form (the normalisation differs between references; Collins–Duncan–Joglekar 1977) Physics , and the coefficient is the β function itself. Under asymptotic freedom ($\mu\, dg/d\mu < 0$) the trace does not become 0. This is the local mechanism of dimensional transmutation (4-07). Before the publication of 2026 there was no counterpart on the side of mathematics, because there was no four-dimensional continuum theory and no place to put this operator (as with the Λ of 18-03). The committee's assessment is that the construction published on 2026-11-10 defined the stress tensor and its trace as renormalised local operators and showed their properties (13-02 The trace anomaly, and not being free). Agreement with perturbation theory at all orders has not been shown (6-04). The 7 rows of Table 0 whose (a) is Settled are agreements of equations, definitions or classical objects, and in the reading from before publication no row has (b) "exists" in four-dimensional continuum quantum theory (if the rows are regraded, the entry "trace anomaly" is a candidate to move; 18-01 § 09).

What this chapter can say

What is said	What is not said
Known The leading term of (3) (excluding $O(1/\beta)$) and the change of scheme by conjugation, the Dashen–Gross ratio, (8) (10)(11), that MK is exact on hierarchical lattices (Ito 1985), the rows whose (a) is Settled.	The existence of the objects in the rows whose (b) is "None". Where the Conditional and Unsettled pairs will go.
Computed Table 1, Figure 1, the numbers in § 04 and § 06. Physics (12), the numbers for asymptotic scaling and scaling, the radial direction.	MK and the Physics labels are not theorems.
That the record's $\delta(b)$ and Symanzik differ in the variable of expansion.	The summability of δ (18-05). Nothing is shown about the flow, the continuum limit or the mass gap.

Sources and reproduction

Item	Kind	Source / tools (level of checking)
Flow (§ 02, § 03)	Known	Checked in abstracts: Wilson, PRB 4 (1971) 3174; Bałaban, CMP 109 (1987) 249, 122 (1989) 355; Gawędzki–Kupiainen, PRL 54 (1985) 2191; Polchinski, NPB 231 (1984) 269; Bauerschmidt–Bodineau, CPAM 74 (2021) 2064; Swendsen, PRL 42 (1979) 859; Gupta et al., PRL 53 (1984) 1721; Dashen–Gross, PRD 23 (1981) 2340. Text: Chatterjee, arXiv:1803.01950.

Item	Kind	Source / tools (level of checking)
MK, hierarchical lattices (§ 04)	Known	Checked in abstracts: Kadanoff, Ann. Phys. 100 (1976) 359; Ito, PRL 55 (1985) 558. Bibliographic data only: Migdal, JETP 42 (1975) 413; Griffiths–Kaufman, PRB 26 (1982) 5022 (hierarchical lattices for spin systems).
Limits, universality, improvement (§ 05, § 06)	Known, Physics	Checked in abstracts: Aizenman–Duminil-Copin, Ann. Math. 194 (2021) 163; Creutz, PRD 21 (1980) 2308; Sommer, NPB 411 (1994) 839; Allton–Teper–Trivini, JHEP 07 (2008) 021; Griffiths, PRL 24 (1970) 1479; Chelkak–Smirnov, arXiv:0910.2045; Chevyrev–Shen, arXiv:2302.12160; Symanzik, NPB 226 (1983) 187; Lüscher–Weisz, NPB 240 (1984) 349; Husung–Marquard–Sommer, EPJC 80 (2020) 200; Uhlenbeck, CMP 83 (1982) 11, 31. Text: Lucini–Teper, JHEP 06 (2001) 050; Lüscher, JHEP 08 (2010) 071; Jaffe–Witten (the Clay problem statement).
§ 07	Known, Physics	Checked in abstracts: 't Hooft, NPB 72 (1974) 461; Guionnet–Maurel-Segala, arXiv:math/0503064; Chatterjee, arXiv:1502.07719; Basu–Ganguly, arXiv:1608.04379; Lévy, arXiv:1112.2452; Susskind–Witten, hep-th/9805114; Heemskerk–Polchinski, arXiv:1010.1264; Fefferman–Graham, arXiv:0710.0919. The original Fefferman–Graham is said to be from 1985 (bibliographic data only; not cross-checked).
§ 08, § 09	Known, Physics	Checked in abstracts: Lüscher, CMP 85 (1982) 39; BPST, PLB 59 (1975) 85; ADHM, Phys. Lett. A 65 (1978) 185; Cao–Chatterjee, arXiv:2111.12813; CCHS, arXiv:2006.04987, 2201.03487; Gribov, NPB 139 (1978) 1; Zwanziger, NPB 209 (1982) 336; Dell'Antonio–Zwanziger, CMP 138 (1991) 291; Collins–Duncan–Joglekar, PRD 16 (1977) 438 (the abstract has no formula, and the normalisation of (12) is not cross-checked; (12) carries the Physics label as a standard form). Text: Lüscher, JHEP 08 (2010) 071.
$\delta(b)$, one step of bookkeeping	Record	The arrangement of the record of this book's study (23-02, 18-05).
Table 1, Figure 1, numbers	Computed	Python 3 on this machine: `mk_flow.py`, `mk_steady.py`, `mk_long.py`, `dg_shift.py`, `sym_delta.py` (the 2.39, 2.58 and 9×10^{-13} of § 06).

Next chapter to read: 18-05 The physical counterparts of our assumptions.

The physical counterparts of our assumptions — are the record's assumptions stronger than the hypotheses of physics?

This chapter is this book's own investigation. The established knowledge it builds on is in 18-03 and 18-04.

What this chapter shows — We set the assumptions that the record of this book's study put in the language of mathematics against the hypotheses of physics, one row at a time, and attach "the same", "a conditional restatement" or "an analogy" in the form of questions. The record's η is fixed to $\eta_{\mathrm{dyn}} = 2 - z$.

Prerequisite chapters — 18-03 (two times, critical slowing down), 18-04 (change of scheme) and 17-12 (a single scale, the diffusive-relaxation-time assumption, the three chains). The two-column grades are in 18-01. The detailed treatment of the relaxation-time assumption is in 23-01, in a later part.

Said first — The correspondences are put as questions and analogies, and none of the assumptions has been proved. No row shows anything about the mass gap or the continuum limit.

The order of this chapter

1. The overall picture — setting the assumptions against the hypotheses of physics, and dividing them into the same, conditional, analogy (Table 0)
2. Fixing the record's η to one — the η in the power form is the dynamic η_dyn = 2 − z
3. The diffusive-relaxation-time assumption and z = 2 — the same power, but uniformity up to constant factors breaks logarithmically depending on the normalisation (Table 1). Detour: why diffusion takes ξ^2
4. The relaxation-time upper-bound assumption and z ≤ 2 — three conditions for the restatement to hold (Figure 1)
5. The relaxation-time lower-bound assumption and the lower bound of critical slowing down — the same inequality, and dimension-4 operators do not reach ξ^2 (Table 2)

6. Summability of the shortfall — step-by-step shifts and changes of scheme, MCRG, kernels with negative weights
7. The equivalence of the physical-unit gap statement — readable as the product of the invariance of mass and z = 2, with cancellation when the premise is dropped (Figure 2, Table 3)
8. A single scale and extra dimensions — dimensional analysis inside the models, which stays an analogy
9. What is known and settled — five rows not counted as results

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01

The overall picture — setting the assumptions against the hypotheses of physics, one row at a time

The record of this book's study wrote the line towards the continuum limit as a sequence of "bookkeeping of steps" (23-01), and put assumptions in the language of mathematics along the way. The answers given against the old hypotheses of physics are of three kinds: **the same** (reducing to the same inequality or definition), **a conditional restatement** (the same once the power form or the choice of dynamics is added), and **an analogy** (similar in form, but no theorem joins them). They correspond to Settled, Conditional and Unsettled in the definition column of 18-01 (the existence column is "None" in every row). The central relaxation-time assumption splits into an upper and a lower half.

$$(\xi^2)\ (C_P \asymp \xi^2)\ =\ (\xi^2)\uparrow\,(C_P \lesssim \xi^2)\ \wedge\ (\xi^2)\downarrow\,(C_P \gtrsim \xi^2)$$

In the formula, (ξ²) is the diffusive-relaxation-time assumption, (ξ²)↑ the upper-bound assumption and (ξ²)↓ the lower-bound assumption.

Table 0 The record's assumptions and the candidates on the side of physics ("stronger" means that the record's assumption is the stronger; γ and ν are the exponents of the susceptibility and the correlation length, 18-03 § 07)

The record's assumption	Candidate on the side of physics	(a), strength
The relaxation-time assumption: $C_P \asymp \xi^2$	Dynamical exponent $z = 2$ (model A)	Unsettled, stronger
The upper-bound assumption: $C_P \lesssim \xi^2$	$z \le 2$ under the power form	Conditional, the same
The lower-bound assumption: $C_P \gtrsim \xi^2$	Lower bound $z \ge \gamma/\nu$	Conditional, stronger
$\sum_k \delta(b_k) < \infty$	Change of scheme, MCRG, error of the projection	Unsettled

The record's assumption	Candidate on the side of physics	(a), strength
Physical-unit gap statement $\Longleftrightarrow \sum \delta < \infty \wedge \eta_{\mathrm{dyn}} = 0$	Renormalisation-group invariance of mass + $z = 2$	Unsettled
A single scale $\Longleftrightarrow$ extra dimensions are visible	Non-separation of KK	Unsettled

02

Fixing the record's η to one — $\eta_{\mathrm{dyn}} = 2 - z$

The record's η pointed at two quantities (18-03 § 07). The **static** η_s is the power of correlations fixed by the scaling dimension Δ of an operator O (not the Δ of the mass gap) as $\eta_s(O) = 2\Delta - (d-2)$, and the record's Lean theorems that deal with static exponents (theorem names and hypotheses in 23-13) are about this one. What enters the power form of C_P and the correlation length ξ is the **dynamic** exponent.

$$C_P \asymp \xi^{\,2-\eta} \quad \text{and} \quad C_P \asymp \xi^{\,z} \quad \Longrightarrow \quad \eta = 2 - z =: \eta_{\mathrm{dyn}} \ (1)$$

This book fixes the η of the power form and of the equivalence of the physical-unit gap statement (§ 07) to η_{dyn}, and uses η_s only for the test-function lower bound (§ 05). The two are equal only within the conventional theory $z = 2 - \eta_s$.

There is one price. The record had put the premise $\eta \geq 0$ of the equivalence as an assumption, grounded on Källén–Lehmann positivity (18-02). What follows from that positivity is $\eta_s \geq 0$ (the unitarity bound) Physics . For the Glauber dynamics of the two-dimensional Ising model, $\eta_s = 1/4$ yet $\eta_{\mathrm{dyn}} = -0.1665$ (18-03 Table 2). The premise $\eta_{\mathrm{dyn}} \geq 0$ is $z \leq 2$, that is, it includes the power form of the upper-bound assumption (§ 04) among the premises.

03

The relaxation-time assumption and $z = 2$ — the same power, but different uniformity up to constant factors

The relaxation-time assumption is the assumption that the slowest relaxation time C_P of Langevin dynamics lines up with the square of the correlation length up to constant factors independent of the lattice spacing (18-03 § 05, 23-01).

$$c\,\xi^2 \ \leq \ C_P \ \leq \ C\,\xi^2 \qquad (0 < c \leq C \text{ independent of the lattice spacing}) \ (2)$$

(2) is the way of writing at the level of a band, and depends on the normalisation of $\mathcal{E}$ ((3) below). By the record's convention the relaxation-time assumption is put at the

level of powers, $\lim \ln C_P / \ln \xi = 2$, and the band level is split off as a separate assumption naming the normalisation (end of § 03, 23-01).

The closest thing in physics is the dynamic scaling hypothesis $\tau \propto \xi^z$ (Halperin–Hohenberg 1969, Hohenberg–Halperin 1977), with the z of Langevin dynamics (model A in that classification) set to 2 Physics . For the free field in the normalisation of the field A ($s^2 = 1$ below), (2) holds exactly (18-03 Table 1).

Question: is the relaxation-time assumption the same as "$z = 2$"? Answer: the relaxation-time assumption is stronger. The hypothesis says only the power, not uniformity up to constant factors. Moreover, (2) includes the normalisation of the Dirichlet form $\mathcal{E}$. For the Gaussian measure $N(0, s^2(K + m^2)^{-1})$ (K is the lattice Laplace operator $-\Delta$ [here Δ is the Laplacian], s^2 the amplitude, m the mass), taking $\mathcal{E}(f) = \mathbb{E}|\nabla f|^2$ makes C_P the largest eigenvalue of the covariance, s^2/m^2, while the correlation length does not depend on s (1-10) Known . Written in the link angle ϑ ($U = e^{i\vartheta \cdot \sigma}$), $s^2 \propto g^2$; written in the field $A = \sqrt{\beta}\,\vartheta$ ($A = 2\vartheta/g$ in this chapter's convention $\beta = 4/g^2$), $s^2 = 1$. The β of this chapter is the SU(2) lattice inverse coupling $\beta = 4/g^2$ (the $2N/g^2$ of the table of notation), not an inverse temperature.

The free approximation here is a model that substitutes a Gaussian field with the mass $1/\xi_{\mathrm{RG}}$ put in by hand, and does not include the gauge directions (zero modes). The bookkeeping length is the correlation length in lattice units fixed by the one-loop running (the record sets the lattice spacing to $a := 1/\xi_{\mathrm{RG}}$, § 07).

$$C_P^{(\vartheta)} = \frac{g^2}{4} C_P^{(A)}, \qquad C_P^{(A)} = \xi^2 \quad \text{(free approximation)} \ (3)$$

The record's $\mathcal{E}$ is written in link angles. Putting in the one-loop $\ln \xi_{\mathrm{RG}} = 3\pi^2\beta/11 + \text{const.}$ (SU(2), 4-06), $C_P/\xi^2 \propto 4/\beta$ falls to 0 like $1/\ln \xi$, and the effective exponent is $z_{\mathrm{eff}} = 2 - \kappa/\beta$ ($\kappa = 4Nb_0 = 11/(3\pi^2)$).

Computed Table 1 Free approximation, SU(2), link-angle normalisation

β	C_P/ξ^2 (1 at $\beta = 2.5$)	z_{eff}
2.5	1.000	1.8514
4	0.625	1.9071
100	0.025	1.9963

In powers $z = 2$, but in the link-angle normalisation the lower side of (2) breaks logarithmically. To set up the relaxation-time assumption one must first decide either to write $\mathcal{E}$ in the normalisation of the field A ($\mathcal{E}_A = \mathcal{E}/\beta$, that is, multiply the angle $\mathcal{E}$ by $g^2/4$; C_P is conversely multiplied by β), or to put the relaxation-time assumption at the level of powers ($\lim \ln C_P / \ln \xi = 2$) and split the band-level statement off as a

separate assumption naming the normalisation. In interacting theories the scale of stochastic time is also renormalised Known (at the level of perturbation theory; 23-00 Entrance — the staircase, and the random walk, the ruler and the clock, 23-01 The line of construction and the diffusive-relaxation-time assumption; there is also an example checked at one loop in supersymmetric Yang–Mills Physics), so there is no guarantee that the logarithm disappears in either normalisation.

In terms of the measurement column of the dictionary (the direction reachable by finite measurement; 18-01 § 02), the lower-bound assumption is on the "Lower bound only" side: exhibiting one test function gives a lower bound on C_P (§ 05). **The upper-bound assumption is on the "Out of reach" side**. An upper bound on C_P is an inequality about every f, so no number of added test functions, and no amount of autocorrelation measurement up to finite times, reaches it (the box "The asymmetry of upper and lower bounds" in 1-10). The upper-bound assumption and the mass gap are not the same statement, but they are of the same type in that both stand on the side of lower bounds.

Detour: why diffusion takes ξ^2

A subsection you may skip.

Smoothing out a distance ξ by a random walk takes ξ^2 steps. The Langevin dynamics of the free field is independent for each Fourier component.

$$d\hat{\phi}_k = -\left(\hat{k}^2 + m^2\right)\hat{\phi}_k \, ds + \sqrt{2}\, d\hat{W}_k(s) \quad (4)$$

s is the Langevin time, not the amplitude s^2 above. $\hat{k}^2 = \sum_\mu 4\sin^2(k_\mu/2)$ is the lattice version of $|k|^2$. The slowest is the rate m^2 at $k = 0$, which as a time is ξ^2. As a metaphor, the longer the wave, the more slowly it is smoothed out. Precisely, this is an equality for the free field; with interactions there is no guarantee they line up, and putting that they do is the relaxation-time assumption.

04

The relaxation-time upper-bound assumption and $z \leq 2$ — three conditions needed for the restatement

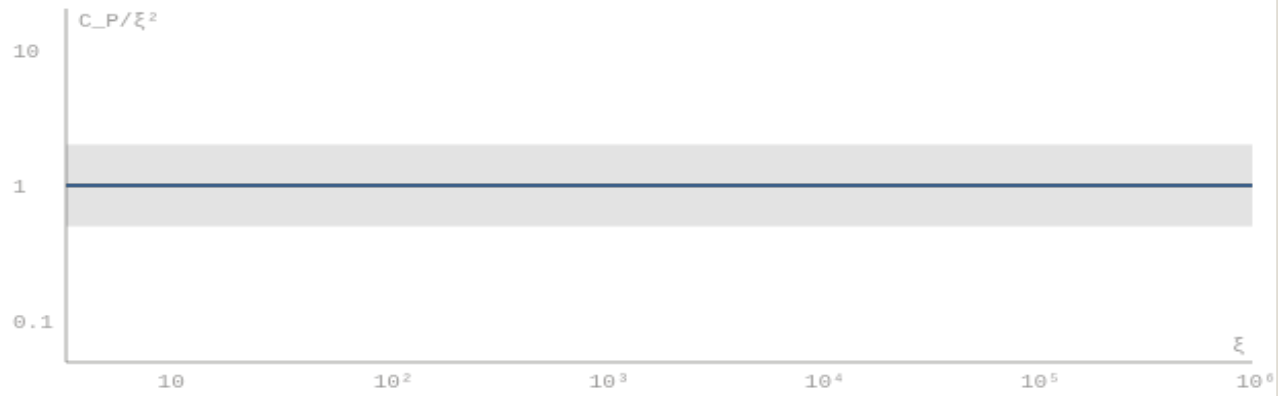
The upper-bound assumption ($C_P \lesssim \xi^2$) is, in the record's reading, the side of "the mass gap is positive" (17-12 § 10). The record restates it as $z \leq 2$. Question: is this a restatement under the dynamic scaling hypothesis? Answer: under three conditions. (i) The power form $C_P \asymp \xi^z$ exists. (ii) The dynamics is reversible Langevin in continuous time ($C_P = \tau_{\text{exp}}$ is an equality only there, 18-03 § 06). (iii) The

normalisation of $\mathcal{E}$ has been decided (§ 03). Writing q for the power of a logarithmic factor, the judgement is as follows.

$$C_P = \xi^{\,z}(\ln \xi)^q \implies \Big((\xi^2)\uparrow \iff z < 2 \text{ or } (z = 2,\ q \le 0)\Big) \quad (5)$$

The hypothesis of Hohenberg–Halperin only assumes the existence of z, and does not claim $z \le 2$. The Glauber dynamics of the Ising model in 18-03 Table 2 has $z > 2$ in both two and three dimensions Physics . Reading the upper-bound assumption as a conjecture of physics needs the condition "$\eta_s = 0$ at the relevant fixed point", and we have not checked any primary source stating it for four-dimensional Yang–Mills.

Figure 1 C_P/ξ^2 when $C_P = \xi^z(\ln\xi)^q$ (the ratio normalised to 1 at $\xi = 10$). Changing z and q with the knobs changes the curve and the judgements of the upper-bound and lower-bound assumptions. The band is constant factors $[1/2, 2]$



z = 2.0000, q = 0.00: C_P/ξ² at ξ = 10⁶ (1 at ξ = 10) = 1.00, pure power form: yes, z ≤ 2: yes, upper-bound assumption holds, lower-bound assumption holds, relaxation-time assumption holds

Computed A schematic drawing of equation (5), drawn on the spot. Without JavaScript it can be read from the next paragraph.

Examples ($\xi \to \infty$): ξ^2 satisfies both; $\xi^2/\ln\xi$ (the angle normalisation) and $\xi^{1.9}$ satisfy the upper-bound assumption only; $\xi^2 \ln\xi$ and $\xi^{2.1665}$ satisfy the lower-bound assumption only. For $\xi^2 \ln\xi$ the power is 2 and yet the upper-bound assumption breaks, so dropping (i) leaves only the one direction "upper-bound assumption $\implies z \le 2$".

05

The relaxation-time lower-bound assumption and the lower bound of critical slowing down — the same inequality, operators that do not reach

The lower-bound assumption ($C_P \gtrsim \xi^2$) is the side that can be shown with test functions: for any f, $C_P \ge \mathrm{Var}(f)/\mathcal{E}(f)$ (1-10). Physics says "local dynamics cannot relax faster than the susceptibility". Taking the magnetisation as f gives $C_P \ge \chi$

(equation (12) of 18-03) Known . Under the two hypotheses of dynamic and static scaling, $z \geq \gamma/\nu = 2 - \eta_S$ (γ is the susceptibility exponent and ν the correlation-length exponent; 18-03) Physics . It is a lower bound regarded as standard in the Monte Carlo literature, and we have not checked its authority.

Question: is the lower-bound assumption the same as this lower bound? Answer: the inequality is the same, but where it reaches differs. Taking f to be a sum of local operators of scaling dimension Δ, the lower bound is fixed by the susceptibility χ_O.

$$\frac{\mathrm{Var}(\sum_x O_x)}{\mathcal{E}(\sum_x O_x)} \asymp \chi_O \sim \begin{cases} \xi^{\,2-\eta_S(O)} & 2\Delta < d \\ \ln \xi & 2\Delta = d \\ \text{bounded} & 2\Delta > d \end{cases} \tag{6}$$

It reaches ξ^2 only when there is an operator with $\eta_S(O) = 0$, that is, of canonical dimension $\Delta = (d-2)/2$. We check this with the four-dimensional free field.

Computed Table 2 Lower bounds on C_P given by test functions for the free field on a four-dimensional lattice

m	ξ	**Dimension 1:** $\sum \phi$ **(**$= C_P$**)**	**Dimension 2:** $\sum \phi^2$	**Dimension 4:** $\sum (\nabla\phi)^2$
1.0	1.04	1	0.0783	0.0537
0.1	10.0	100	0.1719	0.0624
0.01	100.0	10000	0.2657	0.0625

Dimension 2 grows with slope 0.0408 in $\ln \xi$ (continuum estimate $1/(16\pi^2 G(0)) = 0.0409$), and dimension 4 stops at $1/16$ Computed . The infinite-volume integrals were reduced to one dimension and computed, and matched at $m = 0.5$ to 5 significant figures with sums on a 24^4 lattice.

The gauge-invariant local operators of Yang–Mills are of the $\mathrm{tr}\, F^2$ type, of dimension 4, so they fall into the third step of (6) (23-01). The link angle has dimension 1, but under the measure without gauge fixing the correlations between angles of links not sharing an endpoint are exactly 0 after averaging over gauge transformations (the same mechanism as Elitzur's theorem, 18-02). The lower-bound assumption is the demand to push the same inequality up to ξ^2 in a theory with no local order parameter, and is stronger by that much. The record checked that three families of test functions (some by U(1) calculations) do not reach it, but it does not say that the lower-bound assumption is false (23-02).

The lower-bound assumption is, in the record's reading, the side of "the mass in physical units is finite", a condition different from "non-trivial" (there are interactions) in the Clay problem statement. It is satisfied even by a free massive field.

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06
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Summability of the shortfall — do step-by-step shifts fit into a single change of scheme?

The record writes one step that coarsens by a factor b as $\beta \mapsto \beta - \kappa(\ln b - \delta(b))$, and calls $\delta(b)$ the shortfall (23-02, 18-04 § 03). δ is a number measured in the one-loop kernel model, and in the record's measurements in four dimensions it has the form $c_2/b^2 + c_3/b^3 + \cdots$ Computed (that there is no $\ln b/b^2$ term is within the detection limit). Advancing k steps with factors $b_1, b_2, \ldots,$

$$\beta_k = \beta_0 - \kappa \ln\Big(\prod_{j=1}^{k} b_j\Big) + \kappa \sum_{j=1}^{k} \delta(b_j) \quad (7)$$

For an exact step within the same scheme the third term is absent (equation (3) of 18-04), and a change of scheme shifts β once only by c (Dashen–Gross, 18-04 Table 1). If the third term converges, the step-by-step shifts add up to a one-off shift $c = \kappa \sum \delta$, indistinguishable from a single change of scheme.

$$\sum_j \delta(b_j) < \infty \iff \text{the step-by-step shifts fit into a single change of scheme} \quad \text{(within the re}$$

Rough guide: for SU(2), $\Lambda_{\overline{\mathrm{MS}}}/\Lambda_{\mathrm{L}} = 19.82$ corresponds to $\ln 19.82 = 2.987$ in $\sum \delta$ Computed . With b fixed, it is bounded only when $\delta(b) = 0$ Lean (`fixed_step_bounded_iff`. Hypothesis: the error of a step is the same at every step).

Question: which thing in physics is (8)? There are three candidates, and all are analogies. (i) It is not Symanzik's $O(a^2)$. The variables of expansion differ, a against b, and the former does not accumulate step by step (18-04 § 06). (ii) The first SU(3) result measuring the $\Delta\beta$ of gauge theory by the method of MCRG (the Monte Carlo renormalisation group, introduced by Swendsen 1979 for spin systems) was 'no asymptotic scaling' for $K_F < 6.1$ (K_F is the coupling of the Wilson action) (Gupta et al. 1984) Known . This $\Delta\beta$ and the record's decrease per step are, with the same projection, quantities that correspond by definition (the identification is this dictionary's reading). (iii) The error of truncating to the Wilson form: for untruncated perfect actions Hasenfratz–Niedermayer (1994) write 'There exist lattice actions which give cut-off independent physical predictions even on coarse grained lattices'. One reading is that δ is close to this error.

Kernels with negative weights and reflection positivity

The blocking kernel that gave $\delta(2) = 0$ in the record's calculation had a negative weight on the neighbours Computed . If the weights are non-negative, $c_2 > 0$ Lean

(`c2_ge_of_nonneg_weights`. Hypothesis: the measurement that c_2 is an increasing affine function of the amount of weight, positive at weight 0). The abstract of Lüscher–Weisz (1984) writes 'When operators of dimension 6 are added to the standard Wilson action in lattice gauge theories, physical positivity is lost in general.' Known . The pattern, a negative coefficient that removes the leading term taking away positivity, is the same. The difference is the object: theirs is the action (5-03), the record's is the blocking kernel. Whether a kernel with negative weights breaks reflection positivity has not been checked.

07

The equivalence of the physical-unit gap statement — is it the product of "mass is invariant under the renormalisation group" and "$z = 2$"?

The record treats the physical-unit gap statement (the mass gap in physical units) in the form "$a^2 C_P$ lies in a positive band independent of the lattice spacing $a = 1/\xi_{\mathrm{RG}}$", and puts two powers.

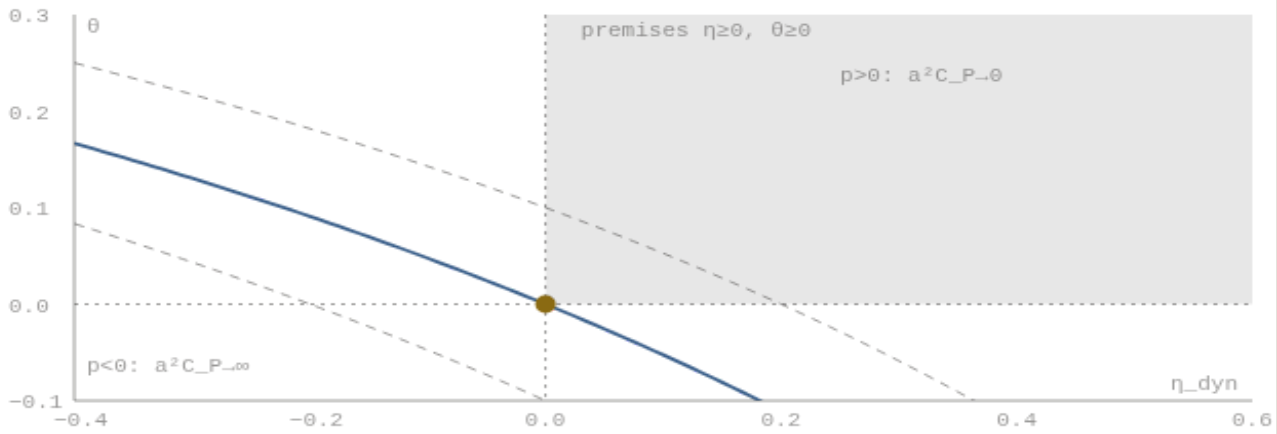
$$C_P \asymp \xi^{\,2-\eta_{\mathrm{dyn}}}, \quad \xi_{\mathrm{phys}} = a\,\xi \propto a^{\theta} \implies a^2 C_P \propto a^{p}, \quad p = \eta_{\mathrm{dyn}} + (2-\eta_{\mathrm{dyn}})\,\theta \ (9)$$

θ is the exponent of accumulation of the shortfall; in the model with b fixed, $\theta = \delta(b)/\ln b$, and if $\sum \delta < \infty$, then $\theta = 0$. If $\eta_{\mathrm{dyn}} \geq 0$ and $\theta \geq 0$, then $p = 0$ is equivalent to both being 0, and this is the content of the record's "physical-unit gap statement $\iff \sum \delta < \infty \wedge \eta = 0$" (the record's η is η_{dyn}, § 02) Lean (`drain_exponent`, `drainExp_eq_zero_iff`. Hypotheses: the power form of (9) and the sign conditions). That the error of the steps is additive and bounded, and that δ is the value in the one-loop kernel model, are also premises.

Question: is this the product of two statements of physics? Answer: within the algebra of the model, it can be read so. $\theta = 0$ is the statement "physical mass is invariant under the renormalisation group, and $m_{\mathrm{lat}}(\beta) \propto a(\beta)\Lambda$". Physics treats $\theta = 0$ as following, if a continuum theory is presupposed, from the Callan–Symanzik equation (4-04) for quantities with no anomalous dimension (dimensional transmutation is in 4-07) Physics . Physics has not made this into a named hypothesis. In mathematics, the lattice correlation-length problem of Chatterjee (2018) (Chatterjee's Problem 5.1; $\xi(\beta) \in (0, \infty)$, and $\xi \to \infty$ as $\beta \to \infty$) asks for the same direction Known . $\eta_{\mathrm{dyn}} = 0$ is $z = 2$, the expectation of § 03 and § 04.

Dropping $\eta_{\mathrm{dyn}} \geq 0$, $\theta^* = -\eta_{\mathrm{dyn}}/(2-\eta_{\mathrm{dyn}})$ gives $p = 0$, and $a^2 C_P$ lies in a constant band, yet neither the relaxation-time assumption nor $\theta = 0$ holds (the same type as Figure 2 of 17-12).

Figure 2 The $(\eta_{\mathrm{dyn}}, \theta)$ plane. Moving the point with the knobs changes where $a^2 C_P$ goes and the judgements. "Play" moves the point left along the solid line $p = 0$, and cancellation happens outside the light shading



η_dyn = 0.000 (z = 2.0000), θ = 0.000: p = 0.000, a²C_P → a constant band, relaxation-time assumption holds, θ = 0 holds

Computed A schematic drawing of equation (9), drawn on the spot. The light shading is the region of the premises, and the dashed lines are $p = \pm 0.2$. Without JavaScript it can be read from Table 3.

Computed Table 3 Examples of cancellation (z from 18-03 Table 2)

Example	z	η_{dyn}	Cancelling θ^*
2D Ising, Glauber	2.1665	−0.1665	0.0769
Free field, Langevin	2	0	0

Repeating the step that halves a 20 times, $a^2 C_P$ grows by about 10 times at $p = -0.1665$, and stays at 1 times at the point of cancellation Computed . The numerical tables alone cannot tell a breaking of the relaxation-time assumption from the exponent of accumulation of the shortfall.

08

A single scale and extra dimensions — an analogy with the non-separation of KK?

This section is easier to follow after reading 17-09 and 17-10 (the cigar, M_{KK}, supergravity). The record has the sentence "a single scale ⟺ extra dimensions are visible (radius $R \sim 1/\Lambda$)". Question: is it the same thing as the non-separation of KK? In the holographic model (Witten's 1998 cigar, 17-09) the background has a single length $R = 1/M_{\mathrm{KK}}$, and multiplying the metric by s^2 multiplies all eigenvalues of the fluctuations by s^{-2}. So $m_n = c_n M_{\mathrm{KK}}$. Since there is only one length scale R, every quantity with the dimension of length squared (both ξ^2 and C_P measured in Langevin time) is proportional to R^2, and the relaxation-time assumption becomes an identity by dimensional analysis inside the model Physics . Precisely, this is counting dimensions after defining C_P in the model. For the same reason the KK modes do not

separate from the glueballs. Sakai–Sugimoto (2005), a model of QCD with quarks, also writes about mesons: 'our D4/D8 model deviates from QCD at the energy scale of M_KK, which is, unfortunately, the same energy scale of the mass of the vector mesons'.

$\Lambda \approx M_{\mathrm{KK}}\, e^{-24\pi^2/(11\lambda)}$ (λ is the 't Hooft coupling at M_{KK} ; the coefficient depends o

At $\lambda = 30,\ 16.6,\ 10,\ 5$, $\Lambda/M_{\mathrm{KK}} = 0.49,\ 0.27,\ 0.12,\ 0.013$ Computed (16.6 is the Sakai–Sugimoto fit converted to $N_c = 3$, the λ in that convention. Putting it into the one-loop formula (10) is a rough guide, and can be off by a factor of 2 by convention). Supergravity can be used on the side of large λ, and Witten writes of the small side 'This looks like a tall order' (17-10).

Answer: an analogy. "One length $\Longrightarrow$ one scale" is obvious inside the model, but there is neither the converse nor a passage to $R \to 0$. The physical name for one scale in pure Yang–Mills is dimensional transmutation (4-07), and the $\Longleftrightarrow$ is the formulation of the record of this book's study.

09

What is known and settled — rows not counted as results

The following rows are treated as known, and are not listed as results in the later parts (this book's study). (1) The correction of direction, that a uniform gap in lattice units is on the trivial side $m_{\mathrm{phys}} \to \infty$, is just the definition that takes the continuum limit where the correlation length diverges, and the direction of the lattice correlation-length problem (Settled, 5-09). (2) $\kappa = 11/(3\pi^2)$ is $4Nb_0$ (Gross–Wilczek, Politzer 1973), and the 5-digit agreement with measurement is agreement within the one-loop model (Settled, 4-06). (3) In the record's one step of bookkeeping, the strong-coupling $\beta' = \beta^4/64$ corresponds only to the decimation step of Migdal–Kadanoff (one MK step including the bond-moving factor b^{d-2} is $\beta' \approx 4\beta^4$, 18-04 § 04), and the weak-coupling $\beta' = \beta\, b^{d-4}$ is dimension counting (Conditional). (4) Reflection positivity and the transfer operator in the reflection-positivity preservation statement (one of the eight statements of construction in 23-01), and the strong-coupling area law, are Osterwalder–Seiler 1978, Lüscher 1977 and Wilson 1974 (Settled, 5-03). (5) The infinite-volume uniqueness statement is the lattice version of uniqueness of the vacuum, and $R \le \tau_{\mathrm{int}} \le C_P$ is error analysis within the same reversible dynamics (Conditional, 18-02, 18-03).

What could not be asserted to be the same are the uniformity of the relaxation-time assumption up to constant factors (§ 03) and that step-by-step accumulation fits into a single change of scheme (§ 06); both go only as far as analogy.

What this chapter can say

What is said	What is not said
Known The type of the variational lower bound (6), the algebra of (3)(7)(9), that the rows of § 09 are known.	A theorem that the record's assumptions are the same as hypotheses of physics. None of the assumptions has been proved.
Computed Tables 1–3, Figures 1 and 2, the numbers in § 06 and § 08 (for the free field and within the model).	Whether four-dimensional Yang–Mills has $z = 2$, or a logarithmic factor. Whether kernels with negative weights break reflection positivity.
Physics The numbers for z (18-03 Table 2), the estimate (10), the testimony on the non-separation of KK.	This chapter shows nothing about the mass gap, the continuum limit or one scale.

Sources and reproduction

Item	Kind	Source / tools (level of checking)
Dynamic scaling, critical slowing down, lower bound	Physics, Known	Halperin–Hohenberg, Phys. Rev. 177 (1969) 952; Hohenberg–Halperin, RMP 49 (1977) 435 (both abstracts only). Sokal, "Monte Carlo Methods in Statistical Mechanics: Foundations and New Algorithms", in DeWitt-Morette, Cartier, Folacci (eds.), *Functional Integration: Basics and Applications* (NATO ASI Series, 1997) 131–192, doi:10.1007/978-1-4899-0319-8_6 (bibliographic data only; the authority for the lower bound $z \geq \gamma/\nu$ not checked). The numbers for z are from the sources of 18-03 Table 2 (by the checks in 18-03). The unitarity bound $\eta_S \geq 0$ (§ 02): original not cross-checked.
Change of scheme, MCRG, perfect actions, improved actions	Known	Dashen–Gross, PRD 23 (1981) 2340 (abstract); Swendsen, PRL 42 (1979) 859 (abstract); Gupta et al., PRL 53 (1984) 1721 (abstract; the quotation checked verbatim in INSPIRE); Hasenfratz–Niedermayer, NPB 414 (1994) 785 (abstract; quotation checked verbatim); Lüscher–Weisz, NPB 240 (1984) 349 (abstract; quotation checked verbatim).
Renormalisation-group invariance of mass, the lattice correlation-length problem	Known, Physics	Callan, PRD 2 (1970) 1541; Symanzik, CMP 18 (1970) 227 (both abstracts; by the checks in 4-04). Chatterjee, arXiv:1803.01950 (text of § 5; by the checks in 18-04).
KK, holography	Physics	Witten, ATMP 2 (1998) 505; Sakai–Sugimoto, PTP 113 (2005) 843 (verbatim by the checks in 17-09 and 17-10).

Item	Kind	Source / tools (level of checking)
The rows of § 09, Elitzur's theorem	Known	Gross–Wilczek, PRL 30 (1973) 1343; Politzer, PRL 30 (1973) 1346 (both bibliographic data only). Osterwalder–Seiler, Ann. Phys. 110 (1978) 440; Lüscher, CMP 54 (1977) 283; Wilson, PRD 10 (1974) 2445 (all abstracts; by the checks in other chapters). Kadanoff, Ann. Phys. 100 (1976) 359 (abstract by the checks in 18-04; main text not obtained, so not cross-checked at the level of formulas); Migdal, Sov. Phys. JETP 42 (1975) 413 (bibliographic data only; not cross-checked at the level of formulas). Elitzur, PRD 12 (1975) 3978 (abstract; by the checks in 18-02).
The record's assumptions	Assumption, analogy	The arrangement of the record of this book's study (23-01, 23-02, 23-13, 17-12).
Tables 1–3, the numbers in § 06 and § 08	Computed	Python 3 on this machine (numpy, scipy): `free_log.py`, `testfn_dim.py`, `cancel.py`.

Next chapter to read: 19-00 Entrance — how to read objects, firm theorems and premises (within Volume 3, from here we enter the objects and the firm theorems). For reading where things stand from a timeline and tables by dimension, see 16-01 How far the world has come. The conventions for η_s and η_{dyn} are in A-01 Table of notation.

Entrance — how to read objects, firm theorems and premises

> This chapter is a guide.

What this chapter shows — What Parts 19 to 22 and Part 28 contain and what they do not. In which part the definitions of the objects, the statements to be shown, the machine-checked theorems and the list of premises are each found. Three ways of choosing a reading order.

Prerequisite chapters — 0-03 (reading the problem statement word by word) and 16-09 (the stages a proof must contain). The language of the lattice is in Part 5.

Said first — These five parts do not show the mass gap. For the existence of four-dimensional Yang–Mills theory (that a theory satisfying the Wightman axioms can be made as a limit of Wilson's lattice gauge theory), the committee's assessment is that the construction published on 2026-11-10 proves it in Lean. That is read in Parts 6 to 13 of Volume 2. Every theorem of Parts 20 to 22 is a statement about a single scale with the lattice spacing fixed: results of known types, machine-checked from the definitions. Most are statements inside the strong-coupling window, at the opposite end from the questions the published construction left open. Holding uniformly across scales is on the side of the premises in Part 28. A form whose rate does not depend on the transverse size exists only for the upper bound on the vortex free energy inside the strong-coupling window (20-06), and even there the prefactor (the transverse area) remains.

The order of this chapter

1. Dividing into three — objects, theorems, premises
2. A map of the parts (Table 1)
3. Reading orders — three ways
4. How to read the labels
5. The relation to Parts 23 to 27 and Part 15 (Table 2)
6. What these five parts do not say

01

Dividing into three

Parts 23 to 27 and Part 15 are written in the order in which the study proceeded. There, statements that have been machine-checked, statements placed as assumptions, and estimates only seen in numbers sit side by side in a single chapter. Parts 19 to 22 and Part 28 take the same material and lay it out again, divided into three.

Division	Contents	Where
Objects	Things defined on the lattice: the measure, twists, the vortex ratio, reflections, the block map, the pushed-forward measure. Definitions only, with no claims	Part 19
Theorems	Equalities and inequalities about the defined objects. The assumptions are only explicit ranges (the range of the coupling, the parity of sizes and so on), and they are machine-checked in Lean 4	Parts 20 to 22
Premises	Statements on the side of "it would suffice if this could be said". Those whose form is settled and those whose form is not yet settled are listed with labels	The last chapter of Part 19, and Part 28

The line between theorems and premises is whether the objects are defined from the lattice. The chapters of theorems carry only what has its objects' definitions written in the chapter, has assumptions that are explicit ranges only, and has been machine-checked and cross-checked by another pair of eyes.

The side of premises divides in two. The statements to be shown on the ultraviolet side (matching, convergence of the full sequence, non-degeneracy, equicontinuity) and the tree of premises supporting them are statements about this book's design (the measure pushed forward by a fixed block map), and are kept as a record of the statements to be shown and sufficient conditions in this book's design. Existence itself was shown by the published construction in a different design (the comparison of the two designs is in Part 14, and the record of this book's ultraviolet attempt in Part 15). The statements to be shown on the infrared side (the band statement and the mass gap) remain open, and they are the main subject of Volume 3.

02

A map of the parts

Table 1 Parts 19 to 22 and Part 28

Part	Title	Contents

Part	Title	Contents
Part 19	Objects, and the statements to be shown	19-01 The lattice and the Wilson measure, 19-02 Twists, the vortex ratio and Polyakov loops, 19-03 Reflections and the vessel of reconstruction, 19-04 The block map and the pushed-forward measure, 19-05 The fermion weight (detour), 19-06 Objects and their Lean definitions — a table of correspondence, 19-07 The statements to be shown
Part 20	Firm theorems — the vessel and strong coupling	Reflection positivity, decay of correlations at strong coupling, the infinite-volume limit, reconstruction and the mass gap (inside the strong-coupling window), the vessel with fermions (detour), 20-06 The area law at strong coupling, and an upper bound on the vortex free energy
Part 21	Firm theorems — the vortex floor and Polyakov loops	Three results on sign systems, the thin-vortex floor, the thick-vortex floor, the minimal action of a vortex, the sign of the vortex ratio, pairs of Polyakov loops, 21-07 The vortex ratio — continuity, limit, matching
Part 22	Firm theorems — the rarity of large plaquettes	22-00 Entrance, 22-01 Rarity of large plaquettes by the Chessboard Estimate, 22-02 What this part says, and what it does not
Part 28	The tree of premises	Three chapters of implications, The list of premises, What depends on design choices, Open places

One step and many steps of the free field, and the linearisation of the block map (one-scale theorems on the ultraviolet side), are in Part 15 (15-12, 15-13) as the record of this book's ultraviolet attempt.

03

Reading orders

- **Knowing first what is to be shown**: skim the definitions of 19-01 to 19-04, then go to 19-07 The statements to be shown. From there, go on to the tree of Part 28.
- **Reading from the firm places**: Part 20 → Part 21 → Part 22. Each chapter runs in the order: objects (with references to Part 19), theorems, the route of the proof, what is not said.
- **Following the course of the study**: Parts 23 to 27 and Part 15. The correspondence between chapters on the same topic is in Table 2.

04

How to read the labels

The labels of these five parts are the same five as in Parts 23 to 27 and Part 15: Proved here Cited Computer-assisted Reading Assumption . Their meanings are the same as in the table of symbols in 15-00. Where the published construction is cited, the labels of Volume 2 (Source, Inferred, Assessment; "Labels" in 6-00) are also used. For statements

machine-checked in Lean, the name of the declaration and its range are written on the same line, outside the labels.

Most rows of the theorems in Parts 20 to 22 carry Cited . It means that a result of a known type has been machine-checked on top of this book's definitions. What machine checking guarantees is that "the written conclusion follows from the written definitions and assumptions"; that the definitions represent the intended objects rests on a cross-check by another pair of eyes.

The premises carry another set of labels (theorem, paper, form settled but unproved, form not yet settled, depends on design, literature). The legend is in 28-00.

05

The relation to Parts 23 to 27 and Part 15

Parts 23 to 27 and Part 15 are kept as the record of the study. In them one can read the roads that stopped, and why. For the same topics, the present arrangement is found as follows.

Table 2 Correspondence of topics

Topic	Record of the study	Present arrangement
Twisted boundary conditions and vortices	Part 25	Definitions in 19-02, theorems in Part 21 and 20-06
Reflection positivity and the chessboard	26-04, 26-05	20-01, Part 22
The crossover band	Part 26	Statements in 19-07, premises in 28-04
The conditional statement and the ledger	Part 27	Part 28
Ultraviolet control	Part 15	Existence in Volume 2 (reading the published construction), the comparison of the two designs in Part 14. One step of the free field in this book's design in 15-12, premises in this book's design in 28-04

06

What these five parts do not say

- They show nothing about the mass gap. The existence of four-dimensional Yang–Mills theory is not a theorem of this book but a result of the published construction (Volume 2).

- The theorems are about a fixed lattice spacing. There is no theorem about the limit in which the lattice spacing is sent to 0.
- The strong-coupling theorems are only about the window where the coupling is small. They say nothing outside the window.
- Inside the strong-coupling window there is a form in which the rate of the upper bound on the vortex free energy does not depend on the transverse size (the prefactor, the transverse area, remains; 20-06). Outside the window, the vortex-floor theorems are all about a fixed transverse size, and there is no estimate uniform in the transverse size.
- The list of premises is a dated, present arrangement. Some premises do not yet have a settled form, and the list will change.

What this chapter can say

Can say	Cannot say
This chapter is a guide and contains no statements. It shows where the contents of Parts 19 to 22 and Part 28 are found	The scope of the statements of each part is in that part's chapter on "what it says and what it does not" (Part 20, Part 21, Part 22) and in 28-06. The scope of one step of the free field is in 15-14

Sources and reproduction

Item	Kind	Source / tools
The map of the parts, the correspondence of topics	Guide	Part 15 and Parts 19 to 28 of this book
The list of machine-checked theorems	Guide	A-04

Words used in this chapter — object, push-forward measure, the tree of premises. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-01 The lattice, link variables and the Wilson measure — the first of the definitions.

The lattice, link variables and the Wilson measure

> This chapter is this book's own investigation. § 01 to § 06 arrange established knowledge, and § 07 is this book's investigation (the conventions on the Lean side and the bridges).

What this chapter shows — The definitions of the most basic objects that the theorems of Parts 20 to 22 talk about: the periodic lattice $(\mathbb{Z}/N)^4$, the link variables $U_\ell \in SU(2)$, the product of Haar measures, the plaquette holonomy U_p and its real part, the Wilson measure and the partition function. We also show here, once, that the partition function is positive and finite. Finally, we gather in one table the fact that on the Lean side the same objects are written in four conventions (strong coupling, vortices, fermions, ultraviolet), and the bridges between them.

Prerequisite chapters — 5-01 Wilson's lattice (introduction). For the language of probability measures, 1-08.

Said first — This is a chapter of definitions, and it states no theorems. The lattice spacing is fixed to 1 and the volume is finite. The continuum limit and the infinite-volume limit are outside this chapter (the space of infinite-volume configurations is in 19-03).

The order of this chapter

1. The lattice and links — the periodic lattice $(\mathbb{Z}/N)^4$ and oriented edges
2. SU(2) as unit quaternions — quaternions of length 1 in place of matrices
3. The Haar measure and its product — the uniform probability on the group
4. Plaquettes and holonomy — the product of four links, and its real part
5. The Wilson measure — the weight $e^{\beta \sum_p \operatorname{Re} U_p}$ and expectations
6. The partition function — positive and finite
7. Four conventions on the Lean side, and bridges — four ways of writing the same objects

---

01

The lattice and links — the periodic lattice $(\mathbb{Z}/N)^4$ and oriented edges

The introduction is in 5-01 § 02. Here we fix the form used by the later parts.

Take a positive integer N, and let the four-dimensional periodic lattice (torus) be

$$\Lambda_N = (\mathbb{Z}/N)^4 \quad (1)$$

The lattice spacing is fixed to 1. We write e_μ for the unit shift in direction μ. A **link** is a pair $\ell = (x, \mu)$, representing the edge from the point x to the point $x + e_\mu$. The point x is called the **base** of the link, and for a link in the time direction the base time is taken as the time of that link (a different word from the "root" [the statements to be shown] of the tree of premises in Part 28). There are $4N^4$ links in all.

The numbering of directions differs between parts; there are two.

- **20-01 to 20-05 of Part 20** (the torus $(\mathbb{Z}/N)^4$ of the same size in all four directions): directions are numbered $\mu \in \{0, 1, 2, 3\}$, with 0 as time. Coordinates are $x = (x_0, x_1, x_2, x_3)$ (the same as the indices of `TorusRP.Lk` in Lean). The formulas of 19-03 are also written in this numbering.
- **Parts 21 and 22, and 20-06** (the lattice of Part 21 is $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, 19-02): directions are numbered $\mu \in \{1, 2, 3, 4\}$, with $1, 2$ transverse and $3, 4$ longitudinal. Coordinates are $x = (x_1, x_2; x_3, x_4)$ (the indices 0 to 3 of `TwistedWilson.Link` in Lean are directions 1 to 4). 20-06 and Part 22 also number the directions 1 to 4 on the torus of the same size in all four directions (the Lean indices 0 to 3 are directions 1 to 4).

The formulas of this chapter are written with pairs of directions $\mu < \nu$, so that they apply to either numbering.

An edge in the reverse direction is not counted as a separate link. When a link is traversed in reverse, the inverse of its link variable is used (§ 02). Because the lattice is periodic, moving by e_i from a point with $x_i = N - 1$ brings us back to $x_i = 0$.

Part 21 uses the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, whose size differs between directions 1, 2 and directions 3, 4. Its definition is placed in 19-02. Links and plaquettes are built in the same way as in this chapter.

Item	Symbol	Number
Lattice points	$x \in (\mathbb{Z}/N)^4$	N^4
Links (oriented edges)	$\ell = (x, \mu)$	$4N^4$
Plaquettes (pairs of directions $\mu < \nu$)	$p = (x; \mu, \nu)$	$6N^4$

For $N = 1$ the same link appears twice in one plaquette. For $N = 2$ there are two links joining the same two points. Both are counted with their separate indices (the Lean definitions do the same).

Link variables and configurations

On each link ℓ we place one element U_ℓ of the group $SU(2)$. This is called the **link variable**, and an assignment to all links

$$U = (U_\ell)_\ell \ \in \ SU(2)^{\{\text{links}\}} \quad (2)$$

is called a **configuration**. The space of configurations is the product of $4N^4$ copies of the compact group $SU(2)$, and is itself compact. When moving from x to $x + e_\mu$ we multiply by $U_{x,\mu}$, and when moving in reverse by $U_{x,\mu}^{-1}$.

> 02

SU(2) as unit quaternions — quaternions of length 1 in place of matrices

This book's Lean definitions write $SU(2)$ not as 2×2 matrices but as **unit quaternions**. For a quaternion $q = a + b\,\mathrm{i} + c\,\mathrm{j} + d\,\mathrm{k}$ ($a,b,c,d \in \mathbb{R}$, $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$) let the norm be $\|q\| = \sqrt{a^2+b^2+c^2+d^2}$, and identify

$$SU(2) \;\cong\; S^3 = \{q \in \mathbb{H} : \|q\| = 1\} \quad (3)$$

The correspondence is given by the following matrix Cited .

$$q = a + b\,\mathrm{i} + c\,\mathrm{j} + d\,\mathrm{k} \;\longmapsto\; \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \quad (4)$$

The determinant of this matrix is $\|q\|^2$ and its trace is $2a$. Hence the following three hold Proved here .

- The product of quaternions corresponds to the product of matrices. The product of unit quaternions is a unit quaternion ($\|pq\| = \|p\|\|q\|$).
- The inverse is the conjugate: $q^{-1} = \bar{q} = a - b\,\mathrm{i} - c\,\mathrm{j} - d\,\mathrm{k}$ (for matrices, the Hermitian conjugate).
- The real part of a quaternion is half the trace: $\mathrm{Re}\,q = a = \frac{1}{2}\,\mathrm{tr}\,q$. For unit quaternions, $|\mathrm{Re}\,q| \le 1$.

The centre is $\{+1, -1\}$. The twist of Part 21 (19-02) is the operation of putting a sign from this centre into the plaquette weight.

When this book writes $\mathrm{Re}\,U$, it means the real part of the quaternion $= \frac{1}{2}\,\mathrm{tr}\,U$. It is the same quantity as $\frac{1}{N}\,\mathrm{Re}\,\mathrm{tr}\,U$ for $SU(N)$ in 5-01 with the N of the group set to 2 (the N of 5-01 is the size of the group, a different quantity from the lattice size N of this chapter).

03

The Haar measure and its product — the uniform probability on the group

The introduction is in 5-01 § 06. On a compact group G there is exactly one probability measure invariant under multiplication from the left. It is called the normalised **Haar measure** and written dU Cited . On a compact group this measure is also invariant under multiplication from the right and under the map taking inverses Cited .

For $SU(2) \cong S^3$, the Haar measure is the uniform surface measure on $S^3 \subset \mathbb{R}^4$, normalised to total mass 1 Proved here . Reason: left multiplication by a unit quaternion is an orthogonal transformation of $\mathbb{R}^4$, so it preserves the uniform surface measure. Since there is only one left-invariant probability measure, the two coincide.

The reference measure on the space of configurations is the product of the Haar measures of the links.

$$dU = \prod_{\ell} dU_\ell \qquad (\text{there are } 4N^4 \text{ links}) \quad (5)$$

This is also a probability measure, and corresponds to choosing each link variable independently and uniformly. Taking inverses on individual links, and maps that rearrange the values of links onto other links, also preserve this product measure. The reflection (19-03) uses this property.

In Lean, the normalised Haar measure is defined as `Measure.haarMeasure ⊤` (the Haar measure taking the value 1 on the whole set). The names are `OSGap.haarProb` and `TwistedWilson.haarProb` (the same definition); that it is a probability measure is `OSGap.haarProb_isProbabilityMeasure`, and left invariance is `OSGap.haarProb_isMulLeftInvariant`.

04

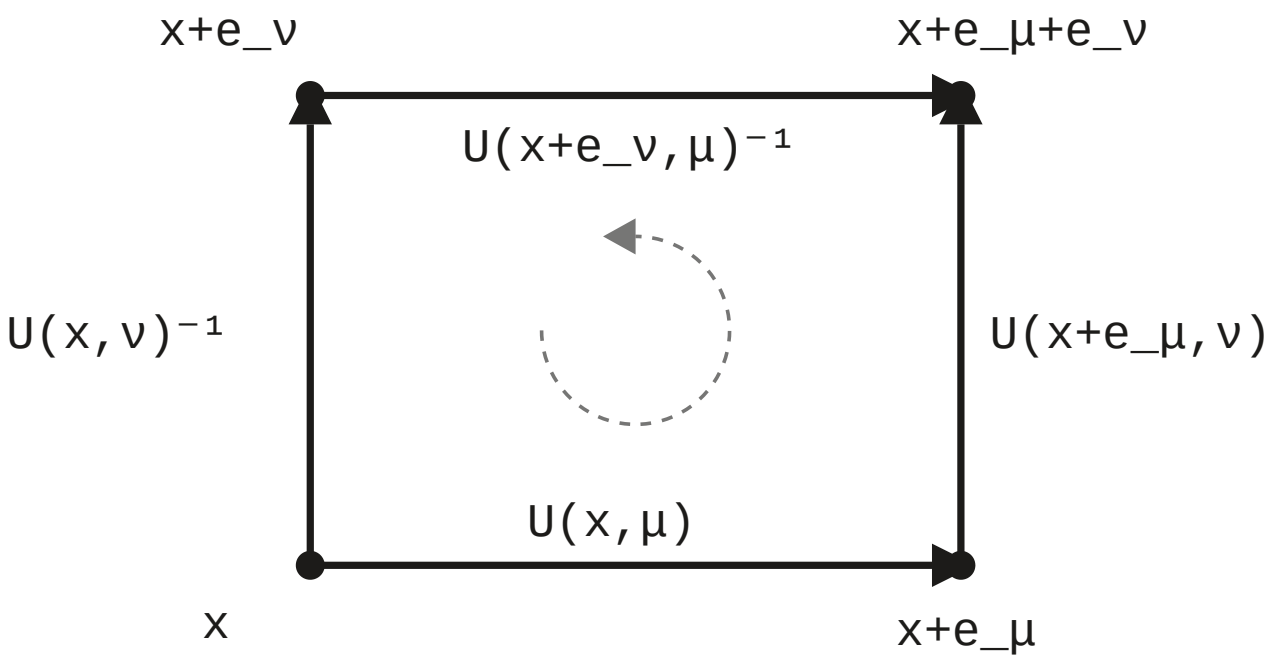
Plaquettes and holonomy — the product of four links, and its real part

The introduction is in 5-01 § 03. A **plaquette** is a pair $p = (x; \mu, \nu)$ $(\mu < \nu)$, the unit square with corners x, $x + e_\mu$, $x + e_\mu + e_\nu$, $x + e_\nu$. There are six pairs of directions ((1,2), (1,3), (1,4), (2,3), (2,4), (3,4) in the numbering of Part 21; (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) in the numbering of Part 20), so there are $6N^4$ plaquettes. The orientation of a plaquette is fixed as the one turning from μ to ν, and plaquettes of the reverse orientation are not counted.

The **holonomy** of a plaquette is the product of the four link variables going once around its boundary.

$$U_p = U_{x,\mu}\, U_{x+e_\mu,\nu}\, U^{-1}_{x+e_\nu,\mu}\, U^{-1}_{x,\nu} \quad (6)$$

Figure 1 The holonomy of the plaquette $p = (x; \mu, \nu)$. Arrows show the orientation of the links. Links opposite to the direction of travel are multiplied as inverses



Proved here The dotted line in the figure is the direction of travel ($x \to x + e_\mu \to x + e_\mu + e_\nu \to x + e_\nu \to x$). The top and left edges are traversed against the orientation of their links, so inverses enter.

U_p is a unit quaternion, and its real part is $\operatorname{Re} U_p = \frac{1}{2} \operatorname{tr} U_p \in [-1, 1]$ Proved here . $\operatorname{Re} U_p = 1$ holds only when $U_p = 1$ (going once around the plaquette changes nothing).

Gauge transformations, and the invariance of the real part

A **gauge transformation** by $g_x \in SU(2)$ at each lattice point sends the link from x to $y = x + e_\mu$ to

$$U_{x,\mu} \longmapsto g_x \, U_{x,\mu} \, g_y^{-1} \quad (7)$$

The holonomy then changes by conjugation, $U_p \mapsto g_x U_p g_x^{-1}$, and the real part $\operatorname{Re} U_p$ does not change Proved here ($\operatorname{Re}(gqg^{-1}) = \operatorname{Re} q$, by cyclicity of the trace). The product of Haar measures is also invariant under gauge transformations Proved here (since each link is only multiplied from the left and the right). In Lean, the gauge transformation is written as `act` (value $g(\text{start})\, U\, g(\text{end})^*$). Gauge-invariant functions are used in Part 20 for reflection positivity through the middle plane (19-03).

05

The Wilson measure — the weight $e^{\beta \sum_p \operatorname{Re} U_p}$ and expectations

The introduction is in 5-01 § 04. Take a real number β. In Parts 19 to 22 and Part 28 of this book, the **weight** of a configuration U is

$$w_\beta(U) = \exp\Big(\beta \sum_p \operatorname{Re} U_p\Big) \quad (8)$$

The sum runs over all $6N^4$ plaquettes. It is related to the Wilson action $S_W(U) = \beta \sum_p \big(1 - \tfrac{1}{2} \operatorname{tr} U_p\big)$ of 5-01 by

$$e^{-S_W(U)} = e^{-6N^4 \beta} \, w_\beta(U) \quad (9)$$

holds Proved here. The constant $e^{-6N^4\beta}$ disappears on normalisation, so the two ways of writing give the same probability measure. β is the same quantity as in 5-01; for $SU(2)$, $\beta = 4/g^2$ Cited.

The **Wilson measure** is the probability measure obtained by multiplying the product of Haar measures by the weight w_β and normalising the whole to 1.

$$d\mu_{N,\beta}(U) = \frac{1}{Z_N(\beta)} \exp\Big(\beta \sum_p \operatorname{Re} U_p\Big) \prod_\ell dU_\ell \quad (10)$$

We fix the notation. We write $\tilde{\mu}_{N,\beta}$ for the unnormalised measure $w_\beta(U) \prod_\ell dU_\ell$ (`TorusRP.wilsonMeasure N β` in Lean), and $\mu_{N,\beta}$ for the normalised (10) (`TorusRP.wilsonProb N β` in Lean). $\tilde{\mu}_{N,\beta} = Z_N(\beta)\, \mu_{N,\beta}$.

The **expectation** of a bounded measurable function F is

$$\langle F \rangle_{N,\beta} = \int F \, d\mu_{N,\beta} = \frac{1}{Z_N(\beta)} \int F(U)\, w_\beta(U) \prod_\ell dU_\ell \quad (11)$$

The sign of β is free as far as the definition goes. Physics usually uses $\beta > 0$ (the more the plaquettes line up at 1, the heavier the weight), but some theorems of Parts 20 and 21 hold including $\beta < 0$. The range is written for each theorem.

The weight (8) is gauge invariant (§ 04). Hence the Wilson measure is also invariant under gauge transformations Proved here. Even without fixing a gauge, the integral is an integral over a finite-

dimensional, compact space.

> May be skipped. In this book the unnormalised measure $w_\beta \prod dU$ is called the "unnormalised Wilson measure". In theorems that say "a certain integral is at least 0", such as reflection positivity, multiplication by a positive constant does not change the conclusion, so many of the Lean statements are written with the unnormalised measure (§ 07). For quantities that take ratios, such as expectations and variances, the normalised measure (10) is used.

06

The partition function — positive and finite

The **partition function** is the normalising constant.

$$Z_N(\beta) = \int \exp\Big(\beta \sum_p \operatorname{Re} U_p\Big) \prod_\ell dU_\ell \quad (12)$$

For the definition (10) to make sense we need $0 < Z_N(\beta) < \infty$. This follows from the following estimate Proved here .

$$e^{-6N^4|\beta|} \;\le\; Z_N(\beta) \;\le\; e^{6N^4|\beta|} \quad (13)$$

Proof: on each plaquette $|\operatorname{Re} U_p| \le 1$, so the absolute value of the exponent is at most $6N^4|\beta|$. Hence the integrand lies between $e^{-6N^4|\beta|}$ and $e^{6N^4|\beta|}$, and the reference measure has total mass 1. (End)

This estimate gets rapidly worse with the volume. It says nothing uniform in the volume. In the infinite-volume limit (Part 20), one deals not with Z itself but with ratios of expectations.

In Lean, the two-sided bounds on the integrand are `TorusRP.wilsonWeight_le` and `TorusRP.wilsonWeight_ge` (the exponent is the number of plaquettes $\times$ $|\beta|$; that the number of plaquettes is $6N^4$ was counted by hand); that the total mass of the unnormalised Wilson measure obtained by integrating them, $\tilde{\mu}_{N,\beta}(\mathbf{whole}) = Z_N(\beta)$, is positive and finite is `TorusRP.wilsonMeasure_univ_pos` and `TorusRP.wilsonMeasure_univ_lt_top`; and that the partition function of the twisted lattice is positive is stated by `TwistedWilson.ZW_pos` (all real β, all pairs of twists).

07

Four conventions on the Lean side, and bridges — four ways of writing the same objects

This book's Lean theorems were written separately in four lines (strong coupling, vortices, fermions, ultraviolet). The object is the same: configurations of the $SU(2)$ lattice and their measure. But the indexing of the lattice, the space of link values and the way of writing the sign of the weight differ slightly. When reading the theorems of any part, check the conventions in this table.

	Strong-coupling side (Part 20)	Vortex side (Part 21)	Fermion side (19-05)	Ultraviolet side (19-04; tl
Lattice	$(\mathbb{Z}/N)^4$. Links `Lk N` = (lattice point) $\times$ (direction). Direction index 0 is time (the same as the numbering 0 to 3 of Part 20; reflection x $0 \mapsto -x0$)	$(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$. `Site N M`, `Link`, `Plaq`. Indices 0 to 3 are directions 1 to 4 (the numbering of Part 21). **Time is index 3 (x_4)** (the reflection $x4 \mapsto -x$ 4 is in 19-03)	Time circle $\mathbb{Z}/2h \times$ space $(\mathbb{Z}/N)^3$. `PhysLink N h`. Direction index 3 is time. The bridge is for $N = 2h$	The box $(\mathbb{Z}/2^m L)^4$ at st `BoxWalk.Site L m`, `Box` (lattice point $\times$ `Fin 4`
Link values	The closed unit ball of quaternions `QBall` ($\|q\| \leq 1$)	The unit sphere `SU2` ($\|q\| = 1$)	The unit sphere `SU2`. Inside the action, mapped to the matrix (4)	The closed unit ball of `ChainBlockMap.Ball`. C `ChainBlockMap.Cfg` (th block map can fall insi domain and codomain the ball; 19-04)
Reference measure	`haarBall`: the Haar measure of the sphere pushed forward to the ball. The product is `Measure.pi`	`haarCfg`: the product of Haar measures	The product of Haar measures	Not fixed on the side of push-forward measure `ChainBlockMap.vP` take probability measures o configuration spaces of each step
Weight	e^{β} action, `action` = $\sum_{\mu<\nu}$ $\mathrm{Re}\, U_p$	$\exp(\beta \sum_p z_p \mathrm{Re}\, U_p$), where $z_p = \pm 1$ is the sign of the twist (19-02)	$e^{-\beta S_W}$, $S_W = \sum_p (1 - \frac{1}{2}$ $\mathrm{Re}\,\mathrm{tr}\, U_p)$	No weight enters the de Wilson measure or ano placed on the box type
Normalisation	Not normalised (`TorusRP.wilsonMeasure`). The normalised form is `wilsonProb`	No name is given to the measure; written with the partition function `TwistedWilson.ZW` and its ratios	Not normalised (`WilsonGauge.gaugeMeasure`)	The measures taken in measures (normalised)
Notation in this book	Unnormalised $\tilde{\mu}_{N,\beta}$, normalised $\mu_{N,\beta}$ ((10)), Z_N (β) ((12))	Z_0, Z_1, θ_M (19-02)	The notation of 19-05	The notation of 19-04

We explain the meaning of the differences in convention in turn.

The closed ball and the sphere. The strong-coupling side places link values in the closed unit ball $\{\|q\| \leq 1\}$, and takes as reference measure "the Haar measure of the sphere pushed forward to the ball" (`haarBall`). Using the ball is a choice made for the construction in Lean. Since the reference measure lives on the sphere, almost every configuration takes values in $SU(2)$, and the values of integrals are the same as under the sphere convention Proved here . The conjugate (`star`) is written in place of the inverse, but the two coincide on the sphere.

The sign of the weight. The weight on the fermion side is $e^{-\beta S_W}$, which by (9) is a positive constant multiple of $e^{\beta \sum \mathrm{Re}\, U_p}$.

Bridges — identifications shown in Lean

Bridge	What it says	Lean
Sum of the actions	On the torus of the same size $N = 2h$ in all four directions, for every configuration, $\sum_{\mu<\nu} \mathrm{Re}\, U_p + S_W = 6(2h)^4$ (through the re-indexing of directions and the embedding of the sphere into the ball)	`WilsonGauge.action_add_wilsonAction`
Agreement of reference measures	Pushing the product of Haar measures on the fermion side forward by `pull` gives the product of `haarBall` on the strong-coupling side	`WilsonGauge.map_pull`
Agreement of measures	`TorusRP.wilsonMeasure (2h) β` on the strong-coupling side equals $e^{6(2h)^4\beta}$ times the push-forward of `gaugeMeasure h β` on the fermion side by the re-indexing `pull` (all real β, $2h > 1$)	`WilsonGauge.wilsonMeasure_eq_map`
Agreement of reflections	Under the same re-indexing, the reflections through lattice-point planes on the two sides correspond	`WilsonGauge.refl_pull` (19-03)
Agreement of measures (strong coupling ↔ vortices)	When $M = N$ and there is no twist ($z_p \equiv 1$), for all $N \geq 1$ and all real β, `TorusRP.wilsonMeasure N β` on the strong-coupling side equals the product of Haar measures `haarCfg N N` on the vortex side multiplied by the density $\exp(\beta \sum_p \mathrm{Re}\, U_p)$ and pushed forward by the re-indexing of links (lattice point $x \mapsto ((x_0, x_1), (x_2, x_3))$, direction indices unchanged) and the inclusion `toBall` of the sphere into the ball. There is no constant factor (both sums over plaquettes run over the $6N^4$ plaquettes with $\mu < \nu$). `toBall` is injective (`WilsonBridge.toBall_injective`)	`WilsonBridge.wilsonMeasure_eq_map`
Expectations and total mass (strong coupling ↔ vortices)	For measurable F, $\int F\, d(\mathrm{wilsonMeasure}) = \int F(\mathrm{toBall}\, U)\, e^{\beta \sum_p \mathrm{Re}\, U_p}\, d(\mathrm{haarCfg})$. With $F = 1$, the total mass is `ZW N N β false false` (converted to a value in $[0, \infty]$)	`WilsonBridge.integral_wilsonMeasure`, `WilsonBridge.wilsonMeasure_univ`
Correspondence aligning time, and reflections (strong coupling ↔ vortices)	The same equality of measures holds for the re-indexing that sends time on the strong-coupling side (index 0) to time on the vortex side (index 3, x_4) and index k to $k - 1$ (lattice point $x \mapsto ((x_1, x_2), (x_3, x_0))$). The reflections through lattice-point planes on the two sides are then the same map ($x_0 \mapsto -x_0$ on the strong-coupling side and $x_4 \mapsto -x_4$ on the vortex side). Time–space plaquettes are sent to plaquettes of reverse orientation (the real part is the same; the holonomy is conjugated)	`WilsonBridgeT.wilsonMeasure_eq_mapT`, `WilsonBridgeT.refl_toBallT`, `WilsonBridgeT.action_toBallT`
The vortex ratio (no background)	With background $n_{12} = 0$ and $M = N$, $Z_{N,N}(\beta; 0, 1)/Z_{N,N}(\beta; 0, 0)$ equals the expectation of $\exp\left(-2\beta \sum \mathrm{Re}\, U_p\right)$ (over the N^2 time–space plaquettes corresponding to $\Pi_{(0,0)}$) under the normalised measure of the strong-coupling side	`WilsonBridgeV.vortex_ratio_eq`
The definition of the Haar measure	`haarProb` on the strong-coupling side and on the vortex side are defined by the same expression `Measure.haarMeasure ⊤` (two definitions in different namespaces; inside the proof of the agreement of measures above, they are unfolded and checked to agree)	`OSGap.haarProb`, `TwistedWilson.haarProb`

When $M = N$ and there is no twist, the strong-coupling side and the vortex side are tied together in Lean as one measure. Pushing forward the product of Haar measures on the vortex side, multiplied

by the weight $\exp(\beta \sum_p \operatorname{Re} U_p)$, by the re-indexing of links and the inclusion of the sphere into the ball gives exactly the Wilson measure of the strong-coupling side (there is no constant factor, and both sums over plaquettes run over $6N^4$ plaquettes). There are two re-indexings. The one that overlays the direction indices unchanged is used for the equalities of measures, expectations and total mass. The one that overlays time on the strong-coupling side (index 0) with time on the vortex side (index 3, x_4) is used in the chapters that use reflection. With the latter, the reflections through lattice-point planes on the two sides become the same map, and time–space plaquettes are sent to plaquettes of reverse orientation. Weights and actions written only in terms of real parts do not change, but when the holonomy itself is transported a conjugation enters. Overlaying the direction indices unchanged is allowed only for the equalities of measures; where reflection is used, the re-indexing that aligns time is used.

What this bridge does not say: there is no equality transporting a twisted measure (background $n_{12} = 1$, or $n_{34} = 1$) as a measure to the strong-coupling side. The θ_M of Part 21 and the expectation $\langle\cdot\rangle_0$ of Polyakov loops are quantities of the weight with background $n_{12} = 1$, not quantities on this bridge (what the bridge transports is only the form that writes the background-free ratio $Z(0,1)/Z(0,0)$ as an expectation). Lattices with $M \neq N$ have no counterpart on the strong-coupling side. These are equalities in finite volume, and they do not restate theorems about the infinite-volume limit or the mass gap.

The Lean definitions on the ultraviolet side consist only of maps and push-forwards on box lattices and ball-valued configurations, and do not contain the Wilson weight. The correspondence between the ultraviolet side and the other three sides is dealt with in 19-04, and is not listed in this chapter's table of bridges.

What this chapter can say

Can say	Cannot say
Proved here $SU(2)$ is the group of unit quaternions, and $\operatorname{Re} q = \frac{1}{2}\operatorname{tr} q \in [-1,1]$	— (definitions and elementary facts)
Cited The normalised Haar measure is unique, and invariant under left and right multiplication and inversion	The only properties named in Lean are that it is a probability measure and left invariance
Proved here $e^{-6N^4\lvert\beta\rvert} \le Z_N(\beta) \le e^{6N^4\lvert\beta\rvert}$. Lean: `TorusRP.wilsonWeight_le`, `wilsonWeight_ge` (the integrand), `TorusRP.wilsonMeasure_univ_pos`, `TorusRP.wilsonMeasure_univ_lt_top`, `TwistedWilson.ZW_pos`	Not an estimate uniform in the volume. The infinite-volume limit is outside this chapter
Proved here The Wilson measure and $\operatorname{Re} U_p$ are gauge invariant	The argument on gauge fixing (used for the reflection through the middle plane in Part 20) is not in this chapter
Lean: by `WilsonGauge.wilsonMeasure_eq_map` ($N = 2h$, all real β), the Wilson measures of the strong-coupling side and the fermion side agree up to a constant multiple	—
Lean: by `WilsonBridge.wilsonMeasure_eq_map` ($M = N$, no twist, all $N \ge 1$ and real β), the Wilson measures of the strong-coupling side and the vortex side are equal under the re-indexing (no constant factor). Under the re-indexing that aligns time, the reflections also agree (`WilsonBridgeT.refl_toBallT`)	A bridge for twisted measures ($n_{12} = 1$ or $n_{34} = 1$) as measures, and $M \neq N$. There is no formula writing the θ_M of Part 21 ($n_{12} = 1$) as an expectation on the strong-coupling side

Sources and reproduction

Item	Kind	Source / tools
Gauge theory on the lattice, the Wilson action	Cited	K. G. Wilson, Phys. Rev. D 10 (1974) 2445. Introduction in 5-01
Existence and uniqueness of the Haar measure, $SU(2) \cong S^3$	Cited	Standard facts. Introduction in 5-01 § 06
The form that writes reflection positivity with the unnormalised measure	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract only)
Definitions of the four conventions, and the bridges	Lean	This book's Lean bundle (bridges strong coupling ↔ vortices: `WilsonBridge.wilsonMeasure_eq_map`, `integral_wilsonMeasure`, `wilsonMeasure_univ`, `toBall_injective`, `WilsonBridgeT.wilsonMeasure_eq_mapT`, `refl_toBallT`, `action_toBallT`, `WilsonBridgeV.vortex_ratio_eq`. Strong-coupling side: `TorusRP.QBall`, `haarBall`, `Lk`, `action`, `wilsonMeasure`, `wilsonProb`. Vortex side: `TwistedWilson.SU2`, `haarCfg`, `hol`, `zt`, `ZW`. Ultraviolet side: `BoxWalk.Site`, `BoxWalk.BoxLk`, `ChainBlockMap.Ball`, `ChainBlockMap.Cfg`, `ChainBlockMap.vP`. Fermion side: `WilsonGauge.wilsonAction`, `gaugeMeasure`, `action_add_wilsonAction`, `wilsonMeasure_eq_map`). The formulas of the definitions were copied by reading the source

Words used in this chapter — lattice, link, link variable, configuration, unit quaternion, Haar measure, plaquette, holonomy, gauge transformation, Wilson measure, unnormalised Wilson measure, partition function. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-02 Twisted boundary conditions, the vortex ratio θ and Polyakov loops — on the same lattice, we define partition functions with the signs of plaquettes flipped one by one, and their ratios.

Twisted boundary conditions, the vortex ratio θ and Polyakov loops

This chapter is this book's own investigation. § 01 to § 04 arrange established knowledge, and § 05 to § 07 are this book's investigation (the definition of the minimal action of a vortex, and the conventions in Lean).

What this chapter shows — The definitions of the objects that the theorems of Part 21 talk about: the torus with transverse size N and longitudinal size M, the twist signs n_{12}, n_{34}, the partition functions Z_0, Z_1, the vortex ratio θ_M, the Polyakov loop $P_b(t)$, the twisted vacuum V, and the minimal action of a vortex $S_{\mathrm{v}}(N,m)$ and S_N. We also state here that the definitions are sound ($|\theta_M| < 1$, $0 \le S_N \le 2N^2$).

Prerequisite chapters — 19-01 (the lattice, plaquettes, the Wilson measure). The introduction to twists is in 25-01 § 01.

Said first — This is a chapter of definitions; it states no estimates of θ_M or of Polyakov loops (the estimates are in Part 21, and in 20-06 inside the strong-coupling window). The transverse size N is a fixed finite number, and nothing uniform in N is said. The lattice spacing is 1.

The order of this chapter

1. The torus with transverse size N and longitudinal size M — a small cross-section and a large plane
2. Twist signs — flipping the sign of plaquette weights
3. The partition functions Z_0, Z_1 and the vortex ratio θ_M — the price of inserting one twist
4. Polyakov loops — the product once around the longitudinal direction
5. The twisted vacuum — a configuration of action 0
6. The minimal action of a vortex — the price of passing one vortex through a block
7. Correspondence with Lean names — conventions on indices and truth values

01

The torus with transverse size N and longitudinal size M — a small cross-section and a large plane

The introduction is in 25-01 § 01. Take positive integers N, M, and let the lattice be

$$\Lambda_{N,M} = (\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2, \qquad x = (x_1, x_2;\, x_3, x_4) \tag{1}$$

Directions 1 and 2 are called **transverse** (size N), and directions 3 and 4 **longitudinal** (size M). The transverse part is a small two-dimensional torus T^2_N, and the longitudinal part is a large plane $(\mathbb{Z}/M)^2$; in Part 21, N is held fixed and M is made large.

Links (x, μ), plaquettes $p = (x; \mu, \nu)$ $(\mu < \nu)$, the holonomy U_p and the product of Haar measures are built in the same way as in 19-01. There are $4N^2M^2$ links and $6N^2M^2$ plaquettes.

Fixing one point $q = (x_3, x_4)$ of the longitudinal plane, the N^2 transverse lattice points sit above it. We write Π_q for the bundle of $(3,4)$ plaquettes based at these N^2 points. Within the plane, Π_q looks like a single point; transversely, it is a collection of plaquettes covering T^2_N (equation (3) of 25-01 § 01).

02

Twist signs — flipping the sign of plaquette weights

On the lattice, 't Hooft's twisted boundary conditions are introduced through signs $z_p \in \{+1, -1\}$ of the plaquette weights Cited . For two twists $n_{12}, n_{34} \in \{0, 1\}$ we set

$$z_p = \begin{cases} (-1)^{n_{12}} & p \text{ is a } (1,2) \text{ plaquette with base } (x_1, x_2) = (0,0) \\ (-1)^{n_{34}} & p \text{ is a } (3,4) \text{ plaquette with base } (x_3, x_4) = (0,0) \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

That is, $n_{12} = 1$ flips the sign of the one plaquette based at the origin in each transverse $(1,2)$ plane (a single T^2_N with the longitudinal point fixed). There are M^2 flipped plaquettes in all. $n_{34} = 1$ flips the one plaquette based at the origin in each longitudinal $(3,4)$ plane (a single plane with the transverse point fixed), and the flipped plaquettes are the N^2 plaquettes of $\Pi_{(0,0)}$.

The twisted weight and partition function are

$$Z_{N,M}(\beta; n_{12}, n_{34}) = \int \exp\Big(\beta \sum_p z_p \operatorname{Re} U_p\Big) \prod_\ell dU_\ell \quad (3)$$

This satisfies $0 < Z < \infty$ Proved here (the same estimate as in 19-01 § 06; $|z_p \operatorname{Re} U_p| \le 1$).

When $N, M \ge 2$, where the flipped plaquettes are placed has no effect on the partition function. Replacing a link variable by $U_\ell \mapsto -U_\ell$ flips the signs of all plaquettes containing that link, and the Haar measure is preserved Proved here. What the partition function sees is only the parity of the number of flips piercing each plane (the flux). The proof is in the Moving-Surface Lemma of 25-01 § 02 (a statement for $N, M \ge 2$; the proof rests on each plaquette containing the same link at most once). 25-01 places the $(1,2)$ flip at $(x_1, x_2) = (N-1, N-1)$, but by this lemma it gives the same partition function.

It is related to the partition function $Z[z] = \int e^{-\beta S^z} \prod dU$ of 25-01 ($S^z = \sum_p (1 - \frac{1}{2} z_p \operatorname{tr} U_p)$) by $Z[z] = e^{-6N^2M^2\beta} Z_{N,M}(\beta; n_{12}, n_{34})$ Proved here. This constant disappears in the ratio of the next section.

03

The partition functions Z_0, Z_1 and the vortex ratio θ_M — the price of inserting one twist

In Part 21 the transverse twist is held fixed as a background at $n_{12} = 1$, and two partition functions are compared according to whether the longitudinal twist n_{34} is present.

$$Z_0 = Z_{N,M}(\beta; 1, 0), \qquad Z_1 = Z_{N,M}(\beta; 1, 1) \quad (4)$$

The **vortex ratio** is defined as

$$\theta_M = \theta_{N,M}(\beta) = \frac{Z_0 - Z_1}{Z_0 + Z_1} \quad (5)$$

The transverse N is held fixed in the notation; when needed we write $\theta_{N,M}$. Since Z_0, $Z_1 > 0$, θ_M is well defined and $|\theta_M| < 1$ Proved here. In terms of the **flux free energy** $\Delta F = \ln(Z_0/Z_1)$ (the term of "the definition of the sector and flux free energies" in 25-01; the same quantity as the vortex free energy F_v of 26-03), $\theta_M = \tanh(\Delta F/2)$ Proved here.

Meaning: flipping the N^2 plaquettes of $\Pi_{(0,0)}$ imposes the condition that one centre vortex (flux -1) pierces the longitudinal plane. θ_M close to 0 means that inserting one vortex hardly changes the partition function, that is, that vortices arise cheaply in the

plane. That θ_M decreases in M at the rate e^{-cM^2} is, through the Tomboulis–Yaffe inequality, the entrance to the argument for confinement Cited (25-01 § 02, 26-01).

$\theta_M \geq 0$ ($Z_1 \leq Z_0$) does not follow from the definition. It is a theorem of Part 21 (for even M), and is not assumed here. The continuity of the vortex ratio in the coupling, its limits, and matching it to a chosen value are in 21-07.

May be skipped. The thin-vortex floor (Part 21) is written in Lean in a form that holds whether the background n_{12} is set to 0 or 1. We write $\theta_M^{(n_{12})}$ for the ratio $(Z_{N,M}(\beta; n_{12}, 0) - Z_{N,M}(\beta; n_{12}, 1))/(Z_{N,M}(\beta; n_{12}, 0) + Z_{N,M}(\beta; n_{12}, 1))$ in that case. θ_M means $\theta_M^{(1)}$.

04

Polyakov loops — the product once around the longitudinal direction

Fix a transverse position $b \in (\mathbb{Z}/N)^2$ and a position $t \in \mathbb{Z}/M$ in direction 4, and multiply in order the M link variables going once around direction 3.

$$P_b(t) = \mathrm{Re}\Big(U_{(b;0,t),3}\, U_{(b;1,t),3} \cdots U_{(b;M-1,t),3}\Big) \quad (6)$$

This is called the **Polyakov loop**. Since the real part is half the trace, $P_b(t) = \frac{1}{2}\,\mathrm{tr}(\cdots) \in [-1, 1]$ Proved here . Being the trace of a product along a closed path, it does not change under gauge transformations Proved here .

We write $\langle\cdot\rangle_0$ for the expectation normalised with the weight of the twist background $n_{12} = 1$, $n_{34} = 0$.

$$\langle F\rangle_0 = \frac{1}{Z_0}\int F(U)\exp\Big(\beta\sum_p z_p \,\mathrm{Re}\,U_p\Big)\prod_\ell dU_\ell \qquad (n_{12}=1,\ n_{34}=0) \quad (7)$$

What Part 21 estimates is the pair of two Polyakov loops separated by T in direction 4, $\langle P_b(0)\, P_{b'}(T)\rangle_0$.

The expectation of a single loop is 0 Proved here : multiplying all direction-3 links at $x_3 = 0$ by -1, every plaquette contains 0 or 2 such links (counting repetitions), so the weight does not change, the Haar measure is preserved, and only $P_b(t)$ changes sign. What carries meaning is therefore the correlation of a pair.

05

The twisted vacuum — a configuration of action 0

Using the unit quaternions i, j, define the configuration V by

$$V_{x,1} = \begin{cases} \mathrm{i} & x_1 = 0 \\ 1 & x_1 \neq 0 \end{cases}, \qquad V_{x,2} = \begin{cases} \mathrm{j} & x_2 = 0 \\ 1 & x_2 \neq 0 \end{cases}, \qquad V_{x,3} = V_{x,4} = 1 \tag{8}$$

This is called the **twisted vacuum**. For the signs (2) with background $n_{12} = 1$, $n_{34} = 0$, on every plaquette

$$z_p \operatorname{Re} V_p = 1 \tag{9}$$

holds Proved here . For example, on the $(1,2)$ plaquette with base $(x_1, x_2) = (0,0)$, $V_p = \mathrm{i}\mathrm{j}\,\mathrm{i}^{-1}\mathrm{j}^{-1} = -1$, which cancels against $z_p = -1$. On the other plaquettes the holonomy is 1. Since the action $1 - z_p \operatorname{Re} U_p$ of each plaquette is at least 0, V gives the minimum (0) of the twisted action. Lean: `TwistBlocks.vac`; (9) is `TwistBlocks.vac_term`.

That i and j anticommute ($\mathrm{i}\mathrm{j} = -\mathrm{j}\mathrm{i}$) is the mechanism by which the transverse twist is absorbed at action 0. That the twisted vacuum is unique up to gauge transformations and global transformations is in 25-01 § 05.

06

The minimal action of a vortex — the price of passing one vortex through a block

We define the number S_N that appears in the exponent of the thick-vortex floor (Part 21). Take a positive integer m, and let the longitudinal size be $M = 2m$. The lines $x_3 \in \{0, m\}$ and $x_4 \in \{0, m\}$ divide the longitudinal plane $(\mathbb{Z}/2m)^2$ into four **blocks** of side m; we use the block at the origin, $b_0 = \{0, \dots, m-1\}^2$.

- **Frame**: the links lying on the boundary lines. Direction-3 links whose x_4 is on a line, direction-4 links whose x_3 is on a line, and links in the transverse directions 1 and 2 whose longitudinal point (x_3, x_4) is on a line.
- **Interior**: the links that are not in the frame and whose longitudinal point lies in b_0.
- **Plaquettes of the block**: all plaquettes whose base has its longitudinal point in b_0 (all six orientations, all transverse positions).

Fix the frame to the twisted vacuum V, let the interior move freely, and flip the sign only of the $(3,4)$ plaquettes above the corner point $(x_3, x_4) = (0,0)$ (the N^2 plaquettes of $\Pi_{(0,0)}$); the infimum of the resulting action is

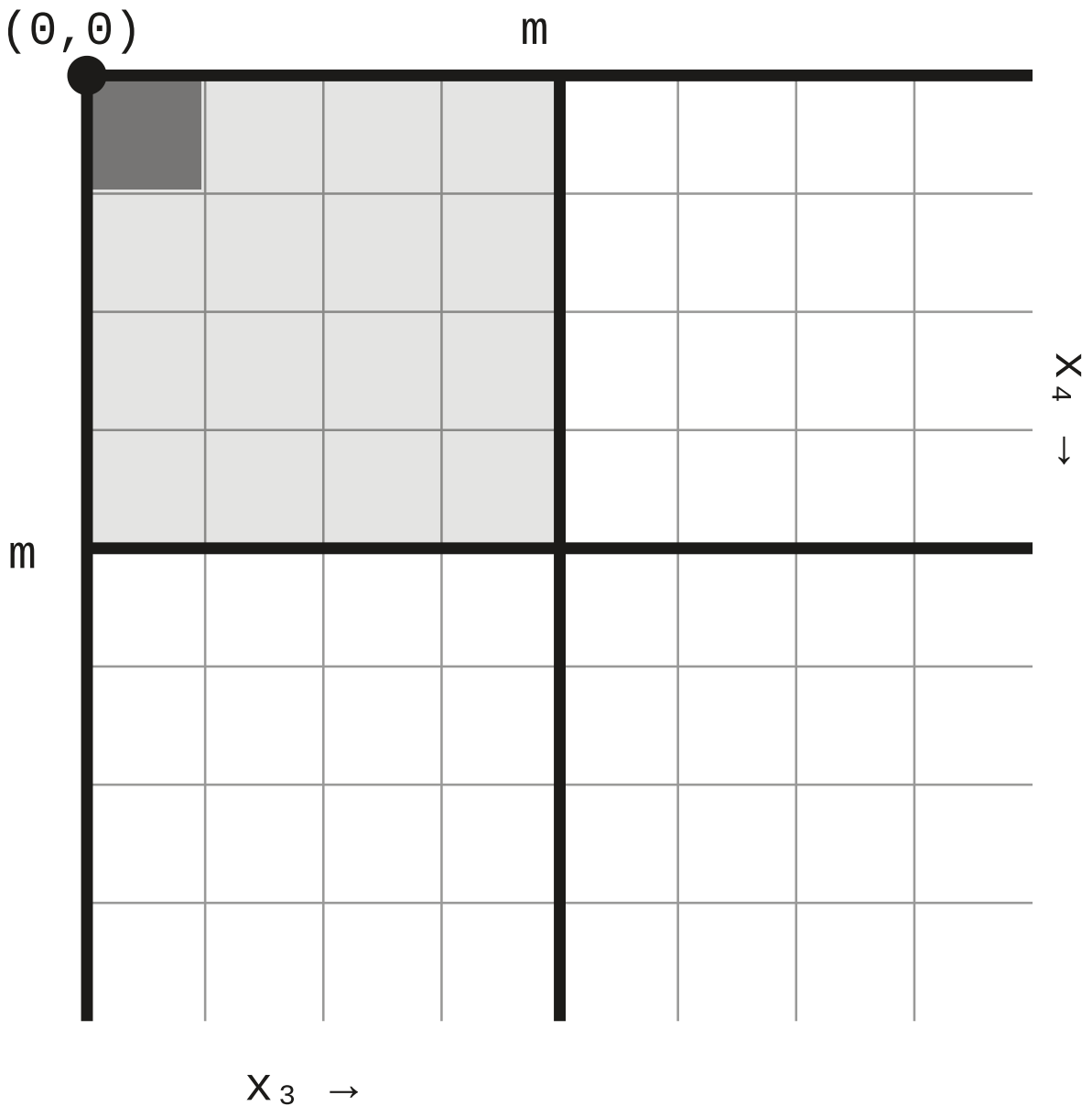
$$S_{\mathrm{v}}(N,m) = \inf_{\text{interior}} \sum_{p \in b_0} \left(1 - z_p\, \eta_p \operatorname{Re} U_p\right), \qquad \eta_p = \begin{cases} -1 & p \in \Pi_{(0,0)} \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

Here z_p are the signs of the background $n_{12}=1$, $n_{34}=0$. The flipped $\Pi_{(0,0)}$ are the same plaquettes as for the longitudinal twist $n_{34}=1$, so $S_{\mathrm{v}}(N,m)$ is "the minimal action of passing one vortex through a block of side m whose boundary is fixed to the twisted vacuum". The infimum over m,

$$S_N = \inf_{m\geq 1} S_{\mathrm{v}}(N,m) \tag{11}$$

is called the **minimal action of a vortex**.

Figure 1 The longitudinal plane $(\mathbb{Z}/2m)^2$ $(m=4)$. Thick lines are the frame (fixed to the twisted vacuum), the grey area is the block $b0$ at the origin, and the black dot is the corner $(0,0)$ whose signs are flipped. Above each point sits the transverse T^2_N



Proved here The figure shows one period of the periodic plane (the right and bottom edges coincide with the left and top edges). The dark square is the square based at the corner point $(0,0)$, above which $\Pi_{(0,0)}$ sits (it refers to the same thing as the black dot in the figure).

We check that the definition is sound. Each term $1 - z_p \eta_p \operatorname{Re} U_p$ lies between 0 and 2, so the infimum is finite and $S_{\mathrm{v}}(N,m) \geq 0$. Taking the interior also to be V, all unflipped plaquettes give 0 ((9)) and each of the N^2 flipped ones gives 2, so $S_{\mathrm{v}}(N,m) \leq 2N^2$. Therefore

$$0 \leq S_N \leq 2N^2 \quad (12)$$

Proved here . Lean: `TwistBlocks.Svortex_le`, `TwistBlocks.SN_bounds`. The infimum in (10) is attained by an interior configuration (because the space of interior configurations is compact; Lean: `TwistBlocks.Svortex_attained`). The upper bound $2N^2$ is the same number as the exponent of the thin-vortex floor; whether S_N is strictly smaller than it, and whether $S_N > 0$, do not follow from the definition. The latter is a theorem of 21-05, and the former for $N = 2, 3, 4$ is a computer certificate in the same chapter.

07

Correspondence with Lean names — conventions on indices and truth values

The Lean definitions on the vortex side correspond to the notation of this chapter as follows. Direction indices start at 0, and 0, 1, 2, 3 are directions 1, 2, 3, 4 of this chapter. Twists are truth values, and `true` is $n = 1$.

This chapter	Lean	Note
$\Lambda_{N,M}$, links, plaquettes	`TwistedWilson.Site N M`, `Link`, `Plaq`	`Site N M = (ZMod N × ZMod N) × (ZMod M × ZMod M)`. Plaquettes are pairs of directions $\mu < \nu$
U_p	`TwistedWilson.hol`	The quaternion product $U_{x,\mu} U_{x+e_\mu,\nu} U^*_{x+e_\nu,\mu} U^*_{x,\nu}$
z_p (equation (2))	`TwistedWilson.zt n12 n34`	The flips are placed at bases $(x_1,x_2) = (0,0)$ and $(x_3,x_4) = (0,0)$
$Z_{N,M}(\beta; n_{12}, n_{34})$	`TwistedWilson.ZW N M β n12 n34`	Not normalised. The reference measure is `haarCfg`
Z_0, Z_1	`ZW N M β true false`, `ZW N M β true true`	
θ_M	`TwistBlocks.thetaM N M β`	The same expression as equation (5)
$P_b(t)$	`HalfSplit.polyLoop N M b t`	Direction 3 is index 2. The product is in the order $k = 0, 1, \ldots, M-1$

This chapter	Lean	Note
$\langle P_b(s) P_{b'}(t) \rangle_0$	`PolyakovT.polyExp N M β b b' s t`	`PolyakovT.corr / ZW N M β true false`. The weight `HalfSplit.wE N M β (fun _ => 1)` of `corr` is the same expression as the weight of Z_0 (`PolyakovT.wE_one`)
V	`TwistBlocks.vac`	
$S_{\mathrm{v}}(N,m)$, S_N	`TwistBlocks.Svortex N m`, `TwistBlocks.SN N`	`Svortex N m = Smin N (2m) m 2 true (0,0) (V on the frame) true`. `SN N` is the infimum over positive integers m

The block action of `Smin` in Lean also includes in its sum the plaquettes lying only on the frame. The difference is a function of the frame alone and has no effect on the infimum over the interior. The "plaquettes of the block" in equation (10) are matched to this convention.

What this chapter can say

Can say	Cannot say
Proved here $0 < Z_{N,M}(\beta; n_{12}, n_{34}) < \infty$, hence $\lvert\theta_M\rvert < 1$. Lean: `TwistedWilson.ZW_pos`	Estimates of the size and sign of θ_M (Part 21)
Proved here For $N, M \geq 2$, the placement of the flips has no effect on the partition function; only the parity of the flux matters (proof in the Moving-Surface Lemma of 25-01 § 02)	The Lean definitions of this chapter fix the placement at the origin. The lemma that moves the placement is not listed as a Lean theorem
Proved here The twisted vacuum V has $z_p \operatorname{Re} V_p = 1$ on every plaquette. Lean: `TwistBlocks.vac_term`	The uniqueness of the vacuum (25-01 § 05) is not dealt with in this chapter
Proved here $\langle P_b(t) \rangle_0 = 0$ (centre transformation)	Not listed as a Lean theorem. Estimates of the pair correlation are in Part 21
Proved here $0 \leq S_N \leq 2N^2$, and the infimum in (10) is attained. Lean: `TwistBlocks.SN_bounds`, `Svortex_attained`	$S_N > 0$ and $S_N < 2N^2$ do not follow from the definition (21-05). Nothing uniform in N is said

Sources and reproduction

Item	Kind	Source / tools
Twisted boundary conditions	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141. Introduction in 25-01
The lattice form with a field of plaquette signs	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117

Item	Kind	Source / tools
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. 26-01
Polyakov loops	Cited	A standard quantity. Its treatment as an order variable at finite temperature is Svetitsky–Yaffe, NPB 210 (1982) 423 (the bibliographic data follow the citation in this book)
Definitions (lattice, twist, Z, θ_M, Polyakov loops, twisted vacuum, S_{v}, S_N) and (9), (12)	Lean	This book's Lean bundle (`TwistedWilson.ZW`, `zt`, `hol`, `haarCfg`, `TwistBlocks.thetaM`, `vac`, `vac_term`, `Svortex`, `SN`, `Svortex_le`, `SN_bounds`, `Svortex_attained`, `HalfSplit.polyLoop`, `PolyakovT.polyExp`, `PolyakovT.corr`). The formulas of the definitions were copied by reading the source

Words used in this chapter — twisted boundary conditions, twist sign, flux, partition function, vortex ratio, flux free energy, Polyakov loop, twisted vacuum, block, frame, minimal action of a vortex. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-03 Reflection, the transfer operator and the vessel of reconstruction — defines the reflection and the vessel that builds from it a Hilbert space and a transfer operator.

2026-11-13 · chapter Yang–Mills and the mass gap Part 19 Objects, and the statements to be shown Reflection, the vessel, the definition of the mass gap · Kiichi, Shiori and Rin

Reflection, the transfer operator and the vessel of reconstruction

This chapter is this book's own investigation. § 01 to § 04 and § 06 arrange established knowledge, and § 05, § 07 and § 08 are this book's investigation (the conventions in Lean for the vessel and for the infinite-volume space).

What this chapter shows — The definitions of the objects that the theorems of Part 20 talk about: the reflection Θ (through lattice-point planes and through middle planes), functions on the positive side, the reflection pairing $\langle F, G\rangle = \int F(\Theta\omega)G(\omega)\,d\mu$, time translation, the vessel of reconstruction (the triple of pairing, translation and constant) and the Hilbert space, vacuum Ω and transfer operator T built from it, the definition of the mass gap $\|T - P\Omega\| \le e^{-m}$, and the space of infinite-volume configurations with the periodic lift.

Prerequisite chapters — 19-01 (the lattice and the Wilson measure). Introductions in 4-10 (reflection positivity and reconstruction), 5-03 (reflection on the lattice and the transfer matrix) and 4-11 (the definition of the mass gap).

Said first — This is a chapter of definitions. Reflection positivity (that the pairing is positive semidefinite), the existence of the infinite-volume limit and estimates of the mass gap are all theorems of Part 20, and are not even assumed here. The definition of the mass gap is a quantity at a fixed lattice spacing, in units of one lattice step. It is not a definition of the mass gap of the continuum limit.

The order of this chapter

1. Reflection — lattice-point planes, and planes through the middle of links
2. The positive side — functions depending on one side of the mirror only
3. The reflection pairing — the integral of a product with the mirrored function
4. Time translation — moving the positive side one step further in
5. The vessel of reconstruction — from pairing, translation and constant to a Hilbert space and a transfer operator
6. The definition of the mass gap — T contracts by at most e^{−m} outside the vacuum

7. The space of infinite-volume configurations — configurations on $\mathbb{Z}^4$ and the periodic lift
8. Correspondence with Lean names — the three mirrors and the names of the vessel

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01

Reflection — lattice-point planes, and planes through the middle of links

The introduction is in 5-03 § 02. The formulas of this chapter are written in the **numbering of Part 20** of 19-01 § 01 (directions $0, 1, 2, 3$, with 0 as time), and the time coordinate x_0 is written t. The lattice is periodic with even size $2M$ in the time direction. The reflections in Lean (`TorusRP.thL`, `MidRP.thM`) are defined only on $(\mathbb{Z}/2M)^4$, of the same size $N = 2M$ in all four directions.

There are two kinds of mirror.

- **Reflection through lattice-point planes**: $t \mapsto -t$. The mirror is the two planes $t = 0$ and $t = M$ passing through lattice points (by periodicity $-M \equiv M$).
- **Reflection through middle planes**: $t \mapsto 1 - t$. The mirror is the two planes $t = \frac{1}{2}$ and $t = M + \frac{1}{2}$ cutting across the middle of the time-direction links.

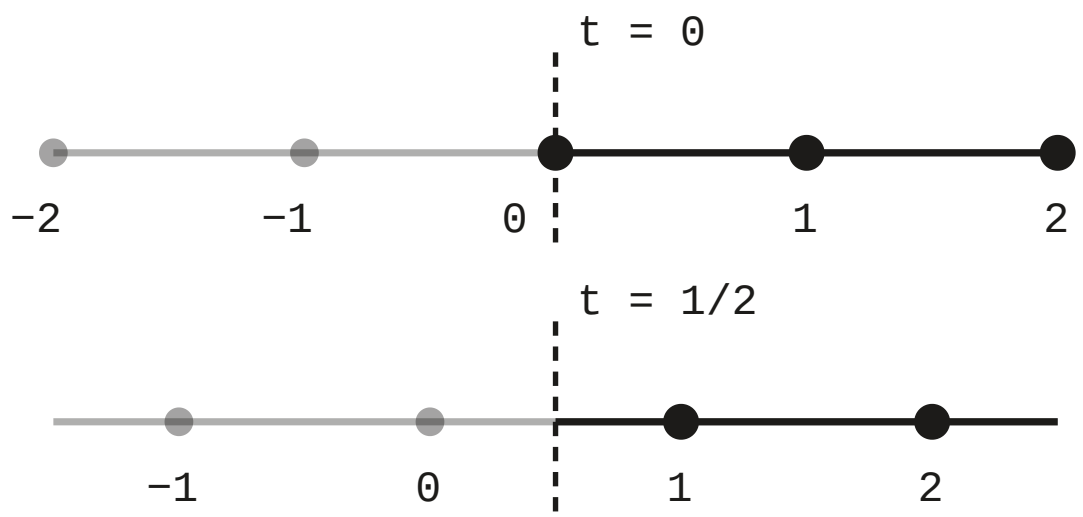
From the map of lattice points $x \mapsto \vartheta x$, the map of link variables is defined as follows. A space-direction link (x, ν) goes to $(\vartheta x, \nu)$ with its value unchanged. A time-direction link has its orientation reversed, so we take the **inverse** of the link it is mirrored to.

$$(\Theta U)_{x,\nu} = U_{\vartheta x,\nu} \quad (\nu \text{ spatial}), \qquad (\Theta U)_{x,0} = \left(U_{\vartheta(x+e_0),0}\right)^{-1} \quad (1)$$

For the reflection through lattice-point planes, $\vartheta(x + e_0) = \vartheta x - e_0$; that is, the link from t to $t + 1$ goes to the link from $-t - 1$ to $-t$. For the reflection through middle planes, the link from t to $t + 1$ goes to the link from $-t$ to $1 - t$, and the time links at $t = 0$ and $t = M$ (the links crossing the mirror) go to their own inverses.

Either Θ is a map on the space of configurations; applied twice it returns to the original, and it preserves the product of Haar measures Proved here (a rearrangement of links and an inverse on each link; 19-01 § 03). Plaquettes go to plaquettes and the value of $\mathrm{Re}\, U_p$ equals the value on the mirrored plaquette, so the Wilson weight is also preserved Proved here (for the lattice-point mirror, Lean: `TorusRP.holQ_refl`. The middle-plane mirror is the lattice-point mirror combined with a translation by one step, and the same follows from translation invariance).

Figure 1 The two mirrors seen along one row in the time direction. Dots are lattice points (space links sit on them), segments are time links. Top: the lattice-point plane $t = 0$. Bottom: the middle plane $t = \frac{1}{2}$. The dark part is the positive side



Proved here In the top row, the space links at the point $t = 0$ on the mirror do not move, and are counted on both the positive and the negative side. In the bottom row there is no lattice point on the mirror, and the time link from 0 to 1 crosses the mirror.

On the lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ of the vortex side (Part 21), the reflection through lattice-point planes in direction 4, $x4 \mapsto -x4$ (M even, mirrors at $x4 = 0$ and $x4 = M/2$), is used with the same rule (1) (in the numbering of Part 21, direction 4 plays the role of time). When $M = N$, under the re-indexing that overlays time on the strong-coupling side with direction 4 on the vortex side, the reflections through lattice-point planes on the two sides are the same map (Lean: `WilsonBridgeT.refl_toBallT`, 19-01 § 07).

02

The positive side — functions depending on one side of the mirror only

One side of the mirror is called the **positive side**, and we consider functions that depend only on links on the positive side. Precisely: fix the set L_+ of links on the positive side; a function F whose values agree whenever two configurations agree on L_+ is said to "depend only on L_+".

Mirror	Links on the positive side L_+
Lattice-point planes ($t = 0, M$)	Space links at times $0 \le t \le M$, and time links whose base time is $0 \le t \le M - 1$ (the closed positive half)

Mirror	Links on the positive side L_+
Middle planes ($t = \frac{1}{2}, M + \frac{1}{2}$)	Space links at times $1 \le t \le M$, and time links whose base time is $0 \le t \le M$ (including the links at $t = 0$ and $t = M$ crossing the mirror)

Θ sends the positive side to the negative side. So if F depends only on the positive side, $F \circ \Theta$ depends only on the negative side (and the mirror) Proved here .

The class of functions used by the theorems of Part 20 is real-valued functions depending only on the positive side that are **bounded and measurable**. For the middle-plane mirror, some theorems use the class with **gauge invariance** (19-01 § 04) added. Which class is used is written for each theorem.

03

The reflection pairing — the integral of a product with the mirrored function

The introduction is in 4-10 § 04. For a finite measure μ on the space of configurations (in Part 20, the Wilson measure) and a reflection Θ, the **reflection pairing** of bounded functions F, G on the positive side is defined as

$$\langle F, G \rangle_\mu = \int F(\Theta\omega)\, G(\omega)\, d\mu(\omega) \quad (2)$$

It is bilinear in F, G. If μ is invariant under Θ, it is symmetric Proved here (change variables by $\omega \mapsto \Theta\omega$ and use $\Theta^2 = \mathrm{id}$; for the Wilson measure, Lean: `TorusRP.wilson_refl_symm`).

Reflection positivity is the property that $\langle F, F \rangle_\mu \ge 0$ for every F on the positive side. This is not a definition but a property of the measure, and that it holds for the Wilson measure is a theorem of Part 20 (20-01). It is not assumed in this chapter.

For the measure $c\mu$ multiplied by a positive constant c, the pairing is also multiplied by c, so whether it is positive semidefinite does not depend on the normalisation. This is why reflection positivity in Lean is written with the unnormalised Wilson measure (19-01 § 05).

04

Time translation — moving the positive side one step further in

Let S be the map shifting a configuration by one step in the time direction, acting on functions by $(\tau F)(\omega) = F(S\omega)$. The direction is chosen so that if F depends on links at times $[0, R]$, then τF depends on links at times $[1, R + 1]$. Then τ sends functions on the positive side to functions on the positive side.

When the measure is translation invariant and $\Theta S = S^{-1}\Theta$ holds, τ is symmetric with respect to the pairing Proved here .

$$\langle \tau F, G\rangle_\mu = \langle F, \tau G\rangle_\mu \quad (3)$$

The operator that this τ induces on the lattice is the **transfer operator** (transfer matrix). 5-03 writes it directly in the temporal gauge (5-03 § 05). Part 20 of this book builds it from the pairing and τ alone, through the vessel of the next section.

05

The vessel of reconstruction — from pairing, translation and constant to a Hilbert space and a transfer operator

The introduction is in 4-10 § 05. What is extracted from the Osterwalder–Schrader reconstruction as the part that does not depend on the measure is called the **vessel of reconstruction**. A vessel is the following tuple on a real vector space V.

Field	Condition	When built from a measure
Pairing B (bilinear form)	Positive semidefinite $B(x,x) \geq 0$, symmetric $B(x,y) = B(y,x)$	The reflection pairing (2)
Translation τ (linear map)	Symmetric with respect to B: $B(\tau x, y) = B(x, \tau y)$	Time translation (§ 04)
Constant $\mathbf{1} \in V$	$B(\mathbf{1},\mathbf{1}) = 1$, $\tau\mathbf{1} = \mathbf{1}$	The constant function 1 (probability measure)
Bounded orbits	For each x there is C with $B(\tau^k x, \tau^k x) \leq C$ for all k	For bounded functions, $C = \|F\|_\infty^2$

From a vessel we build the following.

1. **Hilbert space $\mathcal{H}$**: the completion of V with the semi-inner product B. Vectors with $B(x,x) = 0$ collapse to 0 in the completion. We write $[x]$ for the image of an element x of V.
2. **Vacuum $\Omega = [\mathbf{1}]$**. $\|\Omega\| = 1$.
3. **Transfer operator T**: the continuous extension of $T[x] = [\tau x]$. It can be extended because $B(\tau x, \tau x) \leq B(x,x)$ (τ contracts). This follows from symmetry, positive semidefiniteness and bounded orbits Proved here . Route: by Cauchy–Schwarz, $a_k = B(\tau^k x, \tau^k x)$ is log-convex ($a_{k+1}^2 \leq a_k a_{k+2}$), so the ratio a_{k+1}/a_k increases. If $a_1 > a_0$, then a_k grows exponentially, contradicting boundedness (Lean: `OrbitGrowth.contraction_of_bounded`).

The T built in this way is self-adjoint, $\|T\| \leq 1$, and $T\Omega = \Omega$ Proved here . In Lean, the vessel is written as the structure `OSGap.Datum`, the Hilbert space as `Datum.Hilb`, T as `Datum.T` and Ω as `Datum.Ω`, and the properties are stated by `Datum.contraction`, `Datum.T_symm`, `Datum.T_Ω` and `Datum.norm_Ω`.

The assembly that builds a vessel from a measure is `OSGap.measureDatum`. For a probability measure ν, a reflection and a translation, it takes three fields as assumptions — the pairing is positive semidefinite (reflection positivity), the pairing is symmetric, the translation is symmetric — and fills the remaining fields (constant, bounded orbits) itself. The theorems of Part 20 discharge these three assumptions for the limit of Wilson measures.

May be skipped. Examples satisfying all the fields of a vessel can be made even small. Lean's `OSGap.exampleDatum` is a vessel on the two-dimensional $V = \mathbb{R}^2$ in which, for $|r| \leq 1$, T acts as multiplication by r on the complement of the vacuum (`example_opNorm_eq`). It checks that the fields of the vessel are consistent (non-empty).

06

The definition of the mass gap — T contracts by at most e^{−m} outside the vacuum

The introduction is in 4-11 § 01. Let the projection onto the vacuum be $P_\Omega v = \langle \Omega, v \rangle \, \Omega$. For a real number m, a vessel D is said to **have mass gap** m if

$$\|T - P_\Omega\| \leq e^{-m} \quad (4)$$

(operator norm). Lean: `OSGap.Datum.HasMassGap D m`. The unit is one lattice step, and the lattice spacing is fixed. Since T is a self-adjoint contraction with $T\Omega = \Omega$, $\|T - P_\Omega\| \leq 1$ always holds, and for $m \leq 0$, (4) holds automatically Proved here . It carries meaning when $m > 0$.

Meaning: T does not move Ω and is self-adjoint, so it preserves the orthogonal complement of the vacuum. (4) is the same as $\|T\| \leq e^{-m}$ on that complement Proved here . If $T > 0$ ($T \geq 0$ with no kernel), we can write $T = e^{-H}$, and (4) becomes the statement that the spectrum of H on the complement of the vacuum lies in $[m, \infty)$, that is, that the gap in the sense of 4-11 is at least m. The T of Part 20 has a kernel (20-04). In that case it cannot be written as $T = e^{-H}$, and (4) is read in the form $\|T\| \leq e^{-m}$ on the complement of the vacuum.

(4) is equivalent to truncated correlations decaying at a uniform rate for every x on the positive side.

$$\|T - P_\Omega\| \le e^{-m} \iff \forall x \in V\ \exists C_x\ \forall k:\ B(\tau^k x, \tau^k x) - B(\mathbf{1}, x)^2 \le C_x\, e^{-2mk} \quad (5)$$

Lean: right to left is `OSGap.Datum.hasMassGap_of_decay`, left to right is `OSGap.Datum.decay_of_hasMassGap` (the constant is $C_x = B(x,x) - B(\mathbf{1},x)^2$). This is the form on the vessel of "the gap is equivalent to all correlations decaying uniformly" in 4-11 § 02.

There are two things this definition alone cannot say.

- **Non-degeneracy**: if $\mathcal{H}$ is only the one dimension spanned by Ω, then $T - P_\Omega = 0$ and (4) holds for every m Proved here . So the theorems of Part 20 write separately in their conclusion that there is a vector of length 1 orthogonal to the vacuum (that the complement of the vacuum is not 0).
- $T \ge 0$: the definition does not forbid negative spectrum of T. $T \ge 0$ is also a separate conclusion of Part 20.

07

The space of infinite-volume configurations — configurations on $\mathbb{Z}^4$ and the periodic lift

The introduction is in 5-04 § 01. To compare measures on tori of different sizes, we place them all on one space. Let the space of configurations on all links (x,μ) ($x \in \mathbb{Z}^4$) of $\mathbb{Z}^4$ be

$$\mathcal{C} = \prod_{\ell \text{ a link of } \mathbb{Z}^4} \{q \in \mathbb{H} : \|q\| \le 1\} \quad (6)$$

with the product topology. The closed unit ball is compact, so $\mathcal{C}$ is compact Cited (Tychonoff's theorem). Taking values in the closed ball here is the convention of 19-01 § 07.

The periodic lift: a configuration ω on the torus $(\mathbb{Z}/L)^4$ is made into a point of $\mathcal{C}$ by reading the coordinates of the links of $\mathbb{Z}^4$ modulo L. We write the probability measure on $\mathcal{C}$ obtained by pushing forward the normalised Wilson measure $\mu_{L,\beta}$ (19-01 (10)) by this map, for $L = 2(n+1)$, as

$$\mu^{\text{lift}}_{n,\beta} = (\text{periodic lift})_*\, \mu_{2(n+1),\beta} \qquad (n = 0, 1, 2, \dots) \quad (7)$$

Lean: `WilsonLimit.liftProb β n`. The sequence over all sizes L is `AllN.liftS`.

Convergence of probability measures on $\mathcal{C}$ is measured by **weak convergence** ($\int F$ converges for every bounded continuous F) Cited . That the sequence (7) converges to

one probability measure ν_β in the strong-coupling window is a theorem of Part 20 (20-03); the existence of the limit is not assumed here.

On $\mathcal{C}$, too, the reflection (lattice-point planes $t \mapsto -t$, rule (1)) and time translation are defined by the same formulas. The space of functions on the positive side is

$$W = \{F : \mathcal{C} \to \mathbb{R} \text{ bounded and continuous } : \ F \text{ depends only on finitely many links with bas}$$

W contains the constant 1 and is closed under translation. The subspace W_{gi} restricted to gauge-invariant functions is also used. The vessel of Part 20 has $V = W$ (or W_{gi}), the pairing (2) with respect to ν_β, the translation of § 04, and the constant 1.

08

Correspondence with Lean names — the three mirrors and the names of the vessel

This chapter	Lean	Note
The reflected configuration $\Theta\omega$	`LinkReflection.linkRefl θ σ ω`	$(\Theta\omega)_\ell = \sigma_\ell(\omega_{\theta\ell})$. θ is a rearrangement of links, σ a map on values
The lattice-point mirror $((\mathbb{Z}/2M)^4)$	`TorusRP.thL`, `TorusRP.sig`	Conjugate (`star`) on time links. The positive side is `TorusRP.fixP` (space links on the mirror) and `TorusRP.posP`
The middle-plane mirror	`MidRP.thM`, `MidRP.sigT`	`thM` is the lattice-point mirror combined with a translation by one step. The positive side is `TorusGauge.cr` (crossing links) and `MidRP.posM`
The mirror on the vortex side (direction 4 of Part 21)	`HalfSplit.Th` (`HalfSplit.thL`, `HalfSplit.sgL`)	M even. Links on the mirror are `HalfSplit.fixL`. `HalfSplit.thL` has the same name as `TorusRP.thL` but is a different definition
The mirror on the fermion side	`WilsonGauge.thetaPhys`	Corresponds to the mirror on the strong-coupling side through `WilsonGauge.refl_pull` (19-05)
Vessel, Hilbert space, T, Ω	`OSGap.Datum`, `Datum.Hilb`, `Datum.T`, `Datum.Ω`	The fields of the vessel are `B`, `τ`, `one` and six conditions (`nonneg`, `symm`, `τ_symm`, `one_one`, `τ_one`, `bounded`)
Vessel built from a measure	`OSGap.measureDatum`	Assumption fields: reflection positivity, symmetry of the pairing, symmetry of the translation
Mass gap	`OSGap.Datum.HasMassGap D m`	`‖D.T - vacProj D.Ω‖ ≤ Real.exp (-m)`
$\mathcal{C}$, mirror, translation	`WilsonLimit.Cfg`, `zRefl`, `zShift`	`Cfg` is a map from the links of $\mathbb{Z}^4$ to `QBall`

This chapter	Lean	Note
Periodic lift, (7)	`WilsonLimit.perLift`, `WilsonLimit.liftProb`	The torus size is `WilsonLimit.Nn n = 2(n+1)`
W, W_{gi}	`WilsonLimit.Wsub`, `GaugeZ.Wgi`	`Wgi = Wsub ⊓ Winv` (gauge invariant)

The theorems of Part 20 are stated in a form in which the fields `B`, `τ`, `one` of the vessel are tied, within the conclusion, to formula (2), § 04 and the constant 1. In a form that only says a vessel exists, the fields could be chosen trivially so as to satisfy the mass gap (the same reason as the remark on non-degeneracy in § 06).

What this chapter can say

Can say	Cannot say
Proved here The two reflections are involutions, and preserve the product of Haar measures and the Wilson weight. Lean: `TorusRP.holQ_refl` (lattice-point planes)	Reflection positivity (a theorem of 20-01) is not even assumed in this chapter
Proved here If the measure is invariant under Θ, the pairing is symmetric. For the Wilson measure, Lean: `TorusRP.wilson_refl_symm`	—
Lean: from a vessel, T can be built, and it is self-adjoint, a contraction, and $T\Omega = \Omega$ (`Datum.contraction`, `T_symm`, `T_Ω`)	That the assumptions of the vessel are satisfied for a concrete measure (Part 20)
Lean: the mass gap (4) and the uniform decay of correlations (5) are equivalent (`hasMassGap_of_decay`, `decay_of_hasMassGap`)	(4) includes neither non-degeneracy nor $T \geq 0$, and holds automatically for $m \leq 0$. The lattice spacing is fixed; it is not a definition of the mass gap of the continuum limit
Cited $\mathcal{C}$ is compact	The convergence of the sequence (7) (a theorem of 20-03) is not assumed

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity and reconstruction	Cited	K. Osterwalder, R. Schrader, CMP 31 (1973) 83–112; 42 (1975) 281–305. Introduction in 4-10
Reflections of lattice gauge theory (lattice-point planes, link planes) and the transfer matrix	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract only). Introduction in 5-03

Item	Kind	Source / tools
The definition of the mass gap	Cited	The same definition as 4-11, written on the vessel in units of one lattice step
Tychonoff's theorem, weak convergence of probability measures	Cited	Standard facts
Definitions of the vessel, the mirrors and the infinite-volume space, and the properties of § 05 and § 06	Lean	This book's Lean bundle (`OSGap.Datum` , `measureDatum` , `exampleDatum` , `HasMassGap` , `hasMassGap_of_decay` , `decay_of_hasMassGap` , `LinkReflection.linkRefl` , `TorusRP.thL` , `sig` , `fixP` , `posP` , `WilsonLimit.Cfg` , `liftProb` , `Wsub` , `MidRP.thM` , `posM` , `HalfSplit.Th`). The formulas of the definitions were copied by reading the source

Words used in this chapter — reflection, lattice-point plane, middle plane, positive side, reflection pairing, reflection positivity, time translation, transfer operator, vessel of reconstruction, vacuum, mass gap, non-degeneracy, periodic lift, weak convergence. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-04 The block map and one step of renormalisation — on the same lattice, defines the map to a coarser lattice and the push-forward measure.

2026-11-13 · chapter Yang–Mills and the mass gap Part 19 Objects, and the statements to be shown Objects · Kiichi, Shiori and Rin

The block map and one step of renormalisation — families of paths, averaging, push-forward

> This chapter is this book's own investigation. § 01 is established knowledge.

What this chapter shows — We define the map B from the fine lattice $(\mathbb{Z}/2^{m+1}L)^4$ to the coarse lattice $(\mathbb{Z}/2^{m}L)^4$ through a family of paths, weights, averaging and an operation returning to the group, and set up its linear part C, the push-forward $B_*\mu$, and the multi-step family of the free field $\mathcal{M}$, G_k. The theorems of the second half of Part 15 (15-12, 15-13) talk about these.

Prerequisite chapters — 19-01 Link variables, SU(2) as unit quaternions, 7-05 § 02 The one-step transformation.

Said first — This is a chapter of definitions, and it states no theorems (only the minimum of facts is added, with their Lean names). B is one of many possible constructions, and the weights and the threshold r_0 are design choices (§ 08). The limit of lattice spacing to 0 is not dealt with. This block map is this book's design. The one step of renormalisation in the published construction is of a different design, and it is that construction which showed existence (the comparison of the two designs is in 14-02).

The order of this chapter

1. The idea of one step of renormalisation
2. Two lattices and coarse links
3. The family of paths — weights $(1/8, 3/4, 1/8)$ and six orders
4. The non-linear block map B — returning the average to the ball by a retract
5. The path kernel C — the linear part
6. The push-forward of measures
7. The multi-step family on the free-field side
8. The weights are a design choice

01

The idea of one step of renormalisation

The introduction is in 7-05 § 02 and § 05. One step of renormalisation is the operation of making coarse variables from fine variables by one rule, and carrying the measure to the coarse lattice by that rule Cited . For gauge fields, a coarse link variable is made from the "product along a path" of fine links (under a gauge transformation only the two ends move). However, group elements cannot be added and divided, and the average of path products in general leaves $SU(2)$, so a procedure that returns the average to the group is needed Cited .

The B of this chapter fixes one family of paths, averages the products along each path with weights, and shrinks the value in the radial direction to return it to the unit ball. It is one concrete form of the abstract one step T (15-03).

Below, we write $\mathbb{H}$ for the quaternions and $\mathbb{B} = \{q \in \mathbb{H} : \ |q| \le 1\}$ for their closed unit ball. $SU(2)$ is the sphere $|q| = 1$ (19-01). The conjugate q^* is the inverse on $SU(2)$.

02

Two lattices and coarse links

Take a positive integer L and a step number $m \ge 0$, and set

$$\Lambda_m = (\mathbb{Z}/2^m L)^4, \qquad \Lambda_{m+1} = (\mathbb{Z}/2^{m+1} L)^4 \quad (1)$$

Λ_{m+1} is the fine lattice and Λ_m the coarse lattice. Note that the larger number is the finer one. A coarse link is a pair $b = (y, \mu)$ ($y \in \Lambda_m$, direction $\mu \in \{1, 2, 3, 4\}$), going from y to $y + e_\mu$. In this chapter directions are written 1 to 4, corresponding in order to the Lean indices 0 to 3 (the two numbering conventions are in 19-01 § 01). Since the four directions are treated alike, which direction is called time is not specified. Fine links have the same form (x, ν), $x \in \Lambda_{m+1}$.

A coarse point y sits at the fine point $2y$. The map $y \mapsto 2y$ is well defined as $\mathbb{Z}/2^m L \to \mathbb{Z}/2^{m+1} L$ (since $2^m L \equiv 0$ goes to $2 \cdot 2^m L \equiv 0$) Proved here . On the fine lattice, a coarse link $b = (y, \mu)$ corresponds to the interval $2y \to 2y + 2e_\mu$ of length 2, whose end point $2y + 2e_\mu$ is the image $2(y + e_\mu)$ of the next coarse point $y + e_\mu$ Proved here .

A configuration is a function $U : \{\text{links}\} \to \mathbb{B}$ taking a value in the ball on each link. The Wilson measure lives on configurations with values in $SU(2)$, but since the result of averaging can fall inside the ball, the domain and codomain of the map are taken to be spaces of ball-valued configurations.

Lean: coarse links `BoxWalk.BoxLk L m`, the ball `ChainBlockMap.Ball` (other names in 19-06). A configuration takes values only on links of positive orientation, and the convention is that the value of a reversed link is read as the conjugate U^*.

03

The family of paths

For each coarse link $b = (y, \mu)$, we fix one finite family of fine paths from $2y$ to $2y + 2e_\mu$. Let n_1, n_2, n_3 be the three directions other than μ, in increasing order of number; these are called the **transverse directions**.

- **Offset** $A = (A_1, A_2, A_3)$: for each transverse direction n_t, choose one offset $d_t \in \{-1, 0, +1\}$ (27 ways).
- **Order** o: choose the order in which to move in the three transverse directions, as a permutation of three letters (6 ways).

The path $\gamma_{o,A}(b)$ joins the following three segments.

1. From $2y$, following the order o, step by d_t in the transverse direction n_t, one step at a time (no movement in directions with $d_t = 0$). Call the point reached $z = 2y + \sum_t d_t e_{n_t}$.
2. From z, take two steps in direction μ, arriving at $z + 2e_\mu$.
3. Undo the transverse offsets one step at a time in the reverse of the outgoing order, arriving at $2y + 2e_\mu$.

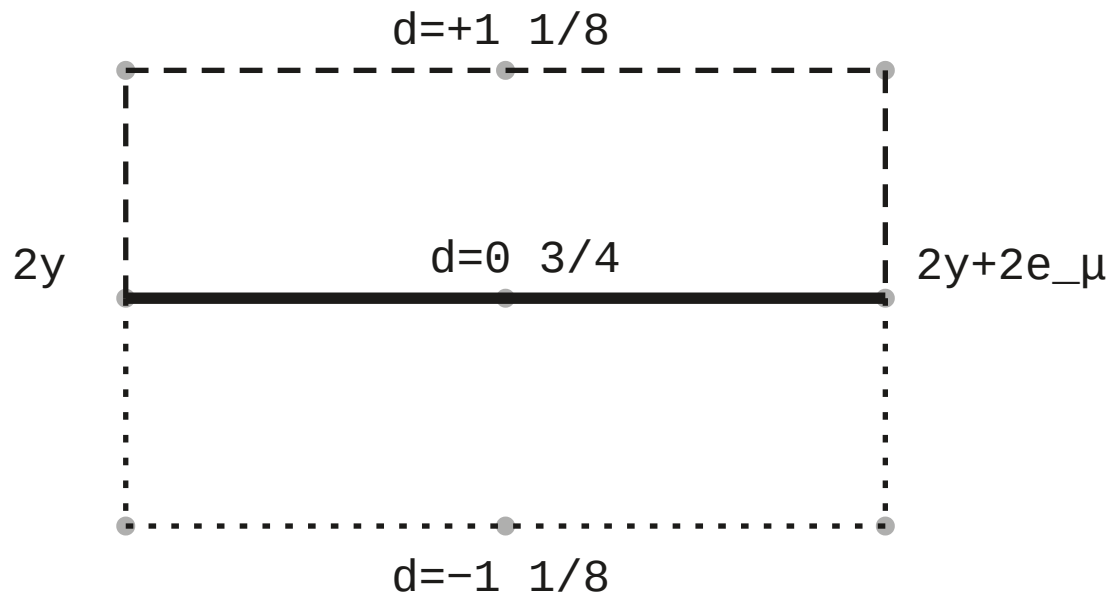
If all transverse offsets are 0, the path is the **straight path** that only takes two steps in μ. The length is $2 + 2\#\{t : d_t \neq 0\}$, at most 8. A step of a path is a pair of a link and its orientation (forward or backward), and a backward step traverses the link from its end point to its start point.

The weight of a path is the product of the per-direction weights $\mathrm{wt}(-1) = \frac{1}{8}$, $\mathrm{wt}(0) = \frac{3}{4}$, $\mathrm{wt}(+1) = \frac{1}{8}$, divided equally among the six orders.

$$w(o, A) = \frac{1}{6} \prod_{t=1}^{3} \mathrm{wt}(d_t), \qquad \sum_o \sum_A w(o, A) = 1. \quad (2)$$

Proved here The weights are non-negative and sum to 1 (the cube of $\sum_d \mathrm{wt}(d) = 1$, divided equally among the six orders). Lean: `ChainBlockMap.sixWeight_total`.

Figure 1 One coarse link ($2y \to 2y + 2e_\mu$) and three paths, drawn for one transverse direction only. There are three transverse directions; the offset is chosen independently in each and the weights are multiplied



↑ transverse n_t → μ (one fine lattice step)

In four dimensions this choice is made independently in three transverse directions, so the weight of the straight path is $(3/4)^3 = 27/64$ Proved here .

Detour: the same path counted twice

May be skipped. There are $6 \times 27 = 162$ indices (o, A), but changing the order of directions with offset 0 does not change the path, so the same path arises from several indices. In Lean this is handled by converting to a finite set of paths with merged weights (the sum of the weights of the indices giving the same path) (`BoxWalk.boxPaths`, `BoxWalk.boxW`). Merging does not change the value of the average. Counting on $\mathbb{Z}^4$, there are $\binom{3}{k} 2^k k!$ paths with k non-zero offsets, $1 + 6 + 24 + 48 = 79$ in all (1, 6, 24 and 48 paths of lengths 2, 4, 6 and 8 respectively) Proved here . On small tori, paths that are distinct on $\mathbb{Z}^4$ can fall onto the same path.

Averaging over the six orders makes the family symmetric under permutations of the three transverse directions. Since the offset weights are symmetric under $d \mapsto -d$, the family is also symmetric under reversal of transverse directions Proved here . The value of a reversed coarse link (from $y + e_\mu$ to y) is read as the conjugate, following the convention of § 02. Forming the average with the family of the same transverse steps, two steps towards $-\mu$ and the same return, and retracting, also gives a value equal to that conjugate, so the convention and the family of paths are compatible (Lean: `BoxWalk.boxMap_rev`).

Lean: paths `WalkCast.walk` (on $\mathbb{Z}^4$), weights `ChainBlockMap.sixWeight`.

04

The non-linear block map B

The product along a path. For a path $\gamma = (s_1, \dots, s_\ell)$ and a configuration U, assign U_s to a forward step s and U_s^* to a backward step, and multiply in the order of the path.

$$U(\gamma) = \prod_{i=1}^{\ell} U_{s_i}^{\pm} \quad \text{(from left to right)}, \qquad U_s^{+} = U_s, \;\; U_s^{-} = U_s^*. \tag{3}$$

The weighted average. The raw value of a coarse link b is the weighted average of path products.

$$q_b(U) = \sum_{o} \sum_{A} w(o, A)\, U\big(\gamma_{o,A}(b)\big) \;\in \mathbb{H}. \tag{4}$$

Proved here If all $|U_\ell| \le 1$, then $|q_b(U)| \le 1$ (the quaternion norm is multiplicative, and the weights are non-negative with sum 1). Lean: `ChainBlockMap.norm_sixAvg_le`.

The operation returning to the ball (retract). Fix a threshold $r_0 > 0$, and set

$$\rho_{r_0}(q) = \frac{q}{\max(|q|,\, r_0)} \tag{5}$$

If $|q| \ge r_0$, then $\rho_{r_0}(q) = q/|q|$ is a unit quaternion (an element of $SU(2)$); if $|q| < r_0$, then $\rho_{r_0}(q) = q/r_0$, with norm $|q|/r_0 < 1$. In either case $|\rho_{r_0}(q)| \le 1$, and ρ_{r_0} is continuous Proved here . For $r_0 = 1$, ρ_1 is the identity on the ball. The Lean definition makes sense for every $r_0 > 0$, but this book takes $0 < r_0 \le 1$ (for $r_0 > 1$, $|q_b| \le 1 < r_0$, so the value always falls inside the ball and never lands in $SU(2)$ Proved here).

Definition (block map). The map $B = B_m : \mathbb{B}^{\{\text{links of } \Lambda_{m+1}\}} \to \mathbb{B}^{\{\text{links of } \Lambda_m\}}$ that assigns to a ball-valued configuration U on the fine lattice the configuration on the coarse lattice

$$B(U)_b = \rho_{r_0}\big(q_b(U)\big) \qquad (b \text{ a coarse link of } \Lambda_m) \tag{6}$$

is called the **non-linear block map** of step m.

Proved here B is continuous. On the "small field" event $|q_b(U)| \ge r_0$, where the average $q_b(U)$ is at or above the threshold, $B(U)_b = q_b(U)/|q_b(U)| \in SU(2)$; below the threshold the value falls inside the ball. Lean: `ChainBlockMap.continuous_blockMap`, `ChainBlockMap.blockMap_of_small_field`.

The Lean definition is `ChainBlockMap.blockMap r₀ hr (BoxWalk.boxPaths L) (BoxWalk.boxW L) m` (`hr : 0 < r₀`). The composed map $B_m \circ B_{m+1} \circ \cdots \circ B_{n-1}$ (from step n to step m) is `ChainBlockMap.down`.

The event below the threshold is a large-field event and is dealt with separately (7-05 § 04, Parts 22 and 28).

05

The path kernel C — the linear part

We look at the form of B when U is near the identity 1. Take a field a of imaginary quaternions on the fine links ($\mathrm{Re}\, a_\ell = 0$; elements of the Lie algebra $\mathfrak{su}(2)$) and a real number ε, and set $U_\ell = \exp(\varepsilon a_\ell) \in SU(2)$ (Lean: `BlockLinearKernel.expCfg`). Expanding the path product in ε, the first-order term is the signed sum of the field along the path. Its weighted average is written

$$(Ca)_b = \sum_o \sum_A w(o,A) \sum_{s \in \gamma_{o,A}(b)} \pm\, a_s \qquad (\text{forward steps } +,\ \text{backward } -) \tag{7}$$

and C is called the **path kernel**. Its Lean name is `BlockLinear.sixSgn`. That for $0 < r_0 \le 1$, $\varepsilon \ge 0$, $|a_\ell| \le M$, $\varepsilon M \le 1/32$, the difference between $B(e^{\varepsilon a})_b$ and $\exp(\varepsilon (Ca)_b)$ is at most $257(\varepsilon M)^2$ is a theorem of Part 15 (15-13, Lean: `BlockLinearKernel.blockMap_exp_linear_le_one`). The retract does not change the linear part.

The form of the coefficients. C is a real linear map acting colour by colour (on the i, j, k components), and can be written with a table of coefficients. On the fine lattice $\Lambda_{m+1} = (\mathbb{Z}/2^{m+1}L)^4$,

$$(Ca)_\mu(X) = \sum_{r \in \Lambda_{m+1}} \sum_{\nu=1}^{4} c(r)_{\mu\nu}\, a_\nu(2X + r), \tag{8}$$

$$c(r)_{\mu\nu} = \sum_o \sum_A w(o,A) \sum_{\substack{s \text{ a step of } \gamma_{o,A} \\ \text{direction } \nu,\ \text{base point} \equiv r}} \pm 1. \tag{9}$$

Here $\gamma_{o,A}$ is the path on $\mathbb{Z}^4$ leaving the origin in direction μ. Lean: the coefficients `LatticeTorus.pathKernel` (in the form with side $2^{m+1}L$), the operator of the form (8) `TorusFourier.Cop` (built from the map $X \mapsto 2X$ and the table of coefficients). The real linear map for a single colour is `PushCov.blockLin`, and that it agrees with `Cop` is the first half of `PushCov.lattice_pushCov`.

Example (constant field). When the field a takes the same value a_ν at every point, a transverse step and its return traverse two links of the same direction with opposite signs and cancel, leaving only the two steps in μ. Hence $(Ca)_\mu = 2a_\mu$. A coarse link is as long as two fine links, so this is the expected form, "proportional to length". Proved here In terms of coefficients, $\sum_r c(r)_{\mu\nu} = 2\delta_{\mu\nu}$. This form of the sum is also the symbol of the path kernel taken at momentum 0 ($k = 0$ in `LatticePath.pathSym_closed`). The contribution of the straight path is $27/64$ each to $c(0)_{\mu\mu}$ and $c(e_\mu)_{\mu\mu}$.

C is only the first order about the identity (for the view of commuting configurations, see 15-05); linearisation about a background field $U \neq 1$ is not defined.

06

The push-forward of measures

For a probability measure μ on configurations of the fine lattice, the **push-forward** $B_*\mu$ is defined by

$$(B_*\mu)(E) = \mu\big(B^{-1}(E)\big) \qquad (E \text{ a measurable set of coarse configurations}) \quad (10)$$

B is continuous and hence measurable, and $B_*\mu$ is a probability measure on coarse configurations Proved here . In other words, it is the distribution of the coarse field $B(U)$ when U follows μ. In Lean it is `Measure.map` (in the form for probability measures, `ProbabilityMeasure.map`).

Multi-step push-forward. We write $\nu_{n,m}$ for the measure μ_n on the fine lattice of step n pushed by the composed map down to step $m \le n$. Pushing one step at a time or all at once gives the same measure: for $m < n$, $(B_m)_*\nu_{n,m+1} = \nu_{n,m}$, and $\nu_{n,n} = \mu_n$ Proved here . Lean: `ChainBlockMap.νP`; the agreement is `ChainBlockMap.hcons_blockMap` and `ChainBlockMap.νP_self`.

The μ_n being pushed is a probability measure on ball-valued configurations of the box Λ_n (the Lean type has the form of a box). The Wilson measure (19-01) and the measure with fermions (19-05) have different lattice types, so what is put in is the measure normalised and re-indexed onto configurations of the box lattice (on the fermion side, `BoxFermion.fermProbBox` in Lean).

The statements to be shown, written with this push-forward measure only, are in 19-07.

Covariance. For linear fields, the covariance of the push-forward is handled in the following form. For a measure μ on a real inner product space E, the **covariance form**

is

$$\operatorname{cov}_\mu(u,v) = \int \langle u,x\rangle\langle v,x\rangle \, d\mu(x) - \int \langle u,x\rangle \, d\mu(x) \int \langle v,x\rangle \, d\mu(x) \quad (11)$$

(Lean: `PushCov.covForm`; it is meaningful when second moments exist). The T with $\operatorname{cov}_\mu(u,v) = \langle u, Tv\rangle$ is called the covariance operator. That pushing by a linear C gives $CTC^\dagger$ is a theorem of 15-13; for the non-linear B it does not take this form.

07

The multi-step family on the free-field side

The "one step of the free field" in Part 15 (15-12) talks about operators obtained by repeating the one-step map $T \mapsto CTC^\dagger$ on the covariance operator of the free field. That this is the covariance of the Gaussian free field pushed by C is in Lean in the one-step, one-colour form (`PushCov.lattice_pushCov`); the multi-step form is a one-line argument repeating the push-forward by a linear map, and is on paper. We set up the objects in two forms, on the lattice side and on the momentum side.

Lattice side. Let the side of the coarse lattice be N, and consider a four-component field $\phi_\nu(X)$ on the torus $(\mathbb{Z}/N)^4$ (the N of this section is the side of the coarse lattice).

- **Free-field covariance** (Feynman gauge) $\mathrm{freeCov}_N$: the inverse of the lattice Laplacian $-\Delta$, taken component by component (0 on the constant component). Lean: `LatticePush.freeCov`.
- **One step**: $T \mapsto CTC^\dagger$ (from operators on $(\mathbb{Z}/2L)^4$ to operators on $(\mathbb{Z}/L)^4$). Lean: `LatticePush.stepOp`.
- **Covariance at step k**: the free-field covariance on $(\mathbb{Z}/2^kN)^4$ pushed down to $(\mathbb{Z}/N)^4$ by k one-step maps, $\mathrm{stageCov}_{k,N}$. Lean: `LatticePush.stageCov`.
- **Plaquette covariance**: the covariance $\mathrm{curl}\circ\mathrm{stageCov}_{k,N}\circ\mathrm{curl}^\dagger$ of the plaquette field $(\mathrm{curl}\,\phi)_{(a,b)}(X) = \big(\phi_a(X+e_b) - \phi_a(X)\big) - \big(\phi_b(X+e_a) - \phi_b(X)\big)$. Lean: `LatticeFace.faceCov`.

Momentum side. Let the lattice momentum be $q = (q_1,\dots,q_4)$, $q_\nu = 2\pi n_\nu/N$.

$$\omega(q) = \sum_\nu 4\sin^2\frac{q_\nu}{2}, \qquad s_\nu(q) = \sin\frac{q_\nu}{2}, \qquad P(q) = 1 - \frac{s(q)s(q)^{\mathsf{T}}}{|s(q)|^2}. \quad (12)$$

$\omega(q)$ is an eigenvalue of $-\Delta$, and $P(q)$ is the projection onto the directions orthogonal to $s(q)$. The symbol of the difference is $1 - e^{iq_\nu} = -2ie^{iq_\nu/2}\sin(q_\nu/2)$, so P is the

projection onto the "physical part" with the gradient (gauge) directions removed Proved here .

The momenta of the fine torus (side $2N$) that fall onto the coarse momentum q under $y \mapsto 2y$ are $r_\zeta(q) = q/2 + \pi\zeta$ ($\zeta \in \{0,1\}^4$, 16 of them); these are called **aliases**. The **one-step map** $\mathcal{M}$ assigns to a family G of 4×4 matrices, one for each momentum,

$$(\mathcal{M}G)(q) = \frac{1}{16} \sum_{\zeta\in\{0,1\}^4} A_\zeta(q)\, G\big(r_\zeta(q)\big)\, A_\zeta(q)^{\mathsf{T}} \tag{13}$$

$A_\zeta(q)$ is a 4×4 real matrix built explicitly from q and ζ. The family at step k is

$$G_0 = P/\omega, \qquad G_k = \mathcal{M}^k(P/\omega). \tag{14}$$

The domain is the momenta with $\omega(q) > 0$ ($\omega = 0$ is the constant component).

Lean holds the momentum as $\theta = q/4$ ($\sin 2\theta_\nu = \sin(q_\nu/2)$). The core names are `OneStepBridge.Mfam` for $\mathcal{M}$ and `TauSplit.Gs` for G_k; other names are in 19-06.

The relation between the two sides: that $\mathrm{stageCov}_{k,N}$ on the lattice side has a symbol of phase form for each momentum (`LatticePush.stageCov_phase`), that the physical part PXP of that symbol equals G_k (`LatticePush.projFam_stage`), and that the plaquette covariance equals the operator built from the momentum side (`LatticeFace.faceCov_eq_posC`) have been checked in Lean, and are used in Part 15 (15-12). What is pushed here is the linear C, not the non-linear B.

08

The weights are a design choice

The B of this chapter was defined with the following choices fixed.

Choice	Value in this chapter	Other possibilities (examples)
Weights per transverse direction $(\alpha, 1-2\alpha, \alpha)$	$\alpha = 1/8$	Other $\alpha \in [0, 1/2]$. $\alpha = 0$ gives only the straight path
Order of transverse moves	Equal average over six orders	Fixed to one order (the symmetry becomes weaker)
Operation returning to the ball	Retract ρ_{r_0}, $0 < r_0 \le 1$	Other thresholds, other ways of projecting

As long as one chooses a family of paths with non-negative weights summing to 1 and a continuous way of returning that maps the ball to the ball, the minimal facts of § 03 to § 06 (values stay in the ball, continuity, the push-forward is a probability measure) stand in the same form Proved here . The threshold-free $q \mapsto q/|q|$ is undefined at $q = 0$

and not continuous, so it does not satisfy this condition. The choice of the width of transverse offsets and of the blocking factor is dealt with in 28-05.

What depends on the choices is the coefficients (9) of the path kernel C, and A_ζ, $\mathcal{M}$, G_k built from them. The concrete constants appearing in the theorems of the second half of Part 15 (the one-step contraction rate, upper bounds on ratios of plaquette covariances, and so on) were computed with the choices of this chapter ($\alpha = 1/8$, factor 2), and cannot be used as they are with other choices. What changes when the weights are changed, and which choice makes the premises easier to satisfy, are left to 28-05 What depends on design choices.

What this chapter can say

Can say	Cannot say
Proved here The weights are non-negative and sum to 1. Lean: `ChainBlockMap.sixWeight_total`	That these weights are better than others (28-05)
Proved here B is a continuous map between ball-valued configurations, and takes values in $SU(2)$ on the event at or above the threshold. Lean: `ChainBlockMap.continuous_blockMap`, `ChainBlockMap.blockMap_of_small_field`	That B maps $SU(2)$ configurations to $SU(2)$ configurations. An estimate of the probability of the event below the threshold
Proved here The convention of reading reversed links as conjugates is compatible with the family of paths. Lean: `BoxWalk.boxMap_rev`	All forms of compatibility with local gauge transformations
Proved here The multi-step push-forward agrees with step-by-step push-forwards ($m < n$). Lean: `ChainBlockMap.hcons_blockMap`	Decay and uniformity of the push-forward measure (19-07, Part 28)
Proved here For a constant field, $(Ca)_\mu = 2a_\mu$	Linearisation about a background field $U \neq 1$

Sources and reproduction

Item	Kind	Source / tools
The one-step transformation; that averages leave the group for gauge fields	Cited	The sources of 7-05 § 05 (T. Bałaban, Comm. Math. Phys. 98 (1985) 17–51, bibliographic data only; J. Dimock, arXiv:1108.1335)
The objects of § 02 to § 07	Lean	Lean definitions from this book's investigation (names in the text and in 19-06). Theorems in 15-12 and 15-13
The number of paths 79, 27/64, the sum of coefficients	Proved here	Counted by hand (formulas in the detour of the text). A script enumerating the paths on $\mathbb{Z}^4$ from the definition gives the same numbers

Words used in this chapter — block map, family of paths, transverse direction, retract, threshold, small-field event, path kernel, push-forward, covariance form, alias, one-step map, covariance at a step, plaquette covariance. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-05 The fermion weight — defines one of the measures to be pushed: the measure whose weight is the determinant of the Wilson–Dirac operator.

The fermion weight — the Wilson–Dirac operator and the measure with a determinant

This chapter is this book's own investigation. § 01 is established knowledge.

What this chapter shows — On the lattice of the time circle $\mathbb{Z}/2h$ and the spatial torus $(\mathbb{Z}/N)^3$, we define the Wilson–Dirac operator $D_m(U)$ and set up the measure $\mu^{\mathrm{wf}}_{\beta,m,j}$ obtained by multiplying the gauge weight $e^{-\beta S_W}$ by $\mathrm{Re}\det D_m(U)^{2j}$. The theorems on the vessel with fermions in Part 20 (20-05) talk about this measure.

Prerequisite chapters — 19-01 Plaquettes and holonomy, the Wilson measure, the Lean conventions, 5-01, the Wilson action.

Said first — This is a chapter of definitions. Fermions appear only in the form of the determinant left after integrating them as Grassmann variables; the Grassmann integral itself is not defined. The mass m and the coupling β are bare parameters on the lattice, and their meaning in the continuum limit (mass renormalisation, chiral symmetry) is not dealt with.

The order of this chapter

1. Why a determinant weight — integrating out fermions leaves a determinant for each gauge configuration
2. The lattice and physical links — a time circle $2h$ and space N^3; the shape of this chapter's lattice
3. The convention for the gauge action — $S_W = \sum_p (1 - \frac{1}{2}\,\mathrm{Re}\,\mathrm{tr}\,U_p)$, and its relation to the weight of 19-01
4. The Wilson–Dirac operator — $r = 1$, antiperiodic in all four directions, the representation of γ
5. The determinant weight — $\mathrm{Re}\det D^{2j}$ is non-negative
6. The measure with fermions — the unnormalised measure, and the probability measure μ^{wf}
7. The range of the mass — the definitions for all m, the reflection-positivity theorem for $m > -1$

8. The relation to the conventions for gauge-field measures — five differences, and the return to the Wilson measure at $j = 0$

01

Why a determinant weight

A quark field on the lattice is represented by Grassmann variables $\psi, \bar{\psi}$ carrying at each point spin (4 components) and colour (2 components for $SU(2)$) Cited . If the action is a quadratic form $\bar{\psi} D(U) \psi$, the integral over the Grassmann variables gives, for each configuration U, the determinant $\det D(U)$. With n_f species (flavours) it is $\det D(U)^{n_f}$ Cited .

The measure after integrating out the fermions in this way is a measure on gauge configurations, with weight $e^{-\beta S_W(U)} \det D(U)^{n_f}$. The determinant can be complex, so making it a probability measure requires the weight to be a non-negative real number. In this chapter the number of flavours is taken even, $n_f = 2j$, and the real part of the weight, $\mathrm{Re} \det D^{2j}$, is used (§ 05 shows that it is non-negative).

The naive difference form of the Dirac operator on the lattice produces extra particles (doublers). Adding a second-order difference term, named after Wilson, makes them heavy Cited . That operator is D. Reflection positivity for lattice gauge theory with fermions was treated by Osterwalder–Seiler Cited (5-03, reflection positivity of the Wilson action).

02

The lattice and physical links

The lattice of this chapter, with time circle of length $2h$ ($h \geq 1$) and spatial side $N \geq 1$, is

$$\Lambda_{N,h}^{\mathrm{wf}} = (\mathbb{Z}/2h) \times (\mathbb{Z}/N)^3 \quad (1)$$

Points are written (t, x), $t \in \mathbb{Z}/2h$, $x \in (\mathbb{Z}/N)^3$. Links are (t, x, μ): for $\mu = 1, 2, 3$, the space link $x \to x + e_\mu$ within the slice at time t; for $\mu = 4$, the time link $(t, x) \to (t + 1, x)$ ($t + 1$ modulo $2h$). These are called **physical links**, and their set is written $\mathcal{L}_{N,h}$.

A configuration is $U : \mathcal{L}_{N,h} \to SU(2)$, with $SU(2)$ the unit quaternions (19-01). The reference measure on the space of configurations is the product $\prod_\ell dU_\ell$ of the Haar probability measures of the links (19-01).

The time direction is given even length $2h$ so that the planes of the reflection $t \mapsto -t$ are the two planes $t = 0$ and $t = h$ (19-03). Taking $2h = N$ gives a torus of the same size in all four directions.

Lean: the type of physical links `FermionSU2Config.PhysLink N hT` (`Fin (2 * hT) × (Fin 3 → Fin N) × Fin 4`; Lean's direction numbers 0, 1, 2 are space and 3 is time), $SU(2)$ is `FermionSU2Config.SU2` (the unit sphere of the quaternions), the reference measure is `Measure.pi fun _ => Measure.haarMeasure ⊤`, and the next time $t \mapsto t + 1 \bmod 2h$ is `FermionSliceBlock.sc`.

03

The convention for the gauge action

The Wilson action of this chapter, counting each unoriented plaquette once, is

$$SW(U) = \sum_{p} \Big(1 - \tfrac{1}{2} \operatorname{Re}\operatorname{tr} U_p\Big), \qquad U_p = U_\mu(z)\, U_\nu(z+e_\mu)\, U_\mu(z+e_\nu)^\dagger\, U_\nu(z)^\dagger \tag{2}$$

and the gauge weight is $e^{-\beta SW(U)}$ ($\beta \in \mathbb{R}$). The plaquettes are the space plaquettes (μ, ν) ($\mu < \nu \le 3$) within each time slice, and the time plaquettes $(\mu, 4)$ joining the slices at times t and $t+1$. The trace is taken by representing quaternions as 2×2 matrices, $q = a + bi + cj + dk \mapsto \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$.

Proved here This representation preserves products, and $\frac{1}{2}\operatorname{tr}$ is the real part of the quaternion. Hence $\frac{1}{2}\operatorname{Re}\operatorname{tr} U_p = \operatorname{Re} U_p$, and with $|\mathcal{P}|$ the number of plaquettes,

$$e^{-\beta SW(U)} = e^{-\beta|\mathcal{P}|}\, e^{\beta \sum_p \operatorname{Re} U_p}. \tag{3}$$

The constant $e^{-\beta|\mathcal{P}|}$ disappears on normalisation, so the normalised measure is the same as under the convention $e^{\beta\sum_p \operatorname{Re} U_p}$ of the Wilson measure of 19-01.

Lean: the action `WilsonGauge.wilsonAction N hT`, the per-plaquette quantity `WilsonGauge.plaqE` ($1 - \frac{1}{2}\operatorname{Re}\operatorname{tr}(ABC^\dagger D^\dagger)$), the quaternion matrix `FermionSU2Config.quatMat`, and links as matrices on the configuration space `WilsonGauge.physSp` (space) and `WilsonGauge.physT` (time).

04

The Wilson–Dirac operator

The fermion field $\psi(t,x)$ is a complex vector carrying at each point 4 spin components and 2 colour components. The operator $D_m(U)$ is a square matrix on $\mathbb{C}^{2h\cdot N^3\cdot 4\cdot 2}$, and

we take Wilson's form with $r = 1$,

$$(D_m(U)\psi)(z) = (m+4)\,\psi(z) - \frac{1}{2}\sum_{\mu=1}^{4}\Big[(1-\gamma_\mu)\,U_\mu(z)\,\psi(z+e_\mu) + (1+\gamma_\mu)\,U_\mu(z-e_\mu)^\dagger\,\psi($$

$U_\mu(z)$ acts on the colour index as a 2×2 matrix, and γ_μ acts on the spin index. $m \in \mathbb{R}$ is the bare mass.

The γ matrices. The spin index is written as a pair $(s, r) \in \{0,1\}^2$. s is ordered by the eigenvalues of γ_4, $+1$ ($s=1$) then -1 ($s=0$), and with the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma 2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma 3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \quad (i = 1,2,3) \tag{5}$$

(blocks in the order $s = 1, 0$, each block 2×2 in r). This is the arrangement exactly as in the Lean definition. They satisfy $\gamma_\mu^2 = 1$, $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ ($\mu \neq \nu$) Proved here .

Boundary conditions. Antiperiodic in all four directions. In space direction i, on the links leaving $x_i = N-1$, and in the time direction, on the link from $t = 2h-1$ to 0, the hopping term is multiplied by the sign -1. The gauge link variables themselves are periodic; the sign is attached to the hopping of the operator.

Lean: the operator `WilsonGauge.physDirac N hT m ω`. Its content is the per-time-slice form `FermionSliceBlock.std4` (the operator (4) with $r = 1$; within a slice, `FermionSliceBlock.stdSlice`), with spatial hopping `FermionSliceBlock.torusX` (boundary signs in `FermionSliceBlock.torusLink`) and temporal hopping (boundary sign `FermionSliceBlock.eps`) inserted. γ is `FermionSliceBlock.gammaT` (γ_4) and `FermionSliceBlock.gammaSp` (γ_i).

The component formula is in Lean: `WilsonGauge.physDirac_apply` says that each component of physDirac has the form (4).

Reading On random configurations, with $m \in \{0.3, -0.7, -4.0\}$ for $2^3 \times 2$ and $m \in \{0.3, -0.7\}$ for $2^3 \times 4$, a literal transcription of `physDirac` was compared with an operator written independently from (4) in another representation of γ (mapped by the change-of-representation matrix), and the component differences were at most 3.6×10^{-15} (numerical record of this book's investigation. Removing the boundary sign in time makes the difference 2.0, which confirms that the comparison is effective).

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05
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The determinant weight

With $2j$ flavours ($j = 0, 1, 2, \ldots$), the fermion weight is

$$F_{m,j}(U) = \mathrm{Re}\big(\det D_m(U)\big)^{2j} \quad (6)$$

For $j = 0$, $F \equiv 1$ Proved here .

Proved here For all $m \in \mathbb{R}$, all j and all configurations, the imaginary part of $(\det D_m(U))^{2j}$ is 0 and its real part is at least 0. Lean: `FermionSliceReflection.wilson_weight_nonneg_std4` (the torus form `FermionSliceReflection.wilson_weight_nonneg_torus`); the form on configurations is `WilsonGauge.fermWeight_nonneg`. Hence (6) is a non-negative real number even before taking the real part, and Re is a matter of notation.

$F_{m,j}$ is a continuous function of the configuration (since the matrix entries are polynomials in the components of U) Proved here . Lean's `WilsonGauge.continuous_fermWeight` is continuity on the split coordinates (next paragraph); continuity on configurations is read together with the continuity of the splitting map. Invariance of the weight under global gauge transformations and under lattice symmetries are theorems of Part 20 (20-05).

Lean's `WilsonGauge.fermWeight` writes (6) on coordinates that split a configuration at the reflection plane into "plane, positive side, negative side", and is the same as the form written with the operator on unsplit configurations (`WilsonGauge.physDirac_eq_split`). The splitting is matched to the definition of the positive side in 19-03.

06

The measure with fermions

With the product of the gauge weight and the fermion weight

$$W_{\beta,m,j}(U) = e^{-\beta S_W(U)}\, F_{m,j}(U) \;\geq 0 \quad (7)$$

as density, we set the measure on the space of configurations to be

$$\mu^{\mathrm{wf},0}_{\beta,m,j}(dU) = W_{\beta,m,j}(U) \prod_{\ell \in \mathcal{L}_{N,h}} dU_\ell, \qquad Z^{\mathrm{wf}}_{\beta,m,j} = \int W_{\beta,m,j}(U) \prod_{\ell} dU_\ell \quad (8)$$

(unnormalised). The partition function Z^{wf} is positive for all β, m, j. This is a theorem of Part 20 (20-05, Lean: `WilsonGauge.partition_pos_wilson_full`; the form with density W is `WilsonGauge.wf_partition_pos`). So

$$\mu^{\mathrm{wf}}_{\beta,m,j} = \left(Z^{\mathrm{wf}}_{\beta,m,j}\right)^{-1} \mu^{\mathrm{wf},0}_{\beta,m,j} \quad (9)$$

is called the **Wilson–fermion measure**. It is a probability measure Proved here .

Lean: the density `WilsonGauge.wfWeight N hT j m β`, the unnormalised measure `WilsonGauge.wfMeasure` (the density enters through the truncation `toNNReal` to non-negative reals; since (7) is non-negative the value does not change: `WilsonGauge.wfWeight_nonneg`), the normalised measure `WilsonGauge.wfProb N hT j m β`, and that it is a probability measure is `WilsonGauge.wfProb_isProbability` ($h \geq 1$).

This measure is also one of the measures to be pushed by the block map (19-04 § 06).

07 The range of the mass

The definitions (4) to (9) make sense for every real m. Non-negativity of the weight and $Z^{\mathrm{wf}} > 0$ also hold for every m. The range narrows on the side of the theorems.

Object or property	Range of the mass m	Where
The operator D_m, the weight $F_{m,j} \geq 0$, the measure μ^{wf}	All $m \in \mathbb{R}$	This chapter
$Z^{\mathrm{wf}} > 0$, invariance under lattice symmetries	All m	20-05
Reflection positivity through lattice-point time planes	$m > -1$	20-05

$m > -1$ includes $m = 0$ and small negative masses. Reflection positivity for $m \leq -1$ is outside the Lean statements of this book. m is a bare parameter, and its relation to the physical mass (tuning to the critical mass) is not defined in this chapter.

08 The relation to the conventions for gauge-field measures

We summarise the differences between the measure of the other chapters of Part 19 (19-01) and the measure of this chapter.

Item	The Wilson measure of 19-01	The measure of this chapter
Lattice	$(\mathbb{Z}/N)^4$, or the torus with transverse size N and longitudinal size M	$(\mathbb{Z}/2h) \times (\mathbb{Z}/N)^3$
Gauge weight	$e^{\beta \sum_p \mathrm{Re}\, U_p}$	$e^{-\beta S_W}$. Differs only by a constant multiple (equation (3))
Haar measure	On the strong-coupling side, the product of the Haar measure of the sphere pushed forward to the closed ball (19-01, the Lean conventions)	The product of Haar probability measures on the unit sphere of $SU(2)$

Item	The Wilson measure of 19-01	The measure of this chapter
Normalisation	Unnormalised $\tilde{\mu}_{N,\beta}$ (`TorusRP.wilsonMeasure`) and normalised $\mu_{N,\beta}$ (`TorusRP.wilsonProb`). The vortex side is written with ratios of partition functions	Both the unnormalised (8) and the normalised (9) are set up
Boundary	Gauge links periodic (twists in 19-02)	Gauge links periodic, fermions antiperiodic in all four directions

Proved here For $j = 0$, $F \equiv 1$, so $\mu^{\mathrm{wf}}_{\beta,m,0}$ is the normalised Wilson measure on the lattice $\Lambda^{\mathrm{wf}}_{N,h}$, and does not depend on m.

The declarations of this chapter are in the namespaces `WilsonGauge`, `FermionSliceBlock` and `FermionSU2Config`, and are definitions separate from `TorusRP.wilsonMeasure` and `TwistedWilson.ZW` of 19-01. When all four directions have the same size (spatial side $N = 2h$, $2h > 1$), it is shown in Lean that pushing forward the pure-gauge measure `WilsonGauge.gaugeMeasure` (a definition without j, whose density is the same as $\mu^{\mathrm{wf},0}$ at $j = 0$ Proved here) by the re-indexing gives a constant multiple of the Wilson measure of the strong-coupling side (`WilsonGauge.wilsonMeasure_eq_map`, 19-01, bridges). There is no theorem connecting the twisted vortex side with the measure of this chapter.

What this chapter can say

Can say	Cannot say
Proved here $\frac{1}{2}\operatorname{Re}\operatorname{tr} U_p = \operatorname{Re} U_p$, and the gauge weight differs from the convention of 19-01 only by a constant multiple (the Lean bridge for $N = 2h$ is in 19-01)	Connecting by Lean theorems to lattices with $N \neq 2h$ or to twisted measures
Proved here For all m and j, $(\det D_m)^{2j}$ is a non-negative real number. Lean: `FermionSliceReflection.wilson_weight_nonneg_std4`	The sign of $\det D_m$ itself (before raising to the power $2j$)
Proved here The normalised μ^{wf} is a probability measure ($Z > 0$ is in 20-05). Lean: `WilsonGauge.wfProb_isProbability`	An odd number of flavours; the Grassmann integral itself
Reading Difference between `physDirac` and an independent transcription of (4) $\leq 3.6 \times 10^{-15}$ (small lattices)	The numerical comparison is for small lattices only. The component formula for general N, h is in Lean (`WilsonGauge.physDirac_apply`)
Proved here At $j = 0$ it returns to the Wilson measure	Mass renormalisation, chiral symmetry, the continuum limit

Sources and reproduction

Item	Kind	Source / tools
The Wilson action	Cited	K. G. Wilson, Phys. Rev. D 10 (1974) 2445 (bibliography of the present book). 5-01
The Wilson term (the second-order difference that makes doublers heavy)	Cited	Bibliographic data only (unverified)
Reflection positivity of lattice gauge theory with fermions	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (bibliographic data only). 5-03
That the Grassmann integral gives a determinant; doublers	Cited	Standard facts of lattice field theory. Textbook bibliographic data unverified
Lattice, action, operator, weight, measure (§ 02 to § 06)	Lean	Lean definitions from this book's investigation (declarations of `WilsonGauge`, `FermionSliceBlock`, `FermionSU2Config`, `FermionSliceReflection`; names in the text). Table of correspondence in 19-06
Numerical cross-check of the operator (§ 04)	Reading	Numerical record of this book's investigation: comparison of the operator transcribed from the Lean text with the operator written independently from (4) (removing the boundary sign in time gives a difference of 2.0, confirming that the comparison is effective)

Words used in this chapter — Wilson–Dirac operator, Wilson term, antiperiodic, physical link, bare mass, flavour, fermion weight, Wilson–fermion measure. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-06 Objects and their Lean definitions — a table of correspondence — makes every object of Part 19 searchable in tables of symbol, meaning, Lean name and the parts that use it.

Objects and their Lean definitions — a table of correspondence

> This chapter is an index.

What this chapter shows — The objects of Part 19 can be looked up from the book's notation to the names of Lean declarations, and from Lean names to the book's notation. Each row gives the anchor of the chapter where the definition is placed, and the parts that state theorems about the object.

Prerequisite chapters — None. Follow the anchor of each row to the defining chapter.

Said first — These tables list only the correspondence of definitions. The names of theorems are in the chapters of each part. Lean names are declaration names; locations (files) are not listed. Where the same object is defined separately in several lines (four: strong coupling, vortices, fermions, ultraviolet), the rows are split by line (the differences in convention are in 19-01 § 07).

The order of this chapter

1. How to read these tables — the meaning of the columns, and namespaces
2. Lattice, links, measures (19-01)
3. Twist, the vortex ratio, Polyakov loops (19-02)
4. Reflection, the vessel, infinite volume (19-03)
5. The block map and the multi-step family (19-04)
6. The fermion weight (19-05)
7. Correspondence between lines (bridges) — identifications closed in Lean
8. List of namespaces — finding the line from the head of a Lean name

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01

How to read these tables

The columns of each table are "Symbol", "Meaning", "Lean declaration", "Defining chapter" and "Parts that use it". Symbols are those of the text; Lean declarations are written in the form `Namespace.name`. Arguments (`N`, `M`, `β` and so on) are omitted in places. "Parts that use it" are the parts that state theorems or premises about the object.

The Lean bundle is written in the four lines of 19-01, the Lean conventions (strong-coupling side, vortex side, fermion side, ultraviolet side). The convention for numbering directions is in 19-01 § 01 (20-01 to 20-05 use 0 to 3 with 0 as time; Parts 21 and 22 and 20-06 use 1 to 4). The lattice on the ultraviolet side is a box of side $2^m L$, and link values are in the closed unit ball (19-04 § 02). Even for the same mathematical object, different lines give different declarations. The bridges connecting the lines in Lean are gathered in 19-01, bridges.

The letter m means different things in different chapters: in 19-02 the side of a vortex block, in 19-04 the step number (the m of Lean's `stageCov m L` is also a step), and in 19-05 the bare mass.

02

Lattice, links, measures (19-01)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
Links of $(\mathbb{Z}/N)^4$	Links of the torus of the same size in all four directions (point × direction)	`TorusRP.Lk`	19-01 #lattice	20
$(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$	Points of the torus with transverse size N and longitudinal size M	`TwistedWilson.Site`	19-01 #lattice, 19-02	20, 21
$U_\ell \in SU(2)$	Link variables. Unit quaternions	`OSGap.SU2`, `TwistedWilson.SU2`, `FermionSU2Config.SU2`	19-01 #su2	15, 20, 21, 22
$\{\lvert q\rvert \le 1\}$	The closed unit ball of quaternions (link values on the strong-coupling and ultraviolet sides)	`TorusRP.QBall`, `ChainBlockMap.Ball`	19-01 #conventions	15, 20

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
dU	The normalised Haar measure and its product	`OSGap.haarProb`, `TwistedWilson.haarProb`, `TorusRP.haarBall`, `TwistedWilson.haarCfg`	19-01 #haar	15, 20, 21, 22
U_p, Re U_p	Plaquette holonomy and its real part	`TorusRP.hol`, `TwistedWilson.hol`	19-01 #plaquette	15, 20, 21, 22
$\sum_p$ Re U_p	The action on the strong-coupling side (positive sign)	`TorusRP.action`	19-01 #wilson-measure	20, 22
$\tilde{\mu}_{N,\beta}$, $\mu_{N,\beta}$	The Wilson measure (unnormalised $\tilde{\mu}$) and the normalised μ. $\tilde{\mu}_{N,\beta} = Z_N(\beta)\mu_{N,\beta}$	`TorusRP.wilsonMeasure`, `TorusRP.wilsonProb`	19-01 #wilson-measure	20, 22
$Z_N(\beta)$	The partition function (total mass of $\tilde{\mu}_{N,\beta}$)	The value of `TorusRP.wilsonMeasure` on the whole space	19-01 #partition	20, 22

03

Twist, the vortex ratio, Polyakov loops (19-02)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
n_{12}, n_{34}, z_p	Presence of twists, and the sign of each plaquette	`TwistedWilson.zt`	19-02 #twist	20, 21
Z_0, Z_1	Partition functions under the background $n_{12} = 1$ ($n_{34} = 0, 1$)	`TwistedWilson.ZW`	19-02 #twist	20, 21
θ_M	The vortex ratio $(Z_0 - Z_1)/(Z_0 + Z_1)$	`TwistBlocks.thetaM`	19-02 #theta	20, 21
$P_b(t)$	The Polyakov loop once around direction 3, and the expectation of a pair	`HalfSplit.polyLoop`, `PolyakovT.polyExp`	19-02 #polyakov	21
$S_{\mathrm{v}}(N, m)$, S_N	The infimum of the vortex action, and its infimum over m (the minimal action of a vortex)	`TwistBlocks.Svortex`, `TwistBlocks.SN`	19-02 #vortex-action	21

04

Reflection, the vessel, infinite volume (19-03)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
Θ	Reflection (lattice-point planes, middle planes). The general form written as a pair of a map on link indices and a map on values	`LinkReflection.linkRefl`; concrete reflections `TorusRP.thL`, `TorusRP.sig` (lattice-point planes), `MidRP.thM` (middle planes), `HalfSplit.Th` (vortex side)	19-03 #reflection	20, 22
Positive side	Links on the positive side of the reflection (lattice-point planes, middle planes), and links within the mirror plane	`TorusRP.posP`, `TorusRP.fixP`, `MidRP.posM`	19-03 #positive-side	20
Pairing	A bilinear form of the form $\int F(\Theta\omega)G(\omega)\,d\mu$	`OrbitGrowth.pairingForm` (the field `OSGap.Datum.B` of the vessel; the vessel built from a measure, `OSGap.measureDatum`)	19-03 #pairing	20
T, Ω, P_Ω	The vessel of reconstruction (pairing form, time translation, constant), and the transfer operator, vacuum and projection onto the vacuum	`OSGap.Datum` (fields `B`, `τ`, `one`); objects built: `OSGap.Datum.Hilb`, `OSGap.Datum.T`, `OSGap.Datum.Ω`; projection `OSGap.vacProj`	19-03 #datum	20
Mass gap m	$\|T - P_\Omega\| \le e^{-m}$	`OSGap.Datum.HasMassGap`	19-03 #mass-gap-def	20, 28
ν_β	The periodic lift to $\mathbb{Z}^4$ of the normalised measure on $(\mathbb{Z}/2(n+1))^4$, and its limit	`WilsonLimit.liftProb`	19-03 #infinite-volume-space	20

05

The block map and the multi-step family (19-04)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
Coarse link $b = (y, \mu)$ of Λ_m	Links of the torus of side $2^m L$. To the fine lattice by $y \mapsto 2y$	`BoxWalk.BoxLk`, `BoxWalk.double`, `BoxWalk.base`	19-04 § 02	15

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
$\gamma_{o,A}(b)$	The family of paths (order o, transverse offset A)	`WalkCast.walk`, `BoxWalk.boxPathsIdx`, `BoxWalk.boxPaths`	19-04 #paths	15
$w(o,A)$	Path weights $\frac{1}{6}\prod_t \mathrm{wt}(d_t)$, $\mathrm{wt} = (1/8, 3/4, 1/8)$	`ChainBlockMap.wt`, `ChainBlockMap.sixWeight`, `BoxWalk.boxW`	19-04 #paths, #weights	15, 28
$U(\gamma)$, $q_b(U)$	The product along a path, and its weighted average	`ChainBlockMap.pathProd`, `ChainBlockMap.sixAvg`, `ChainBlockMap.coarseAvg`	19-04 #block-map	15
ρ_{r_0}	The retract with threshold r_0, $q/\max(\lvert q\rvert, r_0)$	`ChainBlockMap.retract`	19-04 #block-map	15
B	The non-linear block map, and the composed map over steps	`ChainBlockMap.blockMap`, `ChainBlockMap.down`	19-04 #block-map	15, 28
C	The path kernel (the linear part of B about the identity), and its coefficients and operator	`BlockLinear.sixSgn`, `LatticeTorus.pathKernel`, `TorusFourier.Cop`, `PushCov.blockLin`	19-04 #path-kernel	15
$e^{\varepsilon a}$	The exponential configuration of a field of imaginary quaternions	`BlockLinearKernel.expCfg`	19-04 #path-kernel	15
$B_*\mu$, $\nu_{n,m}$	The push-forward measure, and the multi-step push-forward	`Measure.map`, `ChainBlockMap.vP`	19-04 #pushforward	15, 28
cov_μ	The covariance form	`PushCov.covForm`	19-04 #pushforward	15
$\mathrm{freeCov}_N$, $\mathrm{stageCov}_{k,N}$	The covariance of the free field (Feynman gauge), and the covariance at a step, pushed k times by C	`LatticePush.freeCov`, `LatticePush.stepOp`, `LatticePush.pushOp`, `LatticePush.stageCov`	19-04 #family	15
curl, plaquette covariance	The plaquette field, and its covariance	`LatticeFace.curl`, `LatticeFace.faceCov`	19-04 #family	15

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
$\omega(q)$, $P(q)$	$\sum_\nu 4\sin^2(q_\nu/2)$, and the projection onto the physical part	`OneStepET.omF`, `OneStepBridge.Pfam`	19-04 #family	15
Momenta with $\omega > 0$, $r_\zeta(q)$	The domain (coordinates $\theta = q/4$) and the 16 aliases	`OneStepFinal.Qpos`, `OneStepBridge.thR`, `OneStepFinal.rqPos`	19-04 #family	15
$\mathcal{M}$, A_ζ	The one-step map for each momentum, and its matrices	`OneStepBridge.Mfam`, `OneStepBridge.AM`	19-04 #family	15
G_k	$\mathcal{M}^k(P/\omega)$	`TauSplit.Gs`, `TorusLam.Gst`	19-04 #family	15

06

The fermion weight (19-05)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
$\mathcal{L}_{N,h}$	Physical links of time circle $2h \times$ space $(\mathbb{Z}/N)^3$	`FermionSU2Config.PhysLink`	19-05 § 02	20
SW	$\sum_p(1 - \frac{1}{2}\operatorname{Re}\operatorname{tr}U_p)$ (negative-sign convention)	`WilsonGauge.wilsonAction`, `WilsonGauge.plaqE`	19-05 § 03	20
$D_m(U)$	The Wilson–Dirac operator ($r = 1$, antiperiodic in all four directions)	`WilsonGauge.physDirac`, `FermionSliceBlock.std4`	19-05 #wilson-dirac	20
γ_μ	γ_4 and γ_i	`FermionSliceBlock.gammaT`, `FermionSliceBlock.gammaSp`	19-05 #wilson-dirac	20
$F_{m,j}$	$\operatorname{Re}(\det D_m)^{2j}$	`WilsonGauge.fermWeight`	19-05 § 05	20
$W_{\beta,m,j}$, μ^{wf}	The density, and the Wilson–fermion measure (unnormalised, normalised)	`WilsonGauge.wfWeight`, `WilsonGauge.wfMeasure`, `WilsonGauge.wfProb`	19-05 #fermion-measure	15, 20
Pure-gauge measure	The measure with density $e^{-\beta SW}$ (a definition without j, $N = 2h$)	`WilsonGauge.gaugeMeasure`	19-01 (bridges), 19-05 § 08	20

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
Θ (fermion side)	The action of the time reflection $t \mapsto -t$ on configurations	`WilsonGauge.thetaPhys`	19-03 #reflection	20, 22

07

Correspondence between lines (bridges)

Declarations showing in Lean that two definitions from different lines are the same object. The explanation is in 19-01, bridges.

What it connects	What it says	Lean declaration	Range
Strong-coupling side ↔ vortex side (measures)	$\tilde{\mu}_{N,\beta}$ (`TorusRP.wilsonMeasure N β`) equals the measure given by the product of Haar measures on the vortex side times the weight, transported by a one-to-one correspondence given by definition. No constant factor	`WilsonBridge.wilsonMeasure_eq_map`	Only for $M = N$ with no twist. There is no bridge for twisted measures (background $n_{12} = 1$)
Same (expectations, total mass)	The expectations of measurable observables, and the total mass $Z_N(\beta)$, agree on the two sides	`WilsonBridge.integral_wilsonMeasure`, `WilsonBridge.wilsonMeasure_univ`	
Same (correspondence aligning time)	Under the correspondence that overlays time on the strong-coupling side (index 0) with index 3 on the vortex side, the measures also agree, and the reflections through lattice-point planes on the two sides become the same map	`WilsonBridgeT.wilsonMeasure_eq_mapT`, `WilsonBridgeT.refl_toBallT`	
Same (vortex ratio)	Writes the vortex ratio with no background twist ($n_{12} = 0$) as an expectation under the probability measure of the strong-coupling side. It is not the book's θ_M ($n_{12} = 1$)	`WilsonBridgeV.vortex_ratio_eq`	

What it connects	What it says	Lean declaration	Range
Strong-coupling side ↔ fermion side	$\tilde{\mu}_{2h,\beta}$ is $e^{6(2h)^4\beta}$ times the pure-gauge measure `WilsonGauge.gaugeMeasure` transported by re-indexing	`WilsonGauge.wilsonMeasure_eq_map`	Spatial side $N = 2h$, $2h > 1$

Declarations connecting the ultraviolet side with the other lines by a single theorem are not included in this table (for re-indexing of measure types, see 19-04 § 06).

08

List of namespaces

From the head of a Lean name (its namespace), one can find which line the declaration belongs to.

Namespace	Line	Main objects
`TorusRP`, `OSGap`, `LinkReflection`, `MidRP`, `OrbitGrowth`, `WilsonLimit`	Strong-coupling side	The Wilson measure on $(\mathbb{Z}/N)^4$, the vessel, infinite volume
`TwistedWilson`, `TwistBlocks`, `HalfSplit`, `PolyakovT`	Vortex side	Twisted tori, θ_M, vortex action, Polyakov loops
`WilsonBridge`, `WilsonBridgeT`, `WilsonBridgeV`	Bridges between the strong-coupling and vortex sides	Agreement of measures for $M = N$ with no twist
`WalkCast`, `ChainBlockMap`, `BoxWalk`	Ultraviolet side (non-linear)	The family of paths, the block map, push-forward
`BlockLinear`, `BlockLinearKernel`, `LatticeTorus`, `TorusFourier`, `PushCov`	Ultraviolet side (linear part)	The path kernel, the covariance form
`LatticePush`, `LatticeFace`, `OneStepBridge`, `OneStepET`, `OneStepFinal`, `TauSplit`, `TorusLam`	Ultraviolet side (free field)	Covariances at steps, families over momenta
`WilsonGauge`, `FermionSliceBlock`, `FermionSU2Config`, `FermionSliceReflection`	Fermion side	Physical links, the Wilson–Dirac operator, the Wilson–fermion measure

What this chapter can say

Can say	Cannot say
Proved here The correspondence between the objects of Part 19 and the names of Lean declarations (with anchors to the defining chapters)	Identification between lines only within the range of the bridges of § 07 (there is no bridge for twisted measures)

Sources and reproduction

Item	Kind	Source / tools
Names of Lean declarations and namespaces	Lean	Copied by reading the source of the Lean bundle of this book's investigation. The definition and explanation of each object are in 19-01 to 19-05

Words used in this chapter — object, declaration, namespace, line, bridge. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-07 The statements to be shown — a list of the statements to be shown, written with only the objects defined here.

The statements to be shown — written with defined objects only

> This chapter is this book's own investigation. The reading of the problem statement in § 01 and § 08 arranges established knowledge.

What this chapter shows — Statements of "what would have to be shown" towards the continuum limit and the mass gap, written with only the objects defined in Part 19. First we fix how the sequence β_n of bare couplings is chosen (matching), and then write five statements about that sequence: convergence of the full sequence, non-degeneracy, equicontinuity, the mass gap (uniform decay), and the band. Each can be written as a one-line formula, and none uses undefined quantities (effective couplings, activities and so on). Finally, we tabulate the gap to the problem statement and the nearby forms that are already theorems.

Prerequisite chapters — The definitions of the objects are in 19-01 The Wilson measure, 19-02 The vortex ratio, 19-03 The definition of the mass gap and 19-04 The push-forward measure. The problem statement is in 0-03, and the definition of the mass gap in 4-11.

Said first — Every statement in this chapter is set down without a proof Assumption . This book shows none of them. The first four (matching, convergence of the full sequence, non-degeneracy, equicontinuity) are statements on the ultraviolet side, written for this book's design (the measure pushed forward by a fixed block map). The existence of four-dimensional Yang–Mills theory was shown, with a different design, by the construction published on 2026-11-10, and the committee's assessment is that it is proved in Lean (Volume 2; a comparison of the two designs is in Part 14, and the record of this book's ultraviolet attempts in Part 15). The first four are therefore kept as a record of the statements to be shown in this book's design. The last two (uniform decay, with the mass gap as its consequence, and the band) are statements on the infrared side and remain open. They are the main subject of Volume 3. The five statements have content only when read for the sequence β_n fixed by matching. Read in a form where the sequence can be chosen freely, the mass-gap and band statements hold just by setting the coupling to 0 (§ 01). Choices remain in how the matching is done. Of the requirements of the problem statement, fine observables and rotation invariance do not yet have the objects needed to write statements (§ 08).

The order of this chapter

1. Setting — one box, and a sequence of finer lattices
2. Matching — how the sequence of bare couplings is fixed
3. Convergence of the full sequence — for the matched sequence, without passing to subsequences
4. Non-degeneracy — the coarse field stays in small fields
5. Equicontinuity — uniformly in physical position
6. Uniform decay — uniformly in lattice spacing and box (the consequence is the mass gap)
7. The band — the vortex ratio becoming small exponentially in the area
8. The gap to the problem statement — separate stages, and stages whose objects do not yet exist
9. Nearby forms that are already theorems (Table 2)

01

Setting — one box, and a sequence of finer lattices

What the problem statement asks for is a theory on infinitely extended continuous four-dimensional space (0-03 § 03). In this chapter, as a stage before that, we consider bringing the lattice spacing towards 0 on **one box of fixed size**. The objects used in the statements are only those defined in Part 19.

Let the size of the box be a positive integer L. Only the matching of § 02 is done on a box of one fixed **reference size** L_{ref}. The statements of § 03 to § 05 are about a box of one fixed size L (not limited to $L = L_{\mathrm{ref}}$), and in § 06 and § 07 the sizes L and L_0 vary. The lattice at step $n \geq 0$ is

$$\Lambda_n = (\mathbb{Z}/2^n L)^4 \quad (1)$$

and the lattice spacing at step n is read as $a_n = 2^{-n}$. Read this way, the lattice at every step subdivides the same box of side L. The larger the step number, the finer the lattice (the same direction as in 19-04 § 02). The direction playing the role of time is direction 0 in the numbering of Part 20 in 19-01 (direction 1 in the numbering 1 to 4 of 19-04; 19-04 itself does not specify a time direction).

Four objects are used.

Symbol	Contents	Definition
μ_n^β	The normalised Wilson measure on Λ_n (coupling β). Once the sequence of couplings (β_n) is fixed, $\mu_n := \mu_n^{\beta_n}$	19-01. Lean: `TorusRP.wilsonProb`
$\nu_{n,m}$	The measure μ_n pushed by the block map down to step $m \le n$. A probability measure on configurations of the coarse lattice Λ_m	19-04. Lean: `ChainBlockMap.vP`
$\theta_{N,M}(\beta)$	The ratio $(Z_0 - Z_1)/(Z_0 + Z_1)$ of the partition functions without and with the (3,4) twist, on the torus with transverse size N and longitudinal size M, with the background twist $n_{12} = 1$ fixed	19-02. Lean: `TwistBlocks.thetaM`
The vessel and the mass gap	The Hilbert space, transfer operator T and vacuum Ω built from a measure. $\|T - P_\Omega\| \le e^{-m}$ is called a gap of mass m	19-03. Lean: `OSGap.Datum.HasMassGap`

$\nu_{n,m}$ is the theory on the fine lattice of step n, viewed at the fixed resolution of step m. Holding step m fixed and letting $n \to \infty$, we see the limit in which the lattice spacing a_n approaches 0, on the same coarse lattice. The push-forward is a rule for making coarse fields; no decomposition (representation) into an effective action is used.

The space of configurations is compact (a finite product of balls). So at each step m, $\nu_{n,m}$ converges weakly along a subsequence, and by the diagonal argument a subsequence common to all steps can be taken Proved here . **Subsequential limits exist without any premise**, so the statements to be shown have meaning only in forms stronger than that. The following six are all of that form.

Read together with the matching. Each of the five statements of § 03 to § 07 involves the sequence of couplings (β_n). Read in a form where the sequence can be chosen freely, the statements lose their content.

- Setting $\beta_n \equiv 0$, the fine measure is the product of Haar measures. Since the block map is local, coarse variables separated in time are independent and their covariance is 0. So the mass-gap statement (§ 06) holds for this sequence Proved here . Also, at $\beta = 0$ the partition function does not change with or without the twist, $Z_0 = Z_1$, so $\theta = 0$, and the band statement (§ 07) holds too Proved here .
- Taking β_n large pushes the fine measure towards flat configurations, and the non-degeneracy statement (§ 04) is expected to become easier to satisfy.

Conversely, read as "for every sequence", there are expected to be sequences for which they are false (for instance, for a sequence alternating between the strong-coupling end and the weak-coupling end, convergence of the full sequence cannot be expected). So in this chapter we first fix one sequence (β_n) by **matching** in § 02, and write the five of § 03 to § 07 **for that sequence**. Note also that the theorems in the strong-coupling window (§ 09) lie in the direction on which the mass-gap statement holds trivially.

02

Matching — how the sequence of bare couplings is fixed

The coupling β on the lattice is, in the continuum limit, not an "input" but a number adjusted to calibrate the lattice spacing (introduction in 5-05 § 02). Here we write that adjustment with defined objects only.

The matching rule. Choose a step m_0, a bounded continuous function Y_0 of the coarse configuration at step m_0 (for example, the average of $\mathrm{Re}\, U_p$ over coarse plaquettes), and a target value κ_0. For each step $n \ge m_0$, set

$$f_n(\beta) = \int Y_0 \, d\nu_{n,m_0}^{\beta}, \qquad \beta_n \text{ a solution of } f_n(\beta_n) = \kappa_0 \quad (2)$$

Here ν_{n,m_0}^{β} is the push-forward of the Wilson measure with coupling β on the box of reference size L_{ref}.

Existence of solutions. f_n is continuous in β Proved here . Reason: $f_n(\beta)$ is the ratio of two integrals with respect to the product of Haar measures, $\int (Y_0 \circ B)\, e^{\beta S}$ and $\int e^{\beta S}$ ($S = \sum_p \mathrm{Re}\, U_p$, B the composed map from step n to step m_0). The integrands are bounded and continuous in β, and the configuration space is compact, so both integrals are continuous by dominated convergence, and the denominator is positive. Hence if κ_0 lies between two values $f_n(\beta')$, $f_n(\beta'')$, there is a solution in $[\beta', \beta'']$ by the intermediate value theorem Proved here . Whether κ_0 falls in this range for every n depends on the choice of Y_0 and m_0.

Where choices remain. (i) How Y_0, m_0 and κ_0 are chosen. (ii) Since the solution is not necessarily unique, which solution is taken (for example, the largest solution in a fixed interval). These two are design choices, and are listed in the table of 28-05.

The statement to be shown. If solutions exist for every n, the matching rule fixes one sequence. The existence of solutions hinges on checking the end values, which has not yet been done. Beyond that, what must be shown is that the sequence so fixed falls within **a bracket on how the coupling runs**.

Assumption `Matching: there are` β_* `and` $0 < b_- \le b_+$ `such that` $\beta_* + b_- n \le \beta_n \le \beta_* + b_+ n$ `for every` n

It is the form in which β increases by roughly a fixed amount each time the lattice spacing is halved. It corresponds to the logarithmic running $\beta \sim \text{const} \times \log(1/a)$ predicted by one-loop asymptotic freedom, written in terms of $n = \log_2(1/a_n)$ (introduction in 5-05 § 03). In the conditional implication assembled in this book's study (weak ultraviolet existence in a small box), a bracket with $b_\pm$ taken as constant multiples of the one-loop coefficient (the matching bracket for the bare coupling) was placed in a form that follows from an assumption on the effective coupling (the one-step bracket assumption for the coupling) (28-01).

The treatment of the sequence of couplings differs between that implication and Lean. The implication defines and uses the fine measure as "the Wilson measure at the β_n chosen by the matching bracket for the bare coupling" (the implication's β_n is chosen by matching the effective coupling to a target value, a different matching from the rule (2) of this section). The Lean theorems take an arbitrary sequence of bare couplings and do not tie it to that choice.

The convention from here on. The matching is done once, on the box of reference size L_{ref}, and the resulting sequence β_n (the coupling at lattice spacing $a_n = 2^{-n}$ of step n) is used for boxes of every size. In § 03 to § 07, β_n is this sequence, $\mu_n = \mu_n^{\beta_n}$ is the Wilson measure of the box of the size under consideration, and $\nu_{n,m}$ is its push-forward. When the size is varied in § 06 and § 07, the sequence is the same.

03

Convergence of the full sequence — without passing to subsequences

Assumption `Convergence of the full sequence: for the sequence $\beta_n$ fixed by matching, for every step $m$ and every bounded cont`

In other words, at each step m, $\nu_{n,m}$ converges weakly to some measure ν_m without passing to subsequences. Then, since the block map is continuous, push-forward and limit commute, and ν_m satisfies the consistency relation $(B_m)_*\nu_{m+1} = \nu_m$ Proved here . The consistent family (ν_m) gives the "continuum-limit theory" on the box of size L at all resolutions at once.

As in § 01, subsequential limits exist without any premise. The content of this statement is that "every subsequence gives the same limit". If the limit depended on the subsequence, the theory would not be determined uniquely.

This statement does not constrain the sequence of couplings, so it is read for the sequence fixed by the matching of § 02. Without a fixed matching, it is not determined what theory the limit is the limit of.

The conclusion of the conditional implication assembled in this book's study goes only as far as convergence along a subsequence (28-01). That implication has an assumption related to convergence of the full sequence (the summability-of-increments assumption), but it is **not the same as this statement**. The summability-of-increments assumption says that, for a countable dense family of gauge-invariant functions, the step-by-step increments of the values of observables are summable. Being summable, it is stronger than this statement, and the family of functions is also different. The step from gauge-invariant functions to all bounded continuous functions is an argument on paper, going through the invariance of the limiting measure under gauge transformations (28-02).

04

Non-degeneracy — the coarse field stays in small fields

Choose a fixed finite region R_m of the coarse lattice at step m (not the whole box, but a region of fixed physical size). The set of configurations in which the coarse links in R_m are close to unit length and the coarse plaquettes are nearly flat is called the **small-field set** C_m. For example, choosing a threshold $y_m > 0$,

$$C_m = \{U : \ |U_b| \geq 1 - \epsilon_m \text{ for every coarse link } b \text{ of } R_m, \ \ \beta_m (1 - \operatorname{Re} U_p) \leq y_m \text{ for every coarse plaquette } p \text{ of } R_m\} \quad (3)$$

(a closed set). β_m is the m-th term of the sequence fixed by matching ($m \geq m_0$). $\epsilon_m < 1$ and y_m are fixed **before the statement** to be as small as later stages (perturbative expansion) can use (for example, for a fixed angle φ_c setting the small-field threshold, below the floor $y_m \leq \beta_m(1 - \cos\varphi_c)$ at which the angle of plaquette values is at most φ_c). Coarse link values lie in the closed unit ball and $1 - \operatorname{Re} U_p \leq 2$, so taking $\epsilon_m \geq 1$, $y_m \geq 2\beta_m$ makes C_m the whole configuration space, and the statement holds trivially Proved here . How R_m is taken is also a choice. Written for the whole box, η_m could approach 1 with the volume, so it is written in the form of a region.

Assumption `Non-degeneracy: for the sequence $\beta_n$ fixed by matching and $(R_m, \epsilon_m, y_m)$ fixed beforehand, for every step $m$ there is`

Meaning: however fine the lattice pushed from, the coarse field at step m stays in the small-field region with at least a fixed probability. The same lower bound remains for the limit ν_m (C_m is closed), so the limiting measure gives positive weight to the small-field region Proved here . It is an entrance condition for entering later stages (the range where perturbative expansion can be used).

$\eta_m < 1$ is the content of the statement. Allowing $\eta_m = 1$, it holds for any measure. Also, this statement is expected to be of the kind that becomes easier to satisfy the larger the coupling is taken (§ 01). That is why it is read for the sequence fixed by matching.

05

Equicontinuity — uniformly in physical position

The continuum-limit theory should have observables not only at lattice points but at any position in the box. This is the statement for that.

Choose one family of observables. Let D be the countable dense set of positions x in the box $(\mathbb{R}/L\mathbb{Z})^4$ that are lattice points at some step $\mathrm{lev}(x)$ (dyadic rational points), and to each $x \in D$ assign a bounded continuous function G_x of the coarse configuration at step $\mathrm{lev}(x)$ (a fixed, non-constant local observable G placed at position x). The statement is needed for every local observable G. Taking G constant makes the statement hold trivially, so a single G alone has no content.

$$s_n(x) = \int G_x \, d\nu_{n,\mathrm{lev}(x)} \qquad (n \geq \mathrm{lev}(x)) \quad (4)$$

Assumption Equicontinuity: for the sequence β_n fixed by matching, there are $\tau_n \to 0$ and a function $\Omega(r)$ tending to 0 as $r \to 0$ s

Meaning: the expectations of observables have a modulus of continuity in position that is uniform in n (apart from the error τ_n coming from the lattice subdivision). Combined with convergence of the full sequence, the limit $s(x) = \lim_n s_n(x)$ is uniformly continuous on D, and extends continuously to every position in the box Proved here .

The choice of the family of observables (D, G) follows the form of the conditional implication assembled in this book's study; a cross-check of the definition and statement by another pair of eyes has not yet been done (the family of observables and small-field set in 28-04).

06

Uniform decay — uniformly in lattice spacing and box (the consequence is the mass gap)

We write τ_t for the translation by t steps in the time direction of the lattice at step m (the convention of § 01: direction 0 in the numbering of Part 20 in 19-01). Since the box is periodic, τ_t is defined as a map on coarse configurations. In physical time it is a shift of $a_m t$.

Assumption Uniform decay: for the sequence β_n fixed by matching, there is $m_* > 0$ such that, for every size L, every step m,

The time difference t is limited to half the period. In a periodic box, τ_t returns to the identity with period $2^m L$, and when t equals the period the covariance returns to $\mathrm{Var}(F)$ Proved here . The constants are taken in the form $C_F = c_F \|F\|_\infty$, where c_F is determined only by the number of links F depends on, and does not depend on n or L. m_* is a mass in physical units, depending neither on the lattice spacing nor on the box size. This is the core of the statement.

There are choices in the form of the statement. Whether to take constants for each observable (the form of this chapter), or one constant C uniformly as $C\|F\|_\infty\|G\|_\infty$. Whether observables depend on links near time 0, or on links at exactly time 0 (whose number grows with the volume). For obtaining the mass gap of the vessel, either form suffices Proved here .

The implication that this decay gives the mass gap ($\|T - P\Omega\| \leq e^{-m}$ in 19-03) in the vessel of reconstruction of the limiting measure is written in 28-03. Building the vessel also needs reflection positivity and symmetries of the limiting measure. Reflection positivity of the fine lattice is a theorem of Part 20, but carrying it over to the push-forward measure is on the side of the premises (reflection positivity at the coarse level in 28-04, symmetries of the measure).

The mass gap of the problem statement (4-11 § 01) is a gap in the spectrum of the Hamiltonian of the infinite-volume theory. This statement takes, among the steps toward it, the part "in a box of fixed size, uniformly in the lattice spacing" and makes it uniform in the size of the box as well. The step to infinite volume is a separate row of the table in § 08.

In this chapter the statement is written with this book's push-forward measure, but the core of the question (that correlations of observables on the renormalised lattice decay at an exponential rate, in physical units, uniformly in the cutoff and the volume) is open independently of the design. If it were obtained, the mass gap and the uniqueness of the vacuum of the limiting theory would follow from known general theory. What the published construction has goes only as far as estimates at a polynomial rate at long distances (16-03).

07

The band — the vortex ratio becoming small exponentially in the area

The vortex ratio $\theta_{N,M}(\beta)$ (19-02) is the relative difference of the partition functions when one twist is inserted. Since $Z_1 > 0$, $\theta_{N,M} \leq 1$ Proved here . If M is even, $0 \leq \theta_{N,M}$ also holds (a theorem of 21-06). In the convention of 19-02, the free energy of the vortex (flux) is $\Delta F = \ln(Z_0/Z_1)$, and $\theta = \tanh(\Delta F/2)$. So when θ is small, $-\ln\theta$ is nearly equal to minus the logarithm of ΔF, $-\ln(\Delta F/2)$ (introduction in 25-01).

Subdividing a square box of physical size L_0 at the lattice spacing of step n, the number of lattice points on a side is $2^n L_0$ (even for $n \geq 1$). For the coupling, the sequence β_n matched on the reference box in § 02 is used in the same way for every L_0.

Assumption `Band: for the sequence` β_n `fixed by matching, there is` $c > 0$ `such that, for every` L_0 `and all sufficiently large` n`,`

It is written without the logarithm $-\ln\theta$ so that the statement does not break at $\theta = 0$ (the best case). Lowering c slightly, it is the same as the form $\liminf_n(-\ln\theta) \geq c\,L_0^2$.

Meaning: in the continuum limit, the vortex free energy ΔF becomes small exponentially in the area of the box ($-\ln\theta$ grows in proportion to the area). This is the form of confinement in the language of vortices (an exponential in the area); that writing the band in the naive form "θ stays uniformly away from 1" is not enough is in 26-01 § 04. Crossing the transition between the weak-coupling side (small boxes) and the strong-coupling side (large boxes) with a single c is the content of this statement.

The band statement can be written with θ and Z alone, and does not use the push-forward measure. The present book has no arrow from this statement to the mass gap (28-03, 28-06).

08

The gap to the problem statement — separate stages, and stages whose objects do not yet exist

Between what the problem statement (0-03) asks for and the six statements of § 02 to § 07, there are still stages.

Table 1 The requirements of the problem statement and their relation to the statements of this chapter (as of 2026-11-13)

Requirement of the problem statement	Relation to the statements of this chapter	Objects for writing the statement
On $\mathbb{R}^4$ (0-03 § 03)	This chapter uses a box of size L. Enlarging the box is a separate stage	Can be written by adding one definition (restriction to a window in the box, or periodic extension). In the strong-coupling window it can be written with the periodic lift, and is a theorem (20-03)
Local fields, Schwinger functions (4-10 § 02)	The observables of this chapter are functions of coarse configurations at step m, not point fields	**Not yet available**. Definitions of a lattice gradient flow and of flowed observables are needed
Rotation invariance (the Euclidean group)	The symmetries of the lattice are only the hypercubic group	**Not yet available**. Observables on which rotations finer than the hypercubic ones act, or a block map from a rotated lattice to the standard lattice, are needed
Non-triviality (0-03 § 05)	Not in this chapter	Can be written, but in terms of which quantity is undecided (example: when matched at step m_0, the limiting value of the same observable at another step m_1)
Mass gap (0-03 § 06)	The uniform decay of § 06 is the form on a box, uniform in lattice spacing and box	Available (the vessel is in 19-03)
Agreement at short distances (0-03 § 07)	The bracket of § 02 is only about how the coupling runs, and says nothing about the short-distance form of correlations	—
Any compact simple group (0-03 § 02)	This chapter is $SU(2)$ only	—

For the two of fine observables and rotations, the objects must be defined before the form of the statement can be discussed. The list is in 28-06.

09

Nearby forms that are already theorems

The theorems of Parts 20 to 22 are forms close to one of the six statements (forms at another scale, or for another object). None is literally a special case. All are theorems at **a single scale with the lattice spacing fixed** (or for one linearised step), and do not reach the six statements themselves. The right-hand column of each row says where they fall short.

Table 2 The six statements, and the forms that are already theorems (as of 2026-11-13)

Statement	Form that is a theorem	Chapter, Lean	Where it falls short of the statement
Uniform decay	Cited In the strong-coupling window $0 < \beta$, $18 \tanh \beta < 1$: a gap of mass $-\frac{1}{2}\log(18 \tanh \beta)$ in the vessel of reconstruction of the infinite-volume limit measure, and clustering in the four directions. A formalisation of results of a known type	20-04. `InfMain2.wilson_infvol_main2`	β stays inside a small window, not a matched sequence $\beta_n \to \infty$. The lattice spacing is fixed at 1. The mass is in lattice units, and only a lower bound. Strong coupling is the direction on the side where things hold trivially (§ 01); it is an example in which the statement can be satisfied, not a check of its content
Convergence of the full sequence	Cited In the same window, the lifts of the Wilson measures on periodic boxes converge weakly to the same measure without a subsequence, for every sequence of box sizes. A formalisation of results of a known type	20-03. `AllN.wilson_infvol_allN`, `InfVol.wilson_infvol_unique`	What moves is the size of the box, not the lattice spacing (not the direction of n in this chapter). Only inside the window
The band	Cited The thin-vortex floor: for all real β and $N, M \geq 1$, $\theta_{N,M}(\beta) \leq \tanh(N^2\lvert\beta\rvert)^{M^2}$. A formalisation of one side of a known picture	21-02. `TwistedWilson.twisted_vortex_bound`	The form with the lattice spacing and the transverse size N held fixed and M made large. The exponent $-\ln\tanh(N^2\lvert\beta\rvert)$ goes to 0 as $N \to \infty$, and does not reach the form $N = M = 2^n L_0$
The band	Cited The thick-vortex floor: for any $\varepsilon > 0$, for every sufficiently large β and sufficiently large M, $\lvert\theta_{N,M+1}(\beta)\rvert \leq \exp\bigl(-(M+1)^2 e^{-\beta(S_N+\varepsilon)}\bigr)$ (S_N is the minimal action of a vortex, 19-02). A formalisation of one side of a known picture	21-03. `TwistBlocks.thick_vortex_floor`	Likewise the form with N and the lattice spacing held fixed. The thresholds in β and M are not explicit. The exponent goes to 0 as $\beta \to \infty$
The band	Cited The vortex floor at strong coupling (an upper bound on the vortex free energy): inside the strong-coupling window, on a torus of transverse size N and longitudinal size M, the difference of the twisted partition functions is exponentially small in the area of the longitudinal plane, and the rate per unit area depends neither on the transverse size nor on the longitudinal size. The prefactor (the transverse area) remains. A formalisation of a known type	20-06. `VortexStrongMN.vortex_floor_MN_sharp`	The coupling stays inside the strong-coupling window, while matched sequences leave the window. The lattice spacing is fixed at 1. Strong coupling is the direction on the side where things hold trivially (§ 01); it is an example in which the statement can be satisfied, not a check of its content
Non-degeneracy (a nearby form)	Cited Rarity at one scale: for $\beta \geq 81$, on every torus with even side ≥ 4 and for every set X of plaquettes, the probability that all plaquettes of X have $\mathrm{Re}\, U_p < 0$ is $\leq q_1^{\lfloor\lvert X\rvert/6\rfloor}$, with $q_1 < 1$ independent of the volume. A formalisation of the Chessboard Estimate	22-01. `FoldCap.rarity_explicit`	An estimate for the measure of the fine lattice itself (no push-forward, $m = n$), with the threshold fixed at $\mathrm{Re}\, U_p < 0$. It is not an estimate about the coarse plaquettes of $\nu_{n,m}$ ($m < n$), and is not a special case of non-degeneracy either
(none of the statements)	One step of the free field: for the field obtained by pushing the Gaussian free field m times by the linearised blocking, an estimate of the covariance of the field strength (ratio $\leq 640/9$), and a contraction $< 1/170$ over 17 steps, uniform in the step. Machine-checked in Lean	15-12. `LatticeFace.lattice_face_T1h`, `TorusLam.stage_tau_17`	The free field rather than the Wilson measure, and the linear part rather than the nonlinear block map. Depends on the weights $(1/8, 3/4, 1/8)$. Material, for the free field, toward a sufficient condition for the six statements (the premise of ultraviolet stability in 28-04)

The right-hand column of Table 2 says the same one thing in every row. **The theorems are in forms with the lattice spacing held fixed, or in free-field or linearised forms; there is not a single theorem of the form in which $n \to \infty$ under a matched sequence of couplings β_n.** The distance from the six statements to theorems is visible in the list of premises (28-04), as which premise carries which label.

What this chapter can say

Can say	Cannot say
Assumption The matching bracket, and five statements about the sequence fixed by matching (convergence of the full sequence, non-degeneracy, equicontinuity, uniform decay, the band), have been written with the objects of Part 19 only	None of the statements is shown
Proved here If the sequence is chosen freely, the uniform-decay and band statements hold for $\beta_n \equiv 0$	Whether they hold for the sequence fixed by matching
Proved here Limits along subsequences exist by compactness of the configuration space alone	That the limit does not depend on the subsequence (convergence of the full sequence)
Proved here The matching function $f_n(\beta)$ is continuous in β, and if the target value lies between the end values there is a solution	Whether the target value falls in the range for every n. Uniqueness of the solution. That the matched sequence falls within the bracket
Proved here Under convergence of the full sequence, the family of limits is consistent under push-forward	Non-triviality of the limit. How the limit depends on the choices in matching

Can say	Cannot say
Table 2: the strong-coupling window, the vortex floors with lattice spacing and transverse size fixed, rarity at one scale, and one step of the free field are theorems machine-checked in Lean	None is one of the six statements itself (they are forms with the lattice spacing fixed, or free-field or linearised forms)
The gap to the problem statement (Table 1)	For fine observables and rotation invariance, the objects for writing statements do not yet exist. Statements for the stages of infinite volume and non-triviality are not in this chapter

Sources and reproduction

Item	Kind	Source / tools
The problem statement	Cited	A. Jaffe, E. Witten (the official problem statement of the Clay Mathematics Institute, 2000). Reading in 0-03
Osterwalder–Schrader reconstruction	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. Introduction in 4-10
Asymptotic scaling and the lattice β function	Cited	Introduction and sources in 5-05
The form of the six statements	Assumption	This book's study (an inventory of premises, 2026-11-13). The assumptions and conclusions of the conditional implication assembled in this book's study (weak ultraviolet existence in a small box), rewritten in the language of defined objects
Definitions of the objects used	Lean	This book's Lean bundle: `TorusRP.wilsonProb`, `ChainBlockMap.vP`, `TwistBlocks.thetaM`, `OSGap.Datum.HasMassGap`. Each definition is in the corresponding chapter of Part 19
The theorems of Table 2	Lean	The source tables of each chapter (20-03, 20-04, 21-02, 21-03, 15-12, 22-01)

Words used in this chapter — statement to be shown, matching, bare coupling, bracket on how the coupling runs, push-forward measure, convergence of the full sequence, subsequential limit, non-degeneracy, small-field set, equicontinuity, uniform decay, vortex ratio, exponential in the area, band. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-00 Entrance — what can be said at one lattice spacing — reads the theorems of Table 2 in order, from the definitions. To go to the list of premises first, see 28-00.

Entrance — what can be said at one lattice spacing

This chapter is a guide (the entrance to this part).

What this chapter shows — The list of theorems of Part 20 (one line per chapter) and the scope that applies to the whole part. The theorems of this part are statements on a finite lattice with the lattice spacing fixed, or on its infinite-volume limit: the parts that set up the vessel of reconstruction (the Hilbert space, the transfer operator, the vacuum), and the mass gap inside the strong-coupling window.

Prerequisite chapters — 19-00 (how to read objects, firm theorems and premises), and 19-01 and 19-03 (the lattice, the Wilson measure, reflection, the vessel of reconstruction).

Said first — The lattice spacing is fixed. The strong-coupling theorems hold only inside the window $18 \tanh |\beta| < 1$ or $24(1 - e^{-4|\beta|}) < 1$, and say nothing about the continuum limit or weak coupling. Every theorem is a Lean formalisation of a result of a known type.

The order of this chapter

1. The theorems of this part — one line per chapter (Table 1)
2. The scope that applies to the whole part
3. Reading order

01

The theorems of this part — one line per chapter (Table 1)

Chapter	What is said (one line)	Main Lean names
20-01 Reflection positivity	Reflection positivity of the Wilson measure on $(\mathbb{Z}/2M)^4$. For the plane through lattice sites, for all real β; for the plane through the middle of links, for gauge-invariant functions with $\beta \geq 0$, $M \geq 2$; for the planes through lattice sites in all four directions, also for all β. The reflection pairing is symmetric	`TorusRP.wilson_rp`, `MidRP.wilson_rp_mid`, `PeierlsPerm.wilson_rp_dir`, `TorusRP.wilson_refl_symm`
20-02 Decay of correlations at strong coupling	On $(\mathbb{Z}/N)^4$, the covariance of functions separated in the time direction is exponentially small. Windows $18 \tanh \lvert\beta\rvert < 1$ (the form measured by the change link by link) and $24(1 - e^{-4\lvert\beta\rvert}) < 1$ (the form with separated supports). The constants contain no N	`LinkSpec.wilson_cov_dist`, `TorusRP.wilson_strong_coupling`
20-03 The infinite-volume limit	In the window $18 \tanh \lvert\beta\rvert < 1$ ($\beta \neq 0$), the periodic lifts of the normalised Wilson measures on $(\mathbb{Z}/2(n+1))^4$ converge weakly, without passing to a subsequence, to a single probability measure ν_β on $\mathbb{Z}^4$ (the sequence over all sizes $(\mathbb{Z}/(m+1))^4$ has the same limit), and ν_β is invariant under unit translations in the four directions. At limit points, clustering in the four directions when $e^{\lambda} \cdot 18 \tanh \lvert\beta\rvert < 1$	`InfVol.wilson_infvol_unique`, `AllN.wilson_infvol_allN`, `TransI.wilson_infvol_translation`, `WilsonLimit.cluster_limit`

Chapter	What is said (one line)	Main Lean names
20-04 Reconstruction and the mass gap	In the reconstruction of the limit measure in the window $18\tanh\lvert\beta\rvert < 1$: $\lVert T - P\Omega\rVert \le (18\tanh\lvert\beta\rvert)^{1/2}$ (a lower bound on the mass) and the complement of the vacuum is not 0 ($\beta \neq 0$; the same form also in the gauge-invariant sector), and $T \ge 0$ ($\beta \ge 0$). A statement bundling these into one for $0 < \beta$. That the range of T lies in the gauge-invariant sector is a separate theorem	`InfMain2.wilson_infvol_main2`, `WilsonLimit.wilson_massGap_dob_exists`, `GaugeZ.wilson_massGap_gauge_exists`, `FullT.wilson_Tpos_full`, `Bind.T_range_gauge_all`
20-05 The vessel with fermions	On space $(\mathbb{Z}/N)^3$ with a time circle $\mathbb{Z}/2h$, the measure whose weight is the product of the Wilson gauge weight and the Wilson–Dirac determinant weight $\mathrm{Re}[(\det D)^{2j}]$ is positive under reflection in a time plane through lattice sites (all β, mass $m > -1$); the partition function is positive (all β, m, j); the determinant is invariant under global gauge transformations; and the normalised measure is invariant under the generators of the lattice symmetries	`WilsonGauge.site_reflection_positive_wilson_full`, `WilsonGauge.partition_pos_wilson_full`, `GaugeDet.physDirac_det_conj`, `CubicSym.wfProb_lattice_generators`

Chapter	What is said (one line)	Main Lean names
20-06 The area law at strong coupling, and an upper bound on the vortex free energy	On a torus with the same size in all four directions, inside the strong-coupling window, the expectation of the Wilson loop on the boundary of a rectangle is exponentially small in the area (in a finite box, and for the infinite-volume limit measure). On a torus of transverse size N and longitudinal size M, the difference of the twisted partition functions is exponentially small in the area of the longitudinal plane, and the rate per unit area does not depend on the transverse size (the transverse area in the prefactor remains)	`AreaLaw.area_law_wilson`, `AreaLaw.area_law_infvol`, `VortexStrong.vortex_floor`, `VortexStrongMN.vortex_floor_MN_sharp`
20-07 What this part says, and what it does not	The scope of the theorems of this part, and a table of what is not said	—

Table 1. The theorems of Part 20. All the assumptions of each row's statement are in the corresponding chapter. Every row is a formalisation of a result of a known type Cited (reflection positivity, strong-coupling decay and the mass gap: Osterwalder–Seiler; the infinite-volume limit (convergence without a subsequence): of Dobrushin's comparison type; the area law: Osterwalder–Seiler; the upper bound on the vortex free energy: of the type of 't Hooft and Münster).

02

The scope that applies to the whole part

- **The lattice spacing is fixed.** Every theorem is a statement at one lattice spacing (taken to be 1). About the limit in which the lattice spacing goes to 0 (the continuum limit), this part says nothing.
- **Only inside the strong-coupling window.** The theorems on decay, the infinite-volume limit and the mass gap hold only inside the window $18 \tanh |\beta| < 1$ ($|\beta| < 0.0556$ Reading) or $24(1 - e^{-4|\beta|}) < 1$ ($|\beta| < 0.0106$ Reading) (the windows for the area law and for the upper bound on the vortex free energy are in 20-06, for each version of the constants). Depending on the theorem, $\beta \neq 0$ or $0 < \beta$ is added, and $T \geq 0$ needs $\beta \geq 0$. Outside the window we say neither that they hold nor that they fail.

- **The area law is a statement about lattice Wilson loops.** Continuum Wilson loops are present neither in this part nor in the published construction (14-06). The strong-coupling theorems of this part are statements at the opposite end (strong coupling, lattice spacing fixed) from the question the published construction left open.
- **Nothing is said about weak coupling.** What the continuum limit needs is the side $\beta \to \infty$ (weak coupling), the opposite end from the window. The mass gap of this part is the gap of the lattice transfer operator inside the strong-coupling window; it is not a claim about the mass gap of four-dimensional Yang–Mills.
- **Reflection positivity and the vessel with fermions do not depend on the window.** The positivity in 20-01 and 20-05 holds for all real β (the middle plane for $\beta \geq 0$; 20-05 for mass $m > -1$), and is a statement for each finite torus.
- **Only a lower bound on the mass.** The upper bound on $\|T - P\Omega\|$ is a lower bound on the mass; neither the true value of the mass nor an upper bound on it is stated.
- **Formalisations of results of known types.** Every theorem is a result of a type known in the literature, machine-checked in Lean 4 on this book's objects (19-01). We do not present them as new mathematical results.

03

Reading order

The vessel rests on three pillars, read in this order. Reflection positivity (20-01) turns the pairing into an inner product (the middle-plane form, with a gauge average in between, is also used for $T \geq 0$); strong-coupling decay (20-02) makes distant correlations small; and the infinite-volume limit (20-03) produces a single measure independent of the volume. 20-04 assembles these three into the vessel of reconstruction and obtains the mass gap inside the window. 20-05 is the part that sets up the same vessel with a measure carrying a fermion weight, and can be read with only 20-01 as a prerequisite. 20-06 gives the area law and the upper bound on the vortex free energy inside the strong-coupling window, and can be read after 20-03. 20-07 is a chapter of tables only.

After this part, Part 21 gives theorems on a lattice with twisted boundary conditions, and Part 22 theorems on the rarity of large plaquettes at weak coupling; both are likewise statements at one lattice spacing. What would need to be said to reach the mass gap in the continuum limit is in Part 28 (28-00).

What this chapter can say

What is said	What is not said
Cited Each row of Table 1 (the statement of each chapter, machine-checked in Lean)	The continuum limit, weak coupling, outside the window, the true value of the mass
Reading The window values $\lvert\beta\rvert < 0.0556$ and $\lvert\beta\rvert < 0.0106$	These are floating-point values and not part of the theorems

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity of lattice gauge theory, the strong-coupling cluster expansion	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. 5-03, 5-02
Osterwalder–Schrader reconstruction	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. 4-10
The Lean declarations of Table 1	Lean	The source tables of each chapter and 19-06. sorry 0; the axioms are the three standard ones

Words used in this chapter — vessel of reconstruction, reflection positivity, strong-coupling window, infinite-volume limit, transfer operator, mass gap, lattice spacing. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-01 Reflection positivity — three forms — the first pillar of the vessel.

2026-11-13 · chapter Yang–Mills and the mass gap Part 20 Firm theorems — the vessel and strong coupling Reflection positivity · Kiichi, Shiori and Rin

Reflection positivity — three forms

> This chapter is this book's own investigation. § 01 is a summary of established knowledge.

What this chapter shows — For the SU(2) Wilson measure on the finite torus $(\mathbb{Z}/2M)^4$, the fact that the integral of a function times its mirror image is never negative (reflection positivity) has been machine-checked in Lean in three forms. It holds for all real β for the plane through sites, for $\beta \geq 0$ and gauge-invariant functions for the plane through the middle of links, and for all β for the site planes in the four directions. In addition, the reflection pairing is symmetric. This chapter gives the four statements, the route of proof (split the action into three parts; fixing the variables on the plane turns the integral into a square), and when to use which.

Prerequisite chapters — 19-01 (the lattice, the Wilson measure) and 19-03 (reflection, the positive side, the pairing). For an introduction, 5-03 (reflection positivity and the transfer matrix) and 4-10.

Said first — Each is a statement on one finite torus, with the lattice spacing fixed. Nothing is said about the continuum limit. Positivity for the infinite-volume limit measure, and the positivity $T \geq 0$ of the transfer operator of the vessel, are in 20-04; the middle-plane statement, with an average over gauge transformations in between, is the source of $T \geq 0$. Each is a formalisation of a result of a known type (reflection positivity: Osterwalder–Seiler).

The order of this chapter

1. Why reflection positivity is needed — from the pairing to a Hilbert space
2. Two mirrors and the closed positive half (Figure 1)
3. Reflection positivity for the plane through sites — the statement
4. The route of proof — split the action, fix the plane, and get a square
5. The plane through the middle of links — the roles of gauge invariance and β ≥ 0
6. Reflection positivity in four directions, and the symmetry of the pairing
7. When to use which of the three forms

8. What is not said

01

Why reflection positivity is needed — from the pairing to a Hilbert space

For an introduction, see 5-03 § 01 and 4-10. A theory on the lattice is written only as a probability measure; there is no Hilbert space and no operator that advances time. If the pairing $\langle F, G\rangle := \int F(\Theta U)\, G(U)\, d\mu$ of functions depending only on the positive side of the mirror is symmetric and $\langle F, F\rangle \geq 0$, then one can set up the Hilbert space obtained by dividing out the null functions and completing, the transfer operator T coming from time translation, and the vacuum Ω. That this positivity holds in lattice gauge theory is due to Osterwalder–Seiler Cited . This book's definitions of the pairing, the vessel and the mass gap are in 19-03, the pairing and the vessel of reconstruction.

Reflection positivity is the first pillar of the vessel, and the four statements of this chapter establish it in Lean for the Wilson measure on a finite torus. The remaining pillars and the assembly are in 20-02–20-04.

02

Two mirrors and the closed positive half (Figure 1)

The lattice is $(\mathbb{Z}/N)^4$ with $N = 2M$. In this part the directions are $\mu = 0, 1, 2, 3$, with 0 as time. This follows the index convention of Lean on the strong-coupling side and shifts the indices relative to the directions 1–4 of the text of 19-01 (the conventions of 19-01). A link $\ell = (x, \mu)$ goes from x to $x + e_\mu$, and $U_\ell \in SU(2)$ is a unit quaternion (19-01). The weight is $w_\beta(U) = \exp\bigl(\beta \sum_p \operatorname{Re} U_p\bigr)$, and the statements of this chapter are written with the unnormalised Wilson measure $w_\beta \prod_\ell dU_\ell$ (the $\tilde{\mu}_{N,\beta}$ of 19-01). It differs from the normalised measure $\mu_{N,\beta}$ only by the positive constant factor $Z_N(\beta)$, so the sign of the inequalities is the same (19-01).

The reflection in the plane through sites Θ is $x_0 \mapsto -x_0$. Setting $\vartheta x := (-x_0, x_1, x_2, x_3)$,

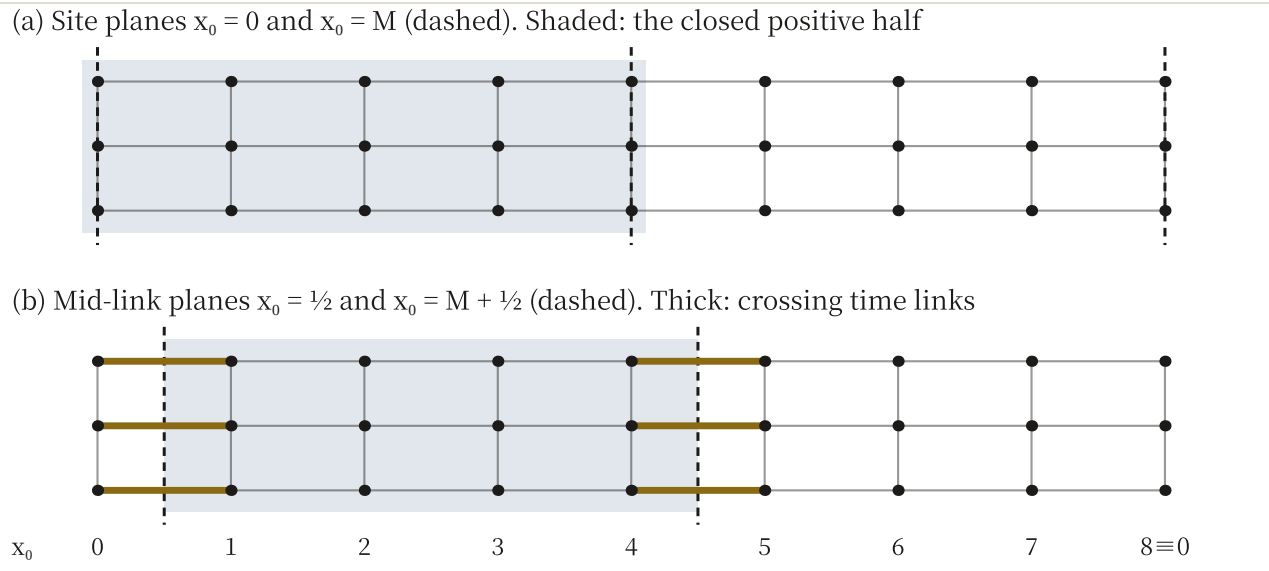
$$(\Theta U)_{(x,\mu)} = \begin{cases} U^{*}_{(\vartheta x - e_0,\, 0)} & (\mu = 0) \\ U_{(\vartheta x,\, \mu)} & (\mu \neq 0) \end{cases} \tag{1}$$

Time links reverse their direction, so they acquire an inverse (the quaternion conjugate $*$). Since $-M \equiv M$, the planes $x_0 = 0$ and $x_0 = M$ stay fixed. The **closed**

positive half Λ_+ consists of the spatial links at times $0, \dots, M$ and the time links whose base time is $0, \dots, M-1$.

The reflection in the plane through the middle of links Θ' is $x_0 \mapsto 1 - x_0$; its planes are $x_0 = \frac{1}{2}$ and $x_0 = M + \frac{1}{2}$. Time links with base time 0 or M **cross** the plane. The closed positive half Λ'_+ consists of the crossing links, the spatial links at times $1, \dots, M$, and the time links with base time $1, \dots, M-1$ (the positive side in 19-03).

Figure 1 A torus with $N = 8$ ($M = 4$) in the time direction, cut along time (horizontal) and one spatial direction (vertical, only 3 points). The shaded region is the closed positive half. (a) The spatial links on the plane belong to the positive half. (b) The thick crossing links straddle the plane; the spatial links at times 0 and 5 are on the negative side.



Proved here The figure draws (1) and the definitions in the text as they stand. Time 8 is the same point as 0 by periodicity.

03

Reflection positivity for the plane through sites — the statement

Statement (reflection positivity for the plane through sites)

Let $N = 2M$ and let β be any real number. If a bounded measurable real-valued function f depends only on the variables of the links of Λ_+, then

$$0 \;\le\; \int f(\Theta U)\, f(U)\, w_\beta(U) \prod_\ell dU_\ell . \qquad (2)$$

Cited A formalisation of Osterwalder–Seiler's reflection positivity for the plane through sites. Lean: `TorusRP.wilson_rp` (hypotheses: $N = 2M$; f measurable, bounded and depending only on Λ_+. β is any real number).

Meaning. The sign of β does not matter, because the sign is used nowhere in the proof (§ 04).

Example. Let C be a closed path in Λ_+, and take the real part of the Wilson loop, $f(U) = \operatorname{Re}\prod_{\ell\in C} U_\ell^{\pm 1}$ $(= \frac{1}{2}\operatorname{tr})$. f depends only on the links of Λ_+, satisfies $|f| \le 1$ and is continuous, so it meets the hypotheses. $f(\Theta U)$ is the real part of the Wilson loop of the mirrored path ϑC. Hence

$$\left\langle \operatorname{Re} W_{\vartheta C} \cdot \operatorname{Re} W_C \right\rangle_{N,\beta} \;\ge\; 0 \qquad (\text{all real } \beta) \quad (3)$$

Proved here (an application of (2); $\langle\cdot\rangle_{N,\beta}$ is the expectation of 19-01).

04

The route of proof — split the action, fix the plane, and get a square

The idea is the same as the "site reflection" of 5-03 § 04; in Lean it is divided into an abstract lemma and a concrete count.

Step 1: split the action into the positive side, the negative side and the plane

Give each plaquette p a weight $w(p)$: 1 for time plaquettes (plaquettes containing direction 0) with base time $0,\dots,M-1$ and for spatial plaquettes at times $1,\dots,M-1$; $\frac{1}{2}$ for spatial plaquettes at time 0 or M (plaquettes lying on the mirror plane); and 0 for the rest. Setting $S_+(U) := \sum_p w(p)\operatorname{Re} U_p$,

$$\sum_p \operatorname{Re} U_p \;=\; S_+(U) + S_+(\Theta U). \quad (4)$$

The mirror maps plaquettes to plaquettes, and the holonomy of a mapped plaquette is a cyclic shift of the conjugate, so its real part does not change. Plaquettes on the positive side are counted once, those on the negative side once as images, and those on the plane twice with a half each. Lean: `TorusRP.weight_factor` ($w_\beta = a(U)\,a(\Theta U)$, $a = e^{\beta S_+}$).

Step 2: make the integrand a pairing of a single function

$g := a\cdot f$ depends only on Λ_+ and is bounded. By (4), $f(\Theta U)\,f(U)\,w_\beta(U) = g(\Theta U)\,g(U)$; the weight disappears and only the product Haar measure remains. The sign of β has not been used so far.

Step 3: fixing the variables on the plane gives a perfect square

Divide the variables into the links on the plane U_0, the interior of the positive side $U_>$, and the negative side $U_<$. Writing $g(U) = G(U_0, U_>)$, we have $g(\Theta U) = G(U_0, V)$, where V is the negative-side variables carried to the positive side by (1). Haar measure is invariant under inversion, so V has the same distribution as $U_>$ and is independent of $U_>$. Therefore

$$\int g(\Theta U)\, g(U) \prod_{\ell} dU_\ell \;=\; \int dU_0 \left(\int G(U_0, U_>)\, dU_> \right)^2 \;\geq\; 0. \quad (5)$$

Lean: `LinkReflection.rp_core` (the abstract form) and `LinkReflection.rp_weighted` (the form with the weight factorised).

Detour: the six conditions of the abstract lemma

May be skipped. `rp_core` assumes the following about an involution θ on the index set, value maps σ_ℓ, and two predicates "on the plane" and "positive". (i) θ is an involution. (ii) Links on the plane are fixed by θ, and σ is the identity on them. (iii) Positive links are not on the plane. (iv) The image of a positive link is not positive. (v) The image of a link that is neither on the plane nor positive is positive. (vi) σ_ℓ preserves the measure on values. From these six, the space of variables is measure-preservingly isomorphic to "on the plane $\times$ positive $\times$ positive (images)", and (5) follows. On the torus, (ii), (iv) and (v) are where $N = 2M$ is used, and (vi) is the invariance of the Haar measure of SU(2) under inversion.

05

The plane through the middle of links — the roles of gauge invariance and β ≥ 0

Gauge transformations g and gauge-invariant functions are as in gauge transformations in 19-01 ($g(x) \in SU(2)$ at each site, $(gU)_{(x,\mu)} = g(x)\, U_{(x,\mu)}\, g(x + e_\mu)^*$).

Statement (reflection positivity for the plane through the middle of links)

Let $N = 2M$, $M \geq 2$ and $\beta \geq 0$. If a bounded measurable real-valued function F depends only on the variables of the links of Λ'_+ and is gauge invariant, then

$$0 \;\leq\; \int F(U)\, F(\Theta' U)\, w_\beta(U) \prod_{\ell} dU_\ell. \quad (6)$$

Cited A formalisation, in the form for gauge-invariant functions, of Osterwalder–Seiler's reflection positivity for link planes. Lean: `MidRP.wilson_rp_mid`

(hypotheses: $N = 2M$, $M \geq 2$, $\beta \geq 0$; F measurable, bounded, depending only on Λ'_+, and invariant under every g with unit values).

The proof has four steps.

Step 1: set the crossing links to 1 by a gauge transformation. The crossing links enter mutually distinct points at times 1 and $M+1$, and no crossing link leaves those points. Taking $g(x)$ to be the value of the crossing link entering that point makes every crossing link 1. For gauge-invariant H, $\int H(U) \prod dU = \int H(U^\circ) \prod dU$ (U° is the configuration with the crossing links set to 1). The change of variables is a skew product, and it preserves the measure because $V \mapsto aVb^*$ (a, b units) preserves Haar on each link. $M \geq 2$ is needed so that the starting points of the crossing links (times $0, M$) and the receiving points (times $1, M+1$) do not overlap. Lean: `GaugeFix.integral_gaugeFix`, `TorusGauge.wilson_gaugeFix`. This is applied to $H = F \cdot (F \circ \Theta') \cdot w\beta$ (Θ' carries gauge transformations to gauge transformations of a different form, and $w\beta$ is gauge invariant).

Step 2: split the action into three parts. $\sum_p \operatorname{Re} U_p = A(U) + A(\Theta' U) + C(U)$, where A is the sum over plaquettes on the positive side and C the sum over crossing plaquettes (time plaquettes having a crossing link as an edge).

Step 3: put the crossing plaquettes in kernel form. If the crossing links are 1, the holonomy of a crossing plaquette with base time 0 is VW^*, involving only the spatial link V at time 1 (positive side) and the spatial link W at time 0 (at the position of the image of V). For a crossing plaquette with base time M, it has the same form with the link at time M (positive side) and the link at time $M+1$. Viewing unit quaternions as points of $\mathbb{R}^4$, $\operatorname{Re}(VW^*) = \sum_{c=0}^{3} V_c W_c$, so

$$C(U^\circ) \;=\; \sum_j \psi_j(U)\, \psi_j(\Theta' U) \tag{7}$$

can be written as a sum of pairs of positive-side coordinates ψ_j (the quaternion components of the spatial links at times 1 and M).

Step 4: a positive-definite kernel. If $\beta \geq 0$, then

$$e^{\beta \sum_j \psi_j(U)\psi_j(\Theta' U)} \;=\; \sum_{n \geq 0} \frac{\beta^n}{n!} \Big(\sum_j \psi_j(U)\psi_j(\Theta' U) \Big)^n \tag{8}$$

is a sum, with coefficients ≥ 0, of terms of the form $h(U)\, h(\Theta' U)$, and each term becomes the integral of a square as in (5) of § 04. Lean: `LinkReflection.rp_kernel`

(abstract reflection positivity with a kernel, $\beta \geq 0$).

The role of gauge invariance. It is for Step 1. A crossing link is mapped to itself by the mirror, so it cannot be fixed and squared as in § 04; instead it is removed by setting it to 1 with a gauge transformation. For a function that does not depend on the crossing links (a function depending only on the interior of the positive side), averaging over two independent gauge transformations and then applying (6) gives a form that does not require gauge invariance (Lean `FullT.full_rp_torus`: normalised measure, $\beta \geq 0$, $M \geq 2$).

The role of $\beta \geq 0$. Step 4 uses that the coefficients $\beta^n/n!$ are ≥ 0. For $\beta < 0$ the kernel $e^{\beta\langle V,W\rangle}$ is not positive definite on S^3: the matrix $\begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$ built from the two points V, $-V$ has determinant $e^{2\beta} - e^{-2\beta}$, which is negative Proved here . This means the tool of the proof cannot be used; (6) says neither that the statement holds nor that it fails for $\beta < 0$.

06

Reflection positivity in four directions, and the symmetry of the pairing

Four directions. Let τ be the swap of directions 0 and ρ, and consider the relabelling of configurations $(P_\tau U)_{(x,\mu)} := U_{(x\circ\tau,\ \tau\mu)}$. The weight w_β is invariant under P_τ (for plaquettes whose orientation is reversed the holonomy is conjugated, and the real part is the same). $\Theta_\rho := P_\tau\,\Theta\,P_\tau$ is the reflection $x_\rho \mapsto -x_\rho$ in the site plane of direction ρ.

Statement (reflection positivity for site planes in four directions)

Let $N = 2M$, $\rho \in \{0,1,2,3\}$, and let β be any real number. If a bounded measurable real-valued f depends only on the closed positive half of direction ρ (links of direction ρ whose base has x_ρ in $0,\dots,M-1$, and links of other directions with x_ρ in $0,\dots,M$), then

$$0 \;\leq\; \int f(\Theta_\rho U)\, f(U)\, w_\beta(U) \prod_\ell dU_\ell\,. \qquad (9)$$

Cited A formalisation, extended to four directions, of reflection positivity for the plane through sites (of Osterwalder–Seiler type). Lean: `PeierlsPerm.wilson_rp_dir` (all ρ, all real β; the invariance of the measure is `PeierlsPerm.wilson_perm_invariant`).

The proof just applies (2) to $f \circ P_T$ and returns by the invariance of the measure. Four directions are needed for the chessboard estimate, which carries a function of one block to all blocks by the mirrors of the four directions and stacks Cauchy–Schwarz direction by direction (22-01).

Statement (symmetry of the reflection pairing)

Let $N = 2M$, let β be any real number, and let f, g be measurable real-valued functions.

$$\int f(\Theta U)\, g(U)\, w_\beta(U) \prod_\ell dU_\ell \;=\; \int g(\Theta U)\, f(U)\, w_\beta(U) \prod_\ell dU_\ell . \quad (10)$$

Cited A formalisation of a known identity following from reflection invariance. Lean: `TorusRP.wilson_refl_symm` (neither boundedness nor a support condition is needed; when not integrable, both sides are 0 by Lean's convention).

There are three ingredients: Θ is an involution, Θ preserves the product Haar measure, and w_β is invariant under Θ (using (4) twice). (10) is the symmetry of the pairing and corresponds to a field of the vessel of reconstruction (19-03).

07

When to use which of the three forms

Form	Range	Class of functions	What it is used for
Plane through sites (time)	All real β, $N = 2M$	Bounded measurable functions on Λ_+	Positive semi-definiteness of the pairing → a Hilbert space. The source of positivity for the infinite-volume limit measure and of the assembly of the vessel (20-04)
Plane through the middle of links	$\beta \geq 0$, $M \geq 2$	Gauge-invariant bounded measurable functions on Λ'_+	The source of the positivity $T \geq 0$ of the transfer operator (20-04; with a gauge average in between)
Site planes in four directions	All real β, all directions	Functions on the closed positive half of direction ρ	The chessboard estimate (22-01)
Symmetry of the pairing	All real β	Measurable functions	Symmetry of the pairing (a field of the vessel)

The site-plane form has the lightest conditions and cares about neither the sign of β nor gauge invariance. In exchange, on its own it gets the transfer operator only as far as the type $T^2 \geq 0$, and cannot give positivity of T itself (the two kinds of “positive” in 5-03 § 03). The middle-plane form pays with $\beta \geq 0$ and gauge invariance and gives

positivity of the type $T \geq 0$. The $T \geq 0$ inside the vessel (20-04, Lean `FullT.wilson_Tpos_full`) is obtained by inserting into (6) the Haar average over all gauge transformations of the torus, extending it to functions on the positive side that do not depend on the crossing links and need not be gauge invariant (`FullT.full_rp_torus`), and passing to the limit.

On the lattice with twisted boundary conditions, the same type of splitting (fix the plane and take a square) is used with the plane through sites (21-06). In Part 21 the role of time is played by direction 4, and that is a separate statement on the lattice of Part 21.

08

What is not said

- **The lattice spacing is fixed and the volume is finite.** Every statement is an equality or inequality on one $(\mathbb{Z}/2M)^4$. Positivity is a property that holds exactly at each volume, and no form uniform in the volume is needed. Nothing is said about the continuum limit.
- **Unnormalised measure.** It differs by a positive constant factor, so the sign is the same. Positivity for the infinite-volume limit measure is in the bundled statement of 20-04 and in the axioms of the vessel (20-04).
- **Real-valued functions only.** Complex-valued functions are outside the statements.
- **Middle plane:** nothing is said about $\beta < 0$, $M = 1$, or F that depends on the crossing links and is not gauge invariant.
- **Lean's convention for values.** In Lean the link values are taken in the closed unit ball of the quaternions, and the reference measure is the image of the Haar measure of SU(2) (the unit sphere). As statements about bounded measurable functions they are the same as statements on SU(2) (the conventions of 19-01).
- **Not a new result.** Reflection positivity of lattice gauge theory is a result of Osterwalder–Seiler; this chapter is its formalisation for the Wilson measure on a finite torus.

What this chapter can say

What is said	What is not said
Cited Reflection positivity for the plane through sites (2): $N = 2M$, all real β, bounded measurable f on Λ_+. Lean `TorusRP.wilson_rp`	The continuum limit or a change of lattice spacing; complex-valued f

What is said	What is not said
Cited Reflection positivity for the plane through the middle of links (6): $M \geq 2$, $\beta \geq 0$, gauge-invariant bounded measurable F. Lean `MidRP.wilson_rp_mid`	$\beta < 0$, $M = 1$, F depending on the crossing links and not gauge invariant
Cited Reflection positivity for site planes in four directions (9): all directions ρ, all real β. Lean `PeierlsPerm.wilson_rp_dir`	A four-direction form of the middle plane
Cited Symmetry of the reflection pairing (10): all real β, measurable f, g. Lean `TorusRP.wilson_refl_symm`	—
Proved here Positivity of pairs of Wilson loops (3): an application of (2)	Estimates in terms of the size of the loop

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity of lattice gauge theory (site planes and link planes) and the transfer matrix	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. Introduction in 5-03; references in A-03
Osterwalder–Schrader reconstruction	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. 4-10
(2), (10) and their parts (splitting the action, abstract reflection positivity)	Lean	`TorusRP.wilson_rp`, `TorusRP.wilson_refl_symm`, `TorusRP.weight_factor`, `LinkReflection.rp_core`, `LinkReflection.rp_weighted`. sorry 0; the axioms are the three standard ones. The list of declarations is in 19-06
(6) and its parts (gauge fixing, reflection positivity with a kernel), the route to $T \geq 0$	Lean	`MidRP.wilson_rp_mid`, `TorusGauge.wilson_gaugeFix`, `GaugeFix.integral_gaugeFix`, `LinkReflection.rp_kernel`; `FullT.full_rp_torus`, `FullT.wilson_Tpos_full`
(9)	Lean	`PeierlsPerm.wilson_rp_dir`, `PeierlsPerm.wilson_perm_invariant`
Checks of the Gram matrix (Z_2 and Z_4 in two and three dimensions, sum over all configurations: positive semi-definite; as a positive control, removing the conjugation of the time links for Z_4 gives a relative smallest eigenvalue of −0.39 to −0.50), Metropolis for the middle plane $((\mathbb{Z}/4)^4$, SU(2); no sign of a negative covariance for pairs of gauge-invariant plaquette averages)	Reading	The numerical record of this book's study (computations copying the Lean definitions; not a substitute for proof)

Words used in this chapter — reflection positivity, plane through sites, plane through the middle of links, closed positive half, crossing link, gauge invariance, gauge fixing, positive-definite kernel, pairing, transfer operator, chessboard estimate. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-02 Decay of correlations at strong coupling — the second pillar of the vessel: it shows, inside the strong-coupling window, that the covariance of distant functions is small.

2026-11-13 · chapter Yang–Mills and the mass gap Part 20 Firm theorems — the vessel and strong coupling Strong coupling ·
Kiichi, Shiori and Rin

Decay of correlations at strong coupling

This chapter is this book's own investigation.

What this chapter shows — When $|\beta|$ is small enough (strong coupling), the covariance of two functions separated in the time direction, under the SU(2) Wilson measure on the torus $(\mathbb{Z}/N)^4$, becomes exponentially small with the distance. This estimate is machine-checked in Lean in two ways, and in neither do the constants contain the volume N. One is the Dobrushin window $18\tanh|\beta| < 1$, which comes from how much the conditional distribution of one link depends on the surrounding links (coefficient $\tanh|\beta|$, row sum at most $18\tanh|\beta|$). The other is the window $24(1 - e^{-4|\beta|}) < 1$, obtained by counting chains of connected plaquettes.

Prerequisite chapters — 19-01 (the Wilson measure), 20-01 (notation). For an introduction, 5-02 (strong coupling) and 5-04 (infinite volume and clustering).

Said first — The lattice spacing is fixed, and the statements hold only inside the strong-coupling windows. The windows are $|\beta| < 0.0556$ Reading and $|\beta| < 0.0106$ Reading, the opposite end from the weak coupling ($\beta \to \infty$) that the continuum limit needs. Nothing is said outside the windows (not even that there is no decay). The distance is only the distance in the time direction, and the statements are in finite volume for each N. Each is a formalisation of a known result of Osterwalder–Seiler or Dobrushin type.

The order of this chapter

1. Two windows, two statements — an overview
2. Expectations and the time distance — the setting
3. The influence one link receives from its surroundings — the coefficient tanh|β| and the count 18
4. Decay with distance in the Dobrushin window — the statement
5. The route of proof — washing out the oscillation of G with heat-bath sweeps
6. The other window — swapping two replicas, and chains of plaquettes
7. Comparing the two windows (Table 1)

8. What is not said

01

Two windows, two statements — an overview

For an introduction, see 5-02 § 01 and 5-04. At $\beta = 0$ the Wilson measure is a product of Haar measures, and different links are independent. When $|\beta|$ is small, the coupling between links is weak, and the correlation of distant links pays a small factor each time it "crosses one coupling", so it decreases exponentially with the distance. This is the decay of correlations at strong coupling; in lattice gauge theory it is a result of Osterwalder–Seiler via the cluster expansion Cited .

The two statements of this chapter give this picture form through two different accountings.

- **Decay with distance** (§ 04): count how much the conditional distribution of one link moves when one surrounding link is changed (Dobrushin's influence coefficients). The window is $18 \tanh |\beta| < 1$.
- **Finite-volume decay** (§ 06): prepare two independent replicas, expand the plaquette weights, and count chains of plaquettes connecting the two supports. The window is $24(1 - e^{-4|\beta|}) < 1$.

Both are finite-volume statements for each N, and the constants on the right-hand side do not contain N. This form, uniform in the volume, is the source of the infinite-volume limit (20-03) and of the mass gap of the vessel of reconstruction (20-04).

02

Expectations and the time distance — the setting

The lattice is the torus $(\mathbb{Z}/N)^4$ ($N \geq 1$); the link variables $U_\ell \in SU(2)$, the weight w_β and the normalised measure $\mu_{N,\beta}$ are as in 19-01. This chapter uses the expectation of 19-01,

$$\langle H \rangle_{N,\beta} := \int H \, d\mu_{N,\beta} = \frac{\int H \, w_\beta \prod_\ell dU_\ell}{\int w_\beta \prod_\ell dU_\ell} \quad (1)$$

and writes the covariance as $\mathrm{Cov}(F, G) := \langle FG \rangle_{N,\beta} - \langle F \rangle_{N,\beta} \langle G \rangle_{N,\beta}$. β is real and its sign does not matter (the windows are written in terms of $|\beta|$).

Time distance. Fix a time $t_0 \in \mathbb{Z}/N$, and use the cyclic distance between the base time x_0 of a link $\ell = (x, \mu)$ and t_0,

$$d(\ell) := \min_{k \in \mathbb{Z}} \left| x_0 - t_0 + kN \right| \qquad (0 \le d(\ell) \le N/2) \tag{2}$$

However far apart things are in the spatial directions, this does not enter d. Two links sharing a plaquette differ in d by at most 1 (because the base times of the four links of a plaquette fit in two consecutive times). This fact, that "the distance changes by at most 1 between neighbours", is where both proofs turn distance into an exponent.

Support of a function. A function F depends only on the links of a finite set A if F takes equal values on any two configurations that agree on A.

03

The influence one link receives from its surroundings — the coefficient tanh|β| and the count 18

With all links other than x fixed at a configuration ω, the conditional distribution of x (the heat-bath kernel of one link) is

$$\gamma_x(dU \mid \omega) = \frac{\exp\left(\beta \sum_{p \ni x} \operatorname{Re} U_p(\omega[x \mapsto U])\right) dU}{\int \exp\left(\beta \sum_{p \ni x} \operatorname{Re} U_p(\omega[x \mapsto U'])\right) dU'} \tag{3}$$

The sum runs only over plaquettes containing x. In four dimensions one link is contained in 6 plaquettes (two, one on each side, for each of the remaining three directions).

Influence coefficients. For two configurations ω, ω' differing only at another link y, measure the difference of the conditional distributions of x. Let n_{xy} be the number of plaquettes containing both x and y. The difference of the local actions $h(U)$ satisfies $|h| \le 2|\beta|\, n_{xy}$, and the difference of two probability measures whose densities are tilted by e^h is bounded by

$$\sup_S \left| \gamma_x(S \mid \omega') - \gamma_x(S \mid \omega) \right| \;\le\; \tanh\left(|\beta|\, n_{xy}\right) \tag{4}$$

Lean: `LinkSpec.spec_coeff` ($\tanh$ in the form $(e^{2a}-1)/(e^{2a}+1)$).

The count 18. From $\tanh(na) \le n \tanh a$ ($a \ge 0$, Lean `LinkSpec.tanhForm_le`) and the fact that each of the 6 plaquettes containing x has 3 links other than x, $\sum_{y \,/=\, x} n_{xy} \le 18$. Hence the row sum satisfies

$$\alpha_0 := \sum_{y \not= x} \sup\bigl|\gamma_x(\cdot \mid \omega') - \gamma_x(\cdot \mid \omega)\bigr| \;\le\; 18 \tanh|\beta| \quad (5)$$

for every x and every N. Lean: `LinkSpec.dobrushin_row`. For $N \ge 3$ there are 18 neighbours, all with $n_{xy} = 1$; for $N = 2$ there are 15 neighbours with $\sum n_{xy} = 18$ Reading (a check of the count).

Meaning. $\alpha_0 < 1$ says that "the distribution of one link is moved, by all the surrounding links together, by less than 1 in total variation". This is the form of Dobrushin's uniqueness condition, and the window $18 \tanh|\beta| < 1$ is $|\beta| < \operatorname{artanh}(1/18) \approx 0.0556$ Reading . As an explicit sufficient condition, $|\beta| < 1/19$ is in Lean (`WilsonLimit.dob_window_strict`).

04

Decay with distance in the Dobrushin window — the statement

Statement (Dobrushin decay of covariance with distance)

Let $N \ge 1$, β real, and $\lambda \ge 0$ with $e^{\lambda} \cdot 18 \tanh|\beta| < 1$. Let $t_0 \in \mathbb{Z}/N$, $r \in \mathbb{N}$, and let the finite set of links A satisfy $d(\ell) \le r$ for all its links. Let F be bounded ($|F| \le c_F$), measurable and depending only on A, and G bounded and measurable. Let $\delta_y \ge 0$ be an upper bound on "the change in G when only link y is changed", that is, $|G(\omega) - G(\omega')| \le \delta_y$ for all ω, ω' agreeing outside y. Then

$$\bigl|\operatorname{Cov}(F, G)\bigr| \;\le\; 2\, c_F\, e^{\lambda r} \sum_{y} e^{-\lambda\, d(y)}\, \delta_y. \quad (6)$$

Cited A formalisation, in sweep form, of Dobrushin's comparison type. Lean: `LinkSpec.wilson_cov_dist` (hypotheses as above; the constants on the right-hand side do not contain N).

Two functions with separated supports. If G depends only on the links of a finite set A', $|G| \le c_G$, and $d \ge r + 2 + k$ for all links of A', one can take $\delta_y = 2c_G$ ($y \in A'$) and $\delta_y = 0$ (otherwise), and

$$\bigl|\operatorname{Cov}(F, G)\bigr| \;\le\; 4\, c_F\, c_G\, |A'|\, e^{-\lambda(k+2)} \quad (7)$$

Proved here (an application of (6); an application of the same type is inside Lean's `TorusRP.wilson_hdec_torus_dob`). Any λ below $-\ln(18 \tanh|\beta|)$ will do; for example, for $\beta = 0.001,\ 0.005,\ 0.01$ that upper limit is $4.02,\ 2.41,\ 1.71$ Reading .

Points to note in the statement. δ is constrained by hypothesis to be at least the true change of G, so it cannot be taken small to make a false claim. If G is constant, $\delta \equiv 0$ and both sides are 0. On the right-hand side, N appears only in the range of the sum and inside the cyclic distance; if the support of δ does not depend on N, the estimate is uniform in N.

05

The route of proof — washing out the oscillation of G with heat-bath sweeps

The idea is "flatten G without touching F". It is divided into five steps.

Step 1: the heat bath preserves the measure. For the heat-bath operator $P_x H(\omega) := \int H(\omega[x \mapsto U])\, \gamma_x(dU \mid \omega)$, $\langle P_x H\rangle_{N,\beta} = \langle H\rangle_{N,\beta}$ (because the conditional distribution of one link is (3); Lean `LinkSpec.hbSweep_invariant` and its iteration `LinkSpec.iter_invariant`).

Step 2: sweep while avoiding A. Let L be a sequence listing every link outside A once, and let $Q := \prod_{x \in L} P_x$ (a sweep). Since F depends only on A, P_x ($x \notin A$) passes through with F still multiplied, and for all K

$$\langle F \cdot Q^K G\rangle_{N,\beta} = \langle FG\rangle_{N,\beta}, \qquad \langle Q^K G\rangle_{N,\beta} = \langle G\rangle_{N,\beta}. \tag{8}$$

Lean: `LinkSpec.iter_mul_invariant`, `iter_invariant`.

Step 3: the flow of oscillation. The upper bounds δ on the link-by-link change of H move, under one application of P_x, as $\delta_x \mapsto 0$, $\delta_y \mapsto \delta_y + \delta_x C_{xy}$ (C_{xy} is the coefficient of (4)). Attaching the weights $\rho_y := e^{-\lambda d(y)}$, since d differs by only 1 between neighbours,

$$\sum_{y \neq x} \rho_y\, C_{xy} \;\le\; \left(e^{\lambda} \cdot 18 \tanh|\beta|\right) \rho_x =: \alpha\, \rho_x \tag{9}$$

(Lean `LinkSpec.row_weighted`). With $a := \sum_{y \in A} \rho_y \delta_y$ and $b := \sum_{y \notin A} \rho_y \delta_y$, one sweep shrinks b by a factor of at most α, and δ on A does not decrease. After K sweeps, $b_K \le \alpha^K b_0$ (Lean `Sweep.iter_bound`). The key point is to start the sweep with A treated as "already processed"; then b does shrink even though a grows by what flows into A.

Step 4: from oscillation to covariance. If $|H - c| \le S$ and $|F| \le c_F$, then $|\operatorname{Cov}(F,H)| \le 2 c_F S$ (Lean `LinkSpec.cov_le`). The total oscillation of $H = Q^K G$ is bounded by $\sum_y \delta_y^{(K)}$, and $\rho \ge e^{-\lambda r}$ on A, so

$$\sum_y \delta_y^{(K)} \;\le\; e^{\lambda r} \sum_y \rho_y \delta_y \;+\; \Big(\sum_y \rho_y^{-1}\Big)\, \alpha^K b_0 . \tag{10}$$

Step 5: $K \to \infty$. Since $\alpha < 1$ and N is fixed, the second term goes to 0, and together with (8) this gives (6). $\sum_y \rho_y^{-1}$ depends on N, but it vanishes in the limit and does not remain on the right-hand side.

Dobrushin's uniqueness theorem itself is not used as a tool. The heat-bath sweeps follow its comparison argument on the finite torus.

A numerical check Reading On a two-dimensional torus of Z_2 gauge theory (time $6 \times$ space 2, sum over all configurations), the ratio of the covariance of pairs of spatial Polyakov loops to the estimate of the form (6) was 6.2×10^{-4} (separation 1) to 3.1×10^{-8} (separation 3) at $\beta = 0.03$, and 3.1×10^{-3} to 3.8×10^{-6} at $\beta = 0.15$, all within the estimate (the numerical record of this book's study). This is not a check on four-dimensional SU(2) itself, and whether the estimate is sharp was not examined.

06

The other window — swapping two replicas, and chains of plaquettes

Statement (finite-volume decay of correlations at strong coupling)

Let $N \ge 1$ and $c := 1 - e^{-4|\beta|}$ with $24c < 1$. Let $t_0 \in \mathbb{Z}/N$, $r, k \in \mathbb{N}$, and let the finite sets of links A, A' satisfy $d \le r$ for all links of A and $d \ge r + 2 + k$ for all links of A'. Let F ($|F| \le c_F$) depend only on A and G ($|G| \le c_G$) only on A', both measurable. Then

$$\big|\mathrm{Cov}(F, G)\big| \;\le\; 2\, c_F\, c_G \cdot 6|A| \cdot \frac{c\,(24c)^k}{1 - 24c} . \tag{11}$$

Cited A formalisation of the type of Osterwalder–Seiler's strong-coupling cluster expansion. Lean: `TorusRP.wilson_strong_coupling` (the window literally as $24(1 - (e^{2|\beta|})^{-2}) < 1$; the constants $2c_F c_G$, $6|A|$, $c/(1 - 24c)$ and the rate $24c$ do not contain N).

The window is $|\beta| < \frac{1}{4} \ln \frac{24}{23} \approx 0.0106$ Reading . The proof has four steps.

Step 1: shift the plaquette weights to be at least 1. With $w_p := e^{\beta \,\mathrm{Re}\, U_p + |\beta|} \in [1, e^{2|\beta|}]$, $\prod_p w_p$ is a positive constant multiple of the Wilson weight, and the expectation (1) does not change.

Step 2: two replicas. Taking two independent configurations U, V, $2Z^2\,\mathrm{Cov}(F,G) = \iint (F(U)-F(V))(G(U)-G(V))\,W(U)W(V)$. Expand $W(U)W(V) = \prod_p (1+\sigma_p)$, $\sigma_p := w_p(U)w_p(V) - 1 \geq 0$, into a sum over sets of plaquettes Γ.

Step 3: separable sets vanish. Γ is called separable if there is a set of links T containing A and disjoint from A' such that every plaquette of Γ lies either inside T or outside T (does not straddle the edge of T). Then swapping the two replicas on T reverses the sign of that term, and these cancel in the sum. What remains are terms containing a minimal set of plaquettes π connecting A and A', and from $\sigma_p/(1+\sigma_p) \leq 1 - e^{-4|\beta|} = c$,

$$\left|\mathrm{Cov}(F,G)\right| \;\leq\; 2c_F c_G \sum_{\pi} c^{|\pi|} \tag{12}$$

This step holds for all β and all N (Lean `SwapDecay.covBound_minimal`, applied in `TorusRP.wilson_covariance`). The sum on the right is not uniform in the volume from this step alone; making it uniform is the counting of the next step.

Step 4: count the chains. A minimal π contains a sequence of plaquettes, consecutive ones sharing a link, from a plaquette touching A to a plaquette touching A' (the counting lemma derives an upper bound from the length of this sequence and the degree of plaquettes). Since 6 plaquettes contain a given link, at most 24 plaquettes (including itself) share support with a given plaquette, and at most $6|A|$ plaquettes touch A. The time distance of neighbouring plaquettes differs by only 1, so a chain has at least $k+1$ plaquettes. There are at most $6|A|\cdot 24^m$ chains of length $m+1$, so $\sum_\pi c^{|\pi|} \leq 6|A| \sum_{m\geq k} c^{m+1} 24^m = 6|A|\, c(24c)^k/(1-24c)$, which is (11) (Lean `TorusRP.card_link_plaqs`, `plaq_degree`, `card_touch_le`, `height_lip`, `SwapDecay.minimalNonsep_sum_le`).

The number 4 comes from twice (two replicas) the range $2|\beta|$ of $\beta\,\mathrm{Re}\,U_p$, and 24 from 4×6. 24 is an upper bound; the true maximum number of plaquettes sharing support with one plaquette is 21 for $N \geq 3$ and 19 for $N = 2$ Reading .

07

Comparing the two windows (Table 1)

	Decay with distance (6)	Finite-volume decay (11)
Window	$e^{\lambda} \cdot 18 \tanh|\beta| < 1$ ($\lvert\beta\rvert < 0.0556$ as $\lambda \to 0$ Reading)	$24(1-e^{-4\lvert\beta\rvert}) < 1$ ($\lvert\beta\rvert < 0.0106$ Reading)
Basis of the accounting	Row sum of one link's influence, $18 \tanh\lvert\beta\rvert$	Number of chains of plaquettes 24^m and weight c^{m+1}

	Decay with distance (6)	Finite-volume decay (11)
Assumption on G	Link-by-link change δ_y (no support needed)	Support A' at time distance at least $r + 2 + k$
Rate (per unit of separation)	Any λ below $-\ln(18 \tanh \lvert\beta\rvert)$	$-\ln(24c)$
Rate at $\beta = 0.001$ Reading	Upper limit 4.02	2.35
Rate at $\beta = 0.005$ Reading	Upper limit 2.41	0.74
Rate at $\beta = 0.01$ Reading	Upper limit 1.71	0.061
Lean	`LinkSpec.wilson_cov_dist`	`TorusRP.wilson_strong_coupling`

Table 1 Comparison of the two statements. The rates are floating-point computations.

The window is about 5.2 times wider on the Dobrushin side Reading , and the upper limit of the rate at the same β is also larger. The advantage of the finite-volume decay is that it handles two functions with separated supports directly, without measuring the link-by-link change of G.

A sense of strength Reading : (11) becomes smaller than the trivial upper bound $c_F c_G$ (Cauchy–Schwarz), with $|A| = 1$, for $k \geq 2$ at $\beta = 0.008$ and for $k \geq 35$ at $\beta = 0.01$. At the edge of the window the estimate has almost no content. On small Z_2 lattices, the true covariance was 10^8 to 10^{22} times smaller than (11) (the numerical record of this book's study).

In the later chapters of this part, the infinite-volume limit (20-03) and the mass gap (20-04) mainly use the Dobrushin window.

08

What is not said

- **Nothing is said outside the windows.** Not even that the decay stops. The windows are only the small end of $|\beta|$, the opposite of the weak coupling ($\beta \to \infty$) needed for the continuum limit. The lattice spacing is fixed.
- **Only distance in the time direction.** The d in (6) and (11) is the cyclic distance of the time coordinate. Clustering in four directions is a separate statement about the infinite-volume limit measure (20-03).
- **Finite volume for each N.** We say that the constants do not contain N, but the infinite-volume measure itself is not in this chapter.

- **The estimates are not sharp.** (11) contains $|A|$ and is almost trivial at the edge of the window. In (6) one supplies δ oneself.
- **This is not the mass gap itself.** Going from decay of covariance to the mass gap of the vessel of reconstruction needs reflection positivity (20-01) and the assembly of the limit (20-04).
- **Not a new result.** Decay of correlations at strong coupling is a known result of the type of Osterwalder–Seiler's cluster expansion and Dobrushin's uniqueness; this chapter is its formalisation for the SU(2) Wilson measure on a finite torus.

What this chapter can say

What is said	What is not said
Cited Dobrushin decay with distance (6): $e^{\lambda} \cdot 18 \tanh \lvert\beta\rvert < 1$, no N in the constants. Lean `LinkSpec.wilson_cov_dist`	Outside the window, distance in spatial directions, the infinite-volume measure itself
Proved here The form for functions with separated supports (7): an application of (6)	—
Cited Finite-volume decay (11): $24(1 - e^{-4\lvert\beta\rvert}) < 1$, time distance $r + 2 + k$, no N in the constants. Lean `TorusRP.wilson_strong_coupling`	Outside the window, distance in spatial directions, a form independent of $\lvert A\rvert$
Reading The window values 0.0556 and 0.0106, the rates of Table 1, the separations below the trivial bound, the comparison on Z_2	Not proofs of upper or lower bounds. The true value of the decay for four-dimensional SU(2)

Sources and reproduction

Item	Kind	Source / tools
The strong-coupling cluster expansion, exponential decay of correlations	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. Introduction in 5-02, 5-04, 7-04
Dobrushin's uniqueness and comparison conditions	Cited	R. L. Dobrushin, Theory Probab. Appl. 13 (1968) 197 (bibliographic data only, not checked). The place of the tool is in 16-10
(4), (5), (6) and their parts (sweeps, the flow of oscillation, the covariance bound)	Lean	`LinkSpec.spec_coeff`, `tanhForm_le`, `dobrushin_row`, `row_weighted`, `cov_le`, `iter_mul_invariant`, `iter_invariant`, `wilson_cov_dist`; `Sweep.iter_bound`; `WilsonLimit.dob_window_strict`. sorry 0; the axioms are the three standard ones. The list of declarations is in 19-06

Item	Kind	Source / tools
(11), (12) and their parts (two replicas, counting plaquettes)	Lean	`SwapDecay.covBound_minimal`, `minimalNonsep_sum_le`; `TorusRP.wilson_covariance`, `card_link_plaqs`, `plaq_degree`, `card_touch_le`, `height_lip`, `wilson_strong_coupling`
Values of the windows and rates, the plaquette counts (19, 21), the separations below the trivial bound	Reading	The numerical record of this book's study (floating point; the plaquette counts by exhaustive enumeration copying the Lean definitions)
Comparison with the true covariance on small Z_2 tori	Reading	The numerical record of this book's study (sum over all configurations; two and three dimensions; no violations)

Words used in this chapter — strong coupling, covariance, time distance, heat bath, influence coefficient, Dobrushin's condition, sweep, two replicas, chain of plaquettes, cluster expansion. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-03 The infinite-volume limit — no subsequence, translation invariance, clustering — from the decay of this chapter, uniform in N, it builds the limit measure without passing to a subsequence.

2026-11-13 · chapter Yang–Mills and the mass gap Part 20 Firm theorems — the vessel and strong coupling The infinite-volume limit · Kiichi, Shiori and Rin

The infinite-volume limit — no subsequence, translation invariance, clustering

> This chapter is this book's own investigation. The explanation of Dobrushin's condition in § 03 introduces established knowledge.

What this chapter shows — In the strong-coupling window $18 \tanh |\beta| < 1$, $\beta \neq 0$: the sequence of Wilson measures on the tori $(\mathbb{Z}/2(n+1))^4$, lifted periodically to $\mathbb{Z}^4$, converges weakly to a single probability measure ν_β without passing to a subsequence. The sequence of all sizes $(\mathbb{Z}/(m+1))^4$ has the same limit. ν_β is invariant under unit translations in the four directions. The limit measure has the clustering property in the four directions (correlations of distant observables decrease exponentially). Each is a formalisation, machine-checked in Lean, of a result of a known type.

Prerequisite chapters — 19-01 The Wilson measure, 19-03 The infinite-volume configuration space and lifts, 20-02 Decay of correlations at strong coupling. The infinite-volume limit and the cluster expansion as established knowledge are in 5-04.

Said first — The lattice spacing is fixed at 1. The window is only $|\beta| < 0.0556$ Reading , and nothing is said about the continuum limit or weak coupling. The limit is a statement about “the sequence of periodic lifts”; nothing is said about other boundary conditions or about uniqueness of the Gibbs measure in the DLR sense. The “unique” part of the statement is trivial from the topology; the content is the convergence without a subsequence.

The order of this chapter

1. The setting — configuration space, periodic lifts, weak convergence: which sequence converges, and in what sense.
2. Four theorems: the theorem on the limit without a subsequence, the theorem on the limit independent of size, translation invariance of the limit, clustering at limit points.
3. Dobrushin's condition — the influence of one link on its neighbours: where the number 18 and the tanh come from.

4. Comparing two sizes — the difference is exponentially small in the distance to the boundary: the core of the proof.
5. From a Cauchy sequence to the limit: cylinder functions determine the measure, and compactness.
6. Translation invariance and clustering: carrying properties of finite boxes over to the limit.
7. Detour: what “unique” says: the uniqueness of $\exists!$, and the content of the convergence.
8. What is not said: the window, boundary conditions, the continuum limit.

01

The setting — configuration space, periodic lifts, weak convergence

The objects are those defined in 19-03. Only the main points are repeated here.

Configuration space. Let $\Lambda = \mathbb{Z}^4 \times \{0,1,2,3\}$ be the set of all links of $\mathbb{Z}^4$; placing one element of SU(2) (a unit quaternion) on each link gives a configuration. The configuration space $\mathcal{C}$ carries the product topology. Each factor is compact, so $\mathcal{C}$ is compact and metrisable. On the Lean side the values are taken in the closed ball $\{q : |q| \le 1\}$, but the measures live on the unit sphere (SU(2)) (the conventions of 19-01).

Periodic lift. From a configuration ω on the torus $(\mathbb{Z}/N)^4$, build the configuration $x_\ell = \omega_{\ell \bmod N}$ on $\mathbb{Z}^4$. Write the probability measure obtained by pushing the normalised Wilson measure (19-01) forward by this map as

$$\mu_n = \mathrm{lift}_*\, P_{N_n,\beta}, \qquad N_n = 2(n+1) \qquad (n = 0,1,2,\dots) \tag{1}$$

Its Lean name is `liftProb β n`. We also use the lift of the torus of every size $m+1$, `liftS β m`.

Weak convergence. A sequence of probability measures μ_n converges weakly to ν if $\int F\,d\mu_n \to \int F\,d\nu$ for every bounded continuous function F on $\mathcal{C}$. In Lean this corresponds to `Tendsto … atTop (𝓝 ν)` in the topology of Mathlib's space of probability measures.

Cylinder functions and supports. A function determined only by the values on a finite set of links S is called a cylinder function with support S. The proof is first carried out for cylinder functions and is finally transferred to measures.

The window and the constant. Throughout this chapter we set

$$q(\beta) = 18 \tanh |\beta| \qquad (\text{in Lean, } \texttt{dobConst}\ \beta)$$ (2)

The window $q(\beta) < 1$ is the same as $|\beta| < \operatorname{artanh}(1/18) = 0.055613$ Reading . The explicit sufficient condition $|\beta| < 1/19 \Rightarrow q(\beta) < 1$ is in Lean as `WilsonLimit.dob_window_strict`.

02

Four theorems

The theorem on the limit without a subsequence

Cited A formalisation of a result of Dobrushin-uniqueness type. Lean: `InfVol.wilson_infvol_unique` ($\beta \neq 0$, $q(\beta) < 1$).

Let $\beta \neq 0$ and $18 \tanh |\beta| < 1$. Then there is a unique probability measure $\nu\beta$ such that the sequence μ_n (all n, without passing to a subsequence) converges weakly to $\nu\beta$.

The theorem on the limit independent of size

Cited A formalisation of a result of the same type. Lean: `AllN.wilson_infvol_allN` ($\beta \neq 0$, $q(\beta) < 1$).

Under the same hypotheses, the limit ν of the sequence μ_n of even sizes is also the limit of the sequence of all sizes `liftS β m` (the tori $(\mathbb{Z}/(m+1))^4$). Moreover, for any sequence of natural numbers s with $s(k) \to \infty$, `liftS β (s k)` also converges weakly to the same ν.

Translation invariance of the limit

Cited A formalisation. Lean: `TransI.wilson_infvol_translation` ($\beta \neq 0$, $q(\beta) < 1$).

The $\nu\beta$ above is invariant under the unit translations $x \mapsto x(\cdot + e_i)$ in the four directions $i = 0, 1, 2, 3$: for every bounded continuous F,

$$\int F(x(\cdot + e_i))\, d\nu\beta(x) = \int F\, d\nu\beta.$$ (3)

Clustering at limit points

Cited A formalisation of a result of the type of exponential mixing under Dobrushin's condition. Lean: `WilsonLimit.cluster_limit` ($\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$; ν is the weak limit of some subsequence of μ_n).

Fix a direction i. Let F, G be bounded continuous cylinder functions such that, for every link of the supports S_F, S_G, the i-th coordinate of its base point lies in $[0, R]$ (the other coordinates are arbitrary). Let G_k be G shifted by k in direction i, $G_k(x) = G(x(\cdot + k e_i))$. Then for $k \geq R$

$$\left| \int F\, G_k \, d\nu - \int F \, d\nu \int G_k \, d\nu \right| \;\leq\; 4 \, \|F\|_\infty \|G\|_\infty \, |S_G| \, e^{-\lambda (k - R)}. \tag{4}$$

The constants do not depend on k, ν, or the other coordinates. The form with $\lambda = -\frac{1}{2}\log q(\beta)$ for ν_β is included in the bundled statement (20-04).

03

Dobrushin's condition — the influence of one link on its neighbours

For an introduction, see 5-04. On a compact configuration space, a sequence of measures always has a convergent subsequence. But the limit need not be unique. To pin it down, one must say that distant influences are weak. Dobrushin's condition measures this “one link at a time”.

Influence coefficients. The conditional distribution of a link x (the distribution of x when all other links are fixed) is determined only by the weights $e^{\beta \operatorname{Re} U_p}$ of the plaquettes containing x. Let C_{xy} be an upper bound on how much the conditional distribution of x moves in total variation when the value of another link y is changed. When x and y share n_{xy} plaquettes, Lean takes

$$C_{xy} = \tanh\bigl(|\beta| \, n_{xy}\bigr) \tag{5}$$

(`coef`). This is because the difference in the local action of x when y is changed lies in $[-2|\beta|, 2|\beta|]$ per shared plaquette (width $4|\beta|$), and tilting an exponential weight by a width w moves the total variation by at most $\tanh(w/4)$ Proved here .

Row sum. In four dimensions one link lies in 6 plaquettes, and each plaquette has 3 other links. So $\sum_y n_{xy} = 18$, and from $\tanh(|\beta| n) \leq n \tanh|\beta|$

$$\sum_{y \neq x} C_{xy} \;\leq\; 18 \tanh|\beta| = q(\beta) \tag{6}$$

holds for every size of torus Proved here . Dobrushin's condition is that this row sum is less than 1, and the window $q(\beta) < 1$ reads directly as “the total influence of one link on all its neighbours is less than 1”.

Weighted form. The proof uses the row sum with weights $\rho_y = e^{\lambda d(y)}$ in the distance d, $\sum_{y \neq x} \rho_y C_{xy} \leq \alpha \rho_x$, $\alpha = e^{\lambda} q(\beta)$. Since d differs by only 1 between two links sharing a plaquette, the weight changes by at most a factor e^{λ} Proved here . The role of the window is that one can take $\lambda > 0$ while keeping $\alpha < 1$.

04

Comparing two sizes — the difference is exponentially small in the distance to the boundary

The core of the proof is that "the expectations on two tori of different sizes are both close to a single number independent of N". We divide it into steps.

Step 1: heat-bath sweeps. On the torus, the operation of redrawing the value of link x from its conditional distribution is called the heat bath P_x. The Wilson measure is invariant under each P_x. Let Q be the operation of redrawing, one at a time in order, the links of a box A around the origin (links whose base point has every coordinate $|\cdot| \leq L$), and repeat Q K times. The oscillation vector δ_y of a function f (the upper bound on the change in f when only y is changed) is carried by the influence coefficients C at each sweep. Measured with the weight $e^{\lambda d(y)}$ (d is the maximum cyclic distance from the origin), the part inside the box shrinks by a factor of at most α each time, and the part leaking outside the box is divided by the weight $e^{\lambda(L+1)}$. The conclusion is that for every configuration ω

$$\left| E_N f - (Q^K f)(\omega) \right| \;\leq\; \left(\alpha^K + e^{-\lambda(L+1)} \right) \sum_y e^{\lambda d(y)} \delta_y \qquad (7)$$

(`LinkSpec.torus_box`). N does not enter the right-hand side.

Step 2: inside the box it is the same as $\mathbb{Z}^4$. If $2(L+1) < N$, the links inside the box $L+1$ map one-to-one under projection to the torus, and plaquettes containing links inside the box do not wrap around. So for F depending only on the links of the box $L+1$, the lift of the sweep on the torus agrees with the same sweep on $\mathbb{Z}^4$: $Q_N^K (F \circ \mathrm{lift}) = (Q_{\mathbb{Z}}^K F) \circ \mathrm{lift}$ (`ZHeat.iter_lift`). The function $Q_{\mathbb{Z}}^K F$ on the right does not depend on N.

Step 3: comparison with a number independent of N. Apply Steps 1 and 2 to a bounded continuous cylinder function F whose support S lies in the box R_0. If $L \geq R_0$ and $n \geq L+1$, then

$$\left| \int F \, d\mu_n - c_{K,L}(F) \right| \;\leq\; \left(\alpha^K + e^{-\lambda(L+1)} \right) \cdot 2 \|F\|_\infty \, |S| \, e^{\lambda R_0} \qquad (8)$$

holds (`InfVol.lift_close`; $\lambda \geq 0$, $\alpha \leq 1$, $\beta \neq 0$). The number compared with, $c_{K,L}(F) = (Q^K_{\mathbb{Z}} F)(\mathbf{0})$, does not depend on n. $\mathbf{0}$ is the configuration in which every link has value 0 (the centre of the closed ball). Any comparison point would do; all that is needed is that it does not depend on n. It need not even be an SU(2) configuration.

Step 4: the difference between two sizes. Applying (8) to two values $n, n' \geq L+1$, subtracting, and letting $K \to \infty$ ($\alpha < 1$) gives

$$\left| \int F \, d\mu_n - \int F \, d\mu_{n'} \right| \;\leq\; 4\|F\|_\infty \, |S| \, e^{\lambda R_0} \, e^{-\lambda(L+1)} \qquad (L \geq R_0,\ L+1 \leq \min(n, n'))$$

Proved here . L is "the distance from the support to where the two tori begin to look different". The difference is exponentially small in this distance; this is what the heading means. This step is a corollary for reading and is not in the Lean chain. The proof of convergence (Step 5) goes directly from (8).

05

From a Cauchy sequence to the limit

Step 5: a Cauchy sequence. Taking $\lambda = -\frac{1}{2}\log q(\beta)$ gives $\alpha = q(\beta)^{1/2} < 1$. For a given ε, make α^K and $e^{-\lambda j}$ small and set $L = \max(R_0, j)$; then by (8) the sequence of expectations $\int F \, d\mu_n$ of a cylinder function is a Cauchy sequence (`InfVol.cauchy_cyl`). The real numbers are complete, so each cylinder function has a limit.

Step 6: cylinder functions determine the measure. If two finite measures on $\mathcal{C}$ give the same integral for every bounded continuous cylinder function, they are equal (`CylDet.ext_of_cyl`). The marginals on finitely many links agree, and by the uniqueness of projective limits (Mathlib) the whole measures agree.

Step 7: convergence without a subsequence. $\mathcal{C}$ is compact and metrisable, so the space of probability measures is also compact. Every subsequence has a convergent subsequence, whose limit agrees with the limit of Step 5 on cylinder functions, and so is the same measure by Step 6. Since every subsequence has a sub-subsequence converging to the same limit, the sequence itself converges (`CylDet.tendsto_of_cyl_cauchy`).

All sizes. The estimate (8) does not use the parity of the size, and holds for the lifts of every size $m+1$ with the same number $c_{K,L}(F)$ in between (`AllN.lift_closeS`). So the sequence of all sizes, and any partial sequence of sizes, have the same limit Proved here .

Step	What it says	Lean name
1	The estimate for box sweeps (7), no N in the constants	`LinkSpec.torus_box`

Step	What it says	Lean name
2	For $2(L+1) < N$, the sweep inside the box agrees with $\mathbb{Z}^4$	`ZHeat.iter_lift`
3	Comparison with a number independent of N (8)	`InfVol.lift_close`
5	Expectations of cylinder functions are Cauchy	`InfVol.cauchy_cyl`
6	Cylinder functions determine the measure	`CylDet.ext_of_cyl`
7	From Cauchy to convergence of the sequence	`CylDet.tendsto_of_cyl_cauchy`

As an example showing that the hypotheses can be met, the case $\beta = 1/20$ is in Lean as `InfVol.infvol_example` ($18 \tanh(1/20) = 0.8993$ Reading).

06

Translation invariance and clustering

Translation invariance. The Wilson measure on each torus is invariant under unit translations of the torus. The lift commutes with translations, so each μ_n is invariant under unit translations of $\mathbb{Z}^4$ (`TransI.lift_shiftI_invariant`). If F is bounded continuous, so is $F(x(\cdot + e_i))$, and taking the limit of both sides under weak convergence gives (3) (`TransI.limit_shiftI_invariant`). The reverse translations are also bijections, so ν_β is invariant under all translations of $\mathbb{Z}^4$ Proved here .

Clustering. The covariance estimate on the torus (the Dobrushin distance form of 20-02) uses about the distance d only that "it changes by at most 1 between two links sharing a plaquette". So replace d by the cyclic distance of the i-th coordinate of the base point (`LinkSpec.coordLip`). The support of F is in $d \le R$ and that of the shifted G_k in $d \ge k$, and if the torus is large enough ($n \ge k + R$)

$$\left|\mathrm{Cov}_{P_N}(F, G_k)\right| \;\le\; 2\|F\|\, e^{\lambda R}\, |S_G|\, e^{-\lambda k} \cdot 2\|G\|$$

Since $F \cdot G_k$, F and G_k are all bounded continuous, the estimate passes to the weak limit and becomes (4) Proved here . Whether the limit point is unique is not used.

Example. For $\beta = 1/20$, $q = 0.8993$ and $\lambda = -\frac{1}{2}\log q = 0.0531$ Reading . The exponential factor $e^{-\lambda(k-R)}$ on the right-hand side of (4) becomes 0.1 when $k - R \approx 43$ Reading . The closer to the edge of the window, the slower the decrease. There is no record of the ratio to the true decrease of correlations.

Detour: what "unique" says

May be skipped. The Lean statement has the form $\exists!\, \nu,\ \mu_n \to \nu$. The topology of weak convergence on the space of probability measures is Hausdorff, so the limit of one sequence, if it exists, is always unique. So the "unique" part is trivial from the topology and adds no

information. The content of the statement is that “the sequence itself converges, without passing to a subsequence”.
Compactness alone only gives “there is a convergent subsequence”. In that case different subsequences could have different limits. The theorem on the limit without a subsequence closes off that possibility for this sequence (if the sequence converges, the limit of every subsequence is the same ν_β; Lean's `CylDet.subseq_limit_eq`).
The sequence is not constant. The lift for $n = 0$ (the 2^4 torus) has period 2, and $x_{\ell+2e_1} = x_\ell$ in every configuration. For $n = 1$ (4^4) these two are different links, and the probability that they agree is 0 Proved here . So $\mu_0 \neq \mu_1$.
Do not read “unique” as uniqueness of the Gibbs measure including limits coming from other boundary conditions. That is outside this theorem (§ 07).

07

What is not said

- **The lattice spacing is fixed at 1.** Nothing is said about the continuum limit, the operation of sending the lattice spacing to 0.
- **The window is narrow.** Only $|\beta| < 0.0556$ Reading . Nothing is said about weak coupling.
- $\beta = 0$ **is outside the statement.** The hypothesis is $\beta \neq 0$ ($\beta = 0$ is the product of Haar measures, and its limit is trivial).
- **Periodic boundary conditions only.** This is convergence of the sequence of periodic lifts; nothing is said about sequences with free or fixed boundary conditions, or about uniqueness of measures satisfying the DLR equations.
- **Translations are the generators, the unit translations.** Invariance under rotations and other lattice symmetries is not in the theorems of this chapter. The invariance under generators in 20-05 is a theorem about a different measure (the measure with fermions on a finite lattice).
- **The clustering estimate is an upper bound only.** (4) is an upper bound on the correlations (a lower bound on the rate of decrease) and does not say how fast the true correlations decrease. λ is chosen within $e^\lambda q(\beta) < 1$, and the $\frac{1}{2}$ in $-\frac{1}{2}\log q$ is a matter of choice.
- **Not a new result.** These are known results of Dobrushin-uniqueness type and Osterwalder–Seiler strong-coupling type, written in Lean on the concrete SU(2) Wilson measure.

What this chapter can say

What is said	What is not said
Cited The theorem on the limit without a subsequence: for $\beta \neq 0$, $18 \tanh \lvert\beta\rvert < 1$, μ_n (size $2(n+1)$) converges weakly, as a sequence, to a single ν_β (`InfVol.wilson_infvol_unique`)	Outside the window, $\beta = 0$, other boundary conditions, uniqueness of the Gibbs measure. "Unique" is trivial from the topology
Cited The theorem on the limit independent of size: the same limit for all sizes $m+1$ and for any $s(k) \to \infty$ (`AllN.wilson_infvol_allN`)	An explicit form of the rate of convergence (an estimate of the form (9) exists, but is not in the statement)
Cited Translation invariance of the limit: invariant under unit translations in four directions (`TransI.wilson_infvol_translation`)	Invariance under rotations and reflections
Cited Clustering at limit points: (4), four directions, at every limit point (`WilsonLimit.cluster_limit`, $\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$)	The true rate of decay, outside the window
Proved here The estimate (9) of the difference between two sizes (a one-line consequence of (8) in Lean)	(9) itself is not a Lean declaration
Reading The window $\lvert\beta\rvert < 0.055613$; for $\beta = 1/20$, $q = 0.8993$ and $\lambda = 0.0531$	Floating-point values (not interval estimates)

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity, the transfer matrix and the strong-coupling type for lattice gauge theory at strong coupling	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract). 5-04
Dobrushin's uniqueness condition (row sum of influence coefficients less than 1)	Cited	The original source of R. L. Dobrushin's uniqueness condition (bibliographic data only, not checked; the year and journal have not been verified in this book)
Definition of lattice Yang–Mills in the language of probability, and an overview of strong-coupling results	Cited	S. Chatterjee, arXiv:1803.01950 (full text)
The four theorems and the steps of proof	Lean	Machine-checked in Lean 4 (all pass the kernel re-check; the axioms are the three standard ones). Declarations in the tables of the text, the table of 20-07, and A-04
The edge of the window, the rates, $18 \tanh(1/20)$	Reading	The numerical record of this book's study (double-precision floating point)

Words used in this chapter — periodic lift, weak convergence, cylinder function, Dobrushin's condition, influence coefficient, heat bath, sweep, clustering, strong-coupling window. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-04 Reconstruction and the mass gap — inside the strong-coupling window — from the $\nu\beta$ of this chapter it assembles the Hilbert space and the transfer operator, and reads off that there are states outside the vacuum and a lower bound on the gap.

2026-11-13 · chapter Yang–Mills and the mass gap Part 20 Firm theorems — the vessel and strong coupling Reconstruction and the mass gap · Kiichi, Shiori and Rin

Reconstruction and the mass gap — inside the strong-coupling window

> This chapter is this book's own investigation.

What this chapter shows — In the strong-coupling window: the transfer operator T on the Hilbert space reconstructed from the infinite-volume measure ν_β satisfies $\|T\| \leq (18 \tanh|\beta|)^{1/2}$ on the orthogonal complement of the vacuum. That complement is not 0 (non-degeneracy). The same holds in the sector of gauge-invariant observables. For $\beta \geq 0$, $T \geq 0$. The range of T lies in the gauge-invariant sector. And a statement bundling these into one. Each is a formalisation, machine-checked in Lean, of a result of a known type.

Prerequisite chapters — 19-03 The vessel of reconstruction, the definition of the mass gap, 20-03 The infinite-volume limit, 20-01 Reflection positivity. Reconstruction and the definition of the mass gap as established knowledge are in 4-10 and 4-11.

Said first — The "mass gap" here is only the upper bound $\|T - P_\Omega\| \leq e^{-m}$, that is, a **lower bound** on the mass. It does not say that the mass is finite ($T \neq P_\Omega$). The lattice spacing is fixed at 1, the window is only $|\beta| < 0.0556$ Reading , and nothing is shown about a mass gap in the continuum limit.

The order of this chapter

1. The vessel — building the Hilbert space and T from the pairing, the translation and the constant: the three data the statements pin down.
2. The mass gap with non-degeneracy: two theorems for two windows.
3. Why non-degeneracy is put in the statement: avoiding a form that holds even for a one-point model.
4. The route of proof: from decay of correlations to the operator norm.
5. The gauge-invariant sector and the range of T: the plaquette observable, and F_0 falling into the kernel.
6. T ≥ 0: reflection in the middle plane and the gauge average.
7. The bundled statement: $0 < \beta$, everything inside the window.

8. What is not said: lower bound only, the window, the lattice spacing.

01

The vessel — building the Hilbert space and T from the pairing, the translation and the constant

The definition of the vessel is in 19-03. For an introduction, see 4-10. Here we write only the form we use.

The space of observables W. The set of bounded continuous functions depending only on finitely many links whose base points have time coordinate from 0 to R (R finite for each function). Equivalently, the cylinder functions on the closed positive half. Its Lean name is `Wsub`.

Three data. Let ν be a probability measure, and set

$$B(F,G)=\int F(\Theta x)\,G(x)\,d\nu(x),\qquad (\tau F)(x)=F(x(\cdot+e_0)),\qquad \mathbf{1}=\text{the constant 1}\ \text{(1)}$$

Θ is the time reflection $x_0\mapsto -x_0$ (time links are shifted by one and reversed in direction; 19-03). When B is positive semi-definite and symmetric, τ is symmetric with respect to B, $\tau\mathbf{1}=\mathbf{1}$, $B(\mathbf{1},\mathbf{1})=1$, and the orbits under translation are bounded, the triple $(B,\tau,\mathbf{1})$ is called a **vessel of reconstruction** (Lean's `OSGap.Datum`).

What is built. Put the semi-inner product B on W, divide out the null elements and complete to get $\mathcal{H}$; write $[F]$ for the image of an element of W, the vacuum $\Omega=[\mathbf{1}]$, T for the continuous extension of τ, and $P_\Omega v=\langle\Omega,v\rangle\Omega$ for the projection onto the vacuum.

Definition of the mass gap. The definition is in 19-03.

$$\texttt{HasMassGap}\ m\quad:\Longleftrightarrow\quad \|T-P_\Omega\|\le e^{-m}.\ \text{(2)}$$

Since $T\Omega=\Omega$ and T is symmetric, this is the same as "$\|T\|\le e^{-m}$ on the orthogonal complement of Ω" Proved here . In one time step, the components other than the vacuum shrink by a factor of at least e^{-m}. m is a lower bound on the mass (the energy gap).

The statements pin down the data. All the theorems below have the form "there is a vessel D such that the B of D is the integral (1), its τ is the translation by one time step, its $\mathbf{1}$ is the constant 1, and ...". All three data fields are fixed in the conclusion, so D is determined by ν Proved here . One cannot satisfy the conclusion by swapping T for an arbitrary operator.

02

The mass gap with non-degeneracy

Set $F_0(x) = \operatorname{Re} x\ell_0$, where ℓ_0 is the link of direction 1 (spatial) based at the origin; $F_0 \in W$. Write $q(\beta) = 18 \tanh|\beta|$ (20-03).

The mass gap with non-degeneracy — the Dobrushin window

Cited A formalisation of results of Dobrushin and Osterwalder–Seiler strong-coupling type. Lean: `WilsonLimit.wilson_massGap_dob_exists` ($\beta \neq 0$, $q(\beta) < 1$).

Let $\beta \neq 0$ and $18 \tanh|\beta| < 1$. Then there are a weak limit ν of some subsequence of μ_n and a vessel D built from ν by (1) such that

$$\|T - P_\Omega\| \le \big(18 \tanh|\beta|\big)^{1/2}, \qquad B(F_0, F_0) - B(\mathbf{1}, F_0)^2 \;\ge\; \tfrac{1}{4} e^{-12|\beta|}, \qquad \exists\, v \perp \Omega,\ \|v\| =$$

The first is **HasMassGap** with $m = -\frac{1}{2}\log q(\beta)$; the second and third are non-degeneracy.

The mass gap with non-degeneracy — at every limit point

Cited A formalisation of a result of the same type. Lean: `WilsonLimit.wilson_massGap_nondegenerate` ($\beta \neq 0$, $24(1 - e^{-4|\beta|}) < 1$; ν is the weak limit of any subsequence of μ_n).

In the window $24(1 - e^{-4|\beta|}) < 1$ ($|\beta| < 0.0106$ Reading), for **every** limit point ν, the vessel D satisfies **HasMassGap**$\big(-\log 24(1 - e^{-4|\beta|})\big)$ and the two non-degeneracy conditions of (3). The estimate in this window comes from the finite-volume decay of 20-02.

The two theorems differ in window and rate. The Dobrushin-window form picks one limit point in its statement, but the same estimate holds at every limit point (`WilsonLimit.wilson_massGap_dob`, § 04 Step 4). Moreover, inside the window the limit point is unique, ν_β (20-03). The narrow-window form reaches a conclusion of the same type by a different route, from the finite-volume decay (20-02), with rate $-\log 24(1 - e^{-4|\beta|})$. § 07 bundles the Dobrushin-window form for the limit ν_β without a subsequence. The narrow-window rate is not in § 07.

An example. For $\beta = 1/20$, $q = 0.8993$, $\|T - P_\Omega\| \le 0.948$, and the lower bound on the mass is $m = 0.053$ Reading. The lower bound for non-degeneracy is $\frac{1}{4}e^{-0.6} = 0.137$ Reading. The closer to the edge of the window, the closer the lower bound on the gap gets to 0.

03

Why non-degeneracy is put in the statement

(2) alone can have thin content. If $\mathcal{H}$ is the one-dimensional space spanned by Ω, then $T = P_\Omega$ and $\|T - P_\Omega\| = 0$. Then (2) holds for every m. This happens, for example, if ν is a point mass. So unless "there is a gap" is paired with "there are states outside the vacuum", it is a statement that holds even for a one-point model. For this reason, this book does not present theorems in the form without non-degeneracy.

Two forms of non-degeneracy. The second item of (3) is "the variance of F_0 measured by the reflection pairing is positive"; the third is "there is a unit vector orthogonal to Ω". The third follows from the second: $y = F_0 - B(\mathbf{1}, F_0)\mathbf{1}$ has $B(\mathbf{1}, y) = 0$ and $B(y, y) > 0$, so $v = [y]/\|[y]\|$ is a unit vector (`OSGap.Datum.exists_unit_orth`). The third rules out a one-dimensional $\mathcal{H}$ at the level of types.

Where the lower bound on the variance comes from. In steps:

1. ℓ_0 is a spatial link at time 0, so it is fixed by Θ, and $F_0 \circ \Theta = F_0$. Hence $B(F_0, F_0) - B(\mathbf{1}, F_0)^2 = \mathrm{Var}_\nu(F_0)$ Proved here .
2. For the Haar measure on SU(2), left invariance under $q \mapsto -q,\ iq,\ jq,\ kq$ gives $E\,\mathrm{Re}\,U = 0$ and $E(\mathrm{Re}\,U)^2 = \frac{1}{4}$. Hence $E(\mathrm{Re}\,U - c)^2 \geq \frac{1}{4}$ for every c (`HaarRe.integral_sq_sub_ge`).
3. The density of the conditional distribution of one link is given by the weights of the 6 plaquettes containing that link, and lies between $e^{\pm 6|\beta|}$. So the conditional variance is at least $\frac{1}{4}e^{-12|\beta|}$, and integrating gives $\mathrm{Var}(F_0) \geq \frac{1}{4}e^{-12|\beta|}$ on every torus (`LinkSpec.wilsonProb_sq_sub_ge`).
4. F_0 and F_0^2 are bounded continuous, so the lower bound passes to the weak limit (`WilsonLimit.var_limit_ge`).

04

The route of proof

The mass-gap part combines a general lemma that turns decay of correlations into an operator norm with the decay estimate for the Wilson measure.

Step 1: the axioms of the vessel. Positive semi-definiteness of B comes from carrying reflection positivity for the plane through sites on each torus (20-01) over to the weak limit. Symmetry of τ comes from translation invariance and its relation to Θ; boundedness of orbits from $|F| \leq \|F\|_\infty$.

Step 2: from decay to the operator norm. On the vessel, if for every $F \in W$ there is a constant C_F with

$$B(\tau^k F, \tau^k F) - B(\mathbf{1}, F)^2 \;\le\; C_F\, r^{2k} \qquad (k = 0, 1, 2, \dots) \tag{4}$$

then $\|T - P_\Omega\| \le r$ (`OSGap.Datum.opNorm_gap`). The left-hand side is $\|T^k[F] - P_\Omega[F]\|^2$ Proved here, so this is an argument that turns an estimate on a dense set into a spectral estimate for a symmetric operator.

Step 3: decay for the Wilson measure. $B(\tau^k F, \tau^k F) = \int F(\Theta\tau^k x) F(\tau^k x)\, d\nu$ is the correlation of two observables about $2k$ apart in time. Applying the Dobrushin distance covariance estimate on the torus (20-02, `LinkSpec.wilson_cov_dist`) gives $C_F\, e^{-2\lambda k}$ with a constant independent of N (`TorusRP.wilson_hdec_torus_dob`). λ is any number with $\lambda \ge 0$ and $e^{\lambda} q(\beta) < 1$ (in Step 4, $\lambda > 0$). Decay of this form is `WilsonLimit.dec_eventually_dob`. The estimate passes to limit points.

Step 4: choice of rate. Applying Step 2 with $r = e^{-\lambda}$ gives `HasMassGap` λ (`WilsonLimit.wilson_massGap_dob`). The $\lambda = -\frac{1}{2}\log q(\beta)$ in (3) is one choice satisfying $e^{\lambda} q = q^{1/2} < 1$, and the $\frac{1}{2}$ has no physical meaning. From the same theorem one can take a rate arbitrarily close to $-\log q(\beta)$.

Step 5: non-degeneracy. The four steps of § 03.

05

The gauge-invariant sector and the range of T

For an introduction, see 4-11. A gauge transformation $g : \mathbb{Z}^4 \to SU(2)$ acts as $(g \cdot x)_\ell = g(b)\, x_\ell\, g(b + e_\mu)^*$ (b the base point of ℓ, μ its direction). Let W_{gi} be the set of elements of W invariant under every g. $F_p = \mathrm{Re}\, U_p$ (the real part of the holonomy of the $(1,2)$ plaquette at the origin) belongs to W_{gi}.

The mass gap in the gauge-invariant sector

Cited A formalisation. Lean: `GaugeZ.wilson_massGap_gauge_exists` ($\beta \neq 0$, $q(\beta) < 1$).

The same form as (3) holds with W replaced by W_{gi} and F_0 by F_p. For some limit point ν, on the vessel over W_{gi}, $\|T - P_\Omega\| \le q(\beta)^{1/2}$, $\mathrm{Var}_\nu(F_p) \ge \frac{1}{4} e^{-12|\beta|}$, and there is a unit vector orthogonal to the vacuum. The lower bound on the variance comes from the fact that when one link U of the plaquette is varied, $U_p = UV$ (V, the product of the other three, is a unit), and the Haar variance of $\mathrm{Re}(UV)$ is the same as that of $\mathrm{Re}\, U$.

Why separate the sector. The witness F_0 of non-degeneracy for the whole W is not gauge invariant. The bundled statement (§ 07) contains $T[F_0] = 0$. That is, $[F_0]$ is a nonzero vector orthogonal to the vacuum, but it is killed by T. This is because a gauge

transformation placing the central element -1 at the site e_0 gives $\int G(\Theta x) F_0(\tau x)\, d\nu = 0$ for every $G \in W$ Proved here . This is the same mechanism as Elitzur's theorem (local gauge symmetry cannot be broken) Cited . Non-degeneracy in the gauge-invariant sector (witness F_p) is the non-degeneracy with content. F_p is not a vector known to be killed by T. However, $T[F_p] \;/= 0$ (that T is not 0 on the complement in this sector) is not in the Lean statement either.

The range of T lies in the gauge-invariant sector

Cited A formalisation. Lean: `Bind.T_range_gauge_all` ($\lambda > 0$, $e^{\lambda} q(\beta) < 1$; ν any limit point).

For the vessel D over W and the vessel D' over W_{gi} (both with data pinned down by (1)), both satisfy **HasMassGap** λ, and there is an isometric linear map $\iota : \mathcal{H}' \to \mathcal{H}$ with

$$\iota \Omega' = \Omega, \qquad \iota T' = T \iota, \qquad T u \in \iota(\mathcal{H}') \text{ for all } u \in \mathcal{H}. \quad (5)$$

The gauge-invariant sector embeds isometrically into the whole Hilbert space, commutes with T, and the range of T lies entirely inside it. The proof uses that taking the Haar average of gauge transformations over the end points of the support of τF gives an element of W_{gi} without changing the values of the pairing. The image of ι is closed and T is continuous, so the statement extends from a dense set to all u.

06

T ≥ 0

Cited A formalisation. Lean: `FullT.wilson_Tpos_full` ($\beta \geq 0$, $\lambda > 0$, $e^{\lambda} q(\beta) < 1$; ν any limit point).

Let $\beta \geq 0$. There is a vessel D over W (with data pinned down by (1)) for which **HasMassGap** λ, the two non-degeneracy conditions of (3), and

$$\langle v, T v \rangle \;\geq\; 0 \qquad (\text{for all } v \in \mathcal{H}) \quad (6)$$

all hold.

Route. $\langle [F], T[F] \rangle = B(F, \tau F) = \int F(\Theta x) F(\tau x)\, d\nu$. Since $\tau \circ \Theta' = \Theta$ (Θ' is the reflection $x_0 \mapsto 1 - x_0$ in the plane through the middle of links), this is the pairing for reflection in the middle plane. Reflection positivity for the middle plane (20-01) is about gauge-invariant functions and uses $\beta \geq 0$. So we insert the Haar average A over gauge transformations on all sites of the torus (G is a bounded measurable function depending only on the positive side). The measure is invariant under gauge

transformations, so $\int G(\Theta'\omega)G(\omega) = \int (AG)(\Theta'\omega)(AG)(\omega)$, and AG is gauge invariant. Thus it is positive on each torus, positive in the limit, and extends from a dense set to all v.

Reading. Combining $T \geq 0$, $T\Omega = \Omega$ and $\|T - P\Omega\| \leq e^{-m}$, the spectrum of T on the complement of Ω lies in $[0, e^{-m}]$ Proved here . Writing $-\log T$, we have $-\log T \geq m$ on the complement (read as $+\infty$ on the kernel of T). Nothing is said about $\beta < 0$.

07

The bundled statement

Cited A formalisation of results of Dobrushin-uniqueness type and Osterwalder–Seiler strong-coupling type. Lean: `InfMain2.wilson_infvol_main2` ($0 < \beta$, $18 \tanh \beta < 1$).

Let $0 < \beta$ and $18 \tanh \beta < 1$, and set $m = -\frac{1}{2} \log(18 \tanh \beta)$. There is a unique probability measure ν_β for which all of the following hold.

1. $\mu_n \to \nu_\beta$ (weak convergence without a subsequence; 20-03).
2. ν_β is invariant under unit translations in the four directions.
3. $\int F(\Theta x) F(x)\, d\nu_\beta \geq 0$ for every $F \in W$ (reflection positivity).
4. On the vessel over W (data pinned down by (1)): $\mathtt{HasMassGap}\ m$, $\mathrm{Var}(F_0) \geq \frac{1}{4} e^{-12\beta}$, existence of a unit vector orthogonal to Ω, $T[F_0] = 0$, and $T \geq 0$.
5. On the vessel over W_{gi} (pinned down likewise): $\mathtt{HasMassGap}\ m$, $\mathrm{Var}(F_p) \geq \frac{1}{4} e^{-12\beta}$, existence of a unit vector orthogonal to Ω, and $T \geq 0$.
6. Clustering in four directions: (4) of 20-03 with rate m.

The “unique” part is trivial from the topology; the content is the convergence in item 1 (the detour in 20-03). That the range of T lies in the gauge-invariant sector ((5) of § 05) is not in this statement. It is the separate theorem `Bind.T_range_gauge_all`.

08

What is not said

- **Only a lower bound on the mass.** $\|T - P\Omega\| \leq e^{-m}$ also holds if T is 0 on the complement (“infinite mass”). It is not said that the mass is finite, or that $T \neq P\Omega$. The number $-\frac{1}{2} \log(18 \tanh \beta)$ is a lower bound, and the $\frac{1}{2}$ is a matter of choice. There is no record of its ratio to the true mass.

- **The window is narrow.** $|\beta| < 0.0556$ Reading (the form for every limit point: $|\beta| < 0.0106$ Reading). Nothing is said about weak coupling or the physical range of couplings.
- **The lattice spacing is fixed at 1.** This is a lower bound on the mass measured in lattice units. Whether a mass gap survives in the continuum limit is outside the theorems of this part.
- **Sign.** The bundled statement needs $0 < \beta$, and $T \geq 0$ needs $\beta \geq 0$. The gap and non-degeneracy alone hold for $\beta \neq 0$ of either sign.
- **Reading of the sector.** W_{gi} is the set of functions that are "cylinder functions and gauge invariant". Whether its closure coincides with the physical space of gauge-invariant states is not said.
- **Not a new result.** These are known results of Osterwalder–Seiler strong-coupling type, written in Lean on the concrete SU(2) Wilson measure.

What this chapter can say

What is said	What is not said
Cited The mass gap with non-degeneracy (Dobrushin window, some limit point): (3) (`WilsonLimit.wilson_massGap_dob_exists`, $\beta \neq 0$)	That the mass is finite; outside the window
Cited The mass gap with non-degeneracy (every limit point, window $24(1 - e^{-4\|\beta\|}) < 1$) (`WilsonLimit.wilson_massGap_nondegenerate`)	As above; the window is narrower still
Cited The mass gap and non-degeneracy in the gauge-invariant sector (witness F_p) (`GaugeZ.wilson_massGap_gauge_exists`)	That the closure of W_{gi} coincides with the physical space of states
Cited $T \geq 0$ (on the whole W) (`FullT.wilson_Tpos_full`, $\beta \geq 0$)	$\beta < 0$
Cited The range of T lies in the gauge-invariant sector, (5) (`Bind.T_range_gauge_all`)	Not included in the bundled statement
Cited The bundled statement (`InfMain2.wilson_infvol_main2`, $0 < \beta$, $18 \tanh \beta < 1$)	The continuum limit, changing the lattice spacing, an upper bound on the mass
Proved here The reading, from $T \geq 0$ and the gap, that the spectrum on the complement lies in $[0, e^{-m}]$	The reading itself is not a Lean declaration
Reading For $\beta = 1/20$: $\|T - P_\Omega\| \leq 0.948$, $m = 0.053$, lower bound for non-degeneracy 0.137	Floating-point values

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity, the transfer matrix and the strong-coupling mass-gap type for lattice gauge theory	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract). 5-03, 5-04
Euclidean axioms and reconstruction	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281 (abstracts). 4-10
Local gauge symmetry cannot break spontaneously (the reading of $T[F_0] = 0$)	Cited	S. Elitzur, Phys. Rev. D 12 (1975) 3978 (abstract)
Dobrushin's uniqueness condition	Cited	Original source: bibliographic data only, not checked (the year and journal have not been verified in this book)
The theorems of this chapter and their parts	Lean	Machine-checked in Lean 4 (all pass the kernel re-check; the axioms are the three standard ones). The list of declarations is in 20-07 and A-04
The numbers for $\beta = 1/20$, the edge of the window	Reading	The numerical record of this book's study (double-precision floating point)

Words used in this chapter — vessel of reconstruction, transfer operator, vacuum, mass gap (lower bound), non-degeneracy, gauge-invariant sector, Elitzur's theorem, reflection positivity, strong-coupling window. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-05 The vessel with fermions — on a finite lattice whose weight includes the Wilson–Dirac determinant, it reads reflection positivity, Z > 0 and the symmetries.

The vessel with fermions

This chapter is this book's own investigation.

What this chapter shows — On the finite lattice $(\mathbb{Z}/N)^3 \times \mathbb{Z}/2h$, the measure given by the Wilson gauge weight $e^{-\beta S_W}$ multiplied by the determinant of the Wilson–Dirac operator, $\mathrm{Re}\left[(\det D)^{2j}\right]$, is positive with respect to reflection in a time plane through sites (mass $m > -1$, all β). The partition function is positive (all β, m, j). $\det D$ is invariant under global gauge transformations. The normalised measure is invariant under the generators of the lattice symmetries. Each is a formalisation, machine-checked in Lean, of a result of a known type.

Prerequisite chapters — 19-05 The Wilson–Dirac operator, the measure with fermions, 20-01 Reflection positivity. Reflection positivity on the lattice and the transfer matrix as established knowledge are in 5-03.

Said first — All are statements on one finite lattice. Nothing is said about the infinite-volume limit, the continuum limit, chiral symmetry, or mass renormalisation. The symmetries are invariance under each generator; closure as a group is outside the statements. There is no mass gap with fermions in this chapter either.

The order of this chapter

1. The objects — the lattice, the action, the determinant weight: which measure.
2. Reflection positivity for the plane through sites: $m > -1$, all β.
3. The route of proof of reflection positivity: how the action splits, and positivity of the plane matrix.
4. Positivity of the partition function: normalisation can be defined.
5. Global gauge invariance of the determinant: conjugation by a constant g.
6. Invariance under the generators of the lattice symmetries: translations, rotations, reflections, time reflection.
7. What is not said: a finite lattice, the continuum limit, chiral symmetry.

01

The objects — the lattice, the action, the determinant weight

The definitions are in 19-05. Here we write only what is needed to read the theorems.

Lattice and links. Space is the three-dimensional torus $(\mathbb{Z}/N)^3$, and time is the circle $\mathbb{Z}/2h$ ($h \geq 1$). The links are the spatial links $x \to x + e_i$ ($i = 1, 2, 3$) at time t, and the time links $t \to t + 1$. Placing an element of SU(2) on each link gives a configuration ω, and the configuration space carries the product of Haar measures over links.

Gauge action. Counting unoriented plaquettes once each, set

$$S_W(\omega) = \sum_p \left(1 - \tfrac{1}{2} \operatorname{Re} \operatorname{tr} U_p\right) \quad (1)$$

and take the weight $e^{-\beta S_W}$. Since $\frac{1}{2} \operatorname{tr} U$ equals the real part of the quaternion, this is a constant multiple of the weight $e^{\beta \sum_p \operatorname{Re} U_p}$ of 19-01 Proved here. The orientation of β is also the same.

The Wilson–Dirac operator. $D = D(\omega)$ is the operator with Wilson-term coefficient $r = 1$ and mass m, with antiperiodic boundary conditions in all four directions (Lean's `WilsonGauge.physDirac`).

$$D = (4 + m)\,\delta - \tfrac{1}{2} \sum_\mu \left[(1 - \gamma_\mu)\, U_\mu(x)\, \delta_{x+\hat{\mu}} + (1 + \gamma_\mu)\, U_\mu(x - \hat{\mu})^\dagger\, \delta_{x-\hat{\mu}}\right] \quad (2)$$

The weight with fermions. Fix a natural number j, and set

$$w(\omega) = e^{-\beta S_W(\omega)} \cdot \operatorname{Re}\left[(\det D(\omega))^{2j}\right] \quad (3)$$

(`wfWeight`). As in 19-05, $(\det D)^{2j}$ is a non-negative real number, and Re is a notational convention. $\det D^{2j}$ corresponds to integrating out, with Grassmann variables, $2j$ flavours of fermions of the same mass. In Lean no Grassmann variables are introduced; the determinant itself is defined as the weight. The normalised measure $P = w\, d\omega / Z$, $Z = \int w\, d\omega$, is `wfProb` (19-05).

02

Reflection positivity for the plane through sites

For an introduction, see 5-03. The time reflection $\Theta : t \mapsto -t \pmod{2h}$ fixes the two site planes at times $t = 0$ and $t = h$. Divide the configuration into three parts: the spatial links on the two planes, c; the positive side a_+ (spatial links with $0 < t < h$, and time

links leaving $0 \le t < h$); and the negative side a_- (the rest, carried by Θ into the form of the positive side, with time links inverted).

Reflection positivity of the Wilson–fermion weight

Cited A formalisation of a result of Osterwalder–Seiler type. Lean: `WilsonGauge.site_reflection_positive_wilson_full` ($h > 0$, $m > -1$, all $\beta \in \mathbb{R}$, all j).

If φ is a bounded measurable real-valued function of (c, a_+), then

$$\int \varphi(c, a_+)\, \varphi(c, a_-)\, e^{-\beta S_W(\omega)}\, \mathrm{Re}\big[(\det D(\omega))^{2j}\big]\, d\omega \;\ge\; 0. \quad (4)$$

$\varphi(c, a_-)$ is φ read on the reflected configuration, that is, $(\Theta\varphi)$. (4) is reflection positivity of the form "$\langle \Theta\varphi \cdot \varphi \rangle \ge 0$", stated for the unnormalised measure. In Lean the integral is written in the split coordinates (c, a_+, a_-), and `WilsonGauge.physDirac_eq_split` says that the operator assembled in those coordinates is the same as the D of the physical configuration. Since $Z > 0$ (§ 04), the sign is the same after normalisation Proved here .

Why the sign of β does not matter. For reflection in the plane through sites, no plaquette straddles the plane. Every plaquette lies on one of the two fixed planes, on the positive side, or on the negative side. So $e^{-\beta S_W}$ splits as "a factor on the planes only $\times$ a factor on the positive side $\times$ a factor on the negative side", and no sign condition on the factors is needed Proved here . For reflection in the plane through the middle of links, $\beta \ge 0$ is needed to expand the factor of the straddling plaquettes as a positive kernel. The theorems of this chapter do not treat that case.

Why $m > -1$. On the fermion side, we use that the block of the operator on the fixed planes (the plane matrix) is positive definite (`FermionSliceBlock.wilsonPlane_posDef`). The plane matrix has the form $(m+1) + \sum_i (1 - \frac{1}{2}(T_i + T_i^\dagger))$ (T_i unitary hops), and each term $1 - \frac{1}{2}(T_i + T_i^\dagger)$ is positive semi-definite, so it is at least $m + 1$ Proved here . $m > -1$ is a sufficient condition for this. Whether it is necessary cannot be told from this theorem.

03 The route of proof of reflection positivity

Step 1: how the action splits. $S_W(\omega) = K(c) + H(c, a_+) + H(c, a_-)$, where H is **the same function** on the positive and negative sides (`WilsonGauge.wilsonAction_split`). A time plaquette on the negative side takes the form of one on the positive side when its time links are inverted and its two times are swapped, and $\mathrm{Re}\,\mathrm{tr}$ is unchanged by the cyclicity of tr and $\mathrm{tr}\, M^\dagger = \mathrm{tr}\, M$ (`WilsonGauge.plaqE_reflect`).

Step 2: abstract reflection positivity. If the weight has the form $k(c)\,w_+(c,a_+)\,w_+(c,a_-)$ ($k \geq 0$, w_+ real and bounded), and the fermion determinant side satisfies operator conditions such as positivity of the plane matrix, then an integral of the form (4) is non-negative (`FermionSliceReflection.site_reflection_positive_phys`). By Step 1 the gauge weight fits the form $k = e^{-\beta K}$, $w_+ = e^{-\beta H}$.

Step 3: the operators agree. The D written in the physical configuration (equation (2)) agrees with the operator assembled in the split configuration (`WilsonGauge.physDirac_eq_split`). This lets the operator conditions of Step 2 be discharged for the concrete D.

An example showing the hypotheses can be met: with $h = 1$, $m = 0$, $\varphi = 1$, all hypotheses hold. The statement is not vacuous.

04

Positivity of the partition function

Cited A formalisation. Lean: `WilsonGauge.partition_pos_wilson_full` ($h > 0$, all β, m, j).

$$Z = \int e^{-\beta S W(\omega)}\ \mathrm{Re}\big[(\det D(\omega))^{2j}\big]\,d\omega\ >\ 0. \tag{5}$$

There is no condition on the mass m. Since $Z > 0$, the normalised probability measure `wfProb` can be defined. $Z > 0$ is not trivial: it contains the content that $\mathrm{Re}\big[(\det D)^{2j}\big]$ does not vanish on every configuration.

05

Global gauge invariance of the determinant

Proved here A Lean formalisation of a known fact (the proof is the sentence below). Lean: `GaugeDet.physDirac_det_conj` (all N, h, m, g, ω; no hypotheses).

For the configuration $(g\cdot\omega)_\ell = g\,\omega_\ell\,g^{-1}$ obtained by conjugating every link by a constant $g \in SU(2)$,

$$\det D(g\cdot\omega) = \det D(\omega). \tag{6}$$

$D(g\cdot\omega)$ is similar to $D(\omega)$, sandwiched by the matrix that multiplies the colour index by the constant g, so the determinant does not change. g is a constant independent of the site. Local gauge transformations, which take a different $g(x)$ at each site, are outside this declaration.

06

Invariance under the generators of the lattice symmetries

Proved here A Lean formalisation of known facts (the route of proof is below). Lean: `CubicSym.wfProb_lattice_generators` ($h > 0$, all N, j, m, β).

The normalised measure P (`wfProb`) is invariant under push-forward by each of the following generators R: $R_* P = P$.

Generator	Action on configurations	Lean name
Global gauge transformation	Conjugate every link by a constant g	`conjPhys g`
Spatial unit translation (three directions)	$x \mapsto x + e_k$	`transPhys k`
Time unit translation	$t \mapsto t + 1$	`tshiftPhys`
90° rotation in a spatial plane	$e_a \mapsto e_b \mapsto -e_a$ ($a \neq b$, six oriented cases)	`rotPhys a b`
Reflection of a spatial axis	$x_k \mapsto N - 1 - x_k$	`reflPhys k`
Time reflection	$t \mapsto 2h - 1 - t$, time links inverted (the reversal about the mid-link plane $t = h - \frac{1}{2}$, different from the site-plane Θ of § 02. What Lean has is invariance of the measure under this reversal, not reflection positivity)	`treflPhys`

Proved here Likewise a formalisation. Lean: `CubicSym.wfProb_hypercubic_generators` (isotropic lattice $N = 2h$, $h > 0$). On a lattice whose spatial and time sizes are equal, in addition to all the above, the measure is also invariant under 90° rotations in a (space, time) plane (`trotPhys a`).

Route. For every generator, invariance of the measure splits into three parts. (i) The product of Haar measures over links is preserved by relabelling links and by inverting some links, $g \mapsto g^{-1}$. (ii) S_W is invariant. (iii) $\det D$ is invariant. (iii) is shown in Clifford form. Write D as $1 \otimes M + \sum_\mu \gamma_\mu \otimes A_\mu$; with a lattice permutation W and a spin matrix S (for rotations $S = 1 + \gamma_b \gamma_a$; for the reflection of axis k, the product of the other three γ), it follows from $(S \otimes W)\, D(\omega)\, (S \otimes W)^{-1} = D(R\omega)$ (`CubicSym.cliffOp_conj`, `CubicSym.det_cliffOp_perm`).

Time reflection and the determinant. For the time reflection too, the determinant itself is invariant: $\det D(\Theta_t \omega) = \det D(\omega)$ (`CubicSym.physDirac_det_trefl`; all N, h, m, ω; isotropy not needed). The spin matrix is $\gamma_1\gamma_2\gamma_3$ Proved here .

The antiperiodic seam. If the coordinate reversal is taken as $z \mapsto N - 1 - z$, the link carrying the antiperiodic sign ($N - 1 \to 0$) is mapped to itself (with its direction reversed). So the lattice permutation W needs no sign correction Proved here . If it is taken as $z \mapsto -z$, the seam moves and a signed permutation is needed.

07

What is not said

- **One finite lattice.** All theorems are statements on fixed N and h. Neither the infinite-volume limit nor reflection positivity there is in this chapter.
- **No continuum limit.** The lattice spacing is fixed. Nothing is said about the continuum limit or mass renormalisation (how to tune m).
- **No chiral symmetry.** The Wilson term breaks chiral symmetry explicitly. Nothing is said about its restoration or the treatment of fermion doubling.
- **Symmetries generator by generator.** Closure as the cubic or hypercubic group (that every element of the group arises as a product of generators) is outside the statements. Rotations involving time are only for the isotropic lattice $N = 2h$. Gauge invariance is for a global constant g; invariance of $\det D$ under local gauge transformations is not among the declarations.
- **Reflection in site planes only.** Reflection positivity for the plane through the middle of links (with fermions) is not in Lean. $m \leq -1$ is outside the range.
- **No gap.** A mass gap for the transfer operator of the measure with fermions is not among the theorems of this chapter.
- **Not a new result.** This writes, in Lean on the concrete operator, reflection positivity of Osterwalder–Seiler type and known facts about lattice symmetries.

What this chapter can say

What is said	What is not said
Cited Reflection positivity of the Wilson–fermion weight (4) (`WilsonGauge.site_reflection_positive_wilson_full`, $m > -1$, all β)	$m \leq -1$, reflection in the plane through the middle of links, infinite volume
Cited Positivity of the partition function (5) (`WilsonGauge.partition_pos_wilson_full`, all β, m, j)	Estimates of the size of Z
Proved here Global gauge invariance of the determinant (6) (`GaugeDet.physDirac_det_conj`, unconditional)	Local gauge transformations

What is said	What is not said
Proved here Invariance of the measure under the generators of the lattice symmetries (`CubicSym.wfProb_lattice_generators`)	Closure as a group, rotations involving time (anisotropic lattices), lifting to the block level
Proved here Invariance under the hypercubic generators on the isotropic lattice (`CubicSym.wfProb_hypercubic_generators`, $N = 2h$)	Closure as a group, lifting to the block level
Proved here Invariance of the determinant under time reflection (`CubicSym.physDirac_det_trefl`)	Reflection positivity for this reversal
—	The continuum limit, chiral symmetry, mass renormalisation, a mass gap with fermions

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity and the transfer matrix for lattice gauge theory (including fermions)	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract). 5-03
Wilson's lattice formulation	Cited	K. G. Wilson, Phys. Rev. D 10 (1974) 2445 (abstract). The original source of the Dirac operator with the Wilson term: bibliographic data only, not checked
The theorems of this chapter and their parts	Lean	Machine-checked in Lean 4 (all pass the kernel re-check; the axioms are the three standard ones). The list of declarations is in 20-07 and A-04
Agreement of the operator formula (2) with the Lean definition, the geometry of the generators' actions	Reading	The numerical record of this book's study (on small lattices, the difference between the operator copying the Lean definition and the textbook formula was compared in double precision; the difference was of rounding size)

Words used in this chapter — Wilson–Dirac operator, Wilson term, antiperiodic boundary conditions, reflection in a site plane, reflection positivity, partition function, global gauge transformation, hypercubic group, generator, chiral symmetry. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-06 The area law at strong coupling, and an upper bound on the vortex free energy — estimates of Wilson loops and of the twisted partition function inside the strong-coupling window.

The area law at strong coupling, and an upper bound on the vortex free energy

This chapter is this book's study.

What this chapter shows — In the window where the coupling $\beta \geq 0$ is small (strong coupling), we read two estimates for the SU(2) Wilson measure on the four-dimensional torus $(\mathbb{Z}/N)^4$ that are machine-checked in Lean. (i) **The area law**: the expectation of the Wilson loop along the boundary of a rectangle with sides R, T is $\leq C\,(Kq)^{\mathrm{area}}$ ($Kq < 1$, K is 441 or 4096, $q = 1 - e^{-2\beta}$), and the constants contain no N, R or T. It holds for rectangles with any corner and orientation, and for the standard rectangle it also holds for the infinite-volume limit measure. (ii) **An upper bound on the vortex free energy**: the difference between the partition function Z_1 with a 't Hooft twist by the centre element -1 and the untwisted Z_0 is small at a rate per unit area independent of N. The prefactor N^2 remains. The proof is made of four parts: the positive mixture, the terms that vanish under reversal of the central sign, the domination lemma, and the connectedness, area lower bound and counting of minimal sets of plaquettes.

Prerequisite chapters — 19-01 The Wilson measure, plaquettes and holonomy, the product of Haar measures; 19-02 the sign of the twist, the vortex ratio; 20-03 The infinite-volume limit. The introductions are 5-02 (strong coupling and the area law) and 25-01 (twisted tori and flux).

Said first — The lattice spacing is fixed at 1, and these are statements only inside the strong-coupling window. Depending on the version of the constants, the window is $\beta < 1.22 \cdot 10^{-4}$ or $\beta < 1.135 \cdot 10^{-3}$ Reading . Nothing is said about the continuum limit, weak coupling ($\beta \to \infty$) or the band. These are statements on the lattice, distinct from "continuum Wilson loops and the area law" Assessment , which the assessment of the published construction lists as an open question. The constants are coarse, and the rates are on the side much smaller than the true values. Each is a formalisation of known results of the type of Osterwalder–Seiler (area law) and of 't Hooft and Münster (vortex free energy) Cited .

The order of this chapter

1. Setting — Wilson loops and the twisted partition function — the closed path on the boundary of a rectangle, q and the window
2. Statement (i) — the area law in finite volume — five versions (Table 1)
3. Statement (ii) — for the infinite-volume limit measure — the limit of the whole sequence, and every limit point
4. Statement (iii) — an upper bound on the vortex free energy — the rate per unit area does not depend on the transverse size
5. Route of the proof — four parts and one sum — the overall flow
6. Parts of the proof (1) — the positive mixture and the domination lemma — a one-step inequality that needs no positivity
7. Parts of the proof (2) — terms that vanish under reversal of the central sign — spanning sets, good sets and bad sets
8. Parts of the proof (3) — connectedness, area lower bound and counting of minimal sets of plaquettes — root, $R\min(T, N-T)$, N^2, $\Delta^{2(n-1)}$
9. Detour: from the parity of row cochains to the area lower bound — a difference equation column by column
10. In numbers — the window, the non-trivial range, comparison with the thin-vortex floor
11. What is said, and what is not

Labels — Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment (details in "Labels" in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are the labels of the whole book; Reading is a numerical reading of this book, and Lean a Lean theorem of this book. The names of Lean declarations are written outside the labels, on the same line as the scope.

01

Setting — Wilson loops and the twisted partition function

The introduction is 5-02. At strong coupling (small β), the expectation of a Wilson loop starts from terms that tile a surface bounded by the loop one plaquette at a time, and is exponentially small in the area. This is the strong-coupling area law, the starting point of the picture of confinement in lattice gauge theory Cited . This chapter reads that estimate as a theorem about the SU(2) Wilson measure on a finite torus.

The measure. The lattice is $(\mathbb{Z}/N)^4$; the link variables $U_\ell \in SU(2)$ (unit quaternions), the plaquette holonomy U_p and its real part $s_p := \operatorname{Re} U_p = \frac{1}{2}\operatorname{tr} U_p \in [-1,1]$ are as in 19-01. The plaquettes are the $6N^4$ unoriented ones with $i<j$. The Wilson measure is the product of Haar probability measures multiplied by the density $e^{\beta\sum_p s_p}$, and expectations are

$$\langle F\rangle_{N,\beta} := \frac{\int F\, d\mu_{\mathrm{W}}}{\int 1\, d\mu_{\mathrm{W}}}, \qquad d\mu_{\mathrm{W}} = e^{\beta\sum_p s_p}\prod_\ell dU_\ell \tag{1}$$

(the ratio of Lean's `wilsonMeasure N β` to its total mass). β follows the same convention as the standard SU(2) $\beta = 4/g^2$.

The boundary of a rectangle. Fix two directions $i \neq j$, a corner a and sides R, T ($1 \le R, T < N$), and let C be the closed path that goes from a R steps in direction i, T steps in direction j, then back R steps in direction i and T steps in direction j. $W(C)$ is the ordered product of the links along this path (links traversed backwards are conjugated), and $\operatorname{Re} W(C) = \frac{1}{2}\operatorname{tr} W(C)$ is gauge invariant. The standard rectangle has its corner at the origin and lies in the plane spanned by directions 1 and 2 (0 and 1 in the Lean indices). Like Parts 21 and 22, this chapter numbers the directions 1 to 4. 20-01 to 20-05 write the directions 0 to 3 as in the Lean indices, and their directions 0 and 1 are directions 1 and 2 of this chapter. Lean: `AreaLaw.rectWord` (standard), `AreaLaw.rectWordG` (general); the product is `wordHol`. For $R = T = 1$ it coincides literally with the boundary of a single plaquette (`AreaLaw.rectWord_one_one`). Since R, $T < N$, the path does not wind around the torus.

The twisted partition function. Write Z_D for the total mass of the weight $e^{\beta\sum_p \eta_p s_p}$ in which the sign of s_p is reversed on a set D of plaquettes ($\eta_p = -1$ for $p \in D$, $\eta_p = 1$ otherwise). On $(\mathbb{Z}/N)^4$, take D to be "the plaquettes of orientation 1, 2 with $x_1 = x_2 = 0$" (x_3, x_4 arbitrary, N^2 plaquettes), and set $Z_0 := Z_\varnothing$ (the ordinary partition function) and $Z_1 := Z_D$. This is the partition function with one 't Hooft twist by the centre element -1 (the same placement of plaquettes as the twist in 19-02). Lean: `VortexStrong.Ztw`, `VortexStrong.twist`.

The window. Throughout this chapter we set

$$q := 1 - e^{-2\beta} \qquad (\beta \ge 0) \tag{2}$$

so $q \approx 2\beta$. The window has the form $Kq < 1$: for $K = 4096$ it is $\beta < 1.22085\cdot 10^{-4}$, and for $K = 441$ it is $\beta < 1.13507\cdot 10^{-3}$ Reading .

02

Statement (i) — the area law in finite volume

Statement (the strong-coupling area law, finite volume)

Let $N \geq 2$, $\beta \geq 0$ and $4096q < 1$. For the boundary C of a standard rectangle with $1 \leq R, T < N$,

$$\left|\langle \operatorname{Re} W(C)\rangle_{N,\beta}\right| \;\leq\; 16\,\frac{(4096\,q)^{R\cdot\min(T,\,N-T)}}{4096\,(1-4096\,q)}. \tag{3}$$

Lean A formalisation of the type of the Osterwalder–Seiler strong-coupling area law. Lean: `AreaLaw.area_law_wilson` (the only assumptions are $\beta \geq 0$, $1 \leq R, T < N$, $4096q < 1$; the constants 16 and 4096 do not depend on N, R, T).

If $2T \leq N$, then $\min(T, N-T) = T$ and the exponent is the area RT (`AreaLaw.area_law_wilson_area`). There are four more versions of the same form (Table 1).

Version	Rectangle	Window	Right-hand side	Lean
Basic	Standard	$4096q < 1$	$16\,(4096q)^{R\min(T,N-T)}/(4096(1-4096q))$	`AreaLaw.area_law_wilson`
Position and orientation	Any corner a, any $i \neq j$	$4096q < 1$	Same	`AreaLaw.area_law_wilson_general`
Tightened constants	Standard	$441q < 1$	$6\,(441q)^{R\min(T,N-T)}/(441(1-441q))$	`AreaLaw.area_law_wilson_sharp`
Tightened, position and orientation	Any corner and orientation	$441q < 1$	Same	`AreaLaw.area_law_wilson_general_sharp`
Maximum of exponents	Standard	$441q < 1$	$6\,(441q)^{\max(R\min(T,N-T),\,T\min(R,N-R))}/(441(1-441q))$	`AreaLaw.area_law_wilson_max_sharp`

Table 1. Five versions of the area law. $\langle\cdot\rangle$ is the normalised expectation of (1). All with $\beta \geq 0$, $1 \leq R, T < N$.

Meaning. The expectation of a Wilson loop is exponentially small in the area enclosed by the loop, at a rate of at least $-\ln(441q)$. The rate and the prefactor depend neither on the lattice size N nor on the size of the rectangle. The versions with position and orientation are corollaries of the invariance of the Wilson measure under translations and permutations of directions (`AreaLaw.wilson_cfgShift_invariant`, `expect_perm`).

On the exponent. If $R \le N/2$ or $T \le N/2$, the exponent of the maximum version is the area RT. When R and T both exceed $N/2$, the exponent is $\max(R(N-T), T(N-R))$, smaller than RT. In this case the $N^2 - RT$ plaquettes of the whole plane of directions 1, 2 with the rectangle removed also have the same boundary mod 2, so the number of plaquettes truly needed can also be smaller than RT (7 plaquettes on two- and three-dimensional lattices with $N = 4$, $R = T = 3$ Reading ; a record of integer programming; not checked on the four-dimensional 4^4). The exponent of the theorem does not exceed that number, and a gap remains. The $N - T$ in $\min(T, N-T)$ is a weakness coming from the tool of the proof (the detour in § 08); it does not mean that $R \times (N - T)$ plaquettes span the boundary C.

03

Statement (ii) — for the infinite-volume limit measure

The space of infinite-volume configurations, the sequence of periodic lifts μ_n (the Wilson measure on the torus $(\mathbb{Z}/2(n+1))^4$ lifted to $\mathbb{Z}^4$, Lean's `liftProb β n`) and weak convergence are as in 20-03. Write $C_{R,T}$ for the boundary of the standard rectangle in $\mathbb{Z}^4$ (Lean's `zRectWord`).

Statement (the strong-coupling area law, infinite volume)

Let $\beta > 0$, $18 \tanh\beta < 1$, $4096q < 1$. Then there is a probability measure ν_β such that the whole sequence μ_n converges weakly to ν_β, and for all $R, T \ge 1$

$$\left| \int \operatorname{Re} W(C_{R,T}) \, d\nu_\beta \right| \;\le\; 16 \, \frac{(4096\,q)^{RT}}{4096\,(1 - 4096\,q)} \cdot \tag{4}$$

Lean A formalisation of the type of the strong-coupling area law. Lean: `AreaLaw.area_law_infvol` (the existence of ν_β is taken from the strong-coupling infinite-volume theorem `InfMain2.wilson_infvol_main2` of 20-03 and 20-04).

The intersection of the two windows is $0 < \beta < 1.22085 \cdot 10^{-4}$, and the window of the area law lies entirely inside the Dobrushin window ($\beta < 0.0556$) Reading .

At every limit point. There is also a form that does not use the existence of the limit. If $\beta \ge 0$, $4096q < 1$, and $\mu_{\varphi(n)} \to \nu$ (weakly) for a strictly increasing φ, then (4) holds for ν (`AreaLaw.area_law_limit_sub`; the case $\varphi = \mathrm{id}$ of the whole sequence is `AreaLaw.area_law_limit`). The proof uses, for $n \ge R + T$, the $2T \le N$ version of the finite-volume (3), and simply keeps the inequality in the limit of the integrals of the bounded continuous function $\operatorname{Re} W(C_{R,T})$.

The infinite-volume version is stated only with the constants 16 and 4096 and the standard rectangle (origin, directions 1, 2). The versions with tightened constants or with arbitrary corner and orientation are in finite volume only.

04

Statement (iii) — an upper bound on the vortex free energy

Statement (the vortex floor at strong coupling, isotropic torus)

Let $N \geq 1$, $\beta \geq 0$, $4096q < 1$. For Z_0, Z_1 on $(\mathbb{Z}/N)^4$ (§ 01),

$$|Z_0 - Z_1| \;\leq\; N^2 \,\frac{(4096\,q)^{N^4}}{4096\,(1-4096\,q)}\,(Z_0 + Z_1). \quad (5)$$

Lean A formalisation of the type of the strong-coupling estimate of 't Hooft's vortex free energy (Münster and others). Lean: `VortexStrong.vortex_floor` (the only assumptions are $\beta \geq 0$, $4096q < 1$). The version with tightened constants `VortexStrong.vortex_floor_sharp` (with $441q < 1$, 4096 replaced by 441); the 441 version with the denominator written as the total mass of the Wilson measure, `VortexStrong.vortex_floor_wilson` ($441q < 1$).

If the coefficient on the right-hand side is ε and $\varepsilon < 1$, then $|Z_0 - Z_1| \leq \frac{2\varepsilon}{1-\varepsilon} Z_0$ (`VortexStrong.vortex_floor_rel`; an arithmetic rewriting).

The same form also exists on the lattice of Part 21 (the torus of transverse size N and longitudinal size M, 19-02), and deals with the same quantity as the vortex ratio θ_M (19-02).

Statement (the vortex floor at strong coupling, torus of transverse size N and longitudinal size M)

Let $N, M \geq 1$, $\beta \geq 0$, $441q < 1$, and let the background twist $n_{12} \in \{0, 1\}$ be arbitrary. For $Z_k := Z_{N,M}(\beta; n_{12}, k)$ $(k = 0, 1)$,

$$|Z_0 - Z_1| \;\leq\; N^2 \,\frac{(441\,q)^{M^4}}{441\,(1-441\,q)}\,(Z_0 + Z_1). \quad (6)$$

Lean A formalisation of the same type. Lean: `VortexStrongMN.vortex_floor_MN_sharp` (the version with constant 9216 is `VortexStrongMN.vortex_floor_MN`).

Since $|Z_0 - Z_1|/(Z_0 + Z_1) = |\theta_M^{(n12)}|$, (6) is the same as

$$|\theta_M^{(n12)}| \;\le\; N^2\, \frac{(441\,q)^{M^2}}{441\,(1 - 441\,q)} \qquad (7)$$

Proved here .

Meaning. Inserting one vortex hardly changes the partition function, and the difference is exponentially small in the area M^2 of the longitudinal plane. **The rate per unit area $-\ln(441q)$ depends neither on the transverse size N nor on the longitudinal size M.** The prefactor N^2 (the number of plaquettes of the twist) remains. The thin-vortex floor (21-02), $|\theta_M| \le \tanh(N^2|\beta|)^{M^2}$, holds for all β, but its rate per unit area $-\ln\tanh(N^2|\beta|)$ approaches 0 as $N \to \infty$. (7) holds only inside the strong-coupling window, but differs in that its rate does not depend on N (comparison in § 09).

The statements of this section are upper bounds on the difference $|Z_0 - Z_1|$, and do not give the direction $Z_1 \le Z_0$ ($\theta_M \ge 0$). That direction is a separate theorem for even longitudinal size M (21-06).

05

Route of the proof — four parts and one sum

The area law (3) and the vortex floors (5), (6) are shown along the same line. We write it for the area law.

1. **Positive mixture (§ 06).** Write the plaquette weight as $e^{\beta s_p} = e^{-\beta}(1 + \sigma_p)$ with $0 \le \sigma_p \le t := e^{2\beta} - 1$, and expand $\prod_p(1 + \sigma_p)$ as a sum $\sum_X \prod_{p\in X} \sigma_p$ over sets X of plaquettes. Numerator and denominator become sums over all X, $\sum_X \int W \prod_X \sigma$ and $\sum_X Z_X$ ($Z_X := \int \prod_{p\in X} \sigma_p\, dU \ge 0$).
2. **Vanishing terms (§ 07).** The transformation that reverses the sign $U_\ell \mapsto -U_\ell$ on a set of links preserves the product of Haar measures. A term in which the boundaries of the plaquettes of X do not span C mod 2 changes sign under this transformation, and is 0.
3. **Domination (§ 06).** From $Z_{X\cup\{p\}} \le t\, Z_X$, the sum of the weights of the X containing a set Y of plaquettes is at most $q^{|Y|} \sum_X Z_X$.
4. **One sum.** Since $|W| \le 1$, the remaining terms are bounded by $q^{|Y|}$ for each "minimal set Y of plaquettes spanning C", giving

$$\big|\langle \operatorname{Re} W(C)\rangle\big| \;\le\; \sum_{Y\colon C\text{-spanning minimal set}} q^{|Y|} \qquad (8)$$

This holds for all $\beta \ge 0$ and all N (Lean `AreaLaw.abs_wordExpect_le`; the family of minimal sets `minSpanC` is placed concretely inside the definition, and is not an assumption).

5. **Geometry of minimal sets of plaquettes (§ 08).** A minimal Y is connected in the sense of sharing links, contains a plaquette touching a fixed link, and has $|Y| \ge R \min(T, N-T)$. The number of connected sets of size n touching the root is at most $16 \cdot 64^{2(n-1)}$, so the right-hand side of (8) is at most $16 \sum_{n \ge A} 4096^{n-1} q^n$ ($A = R \min(T, N-T)$), and summing the geometric series gives (3).

For the vortex floor, step 2 is replaced by "on good sets the twisted and untwisted weights coincide", step 4 by "the sum over bad sets", and the area lower bound of step 5 by "a bad set has at least N^2 plaquettes" (§ 07, § 08).

This line does not use the many-body sums of the cluster expansion. The only inequalities used are $0 \le \sigma_p \le t$ at each point and induction over finite sets. It is accounting of the same family as the "two copies and chains of plaquettes" of strong-coupling decay of correlations (20-02 § 06).

06

Parts of the proof (1) — the positive mixture and the domination lemma

Positive mixture. For $s \in [-1, 1]$,

$$e^{\beta s} = e^{-\beta}\big(1 + \sigma\big), \qquad \sigma := e^{\beta(s+1)} - 1 \in [0,\, e^{2\beta} - 1] \qquad (9)$$

($\beta \ge 0$). Lean: `AreaLaw.exp_eq_mix`, `mixVar_mem`. A product over a finite set is $\prod_p (1 + a_p) = \sum_X \prod_{p \in X} a_p$ (`AreaLaw.prod_one_add_univ`), and the form expanded inside an integral is `AreaLaw.integral_mul_prod_one_add`. That each σ in the expanded terms is at least 0 is the source of treating the weights, in the domination lemma below, as quantities that "increase as more are added".

The one-step inequality. For $Z_X := \int \prod_{p \in X} \sigma_p \prod_\ell dU_\ell$, using $\sigma_p \le t$ pointwise gives

$$Z_{X \cup \{p\}} \;\le\; t\, Z_X \qquad (p \notin X) \qquad (10)$$

(`AreaLaw.ZX_insert_le`).

The domination lemma (Lean: `AreaLaw.sum_superset_le`). If a real-valued function Z on the subsets of a finite set P and $t \geq 0$ satisfy $Z(S \cup \{p\}) \leq t\, Z(S)$ for all S and $p \notin S$, then for all Y

$$\sum_{X \supseteq Y} Z_X \;\leq\; \Big(\frac{t}{1+t}\Big)^{|Y|} \sum_{X} Z_X \qquad (11)$$

Here $t/(1+t) = 1 - e^{-2\beta} = q$ (`AreaLaw.q_eq`).

The core of the proof is one line. Let $a_Y := \sum_{X \supseteq Y} Z_X$; for $p \notin Y$, splitting a_Y into "X containing p" and "X not containing it" gives $a_Y = a_{Y \cup \{p\}} + b$ (b the side not containing it), and using (10) term by term gives $a_{Y \cup \{p\}} \leq t\, b$. Together, $(1+t)\, a_{Y \cup \{p\}} \leq t\, a_Y$, that is, $a_{Y \cup \{p\}} \leq q\, a_Y$. Induction on the size of Y gives (11). $Z \geq 0$ **is not used.** It follows from the splitting and the one-step inequality alone.

07

Parts of the proof (2) — terms that vanish under reversal of the central sign

Reversal. For an $\mathbb{F}_2$-valued function (cochain) T on links, let flip_T be the transformation $U_\ell \mapsto -U_\ell$ on the links with $T_\ell = 1$. -1 is in the centre of SU(2), and $-U$ has the same distribution as U, so flip_T is a measurable involution preserving the product of Haar measures (`AreaLaw.flipZ_preserving`). The product along a path w changes as

$$W(\mathrm{flip}_T\, U) = (-1)^{\langle T,\, \partial w \rangle}\, W(U) \qquad (12)$$

where ∂w is the number of times the path w passes each link, mod 2, and $\langle T, \partial w \rangle$ is its sum (`AreaLaw.wordHol_flip`; no condition on the lattice size is needed). The plaquette holonomy is the product along the path once around the boundary of the plaquette (`hol_eq_wordHol`), so s_p is also multiplied by $(-1)^{\langle T, \partial p \rangle}$.

Vanishing terms (Lean: `AreaLaw.integral_word_mul_prod_eq_zero_of_not_mem_span`). If ∂C is not in the $\mathbb{F}_2$-span of $\{\partial p : p \in X\}$, then for any function g

$$\int \mathrm{Re}\, W(C) \prod_{p \in X} g(s_p) \prod_{\ell} dU_\ell \;=\; 0. \qquad (13)$$

By duality in $\mathbb{F}_2$ linear algebra (`AreaLaw.not_mem_span_iff_exists_sep`), not being in the span is equivalent to "there is a T with $\langle T, \partial p \rangle = 0$ on every plaquette of X and $\langle T, \partial C \rangle = 1$ on C". Reversing by that T, the factor $\prod g(s_p)$ of the integrand is unchanged

and only W changes sign, so the integral equals its own negative and is 0. g is arbitrary, and not even measurability is needed.

The correct criterion is "lies in the span", not "X contains the plaquettes of C". For example, on a periodic torus the boundary of one plaquette can equal, mod 2, the sum of the boundaries of all the remaining plaquettes. For the quaternion group Q_8, summing over all configurations of a two-dimensional 2×2 torus, the terms of the 7 non-spanning X were exactly 0, those of the 9 spanning ones were not 0, and $\{1, 2, 3\}$, which does not contain plaquette 0, was not 0 either Reading .

On the vortex-floor side: good sets and bad sets. The twisted weight can be written $e^{\beta \sum_p (-1)^{m_p} s_p}$ using a mask of signs per plaquette $m : \text{plaquettes} \to \mathbb{F}_2$ (1 on D). The reversal flip_T changes the mask as $m \mapsto m + \delta T$ (δT the coboundary of T, $(\delta T)_p = \langle T, \partial p \rangle$) (`VortexStrong.ZM_gauge`). When there is a T for which the mask equals a coboundary on the set X of plaquettes, that is $m_p = \langle T, \partial p \rangle$ ($p \in X$), we call X a **good** set (Lean `VortexStrong.Good`). On good sets a single reversal makes the twisted and untwisted terms coincide (`VortexStrong.ZM_good`), and only the terms of **bad** sets remain in $Z_0 - Z_1$. Using the domination lemma for both weights,

$$|Z_0 - Z_1| \;\le\; \Big(\sum_{Y\text{: minimal bad set}} q^{|Y|} \Big) (Z_0 + Z_1) \qquad (14)$$

(`VortexStrong.abs_Z_sub_le`). The vortex floor does not use duality, because the definition of good and bad is itself written in terms of the existence of a coboundary.

08

Parts of the proof (3) — connectedness, area lower bound and counting of minimal sets of plaquettes

Connectedness (area law). Call two plaquettes neighbours when they share a link. Split a minimal set Y spanning C into two parts and suppose there is no pair of neighbours between them; then the sum of the boundaries of one part is closed ($\partial\partial = 0$, `AreaLaw.div_plaqBd`) and its support lies in the boundary of the rectangle. The boundary of the rectangle is a simple closed cycle, so such a closed chain is only 0 or ∂C itself (`AreaLaw.rect_closed_cases`, $1 \le R, T < N$). Either contradicts the minimality of Y, so a minimal Y is connected (`AreaLaw.minSpan_rect_conn`). Moreover Y contains a plaquette touching a fixed link of the rectangle (the root).

Connectedness (vortex floor). If a minimal bad set splits into two parts sharing no neighbours, one can glue the cochains of the two sides, "on the support of one" and "outside it", into a cochain for the whole, making the whole good, a contradiction (`VortexStrong.minBad_conn`).

Area lower bound (area law). A set of plaquettes spanning C has at least $R \min(T, N - T)$ plaquettes (`AreaLaw.card_ge_of_spans_rect`). The proof is placed in the detour. The same argument with directions 1 and 2 exchanged gives $T \min(R, N - R)$ as a lower bound too (`AreaLaw.wordBd_swap_rect`), and their maximum is the last version in Table 1.

Area lower bound (vortex floor). The mask of the translate $D_{a,b}$ of D (plaquettes of orientation 1, 2 with $x_1 = a$, $x_2 = b$) differs from the mask of D only by the coboundary of row and column cochains (`VortexStrong.good_twist_iff`). Hence a bad set meets every one of the N^2 mutually disjoint $D_{a,b}$, and has at least N^2 plaquettes (`VortexStrong.card_bad_ge`). The lower bound is attained: the plane $x_3 = x_4 = 0$ of directions 1, 2 (N^2 plaquettes) is a bad set (`VortexStrong.plane_bad`). A bad set contains a plaquette of D, so the roots are the N^2 plaquettes of D. This is the prefactor N^2 in (5).

Counting. If, for a symmetric neighbour relation, each point has at most Δ neighbours, then the number of connected sets of size n containing a fixed root is at most $\Delta^{2(n-1)}$ (`AreaLaw.card_conn_le`). It follows by an induction that inserts "go to a neighbour and come back" detours into a closed walk, without building a spanning tree. In the coarse count there are at most 64 neighbours and at most 16 roots, and $64^2 = 4096$. In the tightened count, the plaquettes containing a given link number 6, the plaquettes sharing a link with a given plaquette number at most 21 including itself ($1 + 4 \cdot 5$), so $21^2 = 441$, and there are 6 roots (`AreaLaw.card_rootP_le`, `card_nbQ_le`). The geometric series

$$\sum_{n \geq A} c_0 \, \Delta^{2(n-1)} q^n = \frac{c_0}{\Delta^2} \cdot \frac{(\Delta^2 q)^A}{1 - \Delta^2 q} \qquad (\Delta^2 q < 1) \tag{15}$$

gives the right-hand sides of (3), (5) and (6) (c_0 is the number of roots).

Detour: from the parity of row cochains to the area lower bound

May be skipped. For the standard rectangle (plaquettes of orientation 1, 2), for a column $0 \leq a < R$ and a row $y \in \mathbb{Z}/N$, consider the cochain $\rho_{a,y}$ that is 1 on "the links of direction 1 whose coordinates in directions 1, 2 are (a, y)" (all values of the other coordinates). Its pairing with plaquette boundaries is $\langle \rho_{a,y}, \partial p \rangle = [p \in Q(a,y)] + [p \in Q(a, y-1)]$ ($Q(a,y)$ is the class of plaquettes of orientation 1, 2 at coordinates (a, y); plaquettes of orientation 1, 3 or 1, 4 are hit twice and cancel), and its pairing with the boundary of the rectangle is $\langle \rho_{a,y}, \partial C \rangle = [y = 0] + [y = T]$. If X spans C, then, with $n(y)$ the number of plaquettes of X in $Q(a,y)$, $n(y) + n(y-1) = [y = 0] + [y = T]$ holds mod 2 for every y. Solving this difference equation, n is odd on all of $[0, T)$ or odd on all of $[T, N)$. Either way column a needs at least $\min(T, N - T)$ plaquettes, and R

columns need at least $R \min(T, N-T)$ Proved here . The solution on the $[T, N)$ side is a solution of the parity equation; it does not mean that the plaquettes on that side span the boundary C.

09

In numbers

Quantity	Value	Label
Window $4096q < 1$	$\beta < 1.22085 \cdot 10^{-4}$	Reading
Window $441q < 1$	$\beta < 1.13507 \cdot 10^{-3}$	Reading
Window of the infinite-volume version (intersection with $18 \tanh \beta < 1$)	$0 < \beta < 1.22085 \cdot 10^{-4}$	Reading
Range in which (3) is non-trivial (right-hand side < 1) for every area ≥ 1	$\beta < 1.2161 \cdot 10^{-4}$	Reading
Range in which the coefficient of (5) is < 1 for every N (4096 version / 441 version)	$\beta < 1.2093 \cdot 10^{-4}$ / $\beta < 1.1025 \cdot 10^{-3}$	Reading
Rate per unit area $-\ln(441q)$: $\beta = 10^{-5}, 10^{-4}, 10^{-3}$	`4.73, 2.43, 0.127`	Reading
Rate of the leading term at small β, $-\ln(\beta/4)$: same β	`12.9, 10.6, 8.29`	Reading

Table 2. Windows and rates. Floating-point calculations. The rate of the leading term is a rough guide read from the average of a single plaquette, $I_2(\beta)/I_1(\beta) = \beta/4 + O(\beta^3)$, not an estimate.

The rates of the upper bounds are much smaller than the rates of the leading term. The constant 441 (or 4096) comes from the coarseness of the count of connected sets, and also determines the width of the window.

Comparison with the thin-vortex floor Reading . With $N = M$, comparing the common logarithms of the upper bounds (7) and the thin-vortex floor $\tanh(N^2\beta)^{N^2}$, (7) becomes smaller (stronger) for $N \geq 30$ at $\beta = 10^{-5}$ and 10^{-4}, and for $N \geq 38$ at $\beta = 10^{-3}$. For example, at $\beta = 10^{-5}$, $N = 100$, the thin floor gives $-1.00 \cdot 10^4$ against $-2.05 \cdot 10^4$ for (7); at $N = 10$, the thin floor gives -300 against -206 for (7). The thin floor holds for all β, whereas (7) holds only inside the window.

Checks on small lattices Reading . We looked at the minimum size of bad sets by exhaustive search and integer programming in a separate script copying the Lean definitions. It was 4 on the three-dimensional 2^3, 4 on the four-dimensional 2^4, and 9 on 3^4 (all N^2); on lattices of transverse size N and longitudinal size M with $(N, M) = (2, 3), (3, 2), (2, 4)$ it was 9, 4, 16 $(= M^2)$, the same values as the Lean lower bound.

10

What is said, and what is not

- **The lattice spacing is fixed at 1.** These are statements about finite tori and their infinite-volume limit measure. Nothing is said about the continuum limit or sequences sending the lattice spacing to 0.
- **Only the strong-coupling window.** $\beta < 1.135 \cdot 10^{-3}$ (tightened constants), and $\beta < 1.22 \cdot 10^{-4}$ for the infinite-volume version and the basic version of the vortex floor. Nothing is said outside the window (nor that the area law fails there). This is the end far from the β of physical interest (the band around 2 to 3).
- **Rectangles only.** Loops along general closed paths are not treated. The infinite-volume version is only for the standard rectangle with the coarse constants.
- **A gap in the exponent.** When R and T both exceed $N/2$, the exponent stays smaller than the number of plaquettes truly needed.
- **The rates are coarse.** The rate per unit area is much smaller than the rate of the leading term at small β, and is not an estimate of the value of the tension. Estimates from below (lower bounds on Wilson loops, upper bounds on the tension) are not said.
- **The prefactor of the vortex floor.** The rates of (5) and (6) do not depend on N or M, but the prefactor N^2 remains. The direction $Z_1 \leq Z_0$ is not said.
- **Reaches neither the band nor the mass gap.** The mass gap inside the strong-coupling window is a separate theorem in 20-04. The band statement continuing to weak coupling (19-07) does not follow from the theorems of this chapter.
- **Distinct from the open question of the assessment.** The question listed by the assessment of the published construction, "construct Wilson-loop observables, connect them to local observables, and show the area law" Assessment , is a question about the continuum theory. This chapter is a strong-coupling statement on the lattice and does not answer that question.
- **Not a new result.** It is a formalisation, for the SU(2) Wilson measure, of the type of the strong-coupling area law (Osterwalder–Seiler) and of the strong-coupling estimate of the vortex free energy ('t Hooft, Münster and others).

The scope of what this chapter says

Said	Not said
Lean The area law in finite volume (3): $\beta \geq 0$, $4096q < 1$, $1 \leq R, T < N$, no N, R, T in the constants. Any corner and orientation; the version with constants 6 and 441; the version with exponent $\max(R\min(T, N-T), T\min(R, N-R))$. Lean `AreaLaw.area_law_wilson`, `area_law_wilson_general`, `area_law_wilson_sharp`, `area_law_wilson_general_sharp`, `area_law_wilson_max_sharp`	Outside the window. General loops. The true exponent when both R, $T > N/2$. Lower bounds on Wilson loops
Lean The area law in infinite volume (4): $\beta > 0$, $18\tanh\beta < 1$, $4096q < 1$, the limit ν_β of the whole sequence. Also at every limit point ($\beta \geq 0$, $4096q < 1$). Lean `AreaLaw.area_law_infvol`, `area_law_limit`, `area_law_limit_sub`	Infinite-volume versions with tightened constants or with arbitrary corner and orientation
Lean The upper bound on the vortex free energy (5): $(\mathbb{Z}/N)^4$, $\beta \geq 0$, $4096q < 1$ (tightened version $441q < 1$), the rate per unit area does not depend on N. Lean `VortexStrong.vortex_floor`, `vortex_floor_sharp`, `vortex_floor_rel`, `vortex_floor_wilson` (the 441 version)	Removal of the prefactor N^2. The direction $Z_1 \leq Z_0$. Outside the window
Lean The version (6) with transverse size N and longitudinal size M: every background n_{12}. The tightened version `vortex_floor_MN_sharp` with $441q < 1$ (the 441 window), the untightened version `vortex_floor_MN` with 9216q < 1 (the 9216 window). Lean `VortexStrongMN.vortex_floor_MN_sharp`, `vortex_floor_MN`	The band statement. The continuum limit
Proved here (7): (6) written in terms of the vortex ratio. The parity argument of the detour	—
Reading Windows, non-trivial ranges, rates (Table 2), comparison with the thin-vortex floor (crossovers at $N = 30, 38$), minimal sets on small lattices	Not proofs of upper or lower bounds. The true rate for four-dimensional SU(2)

Sources and reproduction

Item	Kind	Source / tool
The strong-coupling area law and the foundations of lattice gauge theory	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. K. G. Wilson, Phys. Rev. D 10 (1974) 2445. The introduction is in 5-02
Twisted boundary conditions and the vortex free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141. G. Münster, Nucl. Phys. B 180 (1981) 23–60 (bibliography only; the text not checked). The introduction is in 25-01

Item	Kind	Source / tool
Open question of the published construction (Wilson loops and the area law)	Assessment	The assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), § 8, item 4. How to read it is in 16-02
(3) and the versions of Table 1; parts (mixture, domination, reversal, duality, connectedness, lower bounds, counting, translation and orientation)	Lean	`AreaLaw.area_law_wilson`, `area_law_wilson_area`, `area_law_wilson_general`, `area_law_wilson_sharp`, `area_law_wilson_general_sharp`, `area_law_wilson_max_sharp`, `abs_wordExpect_le`, `exp_eq_mix`, `mixVar_mem`, `prod_one_add_univ`, `ZX_insert_le`, `sum_superset_le`, `q_eq`, `flipZ_preserving`, `wordHol_flip`, `not_mem_span_iff_exists_sep`, `integral_word_mul_prod_eq_zero_of_not_mem_span`, `rect_closed_cases`, `minSpan_rect_conn`, `card_ge_of_spans_rect`, `wordBd_swap_rect`, `card_conn_le`, `card_rootP_le`, `card_nbQ_le`, `wilson_cfgShift_invariant`, `rectWord_one_one`. sorry 0, the three standard axioms, confirmed by re-checking with the kernel, statements checked against the wording of the source
(4)	Lean	`AreaLaw.area_law_infvol`, `area_law_limit`, `area_law_limit_sub`; the existence of the limit is `InfMain2.wilson_infvol_main2` (20-03)
(5), (6) and parts (good sets, bad sets, gluing, translation)	Lean	`VortexStrong.vortex_floor`, `vortex_floor_sharp`, `vortex_floor_rel`, `vortex_floor_wilson`, `ZM_gauge`, `Good`, `ZM_good`, `abs_Z_sub_le`, `good_twist_iff`, `card_bad_ge`, `minBad_conn`, `plane_bad`; `VortexStrongMN.vortex_floor_MN_sharp`, `vortex_floor_MN`
Windows, rates, non-trivial ranges, comparison with the thin-vortex floor	Reading	The numerical record of this book's study (floating point; recomputed with a separate script)
The spanning criterion (Q_8, 2×2); minimum sizes of bad sets and spanning sets	Reading	The numerical record of this book's study (sums over all configurations, exhaustive search, integer programming; no proof of optimality for the four-dimensional 4^4)

Terms used in this chapter — strong coupling, Wilson loop, area law, positive mixture, domination lemma, reversal of the central sign, spanning set, cochain, coboundary, good set, bad set, root, counting of connected sets, 't Hooft twist, vortex free energy. Their meanings are in A-02 Glossary.

Next chapter to read: 20-07 What this part says, and what it does not — the theorems of Part 20 in one table. Other forms of the vortex floor are in 21-02 and 21-07.

What this part says, and what it does not

> This chapter is this book's own investigation. It is a chapter of tables.

What this chapter shows — The theorems of Part 20 gathered into one table, with their scope (range of β, lattice size, window), what they do not say, and the names of their Lean declarations.

Prerequisite chapters — 20-00 Entrance. The content of each row is in 20-01 to 20-06.

Said first — All the theorems of this part are statements on a lattice with the lattice spacing fixed at 1, and are formalisations of results of known types. No row says anything about the continuum limit, weak coupling, or an upper bound on the mass.

The order of this chapter

1. How to read the tables: notation and columns.
2. Reflection positivity (20-01): the three forms and the symmetry of the pairing.
3. Strong-coupling decay (20-02): two estimates in finite volume.
4. The infinite-volume limit and reconstruction (20-03, 20-04): limit, translation, clustering, gap, non-degeneracy, $T \geq 0$.
5. The vessel with fermions (20-05): four theorems on a finite lattice.
6. The area law at strong coupling, and an upper bound on the vortex free energy (20-06): the area law in finite and infinite volume, and the upper bound on the vortex free energy.
7. What the part as a whole does not say: limits common to all rows.

01

How to read the tables

Each row is one theorem. “Scope” copies the hypotheses of the Lean statement and the main form of the conclusion; “What is not said” gives limits specific to that row. Every row is machine-checked in Lean 4. The rows from 20-01 to 20-05 (§ 02– § 05) carry the

label Cited (a formalisation of a result of a known type from the literature). The exception is the rows of § 05 on the determinant and the lattice symmetries, which carry Proved here (a formalisation of known facts whose proof is given in the text). The rows from 20-06 (§ 06) carry the label Lean (a Lean theorem of this book, as in the legend of 20-06). Notation follows Part 19.

- $q(\beta) = 18 \tanh |\beta|$. The window $q(\beta) < 1$ is $|\beta| < 0.0556$ Reading .
- The narrow window $24(1 - e^{-4|\beta|}) < 1$ is $|\beta| < 0.0106$ Reading .
- μ_n: the periodic lift to $\mathbb{Z}^4$ of the Wilson measure on the torus $(\mathbb{Z}/2(n+1))^4$. ν_β: its limit without a subsequence (20-03).
- "Limit point": the weak limit of some subsequence of μ_n.
- W: cylinder functions on the closed positive half. W_{gi}: the gauge-invariant ones among them (20-04).

02

Reflection positivity (20-01)

Theorem	Scope	What is not said	Lean declaration
Reflection positivity for the plane through sites	$(\mathbb{Z}/2M)^4$, all β, bounded measurable f depending only on the closed positive half	The plane through the middle of links; the time direction only	`TorusRP.wilson_rp`
Reflection positivity for the middle plane	$(\mathbb{Z}/2M)^4$, $M \geq 2$, $\beta \geq 0$, gauge-invariant bounded measurable F	F that is not gauge invariant; $\beta < 0$. This theorem alone does not enter the vessel (the $T \geq 0$ of the vessel is a separate theorem in 20-04)	`MidRP.wilson_rp_mid`
Reflection positivity for site planes in four directions	$(\mathbb{Z}/2M)^4$, all β, for each direction ρ	A four-direction form for the plane through the middle of links	`PeierlsPerm.wilson_rp_dir`
Symmetry of the reflection pairing	$(\mathbb{Z}/2M)^4$, all β, measurable f, g	—	`TorusRP.wilson_refl_symm`

03

Strong-coupling decay (20-02)

Theorem	Scope	What is not said	Lean declaration
Finite-volume decay of correlations at strong coupling	Narrow window, for each finite N, no N in the constants, time distance	Distance in spatial directions; outside the window	`TorusRP.wilson_strong_coupling`
Dobrushin decay of covariance with distance	$\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$, finite N, no N in the constants	Outside the window; the true rate of decay	`LinkSpec.wilson_cov_dist`

04

The infinite-volume limit and reconstruction (20-03, 20-04)

Theorem	Scope	What is not said	Lean declaration
The theorem on the limit without a subsequence	$\beta \neq 0$, $q(\beta) < 1$	Other boundary conditions; uniqueness of the Gibbs measure. "Unique" is trivial from the topology; the content is convergence of the sequence	`InfVol.wilson_infvol_unique`
The theorem on the limit independent of size	$\beta \neq 0$, $q(\beta) < 1$, all sizes $m+1$, any $s(k) \to \infty$	The rate of convergence (outside the statement)	`AllN.wilson_infvol_allN`
Translation invariance of the limit	$\beta \neq 0$, $q(\beta) < 1$, unit translations in four directions	Rotations and reflections	`TransI.wilson_infvol_translation`
Clustering at limit points	$\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$, at every limit point, four directions	The true rate of decay	`WilsonLimit.cluster_limit`
The mass gap with non-degeneracy (Dobrushin window)	$\beta \neq 0$, $q(\beta) < 1$; the statement picks one limit point (the same estimate at every limit point: `WilsonLimit.wilson_massGap_dob`); $\|T - P_\Omega\| \leq q(\beta)^{1/2}$, $\mathrm{Var}(F_0) \geq \frac{1}{4} e^{-12\|\beta\|}$	That the mass is finite (lower bound only); the $\frac{1}{2}$ is a matter of choice	`WilsonLimit.wilson_massGap_dob_exi`

Theorem	Scope	What is not said	Lean declaration
The mass gap with non-degeneracy (every limit point)	$\beta \neq 0$, narrow window, every limit point, rate $-\log 24(1-e^{-4\lvert\beta\rvert})$ (not included in the bundled statement)	As above	`WilsonLimit.wilson_massGap_nondege`
The mass gap in the gauge-invariant sector	$\beta \neq 0$, $q(\beta) < 1$; the statement picks one limit point; witness $F_p = \mathrm{Re}\, U_p$	That the closure of W_{gi} coincides with the physical space of states	`GaugeZ.wilson_massGap_gauge_exists`
$T \geq 0$ (on the whole W)	$\beta \geq 0$, $\lambda > 0$, $e^{\lambda} q(\beta) < 1$, every limit point	$\beta < 0$	`FullT.wilson_Tpos_full`
The range of T lies in the gauge-invariant sector	$\lambda > 0$, $e^{\lambda} q(\beta) < 1$, every limit point	Not included in the bundled statement	`Bind.T_range_gauge_all`
The bundled statement	$0 < \beta$, $18 \tanh \beta < 1$. Convergence, translation, reflection positivity, gap and non-degeneracy in both sectors, $T[F_0] = 0$, $T \geq 0$, clustering	The continuum limit; an upper bound on the mass; outside the window	`InfMain2.wilson_infvol_main2`

05

The vessel with fermions (20-05)

The lattice is $(\mathbb{Z}/N)^3 \times \mathbb{Z}/2h$ ($h > 0$), the weight is $e^{-\beta S_W}\, \mathrm{Re}\big[(\det D)^{2j}\big]$, and D is the Wilson–Dirac operator ($r = 1$, antiperiodic in four directions).

Theorem	Scope	What is not said	Lean declaration
Reflection positivity of the Wilson–fermion weight	$m > -1$, all β and j, time plane through sites	$m \leq -1$; the middle plane; infinite volume	`WilsonGauge.site_reflection_positive_wilson_full`
Positivity of the partition function	All β, m, j	The size of Z	`WilsonGauge.partition_pos_wilson_full`

Theorem	Scope	What is not said	Lean declaration
Global gauge invariance of the determinant	Unconditional; constant g	Local gauge transformations	`GaugeDet.physDirac_det_conj`
Invariance under the generators of the lattice symmetries	All N, j, m, β: global gauge, translations in four directions, spatial 90° rotations, reflections of spatial axes, time reflection	Closure as a group; lifting to the block level	`CubicSym.wfProb_lattice_generators`
Invariance under the hypercubic generators	Isotropic lattice $N = 2h$; the above plus (space, time) 90° rotations	Rotations involving time on anisotropic lattices	`CubicSym.wfProb_hypercubic_generators`
Invariance of the determinant under time reflection	Unconditional (all N, h, m, ω); the reversal $t \mapsto 2h - 1 - t$	Reflection positivity for this reversal	`CubicSym.physDirac_det_trefl`

06

The area law at strong coupling, and an upper bound on the vortex free energy (20-06)

Set $q = 1 - e^{-2\beta}$. The window differs for each version of the constants (details in 20-06).

Theorem	Scope	What is not said	Lean declaration
The area law in finite volume	$\beta \geq 0$, $4096q < 1$ (the tightened version: $441q < 1$), the Wilson loop on the boundary of a rectangle; the rate and the prefactor depend neither on the size of the lattice nor on the size of the rectangle	Outside the window, general loops, a lower bound on Wilson loops, continuum Wilson loops	`AreaLaw.area_law_wilson`, `AreaLaw.area_law_wilson_sharp`, `AreaLaw.area_law_wilson_max_shar`

Theorem	Scope	What is not said	Lean declaration
The area law in infinite volume	$\beta > 0$, the intersection of $18 \tanh \beta < 1$ and $4096q < 1$, the limit measure (also at every limit point)	Infinite-volume versions with the tightened constants, or with arbitrary corners and orientations	`AreaLaw.area_law_infvol`, `AreaLaw.area_law_limit_sub`
Upper bound on the vortex free energy	$\beta \geq 0$. On a torus of transverse size N and longitudinal size M, for every background, $441q < 1$ (`VortexStrongMN.vortex_floor_MN_sharp`). On an isotropic torus, $4096q < 1$ (`VortexStrong.vortex_floor`), and the tightened version $441q < 1$ (`VortexStrong.vortex_floor_sharp`). The difference is exponentially small in the area M^2 of the longitudinal plane, and the rate per unit area depends neither on the transverse size nor on the longitudinal size	Removal of the prefactor N^2 (the transverse area), the direction of the sign of the difference, outside the window, the band statement	`VortexStrongMN.vortex_floor_MN_s`, `VortexStrong.vortex_floor`, `VortexStrong.vortex_floor_sharp`

All of these are formalisations of known results of the type of Osterwalder–Seiler (area law) and of 't Hooft and Münster (vortex free energy)Cited . The area law is a statement about lattice Wilson loops; continuum Wilson loops are present neither in this part nor in the published construction.

07

What the part as a whole does not say

- **The lattice spacing is fixed at 1.** No row says anything about the continuum limit or about uniformity when the lattice spacing is changed.
- **Only the strong-coupling window.** The rows on the area law and the upper bound on the vortex free energy (§ 06) also hold only inside the strong-coupling window. The infinite-volume rows hold only for $|\beta| < 0.0556$Reading (the narrow window: $|\beta| < 0.0106$Reading), and nothing is said about weak coupling.
- **Only a lower bound on the mass.** The mass-gap rows are upper bounds on $\|T - P\Omega\|$. It is not said that the mass is finite, nor how it compares with the true mass.
- **Formalisations of known types.** These are known results of Osterwalder–Seiler strong-coupling type, of Dobrushin-uniqueness type, and of the type of the strong-coupling area law and the vortex free energy, written in Lean on the concrete SU(2) Wilson measure. They do not show anything about the mass gap problem for four-dimensional Yang–Mills. They are statements at the opposite end (strong coupling,

lattice spacing fixed) from the questions the published construction left open (convergence and uniqueness, a uniform mass gap, continuum Wilson loops and the area law) (14-06).

The next part treats the vortex floor and Polyakov loops on a twisted lattice, at the same level of “one lattice spacing”.

What this chapter can say

What is said	What is not said
Cited Each row of the tables of § 02– § 05, Lean each row of the table of § 06 (all machine-checked in Lean; scope as in each row)	Changing the lattice spacing, the continuum limit, outside the window, an upper bound on the mass
Reading The edges of the windows, $\lvert\beta\rvert < 0.055613$ and $\lvert\beta\rvert < 0.010640$	Floating-point values (not interval estimates)

Sources and reproduction

Item	Kind	Source / tools
Original sources for the type of each row	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract). The original source of Dobrushin's uniqueness condition: bibliographic data only, not checked. The source tables of each chapter
The declarations of each row	Lean	Machine-checked in Lean 4 (all pass the kernel re-check; the axioms are the three standard ones). A-04
The numbers at the edges of the windows	Reading	The numerical record of this book's study (double precision)

Words used in this chapter — strong-coupling window, limit point, mass gap (lower bound), non-degeneracy, gauge-invariant sector, reflection positivity, generator. Their meanings are in the A-02 Glossary.

Next chapter to read: 21-00 Entrance — vortices in a twisted box — on to Part 21, which treats the vortex floor and Polyakov loops on a twisted lattice.

Entrance — vortices in a twisted box

> This chapter is a guide.

What this chapter shows — a list of the theorems in the chapters of Part 21, the scope that applies to the whole part (what it does not say), and the direction of the word "floor".

Prerequisite chapters — 19-02 the torus of width N and length M, the vortex ratio, Polyakov loops.

Said first — every theorem in this part is a statement on a finite torus at a fixed lattice spacing. Inside the strong-coupling window there is a form whose rate does not depend on the transverse size (the prefactor, the transverse area, remains; 20-06). Outside the window, none of the rates of the floors of this part (upper bounds on θ) is uniform in the transverse size N. The lower bound on the deficit in 21-07 (1/2 in a symmetric box, independent of the transverse size) is a statement in the opposite direction from a floor (on the side of lower bounds on θ). None reaches the band or the mass gap (the statements to be shown are in 19-07). Each is a formalisation of one side of a known picture.

The order of this chapter

1. The theorems of this part
2. The scope that applies to the whole part
3. What a "floor" is
4. A guide to the record of the study

01

The theorems of this part

The objects throughout are the $SU(2)$ Wilson measure on the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ of width N (directions 1 and 2) and length M (directions 3 and 4), its twisted partition

functions $Z_0,\ Z_1$, and the vortex ratio $\theta_M = (Z_0 - Z_1)/(Z_0 + Z_1)$ (definitions in 19-02).

Chapter	Theorem (one line)
21-01 Three results on sign systems	The covering inequality, $\mathbb{Z}_2$ monotonicity, and comparison with $\mathbb{Z}_2$ (abstract implications, with all assumptions shown), and their application to the $SU(2)$ Wilson measure. A representation of θ_M as an average of the sign η of the (3,4) plaquettes
21-02 The thin-vortex floor	For all real β and $N, M \geq 1$, $\lvert\theta_M\rvert \leq \tanh(N^2\lvert\beta\rvert)^{M^2}$. If $\beta \neq 0$, $\lvert\theta_M\rvert \leq e^{-\sigma_0 M^2}$ with $\sigma_0 = -\ln\tanh(N^2\lvert\beta\rvert) > 0$
21-03 The thick-vortex floor — statement and route	For all $N \geq 1$ and any $\varepsilon > 0$, for every sufficiently large β and sufficiently large M, $\lvert\theta_{M+1}\rvert \leq \exp\bigl(-(M+1)^2 e^{-\beta(S_N+\varepsilon)}\bigr)$ (S_N is the minimal action of a vortex; the thresholds are not explicit)
21-04 The thick-vortex floor — parts of the proof	The estimate $\lvert\theta_M\rvert \leq \tanh(K_{\mathrm{eff}})^{k^2}$ ($mk \leq M$) obtained by dividing the longitudinal directions into blocks of side m, and the parts that assemble the thick-vortex floor
21-05 The minimal action of a vortex	For all $N \geq 1$, $S_N > 0$ (a lower bound uniform in the block size m; the value is not explicit). For $N = 2, 3, 4$, $S_N < 2N^2$ (a computer certificate)
21-06 The sign of θ and pairs of Polyakov loops	If M is even, $\theta_M \geq 0$ for all real β ($Z_1 \leq Z_0$). Upper bounds on pairs of Polyakov loops, for example for $M \equiv 0 \pmod 4$, $\lvert\langle P_b(0) P_{b'}(T)\rangle_0\rvert \leq 2\tanh(N^2\lvert\beta\rvert)^{M\,\min(T, M-T)}$
21-07 The vortex ratio — continuity, limit, matching	With the size of the lattice held fixed, the vortex ratio is continuous in the coupling, is 0 at coupling 0, and approaches 1 as the coupling grows. For a chosen target value (at least 0 and less than 1) there is a coupling matched to it (the largest solution). A box with two twists has no flat configuration, and the deficit in a symmetric box is at least 1/2, independent of the transverse size
21-08 What this part says, and what it does not	A table of the scope of each theorem

Every theorem is machine-checked in Lean, and in addition the definitions and statements have been compared against the text by a separate pair of eyes (the computer certificates for $N = 2, 3, 4$ are compared on the side of the corollaries that use them). The names of the declarations and their scope are in each chapter.

02

The scope that applies to the whole part

- **The lattice spacing is fixed.** These are equalities and inequalities about unnormalised partition functions on a finite torus. The continuum limit is not treated.
- **Outside the window, the rate of the floor is not uniform in the transverse size N.** The rates ($\sigma_0 \approx 2e^{-2N^2\lvert\beta\rvert}$ and $e^{-\beta(S_N+\varepsilon)}$) approach 0 as N grows. These are

estimates for a slab with the transverse size held fixed, effectively two-dimensional. Inside the strong-coupling window there is a form whose rate does not depend on the transverse size (the prefactor, the transverse area, remains; 20-06). The lower bound on the deficit in 21-07 (1/2 in a symmetric box, independent of the transverse size) is a statement in the opposite direction from a floor (on the side of lower bounds on θ), not a rate of a floor.

- **Some thresholds are not explicit.** For the thick-vortex floor and the corollaries that rest on it, the thresholds in β and M are given only as existing; no values come out. The positive lower bound on the minimal action of a vortex also has no explicit value.
- **Reaches neither the band nor the mass gap.** Neither the statement of the band, nor the four-dimensional area law, nor the mass gap follows from the theorems of this part. The statements to be shown about the band and the mass gap are in 19-07.
- **A formalisation of one side of a known picture.** Of the known picture of a gas of vortices on a thin twisted slab (said to be the line of Mack–Petkova, Fröhlich and others; bibliography in the table at the end of the chapter, the texts not checked in this book), this part puts the side of the upper estimates into Lean as theorems about the concrete $SU(2)$ Wilson measure. The floor theorems of this part are on the side of the upper estimates; on the opposite side, 21-07 gives only one value, in the limit $\beta \to \infty$ with the lattice held fixed. This book claims no novelty.

03

What a "floor" is

The vortex ratio is tied to 't Hooft's flux free energy $\Delta F = \ln(Z_0/Z_1)$ (the same quantity as in 25-01 and 26-03; 26-03 also calls it the vortex free energy) by $\theta_M = \tanh(\Delta F/2)$. In the picture of confinement, ΔF approaches 0 exponentially in the (3,4) area M^2, and the speed (the rate) corresponds to the tension. The theorems of this part bound θ_M, and hence ΔF, **from above**. That is the same as bounding the rate **from below**, and this lower estimate is what we call a "floor". It is a statement in the direction "the rate is at least this much"; the floor theorems say nothing about an upper limit on the rate (a lower bound on ΔF). On the opposite side, 21-07 gives only one value, in the limit $\beta \to \infty$ with the lattice held fixed (that the slope of the flux free energy converges to the minimal deficit). "Thin" and "thick" differ in whether the rate of the floor is taken from the cost $2N^2$ of the configuration that gathers the vortex onto a single plane, or from the cost S_N of a cheaper thick vortex.

04

A guide to the record of the study

The record of the study of twisted tori, vortices and the band, up to the theorems of this part, is in Part 25 (from the direction of space — a small twisted T^2 and confinement) and Part 26 (the crossover band) of the present book. The view that tools built on structure alone yield no more than the floor is in 26-04. This part does not rewrite them; it points to the chapters where needed.

What this chapter can say

Can say	Cannot say
Proved here The theorems of this part (listed in § 01; statements, assumptions and routes of proof in each chapter)	Anything that moves the lattice spacing; rates of the floor uniform in N; explicit thresholds; the band; the mass gap
Computer-assisted $S_N < 2N^2$ for $N = 2, 3, 4$ (a computer certificate supporting the analytic proof, 21-05)	$S_N < 2N^2$ for general N

Sources and reproduction

Item	Kind	Source / tool
The picture of vortices on a twisted slab, and the comparison of $SU(2)$ with $\mathbb{Z}_2$	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the same bibliography as 26-01; the texts not checked in this book). J. Fröhlich, Phys. Lett. B 83 (1979) 195, bibliography only (not checked). Twisted boundary conditions: G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (25-01)
The theorems of each chapter	Proved here	The routes of proof in the text of each chapter. Machine-checked in Lean 4 (the source table of each chapter)

Words used in this chapter — vortex ratio, flux free energy, floor, thin vortex, thick vortex, minimal action of a vortex, Polyakov loop. Their meanings are in A-02 Glossary.

Next chapter to read: 21-01 Three results on sign systems — we start with the comparison tools that produce the floor.

2026-11-13 · chapter Yang–Mills and the mass gap Part 21 Firm theorems — the vortex floor and Polyakov loops Sign systems ·
Kiichi, Shiori and Rin

Three results on sign systems — the covering inequality, monotonicity, comparison

This chapter is this book's own investigation. The abstract theorems of § 02– § 05 are written as conditional implications.

What this chapter shows — three abstract inequalities (the covering inequality, Z_2 monotonicity, comparison with Z_2) for bounding the SU(2) vortex ratio from above by the vortex ratio of a Z_2 sign system, and their application to the SU(2) Wilson measure. We also set out a representation that writes the vortex ratio θ_M as an average of the signs η of the (3,4) planes.

Prerequisite chapters — 19-01 the product of Haar measures, plaquettes and holonomy; 19-02 the twist, the vortex ratio. We use the language of group characters (homomorphisms, values ± 1).

Said first — § 02– § 05 are abstract theorems in "if ... then" form with every assumption shown. As concrete objects satisfying the assumptions, the only results this chapter counts are the application to the SU(2) Wilson measure (§ 06) and the η representation (§ 07). All three are one-sided comparisons "from the Z_2 side upward"; no lower bound on the SU(2) vortex ratio (no lower bound on the flux free energy) comes out. Nothing here shows monotonicity or a covering inequality for a non-abelian group. The lattice spacing is fixed.

The order of this chapter

1. What the three results are for — the direction of the floor
2. Sign systems — characters, partition functions, twists, base systems
3. The covering inequality
4. Z_2 monotonicity
5. Comparison with Z_2 — couplings of indefinite sign and invariant measures
6. Application to the SU(2) Wilson measure
7. Writing the vortex ratio as an average of signs — the η representation
8. What is not said

9. Detour: reading the covering inequality on the lattice

01

What the three results are for — the direction of the floor

The object of Part 21 is the $SU(2)$ Wilson measure on the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ of width N and length M (19-02). From the ratio of the partition function Z_1 with a twist inserted in one column of (3,4) planes and the partition function Z_0 without it, we form the vortex ratio

$$\theta_M = \frac{Z_0 - Z_1}{Z_0 + Z_1} = \tanh \frac{\Delta F}{2}, \qquad \Delta F := -\ln \frac{Z_1}{Z_0} \quad (1)$$

(definition in 19-02). ΔF is 't Hooft's flux free energy (the same quantity as in 25-01 and 26-03; 26-03 also calls it the vortex free energy). In the picture of confinement, ΔF approaches 0 exponentially in the (3,4) area M^2, and its speed is the tension Cited (the framework of twisted boundary conditions of G. 't Hooft, Nucl. Phys. B 153 (1979) 141).

The three results of this chapter are tools for bounding the ratio Z_1/Z_0 of the $SU(2)$ system **from below** by the ratio of a $\mathbb{Z}_2$ system whose only variables are the signs of the plaquettes. Restated through (1), they bound θ_M and ΔF **from above**, and bound the speed (the rate) at which ΔF decreases in M^2 **from below**. In this part we call this lower estimate a "floor". The $\mathbb{Z}_2$ system we compare with is finally brought down to a system computable by a closed two-dimensional formula (21-02).

The roles of the three are as follows.

Name	What is compared	Use in this part
The covering inequality	A large $\mathbb{Z}_2$ system and a small (base) $\mathbb{Z}_2$ system mapped into it	The pattern for bringing a four-dimensional sign system down to a two-dimensional one
$\mathbb{Z}_2$ monotonicity	Two versions of the same $\mathbb{Z}_2$ system with couplings of different size	That increasing the coupling does not decrease the vortex ratio (a check of the comparison pattern)
Comparison with $\mathbb{Z}_2$	A system on a measure invariant under the action of the centre $\{\pm 1\}$ ($SU(2)$ and others), and a base $\mathbb{Z}_2$ system	Bounding the $SU(2)$ vortex ratio by the vortex ratio of a $\mathbb{Z}_2$ system (§ 05)

Each is a combination of well-known tools of the correlation-inequality type. The pattern of comparing $SU(2)$ with $\mathbb{Z}_2$ is said to be the line of Mack–Petkova Cited

(bibliography in the source table; the text not checked in this book). This book claims no novelty.

02

Sign systems — characters, partition functions, twists, base systems

The three results are written in an abstract form that forgets the lattice. Four things appear.

The group of configurations and the plaquette index. G is a finite group (on the lattice, "a field of signs, one per link", with pointwise product), and ι is a finite set (on the lattice, the plaquettes).

Sign characters. A function $\psi : G \to \mathbb{R}$ is a sign character if it is multiplicative, $\psi(gh) = \psi(g)\psi(h)$, and takes values ± 1 (Lean: `CoveringInequality.IsSignChar`). We take one per plaquette, a family of characters $\chi = (\chi_B)_{B \in \iota}$. On the lattice, $\chi_B(g)$ is the product of the signs of the four links of plaquette B.

Partition functions and twists. For a coupling $c : \iota \to \mathbb{R}$ we set

$$Z_\chi(c) = \sum_{g \in G} \exp\Big(\sum_{B \in \iota} c_B \, \chi_B(g) \Big) \quad (2)$$

(Lean: `CoveringInequality.Z`). A twist is $\zeta : \iota \to \{\pm 1\}$, which replaces the coupling c by $\zeta c = (\zeta_B c_B)_B$. The $-\ln$ of the twisted ratio $Z_\chi(\zeta K)/Z_\chi(K)$ plays the role of the flux free energy of this abstract system.

Base systems and pull-back. Another finite group G_0, a family of characters $\chi^0 = (\chi^0_B)_{B \in \iota}$ (with the **same** index ι), and a map $p : G_0 \to G$ satisfying

$$\chi_B\big(p(t)\big) = \chi^0_B(t) \qquad (\text{for all } B \in \iota,\ t \in G_0) \quad (3)$$

are together called a "base system". p need not be a homomorphism (the proof uses only the multiplicativity of χ_B). On the lattice, p is the map that carries a sign field on a small torus periodically onto a large torus.

The starting lemma is an inequality of the first Griffiths type. For a non-negative coupling $J \geq 0$ and a twist ζ, $Z_\chi(\zeta J) \leq Z_\chi(J)$ Proved here (Lean: `CoveringInequality.Z_flip_le`; the assumption fields are "each χ_B is a sign character", "$J_B \geq 0$", "$\zeta_B = \pm 1$"). The proof expands the product using $e^{c\chi} = \cosh c + \chi \sinh c$ ($\chi = \pm 1$) and shows, by induction on the set of plaquettes, that "the absolute value of the twisted sum does not exceed the untwisted sum". Underneath lies the fact that a sum of a sign character $\sum_g \psi(g)$ is 0 or $|G|$, and never negative.

03

The covering inequality

The covering inequality (Lean: `CoveringInequality.covering_ineq`). We assume the following five.

- (hχ) Each χ_B is a sign character on G.
- (hχ_0) Each χ^0_B is a sign character on G_0.
- (hp) $p : G_0 \to G$ satisfies (3).
- (hK) The coupling is non-negative: $K_B \geq 0$.
- (hζ) $\zeta_B = \pm 1$.

Then

$$Z_\chi(K)\, Z_{\chi^0}(\zeta K) \;\leq\; Z_\chi(\zeta K)\, Z_{\chi^0}(K) \quad (4)$$

The ratio form (`covering_ratio`) and the $-\ln$ form (`vfe_cover_le`) are

$$\frac{Z_{\chi^0}(\zeta K)}{Z_{\chi^0}(K)} \leq \frac{Z_\chi(\zeta K)}{Z_\chi(K)}, \qquad -\ln \frac{Z_\chi(\zeta K)}{Z_\chi(K)} \leq -\ln \frac{Z_{\chi^0}(\zeta K)}{Z_{\chi^0}(K)} \quad (5)$$

with the same five assumptions. Proved here The proof is the two steps below; all three are machine-checked in Lean, and the definitions and statements have been compared by a separate pair of eyes. The tools are of well-known types, Griffiths-type correlation inequalities and change of configuration variables; this book has not checked where in the literature a comparison of this form appears.

The route of proof. Two steps.

1. **The key identity** (Lean: `Z_mul_Z`). In the double sum $\sum_s \sum_t$, replace the configuration on the covering side by $s = u \cdot p(t)$. Since $\chi_B(u\,p(t)) = \chi_B(u)\chi^0_B(t)$, for arbitrary couplings J, J'

$$Z_\chi(J)\, Z_{\chi^0}(J') = \sum_{u \in G} Z_{\chi^0}\left(J\chi(u) + J'\right) \quad (6)$$

   The product of two systems becomes a sum, over u, of single base systems Proved here .

2. **Comparison for each u.** The left side of (4) is $(J, J') = (K, \zeta K)$, the right side $(\zeta K, K)$. The base couplings are $K\chi(u) + \zeta K = \zeta\, a(u)$ and $a(u)$ respectively, where $a_B(u) = \zeta_B K_B \chi_B(u) + K_B \geq 0$. Applying the first Griffiths-type inequality $Z_{\chi^0}(\zeta a) \leq Z_{\chi^0}(a)$ for each u and summing gives (4) Proved here .

No limits and no size conditions are used. It holds for every finite system and every non-negative coupling.

A lattice instance (Lean: `TwistedWilson.z2_four_covers_two`, ratio form `z2_four_ratio`; the only assumption is $K \geq 0$). Take as the covering side the four-dimensional $\mathbb{Z}_2$ gauge theory on the torus of width N and length M (uniform coupling K), and as the base the $\mathbb{Z}_2$ gauge theory on the two-dimensional torus of the length directions (on the (3,4) planes the base character is the sign of a two-dimensional plaquette, not identically 1). With ζ the (3,4) twist,

$$\frac{1-\tanh(N^2K)^{M^2}}{1+\tanh(N^2K)^{M^2}} \;\leq\; \frac{Z_\chi(\zeta K)}{Z_\chi(K)}$$

Proved here . This brings the four-dimensional sign system down to the two-dimensional system of the length directions; it is not itself the detour's "covering that stretches the width n-fold". It is read within the Lean chain, but the proof of the main theorem of the $SU(2)$ thin-vortex floor does not use it; it goes directly from the application of § 06 to the two-dimensional formula.

04

Z_2 monotonicity

$\mathbb{Z}_2$ monotonicity (Lean: `CoveringInequality.mon_ineq`). There are four assumptions.

- (hχ) Each χ_B is a sign character.
- (hK) $K_B \geq 0$.
- (hKK') $K_B \leq K'_B$ (a coupling larger term by term).
- (hζ) $\zeta_B = \pm 1$.

Then

$$Z_\chi(\zeta K')\,Z_\chi(K) \;\leq\; Z_\chi(K')\,Z_\chi(\zeta K), \qquad \text{i.e.} \quad \frac{Z_\chi(\zeta K')}{Z_\chi(K')} \leq \frac{Z_\chi(\zeta K)}{Z_\chi(K)} \quad (7)$$

(the ratio form is `mon_ratio`, with the same assumptions). Increasing the coupling term by term does not increase the twisted ratio, and does not decrease the flux free energy. Proved here The proof follows the route below; it is machine-checked in Lean, and the definitions and statements have been compared by a separate pair of eyes. The pattern is a correlation inequality for ferromagnetic sign systems.

The route of proof. Use (6) with base = cover ($G_0 = G$, p the identity). Bring the couplings that appear into the form $\chi(u)\cdot\big(K' + \zeta\chi(u)K\big)$, remove $\chi(u)$ by the character

shift $Z_\chi(\chi(u)c) = Z_\chi(c)$ (the change of configuration $g \mapsto gu$; Lean: `Z_char_shift`), and apply the first Griffiths-type inequality to $K'_B + \zeta_B \chi_B(u) K_B \geq 0$ Proved here .

This monotonicity is a statement about $\mathbb{Z}_2$ sign systems (non-negative couplings, no twisted background). Whether the same monotonicity in the coupling β holds for the $SU(2)$ Wilson measure is not shown in this chapter (§ 08).

05

Comparison with Z_2 — couplings of indefinite sign and invariant measures

Relaxing the assumption (hK) of the covering inequality, the same form comes out for couplings of indefinite sign.

The form with dominated couplings (Lean: `CoveringInequality.dominated_covering_ineq`). The assumptions are (hχ)(hχ_0)(hp)(hζ) and

- (hJ) $|J_B| \leq K_B$ (the sign of J is arbitrary).

Then

$$Z_\chi(J)\, Z_{\chi^0}(\zeta K) \;\leq\; Z_\chi(\zeta J)\, Z_{\chi^0}(K) \quad (8)$$

The proof is the same as in § 03; the only change is that the base coupling $K_B + \zeta_B J_B$ $\chi_B(u)$ is non-negative by (hJ) Proved here . $J = K$ is the covering inequality, and base = cover with $J = K \leq K'$ is $\mathbb{Z}_2$ monotonicity.

The form on an invariant measure (comparison with $\mathbb{Z}_2$) (Lean: `CoveringInequality.invariant_dominated`). For a configuration space Ω with a measure μ, an action $g \cdot \omega$ of the sign group G on Ω, and functions $J_B : \Omega \to \mathbb{R}$, one per plaquette, we assume the following.

- (hinv) The integral is invariant under the action: for all $g \in G$ and all $f : \Omega \to \mathbb{R}$, $\int f(g \cdot \omega)\, d\mu = \int f(\omega)\, d\mu$.
- (hχ)(hχ_0)(hp) as in § 03.
- (hJ) The coupling changes sign by the character: $J_B(g \cdot \omega) = \chi_B(g)\, J_B(\omega)$.
- (hK) $|J_B(\omega)| \leq K_B$ (for all ω).
- (hζ) $\zeta_B = \pm 1$.
- (hint) For all $c : \iota \to \mathbb{R}$, $\omega \mapsto \exp\left(\sum_B c_B J_B(\omega)\right)$ is μ-integrable.

Then

$$\left(\int e^{\sum_B J_B(\omega)}\, d\mu\right) Z_\chi^0(\zeta K) \;\le\; \left(\int e^{\sum_B \zeta_B J_B(\omega)}\, d\mu\right) Z_\chi^0(K) \quad (9)$$

Cited The pattern is a comparison of a system of a group with centre against a system of the centre group, said to be the line of Mack–Petkova (bibliography in the source table; the text not checked in this book). It is machine-checked in Lean, and the comparison by a separate pair of eyes judged it to hold as an implication.

The route of proof. The key is averaging over the centre (Lean: `card_mul_integral_exp`). From (hinv) and (hJ), for each g, $\int e^{\sum J(\omega)} d\mu = \int e^{\sum \chi(g) J(\omega)} d\mu$, so averaging over g gives

$$|G| \int e^{\sum_B J_B(\omega)}\, d\mu = \int Z_\chi\big(J(\omega)\big)\, d\mu \quad (10)$$

Proved here . The integrand on the right is the partition function of a **sign system** with the configuration ω held fixed, and the coupling $J(\omega)$ is bounded by (hK). So apply (8) for each ω and integrate to obtain (9) (integrability comes from (hint)). In other words, the system on the measure is "a superposition of $\mathbb{Z}_2$ systems with couplings of indefinite sign", and each member of the superposition can be compared with the base $\mathbb{Z}_2$ system.

06

Application to the SU(2) Wilson measure

$SU(2)$ lattice gauge theory satisfies the assumptions of § 05. The correspondence is as follows (notation in 19-01 and 19-02).

Abstract field	The $SU(2)$ Wilson measure	Lean lemma that discharges it
Ω, μ	Link configurations U : links $\to SU(2)$, the product of normalised Haar measures	—
G and its action	A sign per link $g_\ell \in \{\pm 1\}$, $(g \cdot U)_\ell = g_\ell U_\ell$ (multiplying by the centre ± 1)	`TwistedWilson.signAction`
(hinv)	Haar measure is invariant under left multiplication by an element of the centre	`TwistedWilson.haarCfg_inv`
χ_B, (hχ)	The product of the signs of the four links of plaquette B (overlaps kept as overlaps)	`TwistedWilson.chiB_isSignChar`
J_B, (hJ)	$J_B(U) = \beta_B \operatorname{Re} U_B$. Multiplying by signs multiplies $\operatorname{Re} U_B$ by $\chi_B(g)$	`TwistedWilson.hol_smul`
(hK)	$\lvert\operatorname{Re} U_B\rvert \le 1$, so $\lvert J_B\rvert \le \lvert\beta_B\rvert$	`TwistedWilson.abs_re_le_one`
(hint)	The integrand is continuous and bounded	`TwistedWilson.integrable_exp`

The only remaining assumptions are on the side of the base system.

Comparison with $\mathbb{Z}_2$ ($SU(2)$, couplings per plaquette, any base system) (Lean: `TwistedWilson.su2_dominated_general`). If $N, M \geq 1$, β_B are real couplings per plaquette (of arbitrary sign), $\zeta_B = \pm 1$, and the base system (G_0, χ^0, p) satisfies

- each χ^0_B is a sign character on G_0,
- $p : G_0 \to \{\pm 1\}^{\text{links}}$ satisfies $\chi_B(p(t)) = \chi^0_B(t)$,

then

$$\left(\int e^{\sum_B \beta_B \operatorname{Re} U_B}\, dU\right) Z_{\chi^0}\left(\zeta|\beta|\right) \;\leq\; \left(\int e^{\sum_B \zeta_B \beta_B \operatorname{Re} U_B}\, dU\right) Z_{\chi^0}\left(|\beta|\right) \quad (11)$$

(dU is the product of Haar measures, $|\beta| = (|\beta_B|)_B$). Cited This is a formalisation of the comparison said to be the line of Mack–Petkova (the text not checked in this book). It is machine-checked in Lean, and the comparison by a separate pair of eyes judged that it holds conditionally on the three fields of the base system (being characters, the pull-back formula, $\zeta = \pm 1$), and that instances satisfying them exist.

An instance (Lean: `TwistedWilson.su2_dominated_general_instance`, no assumption fields). Take as the base the $\mathbb{Z}_2$ gauge theory on the two-dimensional torus $(\mathbb{Z}/M)^2$ of the length directions. p is the map that sends a sign t of a two-dimensional link to the links in directions 3 and 4 at (x_3, x_4) (independently of the transverse position (x_1, x_2)), and puts $+1$ on the links in directions 1 and 2 (`TwistedWilson.pull`). Taking the coupling $\beta_B = \beta\, z_B$ (z is the sign of the background twist n_{12}) and ζ the sign of the (3,4) twist, the two integrals in (11) become exactly the partition functions $Z_{N,M}(\beta; n_{12}, 0)$ and $Z_{N,M}(\beta; n_{12}, 1)$ of 19-02 (the case $n_{12} = 1$ gives this part's Z_0, Z_1), and

$$Z_{N,M}(\beta; n_{12}, 0) \cdot Z_{\text{base}}\left(\zeta|\beta|\right) \;\leq\; Z_{N,M}(\beta; n_{12}, 1) \cdot Z_{\text{base}}\left(|\beta|\right) \qquad \text{(for all real } \beta,\ N, M \geq$$

follows (the Lean name of the same formula is `TwistedWilson.su2_dominated`). The ratio on the base side is computable by a closed two-dimensional formula, and the thin-vortex floor follows from it (21-02). The background twist n_{12} is absorbed into the sign of J, so the bound $|J_B| \leq |\beta|$ of (hK) is unchanged even with a background. This contrasts with the $\mathbb{Z}_2$ system itself, where the background twist becomes a negative coupling and (hK) of the covering inequality fails.

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07
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Writing the vortex ratio as an average of signs — the η representation

The twist $n34$ was "reverse the sign of one column of (3,4) planes". In fact neither the position of the reversed column nor the way of reversing carries any meaning; only the product of the signs does. The η representation puts this into a formula.

For $\eta : (\mathbb{Z}/M)^2 \to \mathbb{R}$, write $Z[\eta]$ for the partition function in which the coupling of the (3,4) plaquette $((x_1,x_2),(x_3,x_4))$ is multiplied by $\eta(x_3,x_4)$ (the same factor for all N^2 transverse copies; the background n_{12} is left as it is; Lean: `TwistedWilson.ZEta`).

Parity of η (Lean: `TwistedWilson.ZEta_parity`). The only assumption is $\eta_y = \pm 1$ (for all y). Then

$$Z[\eta] = \begin{cases} Z_0 & \prod_y \eta_y = 1 \\ Z_1 & \prod_y \eta_y = -1 \end{cases} \quad (13)$$

(for all real β and $N, M \geq 1$. Here we wrote it in the notation of this part's convention $n_{12} = 1$. For $n_{12} = 0$ the same formula holds with Z_0, Z_1 replaced by $Z_{N,M}(\beta;0,0)$, $Z_{N,M}(\beta;0,1)$. The same goes for (14), whose left side is then $\theta_M^{(n_{12})}$ of 19-02).

θ is the average of η (Lean: `TwistedWilson.theta_eq_eta_average`, no assumption fields).

$$\theta_M = \frac{Z_0 - Z_1}{Z_0 + Z_1} = \frac{\sum_\eta \left(\prod_y \eta_y\right) Z[\eta]}{\sum_\eta Z[\eta]} \quad (14)$$

The sum runs over all $\eta : (\mathbb{Z}/M)^2 \to \{\pm 1\}$ (2^{M^2} of them), and this is a ratio of unnormalised sums. The right side is the expectation of $\prod \eta$ under the joint weight $\exp\left(\beta \sum_B z_B\, \eta_B\, \mathrm{Re}\, U_B\right)$ of the configuration (U, η). Proved here The proof follows the route below; both are machine-checked in Lean, and the definitions and statements have been compared by a separate pair of eyes. That the twist is determined only by its centre class is a standard fact about twisted boundary conditions Cited (G. 't Hooft, Nucl. Phys. B 153 (1979) 141).

The route of proof. Three parts Proved here .

1. Centre gauge invariance (Lean: `ZEta_gauge`). Reversing the links in directions 3 and 4 by a two-dimensional sign field t leaves Haar measure unchanged, and $Z[\eta] = Z[\eta \cdot \chi^{2D}(t)]$ ($\chi^{2D}_y(t)$ is the sign of the two-dimensional plaquette y).
2. Realising plaquette signs (Lean: `TwistedPlane.exists_chi2_eq`). If $\eta = \pm 1$ and $\prod \eta = 1$, there is a t with $\chi^{2D}_y(t) = \eta_y$. Shown by counting.

3. The involution $\eta \mapsto \eta \cdot e_0$ (reverse only plaquette 0) shows that the cases $\prod \eta = 1$ and -1 are equal in number.

(13) follows from 1 and 2, and (14) from (13) and 3. Independence of the position of the reversal (for every position P, "reverse only plaquette P" gives Z_1: `ZEta_flip_any`) follows from the same parts. This representation is the starting point of the thick-vortex floor (21-03).

08

What is not said

- **One side only.** All three point in the direction of bounding the $SU(2)$ ratio Z_1/Z_0 from below, from the $\mathbb{Z}_2$ side. A lower bound on ΔF (a lower bound on θ_M) does not come out of comparisons of this type.
- **Not monotonicity or a covering inequality for a non-abelian group.** § 03 and § 04 are theorems about $\mathbb{Z}_2$ sign systems. For $SU(2)$, neither monotonicity in the coupling β nor a covering inequality across torus sizes is shown. For the finite non-abelian group Q_8, without a twisted background, there are points at weak coupling where both the covering inequality and monotonicity fail Reading (floating-point values of transfer matrices on small three-dimensional lattices. Example: covering number 3, $\beta = 1.25$, $\Delta F_3(1.25) = 1.29498 > \Delta F_1(3.75) = 1.28758$; two independent implementations agree). Q_8 is a finite subgroup of $SU(2)$; whether the same happens for $SU(2)$ is not known.
- **The lattice dictionary is on paper.** The Lean statements of § 03 and § 04 are in abstract form. The dictionary (the detour) that reads a four-dimensional $\mathbb{Z}_2$ torus with the transverse directions stretched n-fold as a pair of a large torus and a base torus is on paper. The only lattice instance in Lean with a non-trivial base is the form at the end of § 03, "bringing four-dimensional $\mathbb{Z}_2$ down to two-dimensional $\mathbb{Z}_2$ of the length directions". Abstract examples whose base character is identically 1 are not counted among the results of this chapter.
- **The ratio of the base system is needed separately.** (11) and (12) go only as far as the comparison with the base $\mathbb{Z}_2$ system; the value of the base ratio is a separate computation (21-02 uses the closed two-dimensional formula).
- **The lattice spacing is fixed.** All are inequalities for finite systems on finite tori; they say nothing about the continuum limit, infinite volume or the mass gap.

The view that tools of this type yield no more than the floor is in the record of the study in 26-04 and 26-06.

Detour: reading the covering inequality on the lattice

You may skip this. In four-dimensional $\mathbb{Z}_2$ gauge theory, suppose a torus Λ covers a small torus Λ_0 n times (stretching the transverse directions periodically). With G the link signs of Λ, G_0 the link signs of Λ_0, p the periodic copy and ι the plaquettes of Λ, (3) is the obvious fact that "the plaquette signs of the copied field equal the plaquette signs of the base". The coupling on the base side is counted by the plaquette index of the cover, so if n plaquettes lie over one plaquette of the base, their couplings add up to nK. The covering inequality (5) then reads

$$\Delta F_{\Lambda}(\beta) \;\leq\; \Delta F_{\Lambda_0}(n\beta)$$

Stretching both transverse directions n-fold gives n^2 plaquettes. This reading is on paper, and has been checked by exhaustive enumeration on small lattices Reading (floating point, with a negative control: a version that miscounts the plaquettes fails). The $SU(2)$ application (§ 06) does not go through this dictionary; it builds the base system on the same lattice (the two-dimensional torus of the length directions) directly in Lean.

What this chapter can say

Can say	Cannot say
Proved here The covering inequality (4)(5), under five assumptions (two sign characters, the pull-back formula, $K \geq 0$, $\zeta = \pm 1$). Lean: `covering_ineq`, `covering_ratio`, `vfe_cover_le`	The reading on the lattice (the dictionary) is on paper. No version for non-abelian groups
Proved here A lattice instance: bringing four-dimensional $\mathbb{Z}_2$ down to two-dimensional $\mathbb{Z}_2$ of the length directions ($K \geq 0$). Lean: `z2_four_covers_two`, `z2_four_ratio`	A Lean instance of the covering that stretches the width n-fold
Proved here $\mathbb{Z}_2$ monotonicity (7): $0 \leq K \leq K'$, $\zeta = \pm 1$. Lean: `mon_ineq`, `mon_ratio`	Monotonicity in β for $SU(2)$. $\mathbb{Z}_2$ systems with a twisted background
Proved here Comparison with $\mathbb{Z}_2$ (8)(9): $\lvert J\rvert \leq K$, invariant measure, integrability. Lean: `dominated_covering_ineq`, `invariant_dominated`	The reverse direction (a lower bound on ΔF)
Proved here The $SU(2)$ application (11), under the three fields of the base system. Lean: `su2_dominated_general`. Instance (12) (no fields): `su2_dominated_general_instance`, `su2_dominated`	The value of the base ratio (21-02). Lattice spacing fixed; no continuum limit
Proved here Parity of η (13) (field $\eta = \pm 1$) and θ as the average of η (14) (no fields). Lean: `ZEta_parity`, `theta_eq_eta_average`	An estimate of the average of η itself (the thick-vortex floor is 21-03)
Reading For Q_8 there are points where the covering inequality and monotonicity fail (no background, small three-dimensional lattices)	The general form of the failure. Floating-point values, not a theorem

Sources and reproduction

Item	Kind	Source / tool
The pattern of comparing $SU(2)$ with $\mathbb{Z}_2$	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the same bibliography as 26-01; the texts not checked in this book). J. Fröhlich, Phys. Lett. B 83 (1979) 195, bibliography only (not checked)
Twisted boundary conditions; the twist being determined only by its class	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (the same bibliography as 25-01)
The three results, the application, the η representation	Proved here	Machine-checked in Lean 4 (declarations in each section; only the three standard axioms, no sorry). The definitions and statements were compared by a separate pair of eyes, which judged the three abstract results "to hold as implications", the application "conditional on the fields of the base system, with satisfying instances", and the η representation "to hold"
The failure for Q_8	Reading	The numerical record of this book's study (transfer matrices on three-dimensional lattices cross-checked by exhaustive enumeration; two independent implementations agree to 4–7 digits. D_4 gives the same values)
The check of the lattice dictionary	Reading	The numerical record of this book's study (sums over all configurations of small four-dimensional $\mathbb{Z}_2$ lattices; fails in a negative control that miscounts the plaquettes)

Words used in this chapter — sign system, sign character, base system, pull-back, covering inequality, $\mathbb{Z}_2$ monotonicity, comparison with $\mathbb{Z}_2$, averaging over the centre, vortex ratio, flux free energy, floor, η representation. Their meanings are in A-02 Glossary.

Next chapter to read: 21-02 The thin-vortex floor — putting the closed two-dimensional formula into the comparison of § 06, we obtain $|\theta_M| \le \tanh(N^2|\beta|)^{M^2}$.

The thin-vortex floor

> This chapter is this book's own investigation.

What this chapter shows — that the vortex ratio of the $SU(2)$ Wilson measure satisfies $|\theta_M| \le \tanh(N^2|\beta|)^{M^2}$ for every real β and all $N, M \ge 1$; its exponential form $|\theta_M| \le e^{-\sigma_0 M^2}$ ($\sigma_0 = -\ln\tanh(N^2|\beta|)$); the meaning of "thin"; and the five steps of the proof.

Prerequisite chapters — 19-02 the torus of width N and length M, the twist, the vortex ratio; 21-01 Three results on sign systems (the application in § 06).

Said first — the lattice spacing is fixed. The rate $\sigma_0 \approx 2e^{-2N^2|\beta|}$ approaches 0 as the width N grows and is not uniform in N. The sign of θ_M ($Z_1 \le Z_0$) is not stated in this chapter (21-06). No rate as $M \to \infty$ (a limit) is defined. Neither the band nor the mass gap is reached (the statements to be shown are in 19-07). This is a formalisation of one side of a known picture.

The order of this chapter

1. Statement — the ratio form, the two-sided form, the exponential form
2. In numbers — the size of the right side, and the gap to the actual ratio
3. What "thin" means — the cost of a vortex gathered onto one sheet
4. The route of proof — five steps
5. Part: the closed two-dimensional Z_2 formula
6. Part: moving the base partition function to two dimensions
7. Arithmetic restatements — a lower bound on the ratio and an upper bound on the flux free energy
8. What is not said

01

Statement — the ratio form, the two-sided form, the exponential form

The object is the $SU(2)$ Wilson measure on the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ of width N (directions 1 and 2) and length M (directions 3 and 4). The weight is $e^{\beta \sum_p z_p \operatorname{Re} U_p}$ ($z_p = \pm 1$ is the sign of the twist), the measure is the product of normalised Haar measures, and partition functions are not normalised (19-01). Write $Z_{N,M}(\beta; n_{12}, n_{34})$ for the partition function with background twist n_{12} and (3,4) twist n_{34}, and let the vortex ratio with background n_{12} be $\theta_M^{(n_{12})} = \big(Z_{N,M}(\beta; n_{12}, 0) - Z_{N,M}(\beta; n_{12}, 1)\big) / \big(Z_{N,M}(\beta; n_{12}, 0) + Z_{N,M}(\beta; n_{12}, 1)\big)$ (the notation of the detour in 19-02). Under this part's convention $n_{12} = 1$, these are Z_0, Z_1 and $\theta_M = \theta_M^{(1)}$. The theorems below also hold for $n_{12} = 0$.

The thin-vortex floor (ratio form) (Lean: `TwistedWilson.twisted_vortex_bound`). For every real β, all $N, M \geq 1$ and $n_{12} \in \{0, 1\}$,

$$\theta_M^{(n_{12})} \;\leq\; \tanh\big(N^2 |\beta|\big)^{M^{\angle}} \quad (1)$$

The only assumption field is $N, M \geq 1$.

The thin-vortex floor (two-sided form) (Lean: `TwistedWilson.twisted_vortex_bound_abs`). In the same range,

$$\big|\theta_M^{(n_{12})}\big| \;\leq\; \tanh\big(N^2 |\beta|\big)^{M^{\angle}} \quad (2)$$

For $\beta < 0$ there are cases with $\theta_M^{(n_{12})} < 0$ ($N = M = 1$, $\beta = -1.5$ in the table of § 02), and there it is (2) that has content.

The thin-vortex floor (exponential form) (Lean: `TwistedWilson.thin_vortex_floor`). If $\beta \neq 0$,

$$\sigma_0 := -\ln \tanh\big(N^2 |\beta|\big) > 0, \qquad \big|\theta_M^{(n_{12})}\big| \;\leq\; e^{-\sigma_0 M^{\angle}} \quad (\text{for all } M \geq 1) \quad (3)$$

The only assumption is $\beta \neq 0$ (at $\beta = 0$, $\tanh 0 = 0$ and σ_0 loses its meaning; in fact $\theta_M^{(n_{12})} = 0$).

Proved here For all three, the route of proof is in § 04– § 06; they are machine-checked in Lean as theorems about the concrete $SU(2)$ Wilson measure, and the definitions and statements have been compared by a separate pair of eyes. The pattern, bounding the $SU(2)$ system by a $\mathbb{Z}_2$ system, is said to be the line of Mack–Petkova Cited (bibliography in the source table; the text not checked in this book). In exhaustive sums over finite groups and $SU(2)$ Metropolis runs, written independently during the comparison, no violation was found Reading .

In the language of this part, (3) is a lower estimate of the rate, "θ_M decreases in M^2 at least at rate σ_0", that is, a **floor** (for the direction, see 21-01 § 01).

02

In numbers — the size of the right side, and the gap to the actual ratio

We list the right side $t = \tanh(N^2|\beta|)$ and the rate $\sigma_0 = -\ln t$ for several N, β Reading (floating-point values).

N	$\lvert\beta\rvert$	$t = \tanh(N^2\lvert\beta\rvert)$	$\sigma_0 = -\ln t$	$2e^{-2N^2\lvert\beta\rvert}$
1	0.25	0.2449	1.407	1.213
1	0.5	0.4621	0.7719	0.7358
1	1	0.7616	0.2723	0.2707
2	0.25	0.7616	0.2723	0.2707
2	0.5	0.9640	0.03664	0.03663
2	1	0.99933	6.709×10^{-4}	6.709×10^{-4}
3	0.25	0.97803	0.02222	0.02222
3	0.5	0.999753	2.468×10^{-4}	2.468×10^{-4}
3	1	$1 - 3.05 \times 10^{-8}$	3.05×10^{-8}	3.05×10^{-8}

Once $N^2|\beta|$ exceeds 1, the rate is almost the same as $2e^{-2N^2|\beta|}$, and it drops by orders of magnitude each time N increases by one. The right side is determined by $N^2|\beta|$ alone, so $(N, \beta) = (1, 1)$ and $(2, 0.25)$ give the same row.

The gap to the actual ratio on small lattices is as follows Reading (for $SU(2)$, Metropolis estimates, from an implementation written independently during the comparison).

(N, M)	$n12$	β	$\theta_M^{(n12)}$ (estimate)	Right side t^{M^2}
(1,1)	1	1.5	0.7033 ± 0.0018	0.9051
(1,1)	0	-1.5	-0.0785 ± 0.0026	0.9051
(2,1)	0	0.5	0.4856 ± 0.0012	0.9640
(1,2)	0	1.0	0.0061 ± 0.0013	0.3364

For the $\mathbb{Z}_2$ system (the same lattice with $SU(2)$ replaced by $\{\pm 1\}$), in the pairs tried with $N = 1$ or $M = 1$, $|\theta_M|$ coincides with the right side Reading (floating-point values of finite sums by variable elimination). Since the system we compare with is a $\mathbb{Z}_2$ system, the bound is nearly an equality where configurations close to $\mathbb{Z}_2$ dominate, and becomes loose where the continuous fluctuations of $SU(2)$ matter.

03

What "thin" means — the cost of a vortex gathered onto one sheet

In closed form the rate σ_0 is

$$\sigma_0 = -\ln\tanh x = \ln\frac{1+e^{-2x}}{1-e^{-2x}} = 2\,\mathrm{artanh}\big(e^{-2x}\big), \qquad x = N^2|\beta| \quad (4)$$

Hence $2e^{-2x} \le \sigma_0 \le 2e^{-2x}/(1-e^{-4x})$, and for large x, $\sigma_0 \approx 2e^{-2N^2|\beta|}$ Proved here (from the positivity of the coefficients of the power series of artanh).

The exponent $2N^2|\beta|$ has the following meaning. Among the (3,4) plaquettes, fix one longitudinal position (x_3, x_4) and consider the configuration in which the signs of all N^2 transverse copies are reversed at once. This is a configuration in which the vortex is gathered onto a single sheet spread out in the transverse directions; we call it a **thin vortex**. Since N^2 plaquettes have $\mathrm{Re}\,U_p$ change from 1 to -1, the cost in action is $2\beta \cdot N^2$.

In the proof, this picture appears directly as a formula. The base system (§ 04) is the $\mathbb{Z}_2$ gauge theory on the two-dimensional torus of the length directions, and the coupling of one of its plaquettes is the sum $N^2|\beta|$ of the couplings of the N^2 transverse plaquettes lying over it. In two-dimensional $\mathbb{Z}_2$ gauge theory, the plaquette signs on a torus are independent apart from a single constraint, "the product of all is 1". Converting (7) (§ 05) into the vortex ratio gives $\theta^{2D} = t^{M^2}$ with $t = \tanh(N^2|\beta|)$; this is the mean of the product of all plaquette signs when each plaquette sign is taken to be an independent ± 1 with mean t. From the probability $q = (1-t)/2 \approx e^{-2N^2|\beta|}$ that a plaquette is reversed, $t^{M^2} = (1-2q)^{M^2} \approx e^{-2qM^2}$ Proved here .

So the right side of (1) says only this much: that the $SU(2)$ vortex decreases in M^2 at least as fast as "a dilute gas of thin vortices, each gathered onto one sheet spanning the full width". An $SU(2)$ vortex can spread out within the continuous group and carry the twist in the length directions at a lower cost. When the cost S_N of such a thick vortex is less than $2N^2$, the rate can be taken larger. That is the thick-vortex floor (21-03; for S_N, see 19-02).

04

The route of proof — five steps

The proof has five steps. Steps 1 and 2 are the application of 21-01 § 06, steps 3 and 4 the computation of the base, and step 5 arithmetic. This section and § 06 are written in the notation Z_0, Z_1 of this part's convention $n_{12} = 1$, but the same steps go through for $n_{12} = 0$.

1. **Averaging over the centre.** Haar measure is invariant under multiplying each link by a sign ± 1, and $\operatorname{Re} U_p$ changes only by the factor of the product of the four signs of the plaquette (`haarCfg_inv`, `hol_smul`). So the $SU(2)$ partition function can be written as "an integral over U of $\mathbb{Z}_2$ sign systems with couplings $J_p(U) = \beta z_p \operatorname{Re} U_p$", with $|J_p| \le |\beta|$.
2. **Comparison with $\mathbb{Z}_2$.** Take as the base system the $\mathbb{Z}_2$ gauge theory on the two-dimensional torus $(\mathbb{Z}/M)^2$ of the length directions, linked by the map that sends the signs of two-dimensional links to the links in directions 3 and 4 independently of the transverse position (`pull`). From the comparison with $\mathbb{Z}_2$ in 21-01 § 05,

$$Z_0 \cdot Z_{\text{base}}(\zeta|\beta|) \le Z_1 \cdot Z_{\text{base}}(|\beta|) \quad (5)$$

   (`su2_dominated`). ζ is the sign of the (3,4) twist.
3. **The base to two dimensions.** Up to a constant factor, the partition function of the base system is that of two-dimensional $\mathbb{Z}_2$ gauge theory with coupling $N^2|\beta|$. The twisted one has the coupling with the sign of plaquette 0 alone changed (`Z_base_const`, `Z_base_flip`; § 06).
4. **The closed two-dimensional formula.** The partition function of two-dimensional $\mathbb{Z}_2$ gauge theory can be written in closed form as $|\text{configurations}| \cdot (\prod \cosh + \prod \sinh)$ (`Z_plane`; § 05).
5. **Arithmetic.** Setting $L = N^2|\beta|$, $C = \cosh(L)^{M^2}$, $S = \sinh(L)^{M^2}$, steps 3 and 4 turn (5) into $Z_0(C - S) \le Z_1(C + S)$, which rearranges to $(Z_0 - Z_1)C \le (Z_0 + Z_1)S$, that is, (1) (`bound_of_base`).

The two-sided form (2) uses step 2 with the roles swapped. Taking the twisted side $n_{34} = 1$ as the "original" system and using `su2_dominated_general` with couplings $\beta_p = \beta\, z_p^{(n_{12},1)}$, $\zeta^2 = 1$ gives $Z_1 \cdot Z_{\text{base}}(\zeta|\beta|) \le Z_0 \cdot Z_{\text{base}}(|\beta|)$ (`su2_dominated_swap`). Applying step 5 to both gives $|Z_0 - Z_1|/(Z_0 + Z_1) \le t^{M^2}$ Proved here. The exponential form (3) is a restatement using $t^{M^2} = e^{M^2 \ln t}$ and the fact that $0 < t < 1$ when $\beta \neq 0$ Proved here.

05

Part: the closed two-dimensional Z_2 formula

Consider $\mathbb{Z}_2$ gauge theory on the two-dimensional torus $(\mathbb{Z}/M)^2$ of the length directions: points y, links (y, d) $(d = 1, 2)$, a sign field on links t, and the sign $\chi_y^{2D}(t)$ of plaquette y (the product of four signs, overlaps kept as overlaps) (Lean: `TwistedPlane.chi2`).

The closed formula (Lean: `TwistedPlane.Z_plane`; the only assumption field is $M \geq 1$). For arbitrary couplings c_y, one per plaquette,

$$Z_\chi^{2D}(c) = \sum_t \exp\Big(\sum_y c_y \chi_y^{2D}(t)\Big) = |\text{configurations}| \cdot \Big(\prod_y \cosh c_y + \prod_y \sinh c_y\Big) \quad (6)$$

For a uniform coupling L and the coupling with the sign of plaquette 0 alone changed (`Z_plane_const`, `Z_plane_flip`),

$$\frac{Z_\chi^{2D}(L \text{ with plaquette 0 reversed})}{Z_\chi^{2D}(L)} = \frac{\cosh^{M^2} L - \sinh^{M^2} L}{\cosh^{M^2} L + \sinh^{M^2} L} = \frac{1 - \tanh(L)^{M^2}}{1 + \tanh(L)^{M^2}} \quad (7)$$

(`TwistedPlane.plane_flip_ratio`). Proved here That two-dimensional lattice gauge theory is solvable is a long-known fact; this is a machine check of its $\mathbb{Z}_2$ case.

Proof. Expanding the product with $e^{c\chi} = \cosh c + \chi \sinh c$,

$$Z_\chi^{2D}(c) = \sum_{A \subseteq \text{plaquettes}} \Big(\prod_{y \in A} \sinh c_y\Big)\Big(\prod_{y \notin A} \cosh c_y\Big) \sum_t \prod_{y \in A} \chi_y^{2D}(t)$$

Every link is an edge of exactly two plaquettes, so if A is neither empty nor everything, there is on the boundary of A a "link with only one side in A", and the change of variables that reverses the sign of that link changes the sign of the inner sum, which is therefore 0 (`sum_FA_eq_zero`). For $A = \emptyset$ the product is 1, and for $A =$ everything each link appears twice, so the product is again 1 (`prod_chi2_univ`). The two remaining terms are (6) Proved here .

06

Part: moving the base partition function to two dimensions

The base system of step 2 keeps the four-dimensional plaquette index and has the two-dimensional sign field t as its variable. Its plaquette character $\chi_B^0(t)$, the product of the signs pulled back to the four links of the four-dimensional plaquette B, is as follows (Lean: `TwistedWilson.chi0_eq`).

$$\chi_B^0(t) = \begin{cases} \chi^{2D}_{(x_3,x_4)}(t) & B \text{ a (3,4) plaquette at longitudinal position } (x_3, x_4) \\ 1 & \text{other plaquettes} \end{cases} \quad (8)$$

Links in directions 1 and 2 carry $+1$, and the signs of links in directions 3 and 4 do not depend on the transverse position, so on plaquettes other than (3,4) the link signs each appear twice and cancel. Hence, if the coupling on plaquettes other than (3,4) is a uniform K,

$$Z_\chi^0(c) = e^A \cdot Z_\chi^{2D}\Big(y \mapsto \sum_{(x_1,x_2)} c((x_1,x_2),y),(3,4)\Big), \qquad A = \sum_{B \in /(3,4)} K \quad (9)$$

(`Z_base`). The couplings of the N^2 transverse plaquettes lying over the longitudinal position y add up to the coupling of the two-dimensional plaquette y. For a uniform $K = |\beta|$ the two-dimensional coupling is $N^2|\beta|$ (`Z_base_const`), and multiplying by the (3,4) twist ζ gives the coupling $N^2|\beta|$ with the sign of plaquette 0 alone changed (`Z_base_flip`). The same constant e^A multiplies both, so it cancels in the ratio Proved here .

What matters here is that the set of plaquettes of the twist $n34$ (the (3,4) plaquettes at longitudinal position $(0,0)$, N^2 transverse copies) is exactly the N^2 plaquettes lying over the two-dimensional plaquette 0 (the definition of the twist in 19-02). The thin vortex of § 03 is this bundle of N^2 plaquettes.

07

Arithmetic restatements — a lower bound on the ratio and an upper bound on the flux free energy

Setting $s = \tanh(N^2|\beta|)^{M^2}$, (1) can also be written in the following two forms. In both, the only assumption field is $N, M \geq 1$, and they hold for every real β. We write them in the notation of this part's convention $n_{12} = 1$; for $n_{12} = 0$ the same formulas hold with Z_0, Z_1 replaced by $Z_{N,M}(\beta;0,0)$, $Z_{N,M}(\beta;0,1)$.

A lower bound on the ratio (Lean: `TwistedWilson.twisted_ratio_lower`).

$$\frac{Z_1}{Z_0} \;\geq\; \frac{1-s}{1+s} \quad (10)$$

This is equivalent to (1) Proved here .

An upper bound on the flux free energy (Lean: `TwistedWilson.vortex_free_energy_bound`).

$$\Delta F = -\ln\frac{Z_1}{Z_0} \;\leq\; \ln\frac{1+s}{1-s} \;\leq\; \frac{2s}{1-s} \quad (11)$$

ΔF is 't Hooft's flux free energy (the same quantity as in 25-01 and 26-03). This is arithmetic from $0 \leq s < 1$ and $\ln x \leq x - 1$; both (10) and (11) are machine-checked in Lean Proved here .

(11) is a form, for each finite M, that bounds ΔF from above by $2s/(1-s) \approx 2e^{-\sigma_0 M^2}$. It is an upper bound on ΔF, not a lower bound ($\Delta F \geq 0$ follows, when M is even, from

the sign in 21-06). The word floor is used in the direction of bounding the speed (the rate) at which ΔF decreases from below.

08

What is not said

- **The sign.** $Z_1 \le Z_0$ ($\theta_M \ge 0$) is not part of the theorems of this chapter. If M is even, $\theta_M \ge 0$ for every real β (21-06). Since there are examples with $\theta_M < 0$ at $\beta < 0$ (the table of § 02), for $\beta < 0$ it is the two-sided form (2) that has content.
- **No rate uniform in N.** $\sigma_0 \approx 2e^{-2N^2|\beta|}$ goes to 0 as $N \to \infty$. This is an estimate for a slab with the width held fixed, effectively two-dimensional, and not a positive lower bound on the four-dimensional tension.
- **The rate as $M \to \infty$ is not defined.** (3) is an inequality for each finite M. We do not have, as a Lean statement, "$\sigma_N(\beta) \ge \sigma_0$" for a tension $\sigma_N(\beta)$ defined by $\lim$ or $\liminf$ (what can be read from (3) is only the estimate for each M).
- **Weak at large $N^2|\beta|$.** The rate is the value for a thin vortex of cost $2N^2|\beta|$; the thick-vortex floor (21-03) can be better at large β ($S_N < 2N^2$ for $N = 2, 3, 4$: 21-05).
- **The lattice spacing is fixed.** Nothing is said about the continuum limit, infinite volume, the four-dimensional area law, the band or the mass gap (the statements to be shown about the band and the mass gap are in 19-07).

A discussion that places this inequality within the view "tools built on structure alone yield no more than the floor" is in the record of the study, 26-04.

What this chapter can say

Can say	Cannot say
Proved here The thin-vortex floor (1): $\theta_M^{(n_{12})} \le \tanh(N^2\|\beta\|)^{M^2}$, every real β, $N, M \ge 1$, $n_{12} \in \{0, 1\}$, no assumption fields. Lean: `TwistedWilson.twisted_vortex_bound`	$Z_1 \le Z_0$ (21-06). For $\beta < 0$ the one-sided form may have no content
Proved here The two-sided form (2): $\|\theta_M^{(n_{12})}\| \le \tanh(N^2\|\beta\|)^{M^2}$, same range. Lean: `TwistedWilson.twisted_vortex_bound_abs`	A lower bound on θ_M
Proved here The exponential form (3): for $\beta \neq 0$, $\sigma_0 > 0$ and $\|\theta_M^{(n_{12})}\| \le e^{-\sigma_0 M^2}$. Lean: `TwistedWilson.thin_vortex_floor`	A rate uniform in N. A definition of the rate as $M \to \infty$, and a statement of its lower bound

Can say	Cannot say
Proved here The lower bound on the ratio (10) and the upper bound on the flux free energy (11). Lean: `twisted_ratio_lower`, `vortex_free_energy_bound`	A lower bound on ΔF
Proved here The closed two-dimensional $\mathbb{Z}_2$ formula (6)(7). Lean: `TwistedPlane.Z_plane`, `plane_flip_ratio`	The two-dimensional ratio for non-abelian groups (not used in this chapter)
Proved here $\sigma 0 = 2\,\mathrm{artanh}(e^{-2N^2\lvert\beta\rvert}) \approx 2e^{-2N^2\lvert\beta\rvert}$	—
Reading The gap between θ_M and the right side on small lattices (§ 02)	The strength on large lattices

Sources and reproduction

Item	Kind	Source / tool
The pattern of comparing $SU(2)$ with $\mathbb{Z}_2$	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the same bibliography as 26-01; the texts not checked in this book). J. Fröhlich, Phys. Lett. B 83 (1979) 195, bibliography only (not checked)
Twisted boundary conditions and the flux free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (the same bibliography as 25-01)
(1)(2)(3)(10)(11) and the parts	Proved here	The route of proof in the text. Machine-checked in Lean 4 (declarations in each section; only the three standard axioms, no sorry). The definitions and statements were compared by a separate pair of eyes and judged "to hold" (reading the definitions line by line)
Numerical checks during the comparison	Reading	Zero violations in independently written exhaustive sums over finite groups and $SU(2)$ Metropolis runs
The tables of § 02	Reading	The numerical record of this book's study (floating-point values of $\tanh$. Metropolis for $SU(2)$ and Q_8: 2000 chains × 250 sweeps, estimating Z_1/Z_0 under the measure of Z_0. Sums by variable elimination for $\mathbb{Z}_2$, Q_8 and the binary tetrahedral group)

Words used in this chapter — thin-vortex floor, thin vortex, vortex ratio, flux free energy, floor, rate, base system, closed two-dimensional Z_2 formula, averaging over the centre. Their meanings are in A-02 Glossary.

Next chapter to read: 21-03 The thick-vortex floor — statement and route — instead of gathering the vortex onto one sheet, we take the rate again from the cost S_N of a cheaper thick vortex.

The thick-vortex floor — statement and route

This chapter is this book's own investigation.

What this chapter shows — that for the SU(2) Wilson measure on a twisted torus of width N and length M, the upper bound on the vortex ratio θ_M, "for every $\varepsilon > 0$, for each sufficiently large β, for sufficiently large M, $|\theta_{M+1}(\beta)| \le \exp(-(M+1)^2 e^{-\beta(S_N+\varepsilon)})$" (the thick-vortex floor), is machine-checked in Lean for all $N \ge 1$ with no assumption fields. How to read the order of quantifiers, the difference from the thin-vortex floor (exponent $2N^2$), and the route of the seven steps of the proof.

Prerequisite chapters — 19-02 (the twisted torus, the vortex ratio θ, the minimal action of a vortex S_N), 19-01 (the Wilson measure), 21-02 The thin-vortex floor. We use the language of limits ("sufficiently large") and the minimum of a continuous function on a compact set.

Said first — this is an upper bound **at a fixed lattice spacing, with the width N held fixed**. Neither the threshold in β nor the threshold in M for each β is explicit. The value of S_N for general N is not in Lean. The reverse direction (a lower bound on the rate) is not shown. It does not reach the band (the statement of 19-07), the four-dimensional area law, the mass gap or the continuum limit. In form it is a formalisation of a result of a known type, belonging to the picture of vortices on a thin twisted slab (the line of Mack–Petkova; conditional independence of blocks with the frame held fixed).

The order of this chapter

1. The objects, restated — the twisted torus, θ_M, how action is counted, S_v(N,m) and S_N
2. Statement — the thick-vortex floor, and how to read the order of quantifiers
3. Meaning — thin vortices and thick vortices — the exponent $2N^2$ drops to S_N. Figure 1
4. A map of the route — the seven steps, and which chapter holds which step
5. First half — dividing into blocks, turning into differences of action — the block-by-block estimate and the Laplace step

6. Second half — swapping the vortex into a cheap window — pigeonhole, compactness, the local model, rigidity, transplanting
7. Assembly — where the order of quantifiers comes from — from ε to m, β, M. Detour: why the twisted background is needed
8. What is not said — thresholds, the value of S_N, the reverse direction, N = 1, the band

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01

The objects, restated

The definitions are in Part 19. Here we restate, one line each, only what the statement needs.

- The lattice is $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$. Directions 1 and 2 are the width (side N), directions 3 and 4 the length (side M) (19-02).
- The background twist $n_{12} = 1$ is always present. Z_0 and Z_1 are the partition functions with twist $n_{34} = 0$ and 1 on the (3,4) planes (the sign of the twist). The measure is the unnormalised Wilson weight $e^{\beta \sum_p z_p \operatorname{Re} U_p}$ times the product of Haar measures.
- The vortex ratio is $\theta_M(\beta) = (Z_0 - Z_1)/(Z_0 + Z_1)$ (19-02). The denominator is positive.
- Action is counted in the form before multiplying by β: $S(U;\eta) = \sum_p \big(1 - z_p\, \eta_p \operatorname{Re} U_p\big) \geq 0$. $\eta_p = \pm 1$ is a sign that multiplies only the (3,4) plaquettes.
- The twisted vacuum V is the configuration with i on the direction-1 links at $x_1 = 0$, j on the direction-2 links at $x_2 = 0$, and 1 elsewhere. Since $ij\,i^{-1}j^{-1} = -1$, even on the twisted (1,2) plaquettes $z_p \operatorname{Re} U_p = 1$, and the action is 0.
- $S_{\mathrm{v}}(N,m)$ is the infimum of the action of the block when, on the lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/2m)^2$, the frame of a block of side m is held fixed at V and the sign of the corner (3,4) plaquette (N^2 transverse copies) is reversed. $S_N = \inf_{m \geq 1} S_{\mathrm{v}}(N,m)$ (19-02).

Leaving the interior at V and reversing the corner plaquette raises the action of each of the N^2 transverse plaquettes from 0 to 2. So $S_{\mathrm{v}}(N,m) \leq 2N^2$ and $S_N \leq 2N^2$.
Proved here Lean: `TwistBlocks.SN_bounds` ($0 \leq S_N \leq 2N^2$, for all $N \geq 1$).

02

Statement

The thick-vortex floor. Fix $N \geq 1$. For any $\varepsilon > 0$, for all sufficiently large β, for all sufficiently large M,

$$|\theta_{M+1}(\beta)| \;\leq\; \exp\Big(-(M+1)^2\, e^{-\beta(S_N+\varepsilon)}\Big). \quad (1)$$

Lean: `TwistBlocks.thick_vortex_floor` (all $N \geq 1$, no assumption fields; the statement is `ThickVortexFloorAbs N`). The form with definitions unfolded is `TwistBlocks.thick_vortex_floor_unfolded`. It is a formalisation of a result of a known type; the type is said to be the line of Mack–Petkova Cited (bibliography in the table at the end of the chapter; the text not checked in this book).

In the language of filters, the Lean statement has the order $\forall \varepsilon > 0,\ \forall^{\mathrm{f}} \beta \in \mathrm{atTop},\ \forall^{\mathrm{f}} M \in \mathrm{atTop}$. Unfolded into quantifiers, the order is as follows.

$$\forall \varepsilon > 0 \;\; \exists \beta_0 \;\; \forall \beta \geq \beta_0 \;\; \exists M_0(\beta) \;\; \forall M \geq M_0(\beta) : \quad (1). \quad (2)$$

Four points to note when reading it.

1. It is of the form in which β_0 **may depend on** ε**, and** M_0 **on** ε **and** β. The values of neither threshold appear in the statement (§ 07).
2. It is **with absolute value**. It bounds $|\theta|$ from above without using the sign of θ. The sign ($\theta \geq 0$ for even M) is a separate theorem (21-06).
3. Writing $M+1$ is only to avoid $M = 0$. Reading it as "for all sufficiently large M" is the same.
4. Only the large side of β. Nothing is said about $\beta < 0$ or about finite β.

Rewriting (1) in terms of the rate, for M with $\theta_M \neq 0$, $-\frac{1}{M^2} \ln |\theta_M| \geq e^{-\beta(S_N+\varepsilon)}$. Setting the slab tension to $\sigma_N(\beta) = \liminf_M -\frac{1}{M^2} \ln |\theta_M|$, we get $\limsup_{\beta\to\infty} \frac{1}{\beta} \ln \frac{1}{\sigma_N(\beta)} \leq S_N$. Proved here This restatement is one line of arithmetic outside Lean.

03

Meaning — thin vortices and thick vortices

The thin-vortex floor (21-02) says $|\theta_M| \leq \tanh(N^2|\beta|)^{M^2}$ for every real β and all $N, M \geq 1$. For large β, $-\ln \tanh(N^2\beta) \approx 2e^{-2N^2\beta}$, so the rate in the exponent is of order $e^{-2N^2\beta}$. **The exponent of the rate is** $2N^2$.

In the thick-vortex floor (1), the rate is $e^{-\beta(S_N+\varepsilon)}$. **The exponent of the rate drops from** $2N^2$ **to** S_N. If $S_N < 2N^2$, the thick-vortex floor is the stronger upper bound

where β is large.

In pictures: a vortex is a single reversal of the sign η of a (3,4) plaquette, threaded through the whole transverse T^2_N.

- **A thin vortex**: the reversal is absorbed at the position of a single (3,4) plaquette. Each of the N^2 transverse plaquettes pays an action of 2, so the cost is $2N^2$.
- **A thick vortex**: the reversal is spread within a block of side m in the directions of the (3,4) planes, and also over the transverse T^2_N. The links are rotated little by little to thin out the burden on each plaquette. The minimal cost is $S_{\mathrm{v}}(N, m)$, and its infimum over $m \to \infty$ is S_N.

Precisely, S_N is the infimum of "the minimal action for threading one vortex through a block whose boundary is held at the twisted vacuum". It is not the minimal action over the whole torus.

Figure 1 Dividing the longitudinal (3,4) plane T^2_M into blocks of side m (left). A single block enlarged (right): the frame (thick lines) is held fixed, and the sign is reversed on the corner plaquette (dark square). The cost of the vortex is paid by spreading this one sheet within the block. Above each point sits the transverse T^2_N.

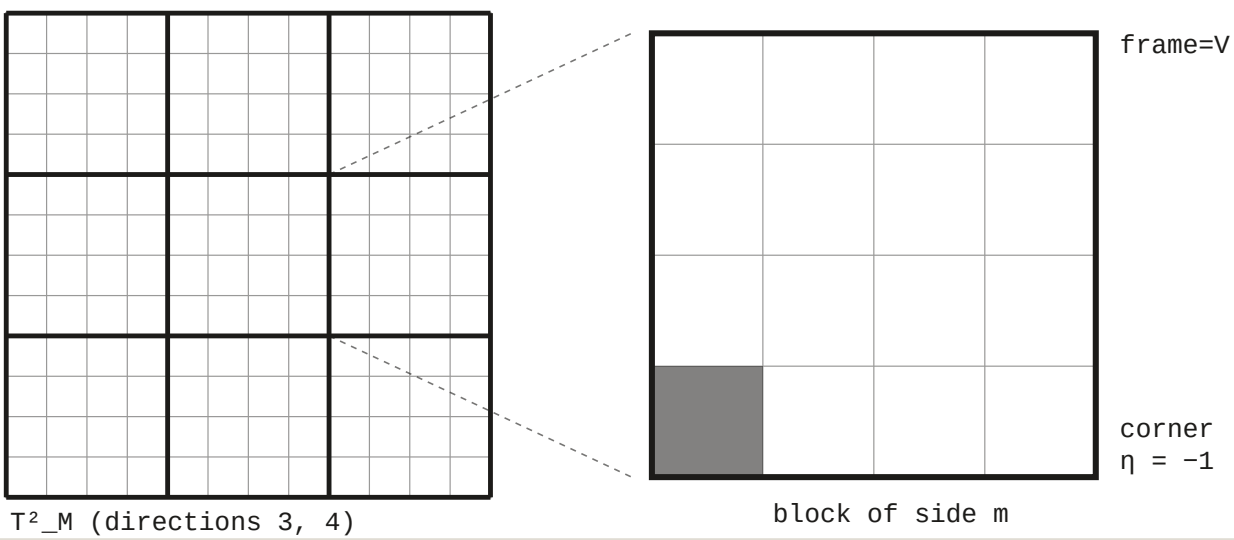

Figure 1 is schematic. No numbers are drawn.

The numbers for S_N are as follows.

N	Thin-vortex exponent $2N^2$	What can be said about S_N	Label / Lean
1	2	$S_1 = 2$ (numerical cooling). No gap	Reading Inside Lean only $S_1 \leq 2$

N	Thin-vortex exponent $2N^2$	What can be said about S_N	Label / Lean
2	8	$S_2 \le 6.67401$, $S_2 < 8$	Computer-assisted 21-05 (`VortexCert.SN_two_le`, `SN_two_lt`)
3	18	$S_3 < 18$ (upper bound 9.09321)	Computer-assisted 21-05 (`VortexCert.SN_three_lt`)
4	32	$S_4 < 32$ (upper bound 12.02057)	Computer-assisted 21-05 (`VortexCert.SN_four_lt`)
General	$2N^2$	$0 < S_N \le 2N^2$	Lean: `VortexPos.SN_pos`, `TwistBlocks.SN_bounds` (21-05)

$S_2 \le 6.67401$ for $N = 2$ is a computer certificate by a configuration with rational entries, checked in the Lean kernel. We use it as support for the analytic proof. For $N = 2$, the rate of the thick-vortex floor $e^{-6.675\beta}$ exceeds the rate of the thin-vortex floor $-\ln\tanh(4\beta)$ (the upper bound becomes smaller) from about $\beta > 0.523$, comparing down to the coefficients Reading . However, since the thresholds of the thick-vortex floor are not explicit, one cannot say that the thick upper bound is smaller at any concrete (β, M). Only the rates as $\beta \to \infty$ can be compared.

04

A map of the route

The proof splits into two halves: "bound $|\theta|$ by a quantity defined block by block, and bound the rate of that quantity as $\beta \to \infty$ by the minimal action of a vortex". Divided into steps, it is as follows.

```
Step 1 block-by-block estimate  |θ| ≤ tanh(K_eff)^{k²}            (first hal
  │                       K_eff = sup of log of the (+)/(−) ratio of a block
Step 2 Laplace step        K_eff ≤ ½(β(κ_c + 2δ) + D)             (first hal
  │                       κ_c = sup of the (+)/(−) difference of the block's
  │
Step 3 canonical signs     the block's sign field goes to a standard form, ac
Step 4 pigeonhole          the block contains a window of small action
Step 5 compactness         "small action" reduced to a claim about "action 0"
Step 6 local model, rigidity  action-0 configurations are gauge transforms of
Step 7 transplant          put a minimising configuration of S_v(N, m₀) into
  │                       ⇒ κ_c ≤ S_N + ε for sufficiently large m
  ▼
Assembly                   take the thresholds in the order ε → m → β₀ → M₀
```

The output of steps 3–7 is "for sufficiently large block side m, $\kappa_c \le S_N + \varepsilon$ for every frame". Inside Lean there is a named intermediate form of this, and from there to (1) is

arithmetic (21-04). The statements and scope of the parts are placed in 21-04. This chapter writes only the flow.

05

First half — dividing into blocks, turning into differences of action

Step 1: the block-by-block estimate

Divide the longitudinal plane T^2_M into $k \times k$ blocks of side m ($mk \le M$; only the last row and column are made wider). The links lying on the boundary lines of the blocks are called the **frame**. Every plaquette contains only frame links and interior links of at most one block. So once the frame is held fixed, the weight splits into a product over blocks.

In each block, compare the partition functions $Z^F_B(\pm)$ of the class where the product of the signs of the (3,4) plaquettes is $+1$ and the class where it is -1. Letting K_{eff} be half the supremum, over blocks and frames, of the logarithm of that ratio,

$$|\theta_M| \le \tanh(K_{\mathrm{eff}})^{k^2}. \quad (3)$$

Lean: `TwistBlocks.block_bound_abs` (every real β, $mk \le M$, arbitrary background n_{12}). The ratio lies within $e^{\pm 2N^2|\beta|}$, so $K_{\mathrm{eff}} \le N^2|\beta|$ (`TwistBlocks.Keff_le`). At $m = 1$ this returns to the form of the thin-vortex floor.

Step 2: the Laplace step

For large β, $Z^F_B(\pm)$ is nearly determined by the minimum $S^F_{\min}(\pm)$ of the block's action. From above it is trivially bounded by $e^{-\beta S_{\min}}$. From below it is bounded by the Haar measure of a small ball around the minimiser. In this way, for $\beta \ge 0$ and $\delta > 0$,

$$K_{\mathrm{eff}} \le \tfrac{1}{2}\big(\beta(\kappa_C + 2\delta) + D(\delta)\big), \qquad \kappa_C = \sup_{b,F} |S^F_{\min}(+) - S^F_{\min}(-)|. \quad (4)$$

Lean: `TwistBlocks.Keff_le_laplace` ($\beta \ge 0$, $\delta > 0$). $D(\delta)$ does not depend on β and is bounded above by a quantity determined by the block size alone (uniform in M). Combined with $-\ln\tanh K \ge e^{-2K}$, (3) and (4) give $|\theta_M| \le \exp(-k^2 e^{-\beta(\kappa_C + 2\delta) - D})$. What remains is to bring κ_C close to S_N.

06

Second half — swapping the vortex into a cheap window

What we want to show is that for sufficiently large m, **for every frame** F, $S^F_{\min}(-) \le S^F_{\min}(+) + S_N + \varepsilon$ (and the reverse) holds. That is, "from a configuration close to the minimum of (+), the class can be reversed at a cost of at most $S_N + \varepsilon$". We show it in five steps.

Step 3: canonical form of the signs

There are many sign fields η within a block. But applying reversals of links by the central $\mathbb{Z}_2$ in the interior, η can be moved to a standard form (all $+1$, or -1 on the corner plaquette alone) without changing the action. The class is determined by the product $\prod \eta$ alone. Lean: `TwistBlocks.canon_holds` (all $N, M \ge 1$, $mk \le M$).

Step 4: choosing a cheap window by pigeonhole

In either class, the block action of a configuration close to the minimum is at most $144N^2m$, since only the plaquettes touching the frame contribute. Inside the block, place T^2 windows of side $m_0 + 2$ so that they share no plaquettes. The sum of the actions of the windows is at most the action of the block, so some window has action at most $144N^2m/T^2$. Taking T large makes the action of this window as small as we like.

Step 5: compactness

The space of configurations around the window is compact, and the action is continuous. So if one can say "for configurations of action 0, the class can be reversed at a cost of at most c", then "for configurations of sufficiently small action, it can be reversed at a cost less than $c + \varepsilon$" follows. This is phrased in terms of cost, without using a distance. The smallness γ has no explicit value (it is a different quantity from the δ of step 2). Lean: `ThickFloor.small_action_cost`.

Step 6: the local model, and rigidity of the twisted torus

Lift the neighbourhood of the window to a single finite model independent of M (the transverse $T^2_N \times$ a square of side $m_0 + 2$). In this model, a configuration of action 0 is, **on the links that appear in plaquettes**, a gauge transform of the local twisted vacuum. The core is a problem on the transverse T^2_N. A flat configuration on T^2_N with one twisted plaquette is determined by two generators a, b, and the twist gives $ab = -ba$. A pair of anticommuting unit quaternions is mapped to (i, j) by conjugation. So the configuration is a gauge transform of the vacuum. Lean: `TwistBlocks.rigidT_all` (all $N \ge 1$), `ThickFloor.anticomm_conj`.

Step 7: transplanting the minimising configuration

Put a minimising configuration of $S_{\mathrm{v}}(N, m_0)$ into the window, matched by a gauge transformation. The plaquettes outside the window do not change. Inside the window the class is reversed once, at a cost of at most $S_{\mathrm{v}}(N, m_0)$. Choosing m_0 so that $S_{\mathrm{v}}(N, m_0) < S_N + \varepsilon/4$, the cost approaches S_N. Lean: `TwistBlocks.thick_vortex_floor_zero_of_rigid` (from rigidity to the zero-action claim; 21-04).

07

Assembly — where the order of quantifiers comes from

The thresholds are fixed in the following order.

1. Give ε.
2. Choose m_0 with $S_{\mathrm{v}}(N, m_0) < S_N + \varepsilon/4$ (existence only, from the definition of the infimum).
3. Obtain $\gamma > 0$ from the compactness of step 5. **Its value is not explicit**.
4. Take a natural number T with $\gamma T > 144 N^2 (m_0 + 3) + \varepsilon/4$, and set the block side to $m = (m_0 + 2)T + 1$. The action of the window of step 4 falls below γ, and for all $M \geq m$, κ_c comes within about ε of S_N.
5. Take β_0 large enough that D and $\ln(4m^2)$ can be absorbed by $\beta\varepsilon/2$. Since D contains the Haar measure of a small ball, **this too is not explicit**.
6. Take M_0. Within this proof $M \geq m$ suffices. But statement (2) is of the form in which M_0 may depend on β, and says no more than that.

So the statement takes the form "$\forall \varepsilon\ \exists \beta_0\ \forall \beta \geq \beta_0\ \exists M_0\ \forall M \geq M_0$", and none of the thresholds can be written as numbers.

Detour: why the twisted background is needed

You may skip this. The background twist $n_{12} = 1$ matters in one place, step 6. Because there is exactly one twisted plaquette, the two generators anticommute. Every pair of anticommuting unit quaternions is mapped to (i, j) by conjugation, and the zero-action configuration is unique up to gauge. Without the twist, the generators of a flat configuration may commute. For example, the configuration with every link equal to 1 and the flat configuration whose holonomy around the direction-1 loop is i have holonomies in different conjugacy classes, so they are not related by a gauge transformation. Proved here In that case the rigidity of step 6 fails, and this route does not go through as it stands.

08

What is not said

- **The thresholds are not explicit.** Neither $\beta_0(\varepsilon)$ nor $M_0(\varepsilon, \beta)$ can be written as numbers. They cannot be compared with numerical values at finite β and M.
- **The value of S_N for general N is not in Lean.** What Lean has is $0 < S_N \leq 2N^2$. $S_N < 2N^2$ exists only as certificates for $N = 2, 3, 4$ (21-05). No upper or lower bound uniform in N is shown.
- **For $N = 1$ it is no better than the thin-vortex floor.** $S_1 = 2 = 2N^2$ Reading , the same exponent as the thin-vortex floor.
- **No reverse direction.** A lower bound on $|\theta_M|$ (an upper bound on the rate, $\sigma_N \leq e^{-\beta(S_N - \varepsilon)}$) is not shown.
- **Only the rate as $\beta \to \infty$.** The prefactor affects β with a power of the order of the number of degrees of freedom of a block. It cannot be used for the band (the statement of 19-07).
- **The lattice spacing is fixed.** It does not reach the continuum limit, the area law for four-dimensional Wilson loops, or the mass gap.

What this chapter can say

Can say	Cannot say
Proved here The thick-vortex floor (1): all $N \geq 1$, quantifiers in the order of (2). Lean: `TwistBlocks.thick_vortex_floor` (no assumption fields)	The values of β_0 and M_0. Comparisons at finite β. A lower bound on the rate
Proved here $S_N \leq 2N^2$. Lean: `TwistBlocks.SN_bounds`	$S_N < 2N^2$ for general N. Estimates uniform in N
Computer-assisted $S_N < 2N^2$ for $N = 2, 3, 4$ (21-05)	That the thick bound is smaller at a concrete (β, M)
Proved here The seven steps of the route (§ 05, § 06). The parts of each step, with Lean names and scope, are in 21-04	A sharp form of the prefactor of step 2 (the degree of the power of β)
Reading $S_1 = 2$; for $N = 2$, the thick rate exceeds the thin rate from about $\beta > 0.523$	Proofs of these numbers

Sources and reproduction

Item	Kind	Source / tool
Twisted boundary conditions and flux	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (bibliography as in 25-01)
Conditional independence of blocks with the frame held fixed; vortex containers	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the texts not checked in this book; the type is the same as the containers of 25-03). T. G. Kovács, E. T. Tomboulis, arXiv:hep-lat/0108017
The thick-vortex floor and the steps of the route (machine-checked in Lean 4)	Proved here	A formalisation of the type above. Lean: `TwistBlocks.thick_vortex_floor` and its chain (parts in 21-04). The three standard axioms, no sorry
$S_1 = 2$, the threshold $\beta > 0.523$	Reading	The numerical record of this book's study (numerical values of S_{v} by cooling, comparison of coefficients with the thin-vortex floor, floating point)
Upper bounds on S_N for $N = 2, 3, 4$	Computer-assisted	21-05 (certificates by configurations with rational entries, checked in the Lean kernel)

Words used in this chapter — thick-vortex floor, thin-vortex floor, vortex ratio θ, twisted vacuum, minimal action of a vortex S_N, block, frame, window, effective coupling K_eff, rigidity, order of quantifiers. Their meanings are in A-02 Glossary.

Next chapter to read: 21-04 The thick-vortex floor — parts of the proof — the Lean statements and scope step by step, and theorems that show the limits of the method.

The thick-vortex floor — parts of the proof

> This chapter is this book's own investigation.

What this chapter shows — the statements and scope of the parts used in the proof of the thick-vortex floor (21-03): the block estimate and the effective coupling K_{eff}, the Laplace step, the lemma on the canonical form of signs, the compactness step, conjugation of anticommuting pairs and rigidity of the twisted torus. The form of the intermediate implications that link the parts, and the fact that all their assumptions are discharged in the final theorem. And a theorem showing that the rate of this method cannot be better than S_N.

Prerequisite chapters — 21-03 § 04 (the map of the route), 19-02 (the minimal action of a vortex), 19-01 (the product of Haar measures). We use the product of quaternions and the minimum of a continuous function on a compact set.

Said first — every part is a statement on a finite lattice with the lattice spacing and the width N held fixed. The intermediate implications are in "if ... then" form, with all assumptions shown. Those assumptions are discharged in the final theorem, but the intermediate implications themselves do not show content. The theorem on the limits of the method says only that an intermediate statement of this proof is false for $S < S_N$. It does not deny the floor itself, nor does it say that the true rate is S_N.

The order of this chapter

1. Blocks, frames, and the partition functions of classes — the conventions for dividing into blocks, and the definitions of $Z_B^F(\pm)$ and K_{eff}
2. The block estimate — $|\theta| \le \tanh(K_{\mathrm{eff}})^{k^2}$ and $K_{\mathrm{eff}} \le N^2|\beta|$
3. The Laplace step — the Haar measure of a small ball, and an upper bound on K_{eff} by κ_c
4. The lemma on the canonical form of signs — bringing a block's sign field to a standard form without changing the action
5. The compactness step — from "if the action is 0" to "if the action is small"
6. Rigidity — conjugation of anticommuting pairs, and zero-action configurations on the twisted torus

7. The form of the intermediate implications, and how the assumptions are discharged — the six links of the chain
8. A theorem showing the limits of the method — for $S < S_N$ the intermediate statement is false

---

01

Blocks, frames, and the partition functions of classes

The lattice is the twisted torus of width N and length M (19-02). Divide the longitudinal plane T^2_M into $k \times k$ blocks. The only condition is $mk \le M$. Blocks have side m; only the blocks in the last row and column have width $M - (k-1)m$.

- **Frame**: the links lying on the boundary lines of the blocks. A direction-3 link is in the frame if its $x4$ is on a line, a direction-4 link if its $x3$ is on a line, and a direction-1 or 2 link if either is on a line.
- **Interior**: the non-frame links belonging to the block.
- **Class**: whether the product $\prod \eta$ of the signs η of the (3,4) plaquettes in the block is $+1$ or -1.

Every plaquette contains only frame links and interior links of at most one block. So once the frame F is held fixed, the weight splits into a product over blocks. Write $Z^F_B(s)$ for the partition function of block b with frame F held fixed, summed over the class $s = \pm 1$ (integrating over the interior links with the product of Haar measures).

$$K_{\text{eff}} = \tfrac{1}{2} \sup_{b,\,F} \left| \ln \frac{Z^F_B(+)}{Z^F_B(-)} \right|. \quad (1)$$

The supremum is taken over all blocks b and all frame values F. Lean: `TwistBlocks.Keff`, `TwistBlocks.ZB`. In Lean, $Z^F_B(s)$ is the integral of the integrand $C_b + sS_b$ obtained by first summing over the signs plaquette by plaquette. It is the same as the sum within the class.

02

The block estimate

The block estimate. For $mk \le M$, every real β, and both backgrounds n_{12},

$$\left|\frac{Z_0 - Z_1}{Z_0 + Z_1}\right| \le \tanh(K_{\mathrm{eff}})^{k^2}, \qquad 0 \le K_{\mathrm{eff}} \le N^2|\beta|. \tag{2}$$

Proved here Lean: `TwistBlocks.block_bound_abs` (no assumption besides $mk \le M$), `TwistBlocks.Keff_le` ($mk \le M$), `TwistBlocks.Keff_nonneg`. This is the form of conditional independence of blocks with the frame held fixed, together with a product of block-by-block suprema.

Cited The type is a formalisation of conditional independence of blocks with the frame held fixed (the line of Mack–Petkova; sources in the table at the end of the chapter).

The proof runs in three lines. (i) With the frame held fixed, the weight of the sum over classes of η becomes a product over blocks. (ii) In each block, the ratio of the weights of classes $\pm$ lies within $e^{\pm 2K_{\mathrm{eff}}}$. Two positive numbers $Z_\pm$ whose ratio lies within $e^{\pm 2K}$ satisfy $|Z_+ - Z_-| \le \tanh K\,(Z_+ + Z_-)$. (iii) By the triangle inequality for integrals, $\tanh$ is multiplied in once for each of the k^2 blocks. $K_{\mathrm{eff}} \le N^2|\beta|$ holds because reversing one class changes only the signs of the N^2 transverse plaquettes.

From (2) and $-\ln\tanh K \ge e^{-2K}$ $(K > 0)$, $|\theta_M| \le \exp(-k^2 e^{-2K_{\mathrm{eff}}})$. Lean: `TwistBlocks.abs_block_bound_exp`.

What is not said. Since K_{eff} also takes the supremum over blocks, when block shapes are mixed it is weaker than the block-by-block product $\prod_b \tanh K_b$. For example, in a small $\mathbb{Z}_2$ model, the right side is $\tanh(1)^4 = 0.3364$ while the block-by-block product is 0.0862 Reading. It does not affect the rate.

03

The Laplace step

A coarse Laplace lower bound. Let ι be a finite set and ν the product of Haar probability measures on $SU(2)^\iota$. The distance is the maximum of the componentwise distances. If f is continuous and L-Lipschitz $(L > 0)$, then for all x_0, $\beta \ge 0$ and $\delta > 0$,

$$e^{-\beta(f(x_0)+\delta)}\,\mu_1(\delta/L)^{|\iota|} \le \int e^{-\beta f}\,d\nu, \qquad \mu_1(r) = \mathrm{Haar}\big(\{g : |g - 1| < r\}\big) > 0. \tag{3}$$

Proved here Lean: `CoarseLaplace.laplace_lower` ($\beta \ge 0$, $\delta > 0$, $L > 0$). The proof only uses $f \le f(x_0) + \delta$ on the ball of radius δ/L around x_0. The measure of the ball does not depend on its centre because left multiplication is an isometry and Haar measure is left-invariant. The prefactor does not depend on β. This is a weak form of Laplace's method that does not make the constants explicit.

We apply this to the action of a block. The action S_B is Lipschitz in the interior links, and from the number n_b of plaquettes in the block the constant can be taken as $L_b = 4n_b + 1$. Let $S^F_{\min}(\pm)$ be the infimum of the action over class $\pm$ and the interior, and set

$$\kappa_C = \sup_{b,F} \left| S^F_{\min}(+) - S^F_{\min}(-) \right|, \qquad D(\delta) = \max_b \Big(\ln \#\{\text{sign fields of block } b\} - \#\{\text{interior li}$$

Then, for $\beta \geq 0$ and $\delta > 0$,

$$K_{\text{eff}} \leq \tfrac{1}{2} \big(\beta(\kappa_C + 2\delta) + D(\delta) \big). \quad (4)$$

Proved here Lean: `TwistBlocks.Keff_le_laplace` ($mk \leq M$, $\beta \geq 0$, $\delta > 0$). When $M < (k+1)m$, $D(\delta)$ is bounded by a quantity $D^*(N, m, \delta)$ determined by N, m, δ alone (`TwistBlocks.Dmax_le`). That is, it is uniform in M. As a trivial upper bound on κ_C there is also $\kappa_C \leq 2N^2$ (`TwistBlocks.kappaC_le_two`, for all m).

What is not said. The value of $\mu_1(r)$ (that it is of order r^3) is not used, and D is not explicit. A sharp form of the prefactor (the degree of the power of β) is not treated. $\beta < 0$ is not treated.

04

The lemma on the canonical form of signs

The class infimum $S^F_{\min}(\pm)$ is the infimum over all sign fields σ in the class and all interior values. The following lemma makes it equal to "the infimum with η held at one value".

The lemma on the canonical form of signs. Let $mk \leq M$. For a block b, a frame F, interior values y, and a sign field σ of the block ($t = 0$ if $\prod \sigma = +1$, $t = 1$ if -1), there are interior values y' such that the action of the block in the standard field (all $+1$ if $t = 0$; -1 on the corner plaquette alone if $t = 1$) equals the action with σ.

Proved here Lean: `TwistBlocks.canon_holds` (statement `TwistBlocks.Canon N`; all $N, M \geq 1$ and $mk \leq M$; no condition on the shape of the block). The construction is a comb-shaped reversal. Let $\rho = \sigma \cdot (\text{standard field})$. In each column, accumulate the product of ρ from bottom to top to determine the signs κ_3 of the direction-3 links, and in the corner row only, accumulate the total product of each column from right to left to determine the direction-4 signs κ_4. Then on each plaquette the product of κ is ρ, and since the product of ρ within the block is 1, $\kappa = 1$ on the frame. On plaquettes spanning directions 1, 2 and directions 3, 4, the same κ appears twice and cancels.

With this lemma, Lean's κ_C equals the version κ with η held fixed (`TwistBlocks.kappaC_eq_kappa`, $mk \le M$).

05

The compactness step

Compactness in the form of cost. Let X be a compact space and $S, \Phi : X \to \mathbb{R}$ continuous. If $\Phi(x) \le c$ at points with $S(x) \le 0$, then for every $\varepsilon > 0$ there is $\gamma > 0$ such that $\Phi(x) < c + \varepsilon$ at points with $S(x) < \gamma$.

Proved here Lean: `ThickFloor.small_action_cost` (closes using Mathlib alone). Proof: $K = \{\Phi \ge c + \varepsilon\}$ is closed and hence compact. If it is empty, $\gamma = 1$ works. If not, S attains a minimum $S(x_0)$ on K, and by assumption $S(x_0) > 0$. Take $\gamma = S(x_0)$.

How it is used: let X be the configuration space of the local model around the window, S the local action, and Φ the infimum of "the action after replacing the interior and reversing one plaquette". There are finitely many positions for the reversal, so Φ is continuous. This reduces "in a window of sufficiently small action, the class can be reversed at cost $S_{\mathrm{V}} + \varepsilon$" to "for a local configuration of action 0, it can be reversed at cost S_{V}" (`TwistBlocks.localVortexL_of_zero`).

What is not said. The value of γ (a different quantity from the δ of § 03). γ depends only on N, m_0 and ε, but is not explicit. This is the main reason the thresholds of the thick-vortex floor are not explicit.

06

Rigidity

Conjugation of anticommuting pairs

Anticommuting pairs of unit quaternions. If unit quaternions A, B satisfy $AB = -BA$, there is a quaternion $g \neq 0$ with $gAg^{-1} = i$ and $gBg^{-1} = j$.

Proved here Lean: `ThickFloor.anticomm_conj`. First, $\mathrm{Re}\, A = \mathrm{Re}\, B = 0$ and the imaginary parts are orthogonal (`ThickFloor.anticomm_unit`). Next, setting $L(h) = h - ihA - jhB - kh(AB)$, from $A^2 = B^2 = -1$ and $AB = -BA$ alone, $L(h)A = iL(h)$ and $L(h)B = jL(h)$. For $e \in \{1, i, j, k\}$, $\sum_e \mathrm{Re}(\bar{e}\, L(e)) = 4$, so $L(e) \neq 0$ for some e. Take $g = L(e)$. Replacing g by $g/|g|$ gives conjugation by a unit quaternion.

Rigidity of the twisted torus

Consider configurations (A, B) on the transverse $T^2_N = (\mathbb{Z}/N)^2$ (A in direction 1, B in direction 2) whose plaquette products equal the sign of the twist (-1 at the origin, $+1$ elsewhere).

Rigidity of the twisted torus. For all $N \geq 1$, every such (A, B) is a gauge transform of the vacuum of T^2_N ($A = i$ at $a_1 = 0$, $B = j$ at $a_2 = 0$, 1 elsewhere). That is, there is $R : T^2_N \to SU(2)$ such that for all a, $R(a)A(a)R(a + e_1)^{-1}$ and $R(a)B(a)R(a + e_2)^{-1}$ equal the values of the vacuum.

Proved here Lean: `TwistBlocks.rigidT_all` (statement `TwistBlocks.RigidT N`, all $N \geq 1$). This is the form "a twisted flat connection is unique up to gauge". Proof: fixing the gauge on a tree of products taken from the front makes the tree links equal to 1. By induction on the flatness of plaquettes, the remaining links are determined by two generators $a = A(N-1, 0)$, $b = B(0, N-1)$ (`TwistBlocks.cfA`, `cfB`). Since there is exactly one twisted plaquette, $ab = -ba$ (`TwistBlocks.cf_comm`). Take the constant u that maps (a, b) to (i, j) by conjugation of anticommuting pairs, and compare with the vacuum gauge-fixed on the same tree (`TwistBlocks.uniqT`). $N = 1$ is a single point, so it is direct (`TwistBlocks.rigidT_one`).

Lifting to the local model

The local model around the window is the product of the transverse T^2_N and a square of side $V + 1$. This model has **links that appear in no plaquette** (the direction-3 and 4 links at the edges of the square, and the direction-1 and 2 links at the corners). They do not affect the action, so they can take arbitrary values even in a configuration of action 0. Local rigidity is therefore stated in the following scope.

Local rigidity. A configuration of local action 0 equals, **on all links that appear in plaquettes**, a gauge transform of the local twisted vacuum.

Proved here Lean: `TwistBlocks.rigidL_of_rigidT` (RigidT $N \Rightarrow$ RigidL N $(V + 1)$, for all V). Fixing the gauge on a maximal tree in directions 3 and 4 makes all direction-3 and 4 links equal to 1, and the direction-1 and 2 links no longer depend on the longitudinal position. What remains is the problem on T^2_N, which closes by the rigidity above. As an equality of functions on all links it is false, so restricting the scope to "links that appear in plaquettes" is the correct form.

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07
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The form of the intermediate implications, and how the assumptions are discharged

The Lean proof is built as a chain of implications that replaces the assumptions one by one with weaker claims. Each link has the form "if ... then the thick-vortex floor". Here we list them with all assumptions shown. $\mathrm{Floor}(S)$ is the statement (1) of 21-03 with S_N replaced by S.

Link	Lean	Assumption (if ...)	Conclusion
1	`thick_vortex_floor_abs_of_H4`	Upper bound on the flipping cost: for some S: for every $\varepsilon > 0$ there is m such that for all $M \geq m$, $\kappa_c(N, M, m, \lfloor M/m \rfloor) \leq S + \varepsilon$	$\mathrm{Floor}(S)$
2	`H4_of_local`	The lemma on the canonical form of signs (`Canon N`), and for all m_0, "if the action of the window is small, the class can be reversed at cost $S_{\mathrm{v}}(N, m_0) + \varepsilon$" (`LocalVortex`)	The upper bound on the flipping cost with $S = S_N$
1+2	`thick_vortex_floor_abs_of_local`	Of the assumptions of link 2, `LocalVortex` for all m_0 (`Canon N` is discharged inside by `canon_holds`)	$\mathrm{Floor}(S_N)$
3	`thick_vortex_floor_abs_of_localL`	The same claim, on the local model independent of M (`LocalVortexL`)	$\mathrm{Floor}(S_N)$
4	`thick_vortex_floor_abs_of_zero`	For all m_0, configurations of local action 0 can be reversed at cost at most $S_{\mathrm{v}}(N, m_0)$ (`ZeroVortexL`)	$\mathrm{Floor}(S_N)$
5	`thick_vortex_floor_abs_of_rigid`	Local rigidity for all m_0 (`RigidL N (m₀+2)`)	$\mathrm{Floor}(S_N)$
6	`thick_vortex_floor_abs_of_rigidT`	Rigidity of the twisted torus (`RigidT N`)	$\mathrm{Floor}(S_N)$

(All names are under `TwistBlocks.`. The `H4` remaining in the Lean names is the name from the source; in the text we call this intermediate statement the "upper bound on the flipping cost".) The theorems connecting the links are as follows.

- `Canon N` in 2 is discharged by `canon_holds` (§ 04). Pigeonhole (`exists_win_le`) is closed within the proof of 2.
- 3 → 1+2: write the answer on the local model back into the interior of the block (`localVortex_of_local`).
- 4 → 3: the compactness step (`localVortexL_of_zero`, § 05).
- 5 → 4: pull a minimising configuration of $S_{\mathrm{v}}(N, m_0)$ back to the local model and transplant it into the window using the rigidity gauge

(`thick_vortex_floor_zero_of_rigid`). The existence of a minimising configuration is `Svortex_attained`.

- $6 \to 5$: the lift to the local model (`rigidL_of_rigidT`, § 06).

The final theorem `TwistBlocks.thick_vortex_floor` is link 6 with `rigidT_all` put in. With this, every assumption of the chain is discharged and no assumption fields remain. As a sanity check of the machinery, we have also checked that the chain contains no intermediate theorem that is vacuous because its assumptions are false. The inner field "local action ≤ 0" of links 4 and 5 is satisfied by the local vacuum (`SL_vacL`), and the inner field "a flat configuration on T^2_N" of link 6 is satisfied by the vacuum of T^2_N (`flatT_nonempty`). These check that the fields are non-empty; they are examples of satisfaction without content.

> The intermediate links 1–6 do not, on their own, show content. For example, link 1 is easily satisfied if S is large. What has content is the pair of the final theorem with its assumptions discharged and the estimates of the value of S_N (21-05).

08

A theorem showing the limits of the method

Consider whether the S in the assumption of link 1 (the upper bound on the flipping cost) can be taken smaller than S_N. That it cannot is shown in Lean.

Comparison of a block with its frame at the vacuum. For all $m \geq 1$, $S_{\mathrm{v}}(N,m) \leq \kappa_c(N, 2m, m, 2)$. Proved here Lean: `TwistBlocks.kappaC_ge_Svortex`. With $M = 2m$, block $(0,0)$, and the frame at the twisted vacuum, the infimum for (+) is 0 and the infimum for (−) is $S_{\mathrm{v}}(N,m)$.

Negation of the intermediate statement. If $S < S_N$, the upper bound on the flipping cost is false. Proved here Lean: `TwistBlocks.H4_fails_below` (assumption $S < S_N$). Applying the upper bound on the flipping cost with $\varepsilon = (S_N - S)/2$ and $M = 2m$ contradicts the comparison above and $S_N \leq S_{\mathrm{v}}(N,m)$.

Three ways to read this.

1. This is **a limit of the method of this proof**. As long as one goes through the intermediate statement, the upper bound on the flipping cost (an upper bound on κ_c), the exponent of the rate obtained cannot be better than S_N.
2. This is **not a denial of the thick-vortex floor**. The floor (1) itself might hold with an S satisfying $S < S_N$; this theorem says nothing about that.

3. This **does not say that the true rate is S_N** either. A lower bound on the rate (the reverse direction) is still missing.

What this chapter can say

Can say	Cannot say
Proved here The block estimate (2): every real β, $mk \le M$. Lean: `TwistBlocks.block_bound_abs`, `Keff_le`	The block-by-block product form when block shapes are mixed
Proved here The coarse Laplace lower bound (3) and the upper bound (4) on K_{eff}: $\beta \ge 0$, $\delta > 0$. Lean: `CoarseLaplace.laplace_lower`, `TwistBlocks.Keff_le_laplace`	The value of D. A sharp form of the prefactor. $\beta < 0$
Proved here The lemma on the canonical form of signs: $mk \le M$. Lean: `TwistBlocks.canon_holds`	—
Proved here Compactness in the form of cost. Lean: `ThickFloor.small_action_cost`	The value of γ
Proved here Rigidity of the twisted torus: all $N \ge 1$. Lean: `TwistBlocks.rigidT_all`, `ThickFloor.anticomm_conj`	Equality on all links of the local model (only on links that appear in plaquettes)
Proved here Links 1–6 (and 1+2) of the intermediate implications, and the discharge of their assumptions in the final theorem (§ 07)	That the intermediate links alone yield content
Proved here The limit of the method: for $S < S_N$, the upper bound on the flipping cost is false. Lean: `TwistBlocks.H4_fails_below`	A denial of the floor itself. That the true rate is S_N
Reading The numbers for the example with mixed block shapes (0.0862 versus 0.3364)	—

Sources and reproduction

Item	Kind	Source / tool
Conditional independence of blocks with the frame held fixed	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the texts not checked in this book; the type is the same as the containers of 25-03). T. G. Kovács, E. T. Tomboulis, arXiv:hep-lat/0108017
Twisted flat connections	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (bibliography as in 25-01)
The Laplace-method lower bound	Proved here	A weak form of Laplace's method that does not make the constants explicit. Proof in § 03 of the text

Item	Kind	Source / tool
The statements of each part (machine-checked in Lean 4)	Proved here	The Lean bundle (block estimate, Laplace, canonical form, compactness, quaternions, rigidity, transplanting, limits of the method). The three standard axioms, no sorry
The example with mixed block shapes	Reading	The numerical record of this book's study (exhaustive enumeration of a small $\mathbb{Z}_2$ model)

Words used in this chapter — block, frame, interior, class, effective coupling K_eff, coarse Laplace lower bound, lemma on the canonical form of signs, compactness in the form of cost, anticommuting pair, rigidity of the twisted torus, local rigidity, intermediate implication, limits of the method. Their meanings are in A-02 Glossary.

Next chapter to read: 21-05 The minimal action of a vortex — positivity, and a certificate for the upper bound — what can be said about the value of the exponent s_N of the floor.

The minimal action of a vortex — positivity, and a certificate for the upper bound

This chapter is this book's own investigation. The upper bounds of § 05 are computer-assisted certificates, placed as support for the analytic proof.

What this chapter shows — we read two facts, machine-checked in Lean, about the quantity S_N (the minimal action of a vortex) that appears in the exponent of the thick-vortex floor. One is that $S_N > 0$ for every $N \geq 1$, and moreover that there is a positive lower bound uniform in the block size m. The other is a certificate for an upper bound showing that, for $N = 2, 3, 4$, S_N is smaller than the thin-vortex value $2N^2$.

Prerequisite chapters — 19-02 the definition of the minimal action of a vortex, the vortex ratio θ_M; 21-02 The thin-vortex floor; 21-03 The thick-vortex floor

Said first — what we could say is only "positivity" and "upper bounds". The value of S_N is not known. The lower bound δ comes from compactness, so it is not an explicit number, nor is it uniform in the width N. The upper bounds are the action of a single configuration at $m = 8$ ($m = 4$ for $N = 4$), not the value of the infimum over m. The lattice spacing is fixed, and nothing is said about the continuum limit or the mass gap.

The order of this chapter

1. The minimal action of a vortex — the main points of the definition, and trivial bounds
2. S_N is positive — statement and meaning
3. The route to positivity — labels and a telescoping product
4. Detour: where the background twist matters
5. Upper bounds for N = 2, 3, 4 — statement
6. How the certificate is made — rational unit quaternions and rounding up
7. Comparison with the thin-vortex floor — comparing rates, and its limits
8. What is not said

01

The minimal action of a vortex — the main points of the definition, and trivial bounds

The definition is in 19-02. Here we repeat only the main points.

The lattice is the torus with transverse part $(\mathbb{Z}/N)^2$ and longitudinal part $(\mathbb{Z}/2m)^2$. In the longitudinal (3,4) plane take the block at the origin $b_0 = \{0,\dots,m-1\}^2$ (the set of longitudinal points at the roots of plaquettes), and consider its product with the transverse torus T^2_N (notation as in 19-02). The frame of the block (the links on the lines $x_3 \in \{0,m\}$ and $x_4 \in \{0,m\}$) is held at the **twisted vacuum** V. V is the configuration whose direction-1 links are the quaternion i at $x_1 = 0$, whose direction-2 links are j at $x_2 = 0$, and which is 1 everywhere else. Under the background twist $n_{12} = 1$, the action of V is 0.

Let $\Pi_{(0,0)}$ be the (3,4) plaquettes at the block corner $(x_3,x_4) = (0,0)$, collected over all N^2 transverse positions, and reverse their sign ($\eta_p = -1$ for $p \in \Pi_{(0,0)}$, $\eta_p = +1$ otherwise). The sum of the actions of all plaquettes whose root lies in b_0 is

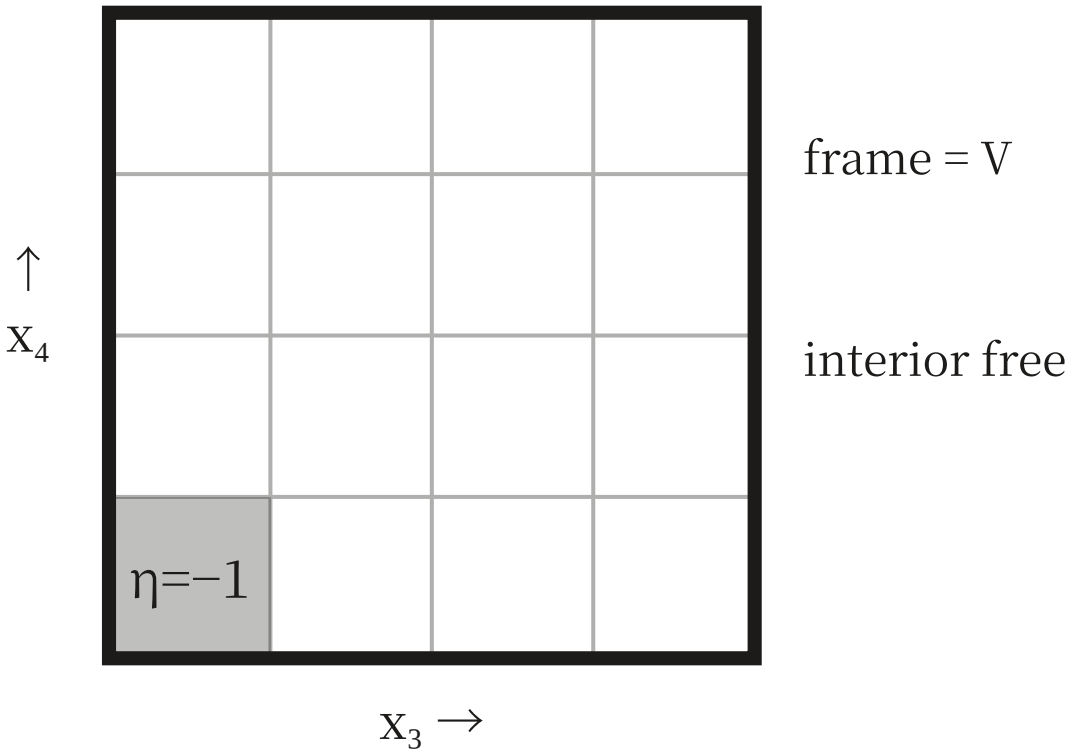
$$S(U;\eta) = \sum_{p:\ \mathrm{root}\in b_0} \left(1 - z_p\,\eta_p\,\mathrm{Re}\,U_p\right) \tag{1}$$

(the notation for action is as in 21-03). z_p is the sign of the background twist. β is not multiplied in. We write the infimum over all configurations of the interior links, and the infimum over block sizes, as

$$S_{\mathrm{v}}(N,m) = \inf_{\text{interior}} S(U;\eta), \qquad S_N = \inf_{m\ge 1} S_{\mathrm{v}}(N,m) \tag{2}$$

The Lean definitions are `TwistBlocks.Svortex` and `TwistBlocks.SN`. Since the frame is the vacuum while one corner sign is reversed, somewhere inside the block a vortex sheet must pass all the way around the transverse torus. $S_{\mathrm{v}}(N,m)$ is the minimal action for threading one such vortex.

Figure 1 A block in the (3,4) plane (example $m = 4$). On the squares of block b_0, the frame lines $x_3, x_4 \in \{0,m\}$ are drawn thick (fixed at the twisted vacuum). The shaded square is the position of the corner reversal $\Pi_{(0,0)}$. Above each point sits the transverse torus T^2_N



There are two trivial bounds. Proved here Since $|\operatorname{Re} U_p| \le 1$, each term of (1) is at least 0, and $S_{\mathrm{v}}(N,m) \ge 0$. Proved here In the configuration with the interior also at V, only the N^2 reversed plaquettes contribute, each $1-(-1)\cdot 1 = 2$, and the rest give 0, so the action is exactly $2N^2$. Hence $0 \le S_N \le 2N^2$ (Lean: `TwistBlocks.Svortex_le`, `TwistBlocks.SN_bounds`). This $2N^2$ is the value of the "thin vortex" that gathers the reversal onto one layer of plaquettes, and corresponds to the exponent of the thin-vortex floor. Reading For $N=1$, $S_1 = 2 = 2N^2$, and a thick vortex gains nothing (the numerical record of this book's study).

S_N appears in the exponent of the thick-vortex floor in the form $e^{-\beta(S_N+\varepsilon)}$. The two facts of this chapter are what can be said about the number in that exponent from below (positivity) and from above (the certificates for $N=2,3,4$). The Lean proof of the thick-vortex floor does not use the two facts of this chapter.

02

S_N is positive — statement and meaning

Positivity of the minimal action of a vortex. For every $N \ge 1$ there is $\delta > 0$, depending only on N, such that

$$\delta \le S_{\mathrm{v}}(N,m) \quad (\text{all } m \ge 1), \qquad \text{hence} \qquad S_N > 0. \ (3)$$

Lean: `VortexPos.Svortex_ge_unif` ($\exists\,\delta>0,\ \forall m\geq 1,\ \delta\leq S_{\mathrm{V}}(N,m)$; all $N\geq 1$, no assumption fields) and `VortexPos.SN_pos` ($0<S_N$).

The order of quantifiers is the point. δ is fixed first and does not depend on m. However large the block is made and however the vortex is spread out, the action does not fall below δ. If it were merely positive for each m, it might approach 0 as $m\to\infty$, and the infimum S_N could be 0.

The value of δ is not known. The proof uses the fact "if the action is small, the configuration is close to the gauge orbit of the vacuum" through **compactness**, not through an explicit estimate of distance. So we can say only that δ exists; no number comes out. Reading Numerically, $S_{\mathrm{V}}(2,m)$ is about 7.0 to 6.67 for $m=3$ to 8 (the numerical record of this book's study), and δ may be much smaller than that.

Cited This theorem is a formalisation, on the lattice, of the known picture that the action of a vortex (a fractional instanton) in a twisted box is positive ('t Hooft's twisted boundary conditions, 25-02) (Lean: `VortexPos.SN_pos`). The only tools are compactness and rigidity.

03

The route to positivity — labels and a telescoping product

We argue by contradiction. Take a configuration of very small action, and show that it is incompatible with the corner reversal.

Step 1 Split into parts. For each point $s=(i,j)$ ($0\leq i,j\leq m$) of the block's (3,4) plane, call the transverse torus T^2_N above it (the direction-1 and 2 links) the **fibre**. Split the action (1) into three kinds of parts.

- Fibre action a_s: the sum of the terms of the (1,2) plaquettes of the fibre.
- Link defect b_λ: how badly the direction-3 or 4 links joining two neighbouring points (N^2 of them) connect the two fibres. For unit quaternions $\|X-Y\|=\sqrt{2(1-\operatorname{Re}XY^*)}$, so $b_\lambda\leq 2N^2\sqrt{2S}$ (S is the action of the block) Proved here . This estimate is also used inside the Lean proof (`VortexPos.linkH_le`, `linkV_le`).
- Square part pP: the sum of the terms of the N^2 plaquettes over the square P of the (3,4) plane.

The fibre actions and square parts are partial sums of (1), so they are at most S.

Step 2 Fibres are close to the vacuum. On the twisted T^2_N, configurations of action 0 lie in the gauge orbit of the twisted vacuum (Lean: `TwistBlocks.rigidT_all`). By compactness, if the fibre action is small, the fibre is close to the gauge orbit of the

vacuum. The only gauge transformations fixing the vacuum are the constants ± 1, and this too is extended by compactness to "if it nearly fixes it, it is close to ± 1".

Step 3 Attach labels. At each point s, take a gauge g_s that brings the fibre to the vacuum. On the frame the fibre is exactly the vacuum, so take $g_s = 1$ there. For each link λ, $g_s^{-1} U_\lambda g_{s'}$ is close to a constant $+1$ or -1; let its sign be the label $c(\lambda) \in \{\pm 1\}$. Links on the frame lines are 1 in the vacuum, so $c = +1$ there.

Step 4 Compatibility square by square. The product around a plaquette over square P, sandwiched by gauges, has the form $h_1 h_2 h_3^* h_4^*$, whose value is close to $\prod_{\lambda \in \partial P} c(\lambda)$. If $\eta(P) \;/= \prod_{\partial P} c$, the plaquette term is at least $2 - 4\varepsilon$ ($3/2$ for $\varepsilon = 1/8$). If all parts are small, then $\eta(P) = \prod_{\partial P} c$ for every square.

Step 5 Telescoping product. Multiplying $\prod_{\partial P} c$ over all squares, each interior link appears twice and the frame links are 1, so the product is 1 Proved here . But η is -1 on exactly one square, the corner, so $\prod_P \eta(P) = -1$. Contradiction.

Step 6 Conclusion. So some part is at least a threshold δ_0. For a fibre or a square, $S \geq \delta_0$; for a link, $2N^2\sqrt{2S} \geq \delta_0$. Together,

$$S_{\mathrm{v}}(N, m) \;\geq\; \delta := \min\!\left(\delta_0,\; \frac{\delta_0^2}{32N^4 + 1}\right) \quad (4)$$

holds for all $m \geq 1$. δ_0 comes from the compactness of step 2, so it is not explicit. The Lean core is a lemma of the form `TwistBlocks.grid_vortex_floor`, and all its part fields are discharged inside `VortexPos.Svortex_ge_unif`.

Detour: where the background twist matters

You may skip this. In step 3 the labels fall into $\{\pm 1\}$ because the only gauge transformations fixing the twisted vacuum are the constants ± 1 (Lean: `TwistBlocks.stab_vacT`). i and j appear on the links of the twisted vacuum, and the only unit quaternions commuting with both i and j are ± 1 Proved here . So the "offset" of each link is close to an element of the centre $\{\pm 1\}$, and the argument rides on $\mathbb{Z}_2$ bookkeeping: compatibility square by square and a telescoping product. The core of the proof is that the corner reversal cannot be cancelled in $\mathbb{Z}_2$ bookkeeping.

04

Upper bounds for N = 2, 3, 4 — statement

The upper side can be stated by exhibiting a single concrete configuration. Since $S_{\mathrm{v}}(N, m)$ is an infimum, the action of any configuration is an upper bound for it. The upper bounds machine-checked in Lean are the following three Computer-assisted .

N	Block side m	Upper bound	Corollary	Lean declarations
2	8	$S_{\mathrm{v}}(2,8) \le 667401/100000$	$S_2 < 8 = 2 \cdot 2^2$	`VortexCert.Svortex_two_eight_le`, `SN_two_le`, `SN_two_lt`
3	8	$S_{\mathrm{v}}(3,8) \le 909321/100000$	$S_3 < 18 = 2 \cdot 3^2$	`VortexCert.Svortex_three_eight_le`, `SN_three_le`, `SN_three_lt`
4	4	$S_{\mathrm{v}}(4,4) \le 1202057/100000$	$S_4 < 32 = 2 \cdot 4^2$	`VortexCert.Svortex_four_four_le`, `SN_four_le`, `SN_four_lt`

None has assumption fields. $S_N \le S_{\mathrm{v}}(N,m)$ follows from the definition of the infimum (Lean: `VortexCert.SN_le_Svortex`).

Put into the thick-vortex floor, the exponent becomes an explicit number. Lean: `TwistBlocks.thick_vortex_floor_two` (no assumption fields) says that, for each sufficiently large β, for sufficiently large M,

$$|\theta_{M+1}(\beta)| \;\le\; \exp\!\left(-(M+1)^2\, e^{-6.675\,\beta}\right) \quad (5)$$

`TwistBlocks.thick_vortex_floor_three` has exponent number 9.094, and `TwistBlocks.thick_vortex_floor_four` has 12.021. The proof just takes the ε of the thick-vortex floor (21-03, `TwistBlocks.thick_vortex_floor`) to be "exponent number $-$ SN". The certificate for the upper bound gives $\varepsilon > 0$.

One caution in reading. What we could say is **only an upper bound**. One cannot write "$S_2 = 6.674$". Since it is a single configuration at $m = 8$ ($m = 4$ for $N = 4$), it is not the value of the infimum over m either. Reading The configuration for $N = 4$ comes from a numerical minimisation at $m = 4$ that has not fully converged, and the true value of S_4 may be smaller.

05

How the certificate is made — rational unit quaternions and rounding up

The certificate is computer-assisted Computer-assisted . It is not a pillar of the analytic proof; it is placed as support for "exhibiting one configuration that gives an upper bound". It is made in four stages.

(a) Make the configuration rational. The interior link values $q = (q_0, \underline{q})$ obtained by numerical minimisation (cooling) are converted to $\underline{\tau} = \underline{q}/(1+q_0)$ and then rounded to $t/2^{16}$ ($t \in \mathbb{Z}^3$). From this, quaternions are built in the Cayley form

$$q = \frac{\left(s^2 - |t|^2,\; 2s\,t\right)}{s^2 + |t|^2}, \qquad s = 2^{16} \quad (6)$$

Proved here Since $(s^2 - |t|^2)^2 + 4s^2|t|^2 = (s^2 + |t|^2)^2$, the norm is **exactly** 1. No square roots are needed (Lean: `VortexCert.normSq_cay`).

(b) Round up plaquette by plaquette. The plaquette term $1 - z_p \eta_p \operatorname{Re}(ABC^*D^*)$ is rational because the four links are rational. Round it **up** in units of 2^{-32} and add the resulting integers. Because the rounding is upward, the integer sum $/2^{32}$ is an upper bound on the action of the block.

(c) Count in the kernel. The integer inequality, for example for $N = 2$ "sum $\times 100000 \le 667401 \cdot 2^{32}$", is checked by a finite computation in the Lean kernel (`decide +kernel`) (Lean: `VortexCert.certN2`, `certN3`, `certN4`). The number of plaquettes is 1536 for $N = 2$, 3456 for $N = 3$, and 1536 for $N = 4$.

(d) Carry over to a real upper bound. The soundness theorem `VortexCert.Svortex_le_of_cert` derives $S_{\mathrm{v}}(N, m) \le b_n / b_d$ from the integer inequality. Its only assumptions are the integer inequality (and positivity of the denominator). Whatever the values in the table, each link is a unit quaternion because of the form (6), the frame values are fixed at V on the side of the definition, and it is shown inside Lean that the background and corner signs agree with the definition. So the configuration of the certificate belongs, **by construction**, to the set over which the infimum (2) is taken.

Reading The actions of the rational configurations are 6.6740076 for $N = 2$, 9.0932068 for $N = 3$ and 12.0205673 for $N = 4$, and the increase due to rounding is less than 4×10^{-7} (the numerical record of this book's study, computed exactly in fractions without rounding). The Lean upper bounds are, in each case, the smallest number on a 10^{-5} grid that passes.

What Lean checks is only the evaluation of integer polynomials. Where the configuration came from (whether the cooling is close to the minimum) has no bearing on the correctness of the upper bound.

06

Comparison with the thin-vortex floor — comparing rates, and its limits

The thin-vortex floor is $|\theta_M| \le \tanh(N^2|\beta|)^{M^2} = \exp(-M^2\sigma_0)$, $\sigma_0 = -\ln\tanh(N^2|\beta|)$. Proved here As $\beta \to \infty$, $\sigma_0 = 2e^{-2N^2\beta}(1 + o(1))$, so the rate at which the thin coefficient decreases is $2N^2$. The rate of the thick-vortex floor (5) is $S_N + \varepsilon$. For $N = 2$, $6.675 < 8$, and as a rate for $\beta \to \infty$, the thick coefficient becomes exponentially larger.

This comparison has three limits.

- **It is not a Lean theorem.** What Lean has is the numerical inequality $6675/1000 < 2 \cdot 2^2$ (`TwistBlocks.thick_vortex_floor_two_rate_lt`) and each of the two floors. "The thick-vortex floor is stronger than the thin-vortex floor" itself is a comparison of numbers set side by side by hand.
- **For small β it is the other way round.** Reading $e^{-6.675\beta}$ exceeds $-\ln\tanh 4\beta$ from $\beta > 0.523$. At $\beta = 1$ it is 1.26×10^{-3} versus 6.71×10^{-4}; at $\beta = 2$, 1.59×10^{-6} versus 2.25×10^{-7}.
- **The thresholds are not explicit.** β_0 and $M_0(\beta)$ of the thick-vortex floor are not explicit, so one cannot say "the thick bound is smaller at a concrete (β, M) with $\beta > 0.523$". What can be said is a comparison of rates as $\beta \to \infty$.

N	Thin-vortex rate $2N^2$	Upper bound on S_N (Lean)	Exponent number of the thick-vortex floor (Lean)
1	2	— (Reading $S_1 = 2$)	—
2	8	6.67401	6.675
3	18	9.09321	9.094
4	32	12.02057	12.021

$S_N < 2N^2$ for general N is not in Lean. For $N = 1$ there is no gain.

07

What is not said

- The value of S_N. What we could say is only $0 < S_N$ (with a non-explicit δ) and the upper bounds for $N = 2, 3, 4$.
- The number δ. It would need a quantitative estimate of rigidity (an explicit relation between action and distance from the vacuum).
- Lower or upper bounds uniform in N. Changing N changes the configuration space of the fibre, and this argument is for each N separately.
- The reverse direction of the thick-vortex floor (a lower bound on θ, an upper bound on the rate). There is no theorem in the reverse direction.
- The band (the statement that the vortex ratio θ becomes small exponentially in the area of the box, that is, that $-\ln\theta$ (corresponding to the slab tension times the area) grows in proportion to the area, stated with a single constant while moving the transverse and longitudinal sizes and the coupling together; 19-07). The quantities of this chapter are in a form with the lattice spacing and the width N held fixed. The distance is examined in 21-08.
- The lattice spacing is fixed; none of the continuum limit, the four-dimensional area law or the mass gap is said.

What this chapter can say

Can say	Cannot say
Lean `TwistBlocks.Svortex_le`, `TwistBlocks.SN_bounds`: $0 \le S_{\mathrm{v}}(N,m) \le 2N^2$, $0 \le S_N \le 2N^2$ (the configuration with the reversal put into the vacuum)	The value of S_N
Lean `VortexPos.Svortex_ge_unif`, `SN_pos`: for every $N \ge 1$ there is $\delta > 0$ uniform in m with $S_N \ge \delta > 0$	The number δ (it comes from compactness). A lower bound uniform in N
Computer-assisted $S_{\mathrm{v}}(2,8) \le 6.67401$, $S_{\mathrm{v}}(3,8) \le 9.09321$, $S_{\mathrm{v}}(4,4) \le 12.02057$, hence $S_N < 2N^2$ $(N = 2,3,4)$	The value of the infimum over m. $S_N < 2N^2$ for general N
Lean `TwistBlocks.thick_vortex_floor_two`, `_three`, `_four`: for each sufficiently large β, for sufficiently large M, floors with exponent numbers 6.675, 9.094, 12.021	The values of β_0 and $M_0(\beta)$
Reading For $N = 2$ the coefficient of the thick-vortex floor exceeds the thin one from $\beta > 0.523$	"The thick-vortex floor is stronger" itself is not a Lean theorem (a comparison of rates)

Sources and reproduction

Item	Kind	Source / tool
Twisted boundary conditions and the flux free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (bibliography as in 25-01)
The action of fractional instantons and the dilute-gas picture	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166 (to the extent checked in 25-02)
The route to positivity (steps 1–6)	Proved here	§ 03 of the text. Lean: `VortexPos.Svortex_ge_unif`, `VortexPos.SN_pos`; core lemma `TwistBlocks.grid_vortex_floor`; rigidity of the fibre `TwistBlocks.rigidT_all`
The certificates for the upper bounds	Computer-assisted	Rational configurations in Cayley form ($s = 2^{16}$), integer sums rounded up plaquette by plaquette in units of 2^{-32}, evaluation in the Lean kernel (`decide +kernel`). Soundness `VortexCert.Svortex_le_of_cert`. `VortexCert.certN2`, `certN3`, `certN4`
Values of the action, $S_1 = 2$, $\beta > 0.523$	Reading	The numerical record of this book's study (a script written out from the Lean definitions, computing the action exactly in fractions without rounding; also checked that lowering the last digit of an upper bound by one makes it fail. The crossing β by floating-point root finding)

Item	Kind	Source / tool
Corollaries of the thick-vortex floor	—	Lean: `TwistBlocks.thick_vortex_floor_two`, `_three`, `_four`; the numerical inequality `TwistBlocks.thick_vortex_floor_two_rate_lt`. The main theorem is in 21-03

Words used in this chapter — minimal action of a vortex $S_{\mathrm{v}}(N, m)$, S_N, twisted vacuum, fibre, link defect, label, telescoping product, Cayley form, rounding up, certificate, comparison of rates. Their meanings are in A-02 Glossary.

Next chapter to read: 21-06 The sign of θ and pairs of Polyakov loops — that the vortex ratio θ is never negative, and that θ bounds correlations of Polyakov loops.

The sign of θ and pairs of Polyakov loops

> This chapter is this book's own investigation.

What this chapter shows — that on a torus of width N and length M, if M is even then $Z_1 \le Z_0$, that is, $\theta_M \ge 0$, for every real β. That with the same reflection tools, the correlation of two Polyakov loops winding around direction 3 is bounded by $2\sqrt{\theta}/(1+\theta)$ at distance $M/2$, and by $2\tanh(N^2|\beta|)^{M\,\min(T,M-T)}$ at a general time separation T. Each is a formalisation, machine-checked in Lean, of an estimate of known type.

Prerequisite chapters — 19-02 the twist, the vortex ratio, Polyakov loops; 19-03 reflection; 21-02 The thin-vortex floor; 21-03 The thick-vortex floor

Said first — the expectation $\langle\cdot\rangle_0$ is with respect to the Wilson measure on a finite lattice with the background twist $n_{12}=1$ and without the (3,4) twist. What can be said is a linear **lower bound** on the energy inside a transverse $N\times N$ tube, not uniform in N. The lattice spacing is fixed. Nothing is said about four-dimensional confinement, the area law or the mass gap. The case of odd M is not included.

The order of this chapter

1. The expectation $\langle\cdot\rangle_0$ and Polyakov loops
2. Splitting the lattice into two halves by a reflection in the time direction
3. θ ≥ 0 — statement and proof
4. Pairs of Polyakov loops at distance M/2, and θ
5. General time separation — repeating the reflection
6. Detour: the range in the parities of M, N and the sign of β
7. How to read it — a linear lower bound on the energy inside a tube
8. What is not said

01

The expectation $\langle\cdot\rangle_0$ and Polyakov loops

The lattice is the torus with transverse part $(\mathbb{Z}/N)^2$ (directions 1 and 2) and longitudinal part $(\mathbb{Z}/M)^2$ (directions 3 and 4). The definitions are in 19-02. Under the background twist $n_{12} = 1$, let Z_0 and Z_1 be the partition functions without and with the (3,4) twist, and write the vortex ratio as $\theta_M = (Z_0 - Z_1)/(Z_0 + Z_1)$ (19-02).

The expectation in this chapter is

$$\langle F\rangle_0 = \frac{1}{Z_0}\int F(U)\, w_0(U)\, dU, \qquad w_0(U) = \exp\Big(\beta \sum_p z_p^{(0)}\, \mathrm{Re}\, U_p\Big) \quad (1)$$

dU is the product of Haar measures, and $z_p^{(0)}$ is the sign of the background twist $n_{12} = 1$ only. **The (3,4) twist is not included.** In Lean the weight is `wE N M β (fun _ => 1)` and the divisor is `ZW N M β true false`.

A Polyakov loop is the real part of the product of the links going once around direction 3, at transverse position $b = (x_1, x_2)$ and direction-4 position t (19-02).

$$P_b(t) = \mathrm{Re} \prod_{k=0}^{M-1} U_3\big((b),(k,t)\big) \quad (2)$$

The real part of a quaternion is $\frac{1}{2}\,\mathrm{tr}$ in the fundamental representation, and $|P_b(t)| \le 1$. In Lean it is `HalfSplit.polyLoop`, and the expectation of a pair is `PolyakovT.polyExp`.

Proved here The expectation of a single loop $\langle P_b(t)\rangle_0$ is 0. Multiplying all direction-3 links at $x_3 = 0$ by -1, every plaquette contains 0 or 2 of the reversed links, so w_0 is unchanged, while $P_b(t)$ alone changes sign. The meaningful quantity is the pair correlation $\langle P_b(0) P_{b'}(T)\rangle_0$.

02

Splitting the lattice into two halves by a reflection in the time direction

When M is even, the reflection $\Theta:\ x_4 \mapsto -x_4$ fixes two planes of lattice points, $x_4 = 0$ and $x_4 = M/2$. Links in directions 1, 2 and 3 only have their positions mapped; links in direction 4 have their orientation reversed and their values inverted (the general form of reflection is in 19-03).

Since the planes pass through lattice points, no square crosses them. The squares on the two planes (those not containing direction 4) are shared half and half between the upper and lower halves. The **half weight** built in this way,

$$a_\eta(U) = \exp\Big(\beta \sum_p c_p \, z_p \, \eta_p \, \mathrm{Re}\, U_p\Big), \qquad c_p \in \{1, \tfrac{1}{2}, 0\} \tag{3}$$

(1 in the upper half, $\frac{1}{2}$ on the planes, 0 below), depends only on the links of the closed upper half. The whole weight then splits as

$$w_\eta(U) = a_{\eta 1}(U)\, a_{\eta 2}(\Theta U) \tag{4}$$

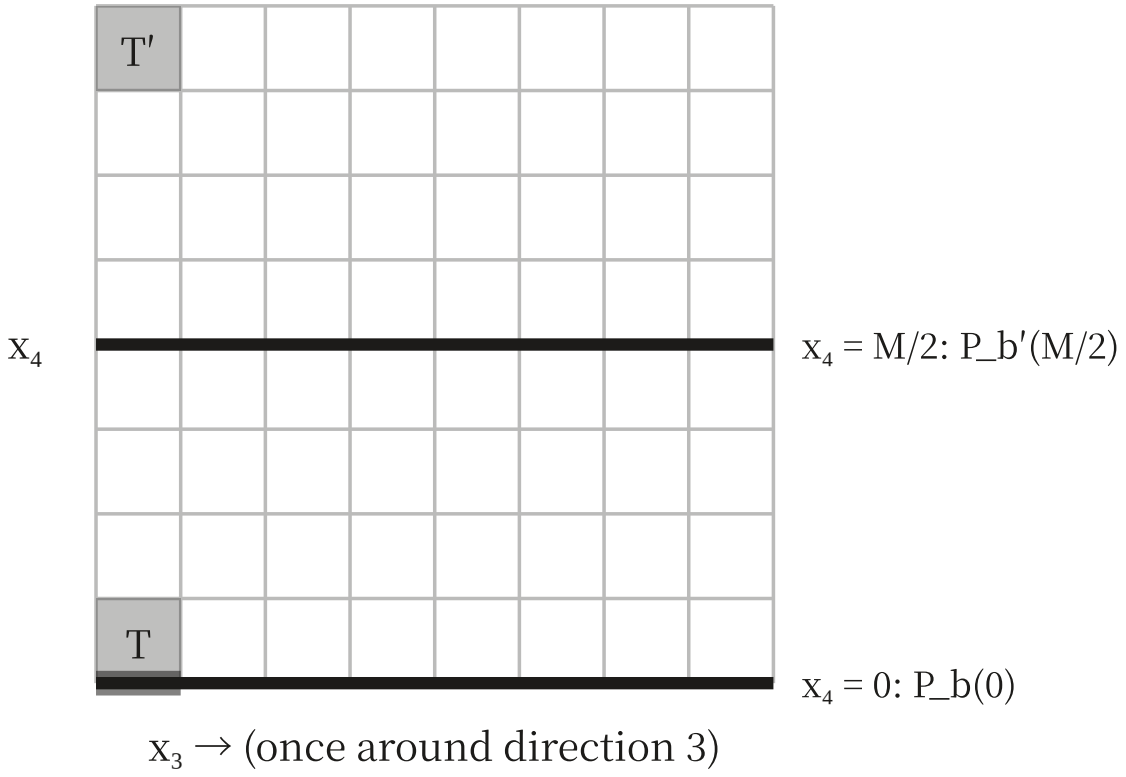
Here η gives the signs of the (3,4) plaquettes as $\eta 1$ in the upper half and as the mirror image of $\eta 2$ in the lower half (Lean: `HalfSplit.half_split`).

The product of Haar measures is invariant under the reflection, and for bounded measurable g depending only on the closed upper half,

$$\int g(\Theta U)\, g(U)\, dU \;\geq\; 0 \tag{5}$$

holds (reflection positivity; Lean: `HalfSplit.half_pairing_nonneg`). By (4) the weight is absorbed into g, so no condition on the sign of β is needed. Cited This form is of the type of Osterwalder–Seiler reflection positivity, and `HalfSplit.half_pairing_nonneg` is its formalisation on the twisted lattice (19-03).

Figure 1 The longitudinal (3,4) plane (example $M = 8$; the transverse T_N^2 omitted). The thick horizontal lines are the reflection planes $x_4 = 0$ and $x_4 = M/2$. Two Polyakov loops lie on them. The shaded squares are the reversal T ($(0,0)$) and its mirror image T' ($(0,-1) = (0, M-1)$). The marked link is the sign reversal c of § 04



The figure only shows the two planes and the positions of the reversals used in § 03 and § 04; it carries no numbers.

03

θ ≥ 0 — statement and proof

The sign of the vortex ratio. If M is even, then for every real β and all $N \geq 1$,

$$Z_1 \leq Z_0, \qquad \text{equivalently} \qquad 0 \leq \theta_M(\beta). \tag{6}$$

Lean: `HalfSplit.Z1_le_Z0` (M even, every real β), `TwistBlocks.theta_nonneg` ($0 \leq \theta_M$ in the same range).

Proof. Let T be the reversal of the sign of the corner (3,4) plaquette $(x_3, x_4) = (0, 0)$ at all transverse positions, and put $g = a_1 - a_T$ into (5). Expanding with (4),

$$0 \leq \int g(\Theta U) g(U)\, dU = Z[1] - Z[T] - Z[T'] + Z[TT'] \tag{7}$$

T' is the reversal at $(0, -1)$, the mirror position of T, and TT' is the two reversals together. Proved here The partition function does not depend on the positions of the reversals; it is determined only by the product of the reversals $\prod \eta$ (positions of reversals can be moved by a change of variables that changes the signs of the links on

a cut; Lean: `TwistedWilson.ZEta_flip_any`, `TwistedWilson.ZEta_parity`). Hence $Z[T] = Z[T'] = Z_1$ and $Z[TT'] = Z_0$, and (7) is $0 \le 2Z_0 - 2Z_1$. □

This estimate removes the absolute value from the thick-vortex floor. Lean: `TwistBlocks.thick_vortex_floor_signed` (all $N \ge 1$) says that for any $\varepsilon > 0$, for each sufficiently large β, for sufficiently large M, if $M+1$ is even,

$$0 \le \theta_{M+1}(\beta) \le \exp\bigl(-(M+1)^2 e^{-\beta(SN+\varepsilon)}\bigr) \quad (8)$$

The thresholds in β and M are not explicit.

Odd M is not included, because the second plane $x_4 = M/2$ does not fall on lattice points. Moreover, for odd M and $\beta < 0$, $\theta \ge 0$ can fail. Reading At $N = 1$, $M = 1$, $\beta = -1.5$ we found $\theta = -0.055 \pm 0.005$ (SU(2) Monte Carlo, the numerical record of this book's study). For $\beta \ge 0$ it is expected to hold regardless of parity, by positivity of the transfer matrix, but this book has not checked it. Proved here Note that $\theta_M < 1$ follows for all M from $Z_1 > 0$ (Lean: `TwistBlocks.theta_lt_one`).

04

Pairs of Polyakov loops at distance M/2, and θ

A bound on pairs of Polyakov loops by θ. If M is even, then for every real β and all transverse positions b, b',

$$\bigl|\langle P_b(0)\, P_{b'}(M/2)\rangle_0\bigr| \;\le\; \frac{2\sqrt{\theta_M}}{1+\theta_M}. \quad (9)$$

Lean: `HalfSplit.polyakov_pair_theta` (M even, every real β). The left side is the expectation (1), with divisor Z_0.

θ is the quantity determined from the flux free energy $\Delta F = \ln(Z_0/Z_1)$ by $\theta = \tanh(\Delta F/2)$ (19-02; 26-03 of the present book also calls ΔF the "vortex free energy"), and (9) says that it is **a quantity that bounds the correlation of loops at long distance from above**. Cited Estimates that bound long loops by ratios of twisted partition functions belong to the line of Tomboulis–Yaffe's argument on the flux free energy, and `HalfSplit.polyakov_pair_theta` is a formalisation, in that line, of the form for pairs of Polyakov loops at $T = M/2$ (the inequality in the original paper is about Wilson loops; the text not checked).

The route.

1. **The central sign reversal** c**.** Multiply the direction-3 links at $(x_3, x_4) = (0,0)$ (N^2 transverse links) by -1. The product of Haar measures is unchanged. The (1,3) and

(2,3) plaquettes contain two reversed links and are unchanged; among the (3,4) plaquettes only the two, T and T', change sign (the marked link in Figure 1). Hence $w_0(cU) = wTT'(U)$.

2. **Parity of the loops.** In $P_b(0)$ only the first link of the loop is reversed, so it changes sign; $P_{b'}(M/2)$ is unchanged. The product $\psi = P_b(0)P_{b'}(M/2)$ satisfies $\psi(cU) = -\psi(U)$, so $\int \psi w_0 = \frac{1}{2}\int \psi\,(w_0 - wTT')$.
3. **Write as sum and difference.** With $a_\pm = a_1 \pm a_T$, (4) gives $w_0 - wTT' = \frac{1}{2}\big[(a_+ \circ \Theta)a_- + (a_- \circ \Theta)a_+\big]$.
4. **Cauchy–Schwarz.** ψ depends only on the links on the two planes (Figure 1), is unchanged by Θ, and $|\psi| \le 1$. Using Cauchy–Schwarz for the reflection pairing with ψ inserted, and putting in $\langle a_\pm, a_\pm\rangle = 2(Z_0 \pm Z_1)$, gives $|\int \psi w_0| \le \sqrt{Z_0^2 - Z_1^2}$ (Lean: `HalfSplit.odd_fix_bound`).
5. **Divide.** $\sqrt{Z_0^2 - Z_1^2}/Z_0 = 2\sqrt{\theta}/(1+\theta)$ Proved here . □

Reading A guide to its strength: in the sum over all configurations for the $\mathbb{Z}_2$ subgroup, $N = 1$, $M = 2$, (9) is an equality. In SU(2) samples the right side was 3–6 times the left side (the numerical record of this book's study).

Putting a floor into (9) gives a floor for loop correlations. From $2\sqrt{\theta}/(1+\theta) \le 2\sqrt{\theta}$ and the thin-vortex floor $\theta \le \tanh(N^2|\beta|)^{M^2}$, $|\langle P_b(0)P_{b'}(M/2)\rangle_0| \le 2\tanh(N^2|\beta|)^{M^2/2}$ (M even, every real β; Lean: `PolyakovT.half_pair_tanh`). From the signed floor (8), Lean: `HalfSplit.polyakov_pair_floor` (all $N \ge 1$) gives, for any $\varepsilon > 0$, for each sufficiently large β, for sufficiently large M, if $M+1$ is even,

$$\left|\langle P_b(0)\,P_{b'}\left(\tfrac{M+1}{2}\right)\rangle_0\right| \;\le\; 2\exp\!\Big(-\tfrac{1}{2}(M+1)^2 e^{-\beta(S_N+\varepsilon)}\Big) \quad (10)$$

(the lattice has length $M+1$). The thresholds are not explicit, and the coefficient $e^{-\beta(S_N+\varepsilon)}$ becomes exponentially small as $\beta \to \infty$.

05

General time separation — repeating the reflection

Pairs of Polyakov loops at general time separation. If $M \equiv 0 \pmod 4$, then for every real β, all $0 \le T \le M$ and all b, b',

$$\left|\langle P_b(0)\,P_{b'}(T)\rangle_0\right| \;\le\; 2\,\tanh(N^2|\beta|)^{\,M\cdot\min(T,\,M-T)}. \quad (11)$$

Lean: `PolyakovT.polyakov_pair_all` ($M \equiv 0 \bmod 4$, every real β).

Cited Decay of correlations inside a tube of finite cross-section is an estimate of known type along the lines of Mack–Petkova and Tomboulis–Yaffe, and `PolyakovT.polyakov_pair_all` is its formalisation.

The case $T = M/2$ follows from § 04. For general T, we move the reflection plane and repeat Cauchy–Schwarz, reducing to convexity of the logarithm of the correlation. Write $G_{bb'}(T) = \int P_b(0) P_{b'}(T)\, w_0\, dU$ (unnormalised).

Step 1 Translation. The direction-4 translation $x_4 \mapsto x_4 + 1$ preserves the product of Haar measures and w_0, so $\int P_b(s) P_{b'}(t)\, w_0$ depends only on $t - s$ (Lean: `PolyakovT.corr_shiftZ`). So the reflection plane can be placed at any time. Also $G_{bb'}(T) = G_{b'b}(M - T)$ (`PolyakovT.Gc_swap`).

Step 2 The weighted reflection pairing. For bounded F, G depending only on the closed upper half, $\langle F, G\rangle_w = \int F(\Theta U) G(U)\, w_0\, dU$ is positive semidefinite because $w_0 = a_1 \cdot (a_1 \circ \Theta)$ by (4), and Cauchy–Schwarz holds (every real β; Lean: `PolyakovT.wpair_nonneg`, `PolyakovT.wpair_cs`).

Step 3 Midpoint convexity for even separations. Taking $F = P_b(u)$, $G = P_{b'}(v)$ and using $P_b(u) \circ \Theta = P_b(-u)$ and translation, for $2u, 2v \le M$,

$$G_{bb'}(u+v)^2 \;\le\; G_{bb}(2u)\, G_{b'b'}(2v), \qquad G_{bb}(2u) \ge 0 \quad (12)$$

(Lean: `PolyakovT.cs_even`; the non-negative side is `PolyakovT.Gc_even_nonneg`).

Step 4 The chord inequality. Proved here If $g \ge 0$ satisfies $g(k)^2 \le g(k-1) g(k+1)$ $(1 \le k \le m-1)$, then $g(k)^m \le g(0)^{m-k} g(m)^k$ (Lean: `LogChord.chord`). Applying this with $g(k) = G_{bb}(2k)$ and $m = M/4$, for $T = 2k \le M/2$, $G(T)$ is bounded by a weighted geometric mean of $G(0) \le Z_0$ and $G(M/2)$.

Step 5 Put in the ends. By § 04 and the thin-vortex floor, $G(M/2)$ is at most $2\, t^{M \cdot M/2}$ Z_0 $(t = \tanh(N^2|\beta|))$ (`PolyakovT.half_pair_tanh`). For even $T \le M/2$ with $b = b'$, this gives (11). For $b \neq b'$, use (12) with $u = v = T/2$ to return to $G_{bb'}(T)^2 \le G_{bb}(T) G_{b'b'}(T)$. For odd $T < M/2$, since $M/2$ is even, $T + 1 \le M/2$, and we use (12) with $u = (T-1)/2$, $v = (T+1)/2$. $T > M/2$ is brought back by the symmetry of step 1.

The constant 2 does not depend on β, N, M or T. Using the trace of the transfer matrix and Hölder's inequality, the θ form $(\theta^{T/M} + \theta^{1-T/M})/(1+\theta)$ is to be expected, but it is not in Lean. (11) is weaker than that in only two respects: the constant 2, and the replacement of θ by t^{M^2} from the thin-vortex floor.

Detour: the range in the parities of M, N and the sign of β

You may skip this. With reflections in planes of lattice points alone, odd separations are not covered when $M \equiv 2 \pmod 4$. Adding reflections in planes through the middle of links covers them, but such a plane crosses squares, so $\beta \geq 0$ is needed (Lean: `PolyakovT.link_rp`, M even ≥ 4, $\beta \geq 0$). If N and M are both even, a change of variables by staggered signs sends $\beta \mapsto -\beta$ without changing the expectation of a pair (`PolyakovT.polyExp_neg`).

Lean declaration	M	N	β
`PolyakovT.polyakov_pair_all`	$M \equiv 0 \bmod 4$	Any	All real numbers
`PolyakovT.polyakov_pair_evenM`	Even, $M \geq 4$	Any	$\beta \geq 0$
`PolyakovT.polyakov_pair_evenNM`	Even, $M \geq 4$	Even	All real numbers

All conclusions have the same form as (11). Not covered are the case of odd M, and the case $M \equiv 2 \pmod 4$ with N odd and $\beta < 0$.

06

How to read it — a linear lower bound on the energy inside a tube

Setting $V(T) = -\frac{1}{M} \ln |\langle P(0)P(T)\rangle_0|$, (11) becomes

$$V(T) \;\geq\; \sigma_0 \cdot \min(T, M-T) - \frac{\ln 2}{M}, \qquad \sigma_0 = -\ln \tanh(N^2|\beta|) \quad (13)$$

A pair of Polyakov loops is a pair of static charges when direction 3 is viewed as time, and $V(T)$ corresponds to the energy of separating the pair to distance T. (13) is **a linear lower bound on the energy inside a transverse $N \times N$ tube**.

- **Only a lower bound.** It says neither that V is linear nor anything about an upper bound.
- **Not uniform in N.** $\sigma_0 \approx 2e^{-2N^2|\beta|}$ goes to 0 as $N \to \infty$. The thicker the tube, the weaker the lower bound.
- **The lattice spacing is fixed.** With N held fixed the system is effectively two-dimensional, and Cited in two-dimensional lattice gauge theory an area law at every β is the usual outcome. This is not a claim about four-dimensional confinement, the area law or the mass gap.

A guide to the non-trivial range. $|\langle P(0)P(T)\rangle_0| \leq \langle P^2\rangle_0$ is trivial. Reading $\langle P^2\rangle_0$ is about 0.25–0.3 (1/4 at $\beta = 0$). The smallest exponent $X = M \min(T, M-T)$ needed for the right side of (11) to fall below 1/4 is as follows Reading .

N	β = 0.1	0.3	0.5	1	2
1	1	2	3	8	57
2	3	12	57	3100	$9.2\cdot10^{6}$
3	7	231	8425	$6.8\cdot10^{7}$	$4.48\cdot10^{15}$

Reading Since $X \le M^2/2$ (with equality at $T = M/2$), for example at $N = 2$, $\beta = 1$, $X \ge 3100$ requires $M \ge 78.74$; with $M \equiv 0 \pmod 4$ in fact $M \ge 80$, and even then the right side falls below 1/4 only for T near $M/2$. At large β and large N, (11) is loose, and that side is covered by forms of the thick-vortex floor such as (10).

07

What is not said

- $\theta \ge 0$ for odd M; pairs of Polyakov loops for odd M.
- The Hölder form itself ((11) is weaker by the constant 2 and by replacing θ with t^{M^2}).
- An upper bound on $V(T)$, the linearity of V, the value of the tension.
- Estimates uniform in the width N. The band (the statement that the vortex ratio θ becomes small exponentially in the area of the box, that is, that $-\ln\theta$ (corresponding to the slab tension times the area) grows in proportion to the area, stated with a single constant while moving the transverse and longitudinal sizes and the coupling together; 19-07) is not reached. The estimates of this chapter are in a form with the lattice spacing and the width held fixed (21-08).
- The values of the thresholds in β and M in (8) and (10).
- The continuum limit; four-dimensional confinement, the area law, the mass gap.

What this chapter can say

Can say	Cannot say
Lean `HalfSplit.Z1_le_Z0`, `TwistBlocks.theta_nonneg`: $0 \le \theta_M$ for even M and every real β	Odd M (Reading there is an example where it fails at $\beta < 0$)
Lean `TwistBlocks.thick_vortex_floor_signed`: at large β and large M ($M+1$ even), $0 \le \theta_{M+1} \le \exp(-(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	The values of the thresholds
Lean `HalfSplit.polyakov_pair_theta`: for even M and every real β, $\lvert\langle P_b(0) P_b'(M/2)\rangle_0\rvert \le 2\sqrt{\theta}/(1+\theta)$	Distances other than $M/2$ (those are the next row)

Can say	Cannot say
Lean `PolyakovT.polyakov_pair_all`: for $M \equiv 0 \bmod 4$, every real β, $0 \le T \le M$, $\lvert\langle P_b(0)P_{b'}(T)\rangle_0\rvert \le 2\tanh(N^2\lvert\beta\rvert)^{M\,\min(T,M-T)}$. `_evenM` ($M$ even ≥ 4, $\beta \ge 0$), `_evenNM` (N, M even, every β)	The Hölder form (weaker by the constant 2 and by $\theta \to t^{M^2}$). Odd M. A tension uniform in N
Lean `HalfSplit.polyakov_pair_floor`: a floor at distance $(M+1)/2$, combined with the thick-vortex floor	The values of the thresholds. Four-dimensional confinement
Lean `PolyakovT.half_pair_tanh`: for even M (including $M = 2$) and every real β, $\lvert\langle P_b(0)P_{b'}(M/2)\rangle_0\rvert \le 2t^{M\cdot M/2}$	Other T for $M \equiv 2 \bmod 4$ (`_evenM` if $M \ge 4$)
Reading The non-trivial range (at $N = 2$, $\beta = 1$: $M \ge 80$, with T near $M/2$)	—

Sources and reproduction

Item	Kind	Source / tool
Reflection positivity (planes of lattice points)	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471 (bibliography in 19-03). Lean: `HalfSplit.half_split`, `HalfSplit.half_pairing_nonneg`
The line of argument bounding long loops by the flux free energy	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313 (the text not checked; the same bibliography as 25-04)
The line of argument on vortices and confinement at finite cross-section	Cited	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117 (the text not checked; the same bibliography as 25-01)
The area law in two-dimensional lattice gauge theory	Cited	A. A. Migdal, Sov. Phys. JETP 42 (1975) 413 (bibliography only)
Twisted boundary conditions and the flux free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (25-01)
$\theta \ge 0$, pairs at distance M/2 (the routes of § 03 and § 04)	Proved here	The text. Lean: `PolyakovT.half_pair_tanh`, `TwistBlocks.theta_lt_one`, `HalfSplit.Z1_le_Z0`, `TwistBlocks.theta_nonneg`, `HalfSplit.odd_fix_bound`, `HalfSplit.polyakov_pair_theta`, `TwistBlocks.thick_vortex_floor_signed`, `HalfSplit.polyakov_pair_floor`
General time separation (steps 1–5 of § 05)	Proved here	The text. Lean: `PolyakovT.cs_even`, `PolyakovT.Gc_even_nonneg`, `LogChord.chord`, `PolyakovT.polyakov_pair_all`, `PolyakovT.link_rp`, `PolyakovT.polyakov_pair_evenM`, `PolyakovT.polyExp_neg`, `PolyakovT.polyakov_pair_evenNM`

Item	Kind	Source / tool
The example with $\theta < 0$ for odd M, the strength ratios, the table of the non-trivial range	Reading	The numerical record of this book's study (SU(2) Monte Carlo and weighted averages over uniform samples, sums over all $\mathbb{Z}_2$ configurations, floating-point computation of $2t^X < 1/4$)

Words used in this chapter — expectation $\langle\cdot\rangle_0$, Polyakov loop, vortex ratio θ_M, reflection positivity, half weight, central sign reversal, midpoint convexity for even separations, chord inequality, staggered signs, linear lower bound on the energy inside a tube. Their meanings are in A-02 Glossary.

Next chapter to read: 21-07 The vortex ratio — continuity, limit, matching — for the vortex ratio with the lattice held fixed, its continuity and limits in the coupling, and matching it to a chosen value.

The vortex ratio — continuity, limit, matching

This chapter is this book's study.

What this chapter shows — For the vortex ratio $\theta_{N,M}(\beta) = (Z_0 - Z_1)/(Z_0 + Z_1)$ of the SU(2) Wilson measure on a torus of transverse size N and longitudinal size M, we read five things that are machine-checked in Lean with the lattice held fixed. (1) $\beta \mapsto \theta$ is continuous and $\theta(0) = 0$. (2) A box with two twists has no configuration in which the holonomy of every plaquette equals the sign of the twist (no flat configuration). (3) The deficit (the loss of action) of that box has a positive lower bound, and in a symmetric box it is at least $1/2$, independent of N. (4) As $\beta \to \infty$, $\theta \to 1$, and the slope of the flux free energy converges to the minimal deficit. (5) For every κ_0 with $0 \le \kappa_0 < 1$ there is a $\beta \ge 0$ with $\theta(\beta) = \kappa_0$, and among the solutions there is a largest one (matching).

Prerequisite chapters — 19-02 the torus of transverse size N and longitudinal size M, the sign of the twist, the vortex ratio, the twisted vacuum; 21-02 The thin-vortex floor. The introduction is in 25-01.

Said first — The lattice spacing is fixed at 1, and nothing is said about the continuum limit. Every theorem is a statement **with the lattice sizes N, M held fixed**. The speed of $\theta \to 1$, the constants in the estimates coming from the lower bound on the deficit, and the matching solution β^* depend on N, M, and nothing is said about sequences in which N and β move together (matched sequences). Nor do we say anything about the monotonicity of θ, the uniqueness of the matching solution, the continuity of β^* in κ_0, or that the largest solution for $\kappa_0 = 0$ is 0. Nor do we compare the true value of the minimal deficit with the value in the continuum theory. Each is a formalisation of the lattice form of the known picture of 't Hooft's twisted box Cited .

The order of this chapter

1. Setting — the vortex ratio, the flux free energy, the deficit — the background with two twists, and its minimal deficit
2. Statement — continuity and the value at β = 0 — θ(0) = 0
3. Statement — a twisted box has no flat configuration — with only one twist there is one

4. Statement — a lower bound on the deficit — $2N^2M^2/(N^2+M^2)^2$, and $1/2$ in a symmetric box
5. Statement — $\theta \to 1$ as $\beta \to \infty$ — the form of the lower bound, and the slope of the free energy
6. Statement — the largest matching solution — every $\kappa_0 \in [0, 1)$, the lower envelope from the right
7. Route of the proof — discrete Stokes, anticommutation of quaternions, positivity of Haar measure on open sets, intermediate values
8. As questions — how things map when the lattice is doubled — what is not a theorem
9. What is said, and what is not

Labels — Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are the labels of the whole book (Reading is a numerical reading, Lean a Lean theorem of this book).

01

Setting — the vortex ratio, the flux free energy, the deficit

The lattice is the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ ($N, M \geq 1$) of transverse size N (directions 1, 2) and longitudinal size M (directions 3, 4); the weight is $e^{\beta \sum_p z_p \operatorname{Re} U_p}$, and the reference measure is the product of normalised Haar measures (19-02). With the background twist $n_{12} = 1$, write the partition functions without and with the longitudinal twist n_{34} as

$$Z_0 = Z_{N,M}(\beta; 1, 0), \qquad Z_1 = Z_{N,M}(\beta; 1, 1) \quad (1)$$

and write the vortex ratio and the flux free energy as

$$\theta = \theta_{N,M}(\beta) = \frac{Z_0 - Z_1}{Z_0 + Z_1}, \qquad \Delta F = \ln \frac{Z_0}{Z_1} = \ln \frac{1+\theta}{1-\theta} \quad (2)$$

(19-02; the equality on the right is Lean `ThetaLimit.phi_theta_eq`). Lean: `TwistBlocks.thetaM N M β`; Z_0 and Z_1 are `ZW N M β true false` and `ZW N M β true true`. This chapter deals mainly with $\beta \geq 0$.

The deficit. In the background with two twists $(n_{12}, n_{34}) = (1, 1)$, define the deficit of a configuration ω as

$$D(\omega) := \sum_p \big(1 - z_p \operatorname{Re} U_p(\omega)\big) \ \geq 0 \quad (3)$$

(z_p is the sign of the twist of this background). It is the loss from the maximum of the action (the number of plaquettes) to the actual action. $D(\omega) = 0$ means $U_p = z_p$ on every plaquette, that is, a flat configuration matching the twist. Write the minimal deficit as

$$D_{\min}(N, M) := \inf_{\omega} D(\omega) \quad (4)$$

Lean: `TwistFlat.defect11`, `ThetaLimit.Smin`.

Caution. $D_{\min}$ is the minimal deficit over the whole twisted box. It is a different quantity from the minimal action of a vortex S_N (19-02, 21-05; the quantity in which the boundary of a block of side m is held at the twisted vacuum).

02

Statement — continuity and the value at β = 0

Statement (continuity of the vortex ratio)

Let $N, M \geq 1$. $\beta \mapsto Z_{N,M}(\beta; n_{12}, n_{34})$ is continuous on the whole real line, and $\beta \mapsto \theta_{N,M}(\beta)$ is continuous too. At $\beta = 0$, $Z = 1$ and $\theta = 0$.

Lean Continuity of a bounded exponential integral. Lean: `ThetaTuning.ZW_continuous`, `thetaM_continuous`, `ZW_zero`, `thetaM_zero` (the only assumption is $N, M \geq 1$).

The proof is by dominated convergence. In the range $|\beta| \leq |\beta_0| + 1$, the integrand is bounded by $\exp((|\beta_0| + 1) \cdot \text{number of plaquettes})$. The denominator $Z_0 + Z_1$ is positive, so θ is continuous too. At $\beta = 0$, the reference measure is a probability measure, so $Z = 1$.

The range of values is $-1 < \theta \leq 1$ (`ThetaLimit.theta_gt_neg_one`, `theta_le_one`), and if the longitudinal size M is even, $\theta \geq 0$ (21-06, `TwistBlocks.theta_nonneg`). That $\theta > 0$ for $\beta > 0$ is not in Lean.

03

Statement — a twisted box has no flat configuration

Statement (a box with two twists has no flat configuration)

Let $N, M \geq 1$ (including the degenerate lattices $N = 1$, $M = 1$). In the background $(n12, n34) = (1, 1)$, there is no SU(2) configuration with $z_p \operatorname{Re} U_p = 1$ on every plaquette.

Lean The lattice form of the known type ('t Hooft): "a box with one twist in directions 1, 2 and one in directions 3, 4 has no flat connection". Lean: `TwistFlat.no_flat_twist11` (no assumptions).

In the background $(1, 0)$ with only one twist, the statement of the same form is false. The twisted vacuum (19-02: the unit quaternion $\mathbf{i}$ on the links of direction 1 at $x_1 = 0$, $\mathbf{j}$ on the links of direction 2 at $x_2 = 0$, and 1 elsewhere) is a configuration with deficit 0 (`ThetaLimit.flat_exists_10`). Because of this contrast, the statement does not hold trivially merely because of how the signs of the twist are placed.

Meaning. In a box with two twists, every configuration contributing to the twisted partition function Z_1 pays a positive loss of action. On the other hand, Z_0 (one twist) has configurations with zero loss. This difference is the source of $Z_1/Z_0 \to 0$, that is $\theta \to 1$, as $\beta \to \infty$ (§ 05).

04

Statement — a lower bound on the deficit

Statement (the minimal deficit is positive and is attained)

Let $N, M \geq 1$. There is a configuration $\omega^\star$ with $0 < D(\omega^\star) \leq D(\omega)$ (for all ω). Hence $D_{\min}(N, M) > 0$.

Lean From compactness and § 03. Lean: `TwistFlat.defect11_floor`, `minDefect11_pos` (no assumptions).

Statement (a uniform lower bound on the deficit)

For all $N, M \geq 1$ and all configurations ω,

$$D(\omega) \;\geq\; \frac{2N^2M^2}{(N^2+M^2)^2}. \tag{5}$$

In particular, in a symmetric box $N = M$, $D(\omega) \geq 1/2$, and $D_{\min}(N, N) \geq 1/2$ holds independently of N.

Lean A coarse uniform version, on the lattice, of the known type of lower bound on the action of a twisted box. Lean: `TwistFloor.defect11_ge_two`, `defect11_sym_half`, `defect11_uniform_half`, `Smin_sym_half` (no assumptions).

In the background with one twist, there is no positive lower bound of the same form (`TwistFloor.no_floor_10`).

The core of (5) is an inequality for quaternions. For unit quaternions a, b, c, d,

$$\|ab + ba\| + \|cd + dc\| + \|ac - ca\| + \|bc - cb\| \;\geq\; 2 \quad (6)$$

(Lean `QuadFloor.quad_floor_two`). The left-hand side is the sum of the failures of four relations, "a, b anticommute, c, d anticommute, and c commutes with both a and b", and in SU(2) these four are incompatible. The constant 2 is attained, for example at $(a, b, c, d) = (\mathbf{i}, \mathbf{j}, 1, \mathbf{k})$, and cannot be raised.

In numbers Reading . The minimal deficit for $N = M = 1$ is numerically 2.000; for $N = M = 2$ and 4 the deficits of configurations found by cooling are 7.84 and 9.80, and these are upper bounds on the minimum. There is still a gap of a factor of 16 to 20 from the lower bound $1/2$. The lower limit of the action of the corresponding sector in the continuum theory is $\pi^2 \approx 9.87$ in units of $\beta = 4/g^2$, from the self-dual estimate and the fractional topological charge $1/2$ Cited . How $D_{\min}(N, N)$ changes with N is not known.

05

Statement — θ → 1 as β → ∞

Statement (the limit of the vortex ratio)

With $N, M \geq 1$ held fixed, $\theta_{N,M}(\beta) \to 1$ as $\beta \to \infty$. Moreover, for all $\beta \geq 0$,

$$\theta_{N,M}(\beta) \;\geq\; 1 - \frac{2}{v}\, e^{-D_{\min}\beta/2} \quad (7)$$

where $v > 0$ is the Haar volume of a certain open set containing the twisted vacuum in the background with one twist (it does not depend on β, depends on N, M, and has no explicit value).

Lean From the lower bound on the deficit and the positivity of Haar measure on open sets. Lean: `ThetaLimit.tendsto_theta_one`, `theta_lower`, `Z11_le`, `Z10_ge`,

`volV_pos`, `ratio_tendsto_zero`.

Statement (the slope of the flux free energy)

With $N, M \geq 1$ held fixed,

$$\frac{\Delta F}{\beta} = \frac{1}{\beta} \ln \frac{Z_0}{Z_1} \longrightarrow D_{\min}(N, M) \qquad (\beta \to \infty). \tag{8}$$

In a symmetric box, this limit is at least $1/2$, independent of N.

Lean Estimates from both sides using the same parts. Lean: `ThetaLimit.tendsto_logratio_div`, `TwistFloor.logratio_limit_half`.

Meaning. With the lattice held fixed and the coupling made weaker, the sector with two twists loses weight exponentially by the amount of the deficit, and the vortex ratio approaches 1. The slope of that exponent is exactly the minimal deficit. This is $\beta \to \infty$ with the lattice held fixed, not the continuum limit (the direction in which κ_0 is held fixed and the lattice is made finer).

06

Statement — the largest matching solution

Statement (matching the vortex ratio)

Fix $N, M \geq 1$. For every κ_0 with $0 \leq \kappa_0 < 1$ there is a $\beta \geq 0$ with $\theta_{N,M}(\beta) = \kappa_0$, and the set of these solutions has a largest element $\beta^*(\kappa_0; N, M)$ within $\beta \geq 0$. If $\kappa_0 > 0$, then $\beta^* > 0$.

Lean Continuity, the intermediate value theorem and compactness. Lean: `ThetaLimit.tuning_all`, `tuning_all_pos` (the only assumption is $0 \leq \kappa_0 < 1$). An example where the assumptions can be met: for $N = M = 2$, a positive solution for $\kappa_0 = 1/2$ (`ThetaLimit.sanity_half`).

As a weak form there is also the statement that if $0 \leq \kappa_0 \leq \theta(\beta_1)$, there are a solution and a largest solution in $[0, \beta_1]$ (`ThetaTuning.tuning_exists`, `tuning_max_exists`). By § 05 such a β_1 can be taken for every $\kappa_0 < 1$, which gives the full form above (`ThetaLimit.exists_theta_gt`).

The lower envelope from the right. Whether θ is monotone in β is not known. We therefore use $\theta(\beta) := \inf_{\beta' \geq \beta} \theta(\beta')$. θ is increasing; at the largest solution β^*, $\theta(\beta^*) = \kappa_0$; for $\beta > \beta^*$, $\theta(\beta) > \kappa_0$; and β^* is also the largest among the solutions of $\theta = \kappa_0$ (`ThetaLimit.thetaLow`, `thetaLow_mono`, `envelope_max_sol`). Even for a θ that is not monotone, the largest solution is characterised by this monotone function. We do not claim continuity of θ.

What the matching is for. The matching in the statements to be shown (19-07) is a statement about how to fix the bare coupling along a sequence of ever finer lattices. The matching in this chapter is the part, lattice by lattice, of one candidate for it, "on each lattice, match the vortex ratio to a chosen value κ_0": the existence on each lattice. What holds as a sequence (how β^* depends on N) is not said.

07

Route of the proof

No flat configuration (§ 03). In two steps.

1. **Discrete Stokes.** Suppose quaternion fields A, B on an additive group G satisfy at each point the exchange relation $A(x)B(x+e) = s(x)\, B(x)A(x+f)$ ($s(x)$ a sign), and $Le = 0$, $L'f = 0$. Then the product along a line going once around in direction e and the product along a line going once around in direction f commute up to the product of all the s on the cut (Lean `FlatStokes.torus_commutator`). This is shown by a two-stage induction, first column by column, then sheet by sheet. For a configuration with deficit 0, $U_p = z_p$ on each plaquette is precisely this exchange relation.
2. **A contradiction in the quaternions.** For the four loops W_1, W_2 (transverse) and W_3, W_4 (longitudinal) through the origin, the product of the signs on the cut is -1 for the cuts in directions 1, 2 and in directions 3, 4, and $+1$ for the cuts in directions 1, 3 and 2, 3. Hence $W_1W_2 = -W_2W_1$, $W_3W_4 = -W_4W_3$, and W_3 commutes with both W_1 and W_2. Only real numbers commute with an anticommuting pair (W_1, W_2) of unit quaternions, so $W_3 = \pm 1$, which contradicts $W_3W_4 = -W_4W_3$ (`FlatStokes.no_twisted_quadruple`). Only four pairs of directions are used.

The lower bound on the deficit (§ 04). Extend (6) to a quantitative form. If the fields are unit-valued and the failure of the exchange relation is at most $d(x)$ at each point, then the failure of the loops to commute is at most the sum of d over the cut (`FlatStokes.approx_torus_commutator`; each sheet peeled off adds one error). The failure on a plaquette is $\|U_p - z_p\| = \sqrt{2(1 - z_p \operatorname{Re} U_p)}$. Applying (6) to the four loops at each base point, summing over the base points, recounting by translation and using

Cauchy–Schwarz twice gives (5). When $N \neq M$, the weights of the cuts in directions 1, 2 (N^2 sheets) and in directions 3, 4 (M^2 sheets) do not balance, and the bound becomes $2/(\rho + 1/\rho)^2$ in the ratio $\rho = N/M$. We take its approach to 0 as $\rho \to 0$ or $\rho \to \infty$ to be a weakness of the method Assumption . The only grounds are small examples: on the degenerate lattice $N = 1$, the minimal deficit for $M = 1, 2, 3$ stays numerically at 2.000 and does not drop Reading .

$\theta \to 1$ **(§ 05).** For $\beta \geq 0$, action = number of plaquettes $-D$, so $Z_1 \leq e^{\beta(|P| - D_{\min})}$ ($|P|$ is the number of plaquettes). On the other hand, the action in the background with one twist is continuous and takes its maximum $|P|$ at the twisted vacuum, so the set of configurations whose action exceeds $|P| - D_{\min}/2$ is an open set containing the twisted vacuum. The product of Haar measures puts positive weight on non-empty open sets, so its volume v is positive, and $Z_0 \geq v\, e^{\beta(|P| - D_{\min}/2)}$. Taking the ratio, $Z_1/Z_0 \leq v^{-1} e^{-D_{\min}\beta/2}$, and $\theta = 1 - 2Z_1/(Z_0 + Z_1) \geq 1 - 2Z_1/Z_0$ gives (7). (8) uses the same parts from above and below, with an open set around the minimiser. No precise Laplace-type estimate is needed.

Matching (§ 06). Since $\theta(0) = 0$, θ is continuous and $\theta \to 1$, the intermediate value theorem gives a solution. The set of solutions is closed, and there are no solutions beyond some b_0 ($\theta > \kappa_0$), so as a bounded closed set it has a largest element.

08

As questions — how things map when the lattice is doubled

What follows is not a theorem; these are questions lying beyond the theorems of this chapter.

- **How doubling maps.** With the coupling matched to $\theta = \kappa_0$ in a symmetric box of side N (the largest solution $\beta^*(\kappa_0; N)$), measuring the vortex ratio of a box of side $2N$ determines a value $\Sigma(\kappa_0; N) := \theta_{2N}(\beta^*(\kappa_0; N))$. Whether $\Sigma(\cdot; N)$ approaches a single function as $N \to \infty$, and if so how that function behaves between 0 and 1, remains a question. For two-dimensional SU(2), the corresponding function can be written by a formula on the continuum side, and it behaves like κ_0^4 for small κ_0 Reading (a formula using Bessel coefficients, and numbers; no proof is given). The Monte Carlo numbers on small four-dimensional lattices have large errors and are not at a stage where anything can be said.
- **The limit of the fixed-lattice direction.** What § 05 gives is only that the slope of ΔF as $\beta \to \infty$ with the lattice held fixed is $D_{\min}(N, N)$, and that this is at least $1/2$. If $D_{\min}(N, N)$ and $D_{\min}(2N, 2N)$ differ, no statement bounding $\Delta F_N - \Delta F_{2N}$ uniformly can be made in the fixed-lattice form. Even if they are equal, a

comparison of the terms slower than linear in β remains. The cooling numbers are all upper bounds, so whether they differ is not known.

- **Relation to the band.** The band statement (19-07; the correct form is in 26-01) says that the vortex ratio becomes exponentially small in the longitudinal area, uniformly in the lattice spacing and the size of the box. The theorems of this chapter are all statements with the lattice held fixed, and do not reach the band statement.

09

What is said, and what is not

- **Statements with the lattice held fixed.** v, $D_{\min}$, β^* and the speed of convergence in (7) depend on N, M. What happens to θ along sequences in which N and β move together is not said.
- **The shape of θ.** Monotonicity, $\theta > 0$ for $\beta > 0$, uniqueness of the matching solution, continuity of β^* in κ_0, and that the largest solution for $\kappa_0 = 0$ is 0, are not said.
- **The value of the deficit.** The true value of $D_{\min}$, its comparison with π^2, and its dependence on N are not said. The lower bound $1/2$ is a coarse uniform version. For $N \neq M$, the lower bound approaches 0 when the aspect ratio becomes lopsided.
- **Reaches neither the band nor the mass gap.** The lattice spacing is fixed, and the continuum limit is not treated.
- **Not a new result.** That a twisted box has no flat connection, the lower bound on its action, and the behaviour of the vortex free energy at weak coupling are a known picture; this chapter is its formalisation on the lattice.

The scope of what this chapter says

Said	Not said
Lean Continuity and $\theta(0) = 0$, $-1 < \theta \leq 1$. Lean `ThetaTuning.ZW_continuous`, `thetaM_continuous`, `ZW_zero`, `thetaM_zero`; `ThetaLimit.theta_gt_neg_one`, `theta_le_one`	Monotonicity; $\theta > 0$ for $\beta > 0$
Lean The background $(1, 1)$ has no flat configuration (all $N, M \geq 1$, no assumptions). Lean `TwistFlat.no_flat_twist11`. Contrast: `ThetaLimit.flat_exists_10`	—

Said	Not said
Lean The lower bound $2N^2M^2/(N^2+M^2)^2$ on the deficit, and $1/2$ in a symmetric box. The minimum is positive and attained. Lean `TwistFloor.defect11_ge_two`, `defect11_sym_half`, `defect11_uniform_half`, `Smin_sym_half`; `TwistFlat.defect11_floor`, `minDefect11_pos`; parts `QuadFloor.quad_floor_two`, `FlatStokes.approx_torus_commutator`	The true value of $D_{\min}$ and its dependence on N; comparison with π^2
Lean $\theta \to 1$, the lower bound (7), $\Delta F/\beta \to D_{\min}$ (limit $\geq 1/2$ in a symmetric box). Lean `ThetaLimit.tendsto_theta_one`, `theta_lower`, `tendsto_logratio_div`; `TwistFloor.logratio_limit_half`	That the speed is uniform in N; uniform estimates at finite β
Lean A largest solution β^* for every $\kappa_0 \in [0,1)$; $\beta^* > 0$ for $\kappa_0 > 0$; a characterisation by the lower envelope from the right. Lean `ThetaLimit.tuning_all`, `tuning_all_pos`, `envelope_max_sol`	Uniqueness; continuity in κ_0; dependence of β^* on N; properties of matched sequences
Reading Numerical values of the minimal deficit (2.000; upper bounds 7.84 and 9.80), and the behaviour of the doubling function in two dimensions	Not proofs of upper or lower bounds. The doubling function in four dimensions

Sources and reproduction

Item	Kind	Source / tool
Twisted boundary conditions; flat connections and the lower bound on the action in a twisted box	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141. G. 't Hooft, Commun. Math. Phys. 81 (1981) 267 (bibliography only; the text not checked). The introduction is in 25-01
Continuity in § 02, and the weak form of matching	Lean	`ThetaTuning.ZW_continuous`, `thetaM_continuous`, `ZW_zero`, `thetaM_zero`, `tuning_exists`, `tuning_max_exists`
The statements and parts of § 03–§ 06	Lean	`TwistFlat.no_flat_twist11`, `defect11`, `defect11_floor`, `minDefect11_pos`; `FlatStokes.torus_commutator`, `no_twisted_quadruple`, `approx_torus_commutator`; `QuadFloor.quad_floor_two`; `TwistFloor.defect11_ge_two`, `defect11_sym_half`, `defect11_uniform_half`, `Smin_sym_half`, `logratio_limit_half`, `no_floor_10`; `ThetaLimit.Smin`, `Z11_le`, `Z10_ge`, `volV_pos`, `theta_lower`, `tendsto_theta_one`, `exists_theta_gt`, `tuning_all`, `tuning_all_pos`, `sanity_half`, `ratio_tendsto_zero`, `flat_exists_10`, `phi_theta_eq`, `tendsto_logratio_div`, `thetaLow`, `thetaLow_mono`, `envelope_max_sol`. sorry 0, the three standard axioms, confirmed by re-checking with the kernel, statements checked against the wording of the source
$\theta \geq 0$ (M even)	Lean	`TwistBlocks.theta_nonneg` (21-06)

Item	Kind	Source / tool
Numerical values of the minimal deficit, upper bounds from cooling, the doubling function in two dimensions	Reading	The numerical record of this book's study (quaternion minimisation, cooling, numerical values of the formula with Bessel coefficients; the cooling values are upper bounds)

Terms used in this chapter — vortex ratio, flux free energy, 't Hooft twist, deficit, flat configuration, twisted vacuum, discrete Stokes, anticommutation, positivity of Haar measure on open sets, matching, largest solution, lower envelope from the right. Their meanings are in A-02 Glossary.

Next chapter to read: 21-08 What this part says, and what it does not — the theorems of Part 21 in one table. The vortex floor in the strong-coupling window, with a rate independent of the transverse size (the prefactor N^2 remains), is in 20-06.

What this part says, and what it does not

> This chapter is this book's own investigation. The table of § 01 is a list of the theorems of Part 21.

What this chapter shows — the theorems of Part 21 in a single table, with their scope, what they do not say and their Lean declarations. After that, a short look at how far the floors of this part are from the band.

Prerequisite chapters — 21-00 Entrance and each chapter of Part 21. The definitions of the objects are in 19-02

Said first — apart from the first, abstract implication, every theorem of this part is an equality or inequality at a fixed lattice spacing on a finite lattice of transverse size N and longitudinal size M. Inside the strong-coupling window there is a form whose rate does not depend on the transverse size (the prefactor, the transverse area, remains; 20-06). Outside the window, none of the rates of the floors of this part (upper bounds on θ) is uniform in the transverse size N. The lower bound on the deficit in 21-07 (1/2 in a symmetric box, independent of the transverse size) is a statement in the opposite direction from a floor (on the side of lower bounds on θ). Nothing is said about the continuum limit, four-dimensional confinement or the mass gap.

The order of this chapter

1. The list of theorems — scope and what is not said
2. What depends on the width N
3. The distance from the floor to the band

01

The list of theorems — scope and what is not said

The object throughout is the SU(2) Wilson measure on the torus with transverse part $(\mathbb{Z}/N)^2$ and longitudinal part $(\mathbb{Z}/M)^2$, with background twist $n_{12} = 1$ (19-02). We write

$\theta_M = (Z_0 - Z_1)/(Z_0 + Z_1)$ and $t = \tanh(N^2|\beta|)$. Each is a formalisation of a result or picture of known type (the Cited rows of each chapter).

Only the first row is an abstract implication with assumption fields. The other rows are theorems about concrete objects, whose assumption fields are only the ranges written in the table.

Theorem	Scope	What is not said	Lean declarations
Three results on sign systems (the covering inequality, $\mathbb{Z}_2$ monotonicity, comparison with $\mathbb{Z}_2$)	Abstract groups and sign characters. Implications with a column of assumptions	On their own they are not estimates for a concrete measure (the application is inside the next row)	`CoveringInequality.covering_ineq`, `mon_ineq`, `dominated_covering_ineq`, `invariant_dominated`
The thin-vortex floor (ratio form)	All real β, $N, M \geq 1$, any background twist: `twisted_vortex_bound` is the signed form $(Z_0 - Z_1)/(Z_0 + Z_1) \leq t^{M^2}$, and `_abs` the absolute-value form $\lvert Z_0 - Z_1\rvert/(Z_0 + Z_1) \leq t^{M^2}$	The sign ($Z_1 \leq Z_0$ is a separate row). A rate uniform in N	`TwistedWilson.twisted_vortex_bound`, `twisted_vortex_bound_abs`
The thin-vortex floor (exponential form)	$\beta \neq 0$: $\sigma_0 = -\ln t > 0$, $\lvert\theta_M\rvert \leq e^{-\sigma_0 M^2}$	$\sigma_0 \to 0$ ($N \to \infty$)	`TwistedWilson.thin_vortex_floor`
The block estimate	$mk \leq M$, all real β: $\lvert\theta_M\rvert \leq \tanh(K_{\text{eff}})^{k^2}$	When blocks of different shapes are mixed, it is weaker than the product over blocks	`TwistBlocks.block_bound_abs`
The thick-vortex floor	All $N \geq 1$, any $\varepsilon > 0$, for every sufficiently large β and sufficiently large M: $\lvert\theta_{M+1}\rvert \leq \exp(-(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	The values of the thresholds in β and M. The value of S_N. The opposite direction (a lower bound on the rate)	`TwistBlocks.thick_vortex_floor`

Theorem	Scope	What is not said	Lean declarations
Positivity of the minimal action of a vortex	All $N \geq 1$: $\delta \leq S_{\mathrm{v}}(N, m)$ with a $\delta > 0$ uniform in m, and $S_N > 0$	The number δ (not explicit). A lower bound uniform in N	`VortexPos.Svortex_ge_unif`, `VortexPos.SN_pos`
Certificates for the upper bounds Computer-assisted	$S_{\mathrm{v}}(2,8) \leq 6.67401$, $S_{\mathrm{v}}(3,8) \leq 9.09321$, $S_{\mathrm{v}}(4,4) \leq 12.02057$, hence $S_N < 2N^2$ $(N = 2,3,4)$	The value of S_N (only upper bounds, from a single configuration). General N	`VortexCert.Svortex_two_eight_le`, `Svortex_three_eight_le`, `Svortex_four_four_le`, `SN_two_lt`, `SN_three_lt`, `SN_four_lt` (soundness: `VortexCert.Svortex_le_of_cert`)
The thick-vortex floor with explicit exponents	$N = 2,3,4$, for every sufficiently large β and sufficiently large M: exponent numbers 6.675, 9.094, 12.021	The values of the thresholds. "Stronger than the thin-vortex floor" is a comparison of rates and not a Lean theorem (Reading from $\beta > 0.523$ for $N = 2$)	`TwistBlocks.thick_vortex_floor_two`, `_three`, `_four`
The sign of the vortex ratio	M even, all real β: $Z_1 \leq Z_0$, $\theta_M \geq 0$	M odd (Reading examples where it fails for $\beta < 0$)	`HalfSplit.Z1_le_Z0`, `TwistBlocks.theta_nonneg`
The signed thick-vortex floor	In the range of the thick-vortex floor, with $M+1$ even: $0 \leq \theta_{M+1} \leq \exp(-(M+1)2_e{-}\beta(S_N+\varepsilon))$	The values of the thresholds	`TwistBlocks.thick_vortex_floor_signed`
Continuity and matching of the vortex ratio	With the size of the lattice held fixed, for all real couplings: the vortex ratio is continuous in the coupling, is 0 at coupling 0, and approaches 1 as the coupling grows. For every target value at least 0 and less than 1 there is a coupling matched to it (the largest solution)	Monotonicity, uniqueness of the solution, sequences in which the transverse size and the coupling move together	`ThetaTuning.thetaM_continuous`, `ThetaLimit.tendsto_theta_one`, `ThetaLimit.tuning_all`

Theorem	Scope	What is not said	Lean declarations
A lower bound on the deficit of a twisted box	A box with twists in two orthogonal planes has no flat configuration. The deficit of a symmetric box is at least 1/2, independent of the transverse size	The true value of the minimal deficit	`TwistFlat.no_flat_twist11`, `TwistFloor.defect11_sym_half`
Estimate of pairs of Polyakov loops by θ	M even, all real β, all b, b': $\lvert\langle P_b(0)P_{b'}(M/2)\rangle_0\rvert \le 2\sqrt{\theta}/(1+\theta)$	Only the distance $M/2$	`HalfSplit.polyakov_pair_theta`
Pairs of Polyakov loops at a general time separation	$M \equiv 0 \bmod 4$, all real β, $0 \le T \le M$: $\lvert\langle P_b(0)P_{b'}(T)\rangle_0\rvert \le 2t^{M\min(T,M-T)}$	A Hölder form (weaker by the constant 2 and by replacing θ with t^{M^2}). M odd. Linearity; an upper bound	`PolyakovT.polyakov_pair_all`
The same (other parities)	M even ≥ 4 with $\beta \ge 0$ / N, M even ($M \ge 4$) with all β	M odd. $M = 2$ (only $T = M/2$, in the next row). $M \equiv 2 \bmod 4$ with N odd and $\beta < 0$	`PolyakovT.polyakov_pair_evenM`, `polyakov_pair_evenNM`
Pairs at distance $M/2$ and the thin-vortex floor	M even (including $M = 2$), all real β: $\lvert\langle P_b(0)P_{b'}(M/2)\rangle_0\rvert \le 2t^{M\cdot M/2}$	Other time separations (the two rows above)	`PolyakovT.half_pair_tanh`
Pairs of Polyakov loops combined with the thick-vortex floor	For every sufficiently large β and sufficiently large M, $M+1$ even: the pair at distance $(M+1)/2$ is $\le 2\exp(-\frac{1}{2}(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	The values of the thresholds. Four-dimensional confinement	`HalfSplit.polyakov_pair_floor`

Common to all rows: the lattice spacing is fixed, and none of the continuum limit, the four-dimensional area law or the mass gap is said. The Polyakov-loop estimates are linear **lower bounds** on the energy inside a transverse $N \times N$ tube.

```
02
```

What depends on the width N

Every theorem of this part has the form of holding N fixed and moving M and β. We list how each depends on N.

Quantity	Dependence on N	What is known
The rate of the thin-vortex floor $\sigma_0 = -\ln\tanh(N^2\lvert\beta\rvert)$	Proved here $\approx 2e^{-2N^2\lvert\beta\rvert}$, 0 as $N \to \infty$	Explicit
The exponent of the thick-vortex floor S_N	$0 < S_N \leq 2N^2$ (`VortexPos.SN_pos`, `TwistBlocks.SN_bounds`)	Value unknown. Upper bounds only for $N = 2, 3, 4$
The thresholds of the thick-vortex floor $\beta_0(N)$, $M_0(\beta, N)$	Unknown	Not explicit
The positivity lower bound $\delta(N)$	Unknown	Not explicit
The rate of the vortex floor at strong coupling (20-06; a theorem of Part 20)	Independent of it (only inside the strong-coupling window; the transverse area in the prefactor remains)	Explicit. Nothing is said outside the window
The lower bound on the tension inside a tube ((13) of 21-06)	Same as σ_0; goes to 0 as $N \to \infty$	Explicit
The minimal deficit $D_{\min}(N, N)$ of a twisted box (21-07; not a floor, but on the side of lower bounds on θ)	The lower bound $1/2$ does not depend on N. v and β^* depend on N	The lower bound is explicit. The true value is unknown

Outside the strong-coupling window, the explicit quantities of the floors go to 0 with N. For the rate coming from S_N, the dependence on N is unknown, and the thresholds are not explicit either.

03

The distance from the floor to the band

The floors of this part are **upper bounds** on θ, for each width N. They bound the slab tension $\sigma_N(\beta) = \liminf_M -\frac{1}{M^2}\ln\lvert\theta_M\rvert$ (21-03) from below, for each N.

The statement of the band is in 19-07. Assumption For a sequence of bare couplings β_n fixed by matching, there is $c > 0$ such that for every L_0, for all sufficiently large n, $\theta_{2^n L_0, 2^n L_0}(\beta_n) \leq \exp(-c\,L_0^2)$. It is the form that refines the transverse and longitudinal sizes together, moves the coupling as well, and bounds the vortex ratio exponentially in the area with a single constant c.

The floors of this part concern holding the lattice spacing and the transverse size N fixed and moving the longitudinal size M, and do not reach this form. To state the

band, one needs estimates uniform in the transverse and longitudinal sizes and in the coupling. As the table of § 02 shows, every floor theorem of this part is for each N separately and contains no such uniformity (the lower bound on the deficit in 21-07 does not depend on the transverse size, but it is a statement on the side of lower bounds on θ, not a floor). The form in 20-06 inside the strong-coupling window has a rate independent of the transverse size, but since the coupling stays inside the window, it does not reach matched sequences on the side where the coupling grows large.

In this book, this uniform estimate is placed on the side of the premises. The list of premises and the form of the implications are in Part 28.

What this chapter can say

Can say	Cannot say
Each row of the table of § 01 (machine-checked in Lean; the first row is an abstract implication)	The "what is not said" column of each row
Proved here The rate of the thin-vortex floor and the lower bound on the tension inside a tube go to 0 as $N \to \infty$	Estimates uniform in N and in the coupling (the statement of the band, 19-07)

Sources and reproduction

Item	Kind	Source / tool
The theorems of each row	—	The "Sources and reproduction" table of each chapter (21-01–21-07). Lean declarations in the table of § 01
$\sigma_0 \approx 2e^{-2N^2\lvert\beta\rvert}$	Proved here	$\tanh x = 1 - 2e^{-2x} + O(e^{-4x})$ and $-\ln(1-y) = y + O(y^2)$

Words used in this chapter — vortex floor, vortex ratio θ_M, minimal action of a vortex S_N, Polyakov loop, band, estimate uniform in the width. Their meanings are in A-02 Glossary.

Next chapter to read: 22-00 Entrance — the rarity of large plaquettes — on to the theorem on rarity at one scale, on the weak-coupling side.

Entrance — the rarity of large plaquettes

> This chapter is a guide.

What this chapter shows — The list of theorems of Part 22, the scope that applies to the whole part, and the idea of the tool, the Chessboard Estimate. We also set side by side how the published construction uses the same Chessboard Estimate (as a seminorm measuring the size of observables) and how this part uses it (as an estimate of the probability of rare events).

Prerequisite chapters — 19-01 the Wilson measure, 20-01 Reflection positivity, 26-05 the Chessboard Estimate and simultaneous flips.

Said first — The theorems of this part are statements at one scale, about the Wilson measure itself on a fine lattice at a fixed lattice spacing. They are uniform in the volume (the size of the torus), but say nothing about the measure at a coarse scale after renormalisation. The threshold $\beta \geq 81$ is a value chosen for the convenience of the proof, far from the band of physical interest. Each is a formalisation of a known type, the chessboard Peierls argument from reflection positivity Cited .

The order of this chapter

1. List of the theorems of this part — Table 1
2. The Chessboard Estimate as a tool — uniform patterns and the geometric mean
3. Two uses — measuring size, or measuring rarity — the difference from the published construction
4. The scope that applies to the whole part
5. Order of reading

Labels — Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment (details in "Labels" in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are the labels of the whole book; Reading is a numerical reading of this book, and Lean a Lean theorem of this book.

01

List of the theorems of this part

Measure the "deviation" of a plaquette p by $1 - \operatorname{Re} U_p \in [0, 2]$, and call a plaquette exceeding a threshold y a **large plaquette** (for $y = 1$, $\operatorname{Re} U_p < 0$). The statement of this part is that the probability of configurations in which many large plaquettes appear at once is exponentially small in the number of plaquettes, with a rate that does not depend on the size of the torus.

Theorem	Scope	Gist of the statement	Lean
The Chessboard Estimate	Side $N = 2m$ ($m \geq 1$), all real β	The expectation of a product of functions placed block by block is at most the geometric mean of the expectations of the uniform patterns	`FoldWilson.wilson_chessboard_even`
Rarity uniform in the volume	$\beta \geq 81$, $m \geq 2$	The probability that $\operatorname{Re} U_p < 0$ on all of a set X of plaquettes is $\leq q_1^{\lfloor \lvert X \rvert / 6 \rfloor}$, with $q_1 < 1$ independent of N and X	`FoldCap.rarity_explicit`
The form with rate $1/4$	$0 < y \leq 2$, at or above some β_0, $m \geq 1$	Probability $\leq q_4^{\lfloor \lvert X \rvert / 24 \rfloor}$	`FoldQuarter.rarity_uniform4`

Table 1. The theorems of Part 22. Statements, proofs and numbers are in 22-01; the scope and what is not said are in 22-02.

02

The Chessboard Estimate as a tool

The introduction is in 26-05 and 20-01. For a reflection-positive measure, the expectation of the product of a function with its mirror image is at least 0, and from this follows an inequality of Cauchy–Schwarz form, the "folding inequality". Repeating

it for every mirror, one divides the torus into blocks, and for the expectation of a product with a function f_w placed in each block,

$$\left|\left\langle \prod_{w} f_w \right\rangle\right| \;\le\; \prod_{w} \left\langle \prod_{w'} f_w^{(w')} \right\rangle^{1/|\mathrm{blocks}|} \quad (1)$$

holds (the Chessboard Estimate of Fröhlich–Israel–Lieb–Simon Cited). The $\prod_{w'} f_w^{(w')}$ on the right is the "uniform pattern" in which the same function f_w is carried by reflections to every block. The point is that any placement on the left is bounded by the geometric mean of the expectations of uniform patterns.

Each factor on the right is a quantity determined by a single function f,

$$\|f\|_{\mathrm{cb}} := \left\langle \prod_{w'} f^{(w')} \right\rangle^{1/|\mathrm{blocks}|} \quad (2)$$

and we call it the **chessboard root**. Since it is a quantity per block, it takes a form independent of the volume even as the volume grows.

03

Two uses — measuring size, or measuring rarity

How the published construction uses it. From the reflection positivity of the Wilson measure, the published construction derives the chessboard inequality (`Cells.Tiling.tiles` , `YM/Cells/Chess.lean:80`) and the triangle inequality for the chessboard root, $\|F+P\|_{\mathrm{cb}} \le \|F\|_{\mathrm{cb}} + \|P\|_{\mathrm{cb}}$ (`Cells.Tiling.adds` , same file `:108`) Source . The root `rootTest` (`YM/Cells/Tiling.lean:226`) is a quantity of the form (2) Source . Because of the triangle inequality, the root can be used like a seminorm on the space of observables Inferred .

With this seminorm, one of the estimates placed at the interface between the ultraviolet side and the infrared side is written. Measured by the chessboard root, renormalised local observables are at most $A \cdot a_m^{-d-\varepsilon}$, uniformly in all volumes (`Extraction.Wcmain` , `YM/Extraction/Center.lean:38`) Source . This becomes the input of local boundedness and equicontinuity for taking an infinite-volume subsequence (12-02, 12-03) Source . This is not the only input received from the ultraviolet side Source .

How this part uses it. This part uses the same Chessboard Estimate to estimate **the probability of rare events**. If the function placed is the indicator of a plaquette, "this plaquette is large", the uniform pattern becomes the event "that plaquette is large at every location". Its weight is exponentially small compared with the partition function, because the action loses heavily (energy), and the partition function is bounded below

by the weight of configurations with every link near the identity (entropy). The $1/|\text{blocks}|$-th power of the ratio becomes the rate per plaquette. This is the chessboard Peierls argument Cited .

	Published construction	This part
Function placed	Renormalised local observables	Indicator that a plaquette is large
What the root measures	The size of observables (of order $a_m^{-d-\varepsilon}$)	The rarity of events (rate per plaquette < 1)
Where it is used	Local boundedness and equicontinuity → subsequential limits	That configurations with many large plaquettes are rare
Mirrors	For the original Wilson measure, planes through lattice sites and planes through midpoints of edges, four axes, $\beta \geq 0$ Source	The same two kinds (the middle plane for gauge-invariant functions, $\beta \geq 0$, $m \geq 2$)
Triangle inequality	Present Source	Not used

Table 2. Two uses of the Chessboard Estimate.

In the language of the root, the theorems of this part can presumably be restated in the form "the root of the indicator that a plaquette of one orientation is large is at most $q < 1$" Inferred (steps 2–5 of the proof in 22-01 bound this quantity). The published construction does not use the chessboard to estimate the probability of sets of large plaquettes Inferred (within the range of a search by name). Large fields are handled by another mechanism within each step of renormalisation (22-02).

04

The scope that applies to the whole part

- **One scale.** These are statements about the Wilson measure itself on a fine lattice at a fixed lattice spacing. Nothing is said about the measure at a coarse scale pushed forward by the block map (19-04).
- **Uniform in the volume, not uniform in the lattice spacing.** The rate does not depend on N, but β is the coupling of a single lattice and has nothing to do with sequences sending the lattice spacing to 0.
- **The weak-coupling side, with an explicit threshold.** $\beta \geq 81$ ($m \geq 2$). For the form with rate $1/4$, only the existence of the threshold is given. Nothing is said about the band of physical interest (β around 2 to 3).
- **Even sides, no twist.** Only even sides, and only the measure without a background twist.

- **A formalisation of a known type.** A formalisation of the chessboard Peierls argument (of Fröhlich–Israel–Lieb–Simon type) for the SU(2) Wilson measure. No novelty is claimed.

05

Order of reading

22-01 (statements, routes of proof, numbers) → 22-02 (what is said and what is not, and the relation to how the published construction handles large fields). How this rarity was planned to be used on the side of the premises is in 28-04.

The scope of what this chapter says

Said	Not said
Lean The three theorems of Table 1 (details in 22-01)	Coarse scales, $\beta < 81$, odd sides, measures with a twist
Source The published construction has the chessboard inequality and the triangle inequality for the root, and uses an estimate of the size of observables measured by the root as an input to the infinite-volume limit	Where that estimate enters the dependency chain of the published theorems (read in 12-03)
Inferred The published construction does not use the chessboard to estimate the probability of large plaquettes. The theorems of this part can presumably be restated in the language of the root	A Lean form of the restatement

Sources and reproduction

Item	Kind	Source / tool
The Chessboard Estimate and the chessboard Peierls argument	Cited	The standard argument of J. Fröhlich, R. Israel, E. H. Lieb, B. Simon (1978). Checked against the summary in M. Biskup, arXiv:math-ph/0610025 (the text of the original paper not obtained). Same as 22-01
The chessboard and the root in the published construction	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Cells/Chess.lean:80` `tiles`, `:108` `adds`; `YM/Cells/Tiling.lean:226` `rootTest`; `YM/Extraction/Center.lean:38` `Wcmain`
The theorems of Table 1	Lean	Lean declarations of this book's study (names in Table 1; details of the sources in 22-01)

Terms used in this chapter — large plaquette, Chessboard Estimate, uniform pattern, chessboard root, seminorm, Peierls argument, reflection positivity, uniform in the volume. Their meanings are in A-02 Glossary.

Next chapter to read: 22-01 Rarity of large plaquettes — the chessboard estimate — the statements and routes of proof of the three theorems of this part.

Rarity of large plaquettes — the chessboard estimate

This chapter is this book's own investigation. § 01 is established knowledge.

Scope — The theorems of this chapter are statements about the SU(2) Wilson measure itself at one scale (the fine lattice with the lattice spacing fixed). They say nothing about the blocked measure at a coarse scale, about fields with a background (such as twists), or about estimates uniform across steps. They are uniform in the volume (the size of the torus), but not in the lattice spacing. A summary of the scope of the whole part and of what is not said is in 22-02.

What this chapter shows — For the Wilson measure on the torus $(\mathbb{Z}/N)^4$ with even side $N = 2m$, we read three things: (1) a chessboard estimate for the expectation of products of functions of the links of a unit cube placed block by block (all β, all $m \geq 1$); (2) for $\beta \geq 81$ and $m \geq 2$, the probability that every plaquette of a set X of plaquettes has $\operatorname{Re} U_p < 0$ is $\leq q_1^{\lfloor |X|/6 \rfloor}$, with $q_1 < 1$ independent of N and X; (3) a form with rate $1/4$ that includes $m = 1$.

Prerequisite chapters — 22-00 Entrance (the list of theorems of this part, and the scope that applies to the whole part); 19-01 The Wilson measure, plaquettes and holonomy, the product of Haar measures; 20-01 Reflection positivity; 26-05 § The Chessboard Estimate.

Said first — The scope is as written directly under the title. There are three further points. (1) The q_1 of (4) is written with the true volume $h(1/20)$ of the cap, but what Lean pins down as a number goes only as far as $h(1/20) \geq 83^{-4}$, and the upper bound $e^{-0.88\beta} 83^{16}$ on q_1 is about $2.5 \cdot 10^{12}$ times the true q_1 Reading . The threshold 81 is determined by this upper bound; with the true h it would be about 48 Reading . (2) The threshold $\beta \geq 81$ is a value of convenience for the proof, far from the β of physical interest (the band 2–3). (3) The probability is a ratio to the partition function Z, not a ratio to any other quantity (such as a twisted partition function or a partition function restricted to small fields). Every theorem is a formalisation of a known type, the chessboard Peierls argument from reflection positivity Cited .

The order of this chapter

1. The chessboard and the Peierls argument — spreading an event into a uniform pattern by reflections, and comparing energy with entropy
2. Setting — large plaquettes and caps — the event $\mathrm{Re}\, U_p < 1 - y$, the volume $h(r)$ of a cap
3. The chessboard estimate — every even side, every β
4. Rarity uniform in the volume — $\beta \geq 81 - q_1^{\lfloor |X|/6 \rfloor}$, $q_1 < 1$
5. The route of the proof of rarity — reflection positivity → uniform pattern → an upper bound on the action and a lower bound on the partition function → pigeonhole
6. Detour: the bookkeeping behind $48r^2$ and $h(r)^4$ — the numbers of plaquettes and links per lattice site
7. The form with rate $1/4$ — including $m = 1$ — the threshold exists, nothing more
8. Strength — in numbers — the lower bound on the volume of the cap and its true value, the values of the thresholds
9. What is not said — coarse scales, backgrounds, $\beta < 81$, other ratios

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01

The chessboard and the Peierls argument

The introduction is 26-05 § The chessboard estimate and 20-01. For a reflection-positive measure, the expectation of a product of functions placed block by block is bounded by the geometric mean of the expectations of the "uniform patterns" in which each function is carried by reflections to all the blocks (the chessboard estimate, Fröhlich–Israel–Lieb–Simon)Cited .

Using this to estimate rare events is the chessboard Peierls argumentCited . When the event to be estimated (a plaquette at some place is large) is carried by reflections to all the blocks, it becomes the event of a "uniform pattern" in which the plaquettes are large at every place. The weight of the uniform-pattern event loses a lot in the action, so it is exponentially small compared with the partition function Z. The $1/|\text{number of blocks}|$-th power of that ratio becomes the rate per plaquette. The key point of this argument is that the rate does not depend on the volume.

The theorems of this chapter formalise this argument for the SU(2) Wilson measure. They are not results of a new type.

02

Setting — large plaquettes and caps

The lattice is $(\mathbb{Z}/N)^4$, $N = 2m$. The measure is the Wilson measure of 19-01 (not normalised),

$$d\mu_W(U) = \prod_{\ell} dh(U_\ell)\ \exp\Big(\beta \sum_{p} \operatorname{Re} U_p\Big), \qquad Z = \mu_W(\text{whole space}), \tag{1}$$

where h is the Haar probability measure of $SU(2)$, and the sum over plaquettes is over the $6N^4$ unoriented plaquettes $(x; \mu < \nu)$. Lean: `TorusRP.wilsonMeasure`; Z is `FoldPeierls.Zw` (the total mass). This chapter numbers the directions 1 to 4 (the Lean indices 0 to 3 are directions 1 to 4; 19-01 § Numbering of directions).

Large plaquettes. For a threshold $y \in (0, 2]$, a plaquette p is called "large" when $1 - \operatorname{Re} U_p > y$. For $y = 1$ this is $\operatorname{Re} U_p < 0$. Lean: `FoldEntropy.bigP N y ω p` (an indicator function, strict inequality). $\operatorname{Re} U$ is $\frac{1}{2}\operatorname{tr} U$, and $1 - \operatorname{Re} U = |U - 1|^2/2$ (regarding U as a unit quaternion).

Caps. We write $h(r)$ for the Haar volume of the open cap $\{U \in SU(2) : |U - 1| < r\}$ around the identity ($|\cdot|$ is the quaternion norm). Lean: `FoldEntropy.ballQ r`; $h(r)$ is `haarBall.real (ballQ r)`.

For a set X of plaquettes, $\int \prod_{p \in X} \mathbf{1}[p \text{ is large}]\, d\mu_W / Z$ is the probability of the event "all plaquettes of X are large". No condition is put on plaquettes outside X.

03

The chessboard estimate

The blocks are unit cubes, one for each point w of $(\mathbb{Z}/N)^4$, N^4 in all. A bounded measurable function that depends only on the links inside the cube at the origin (links whose starting point has all coordinates 0 or 1 and whose coordinate in their own direction is 0) is called "a function of the block at the origin". Ψ_w is a map of configurations which, for each coordinate direction, composes a translation if the coordinate of w is even and a reflection if it is odd. For a function f of the block at the origin, $f \circ \Psi_w$ is the function carried to block w.

The chessboard estimate Lean: `FoldWilson.wilson_chessboard_even`. For $N = 2m$ ($m \geq 1$), every real β, a family (f_i) of functions of the block at the origin, and an assignment $F : (\mathbb{Z}/N)^4 \to I$,

$$V(F) = \int \prod_{w} f_{F(w)}\big(\Psi_w U\big)\, d\mu_W(U), \qquad |V(F)|^{N^4} \;\le\; \prod_{w} V\big(\text{the constant assignment } F(\cdot$$

Meaning. Whatever function is placed on whatever block, the expectation of the product is bounded by the geometric mean of the expectations of the uniform patterns in which each function is placed on all the blocks. The side need not be a power of 2. For $f \equiv 1$ both sides are Z^{N^4} and equality holds.

The route of the proof has three stages. (i) On a ring $\mathbb{Z}/2m$ in one direction, derive $|V(F)|^{2m} \le \prod_t V(\text{constant } F(t))$ from the "folding inequality" $V(F)^2 \le V(F^+)V(F^-)$ for reflections in all planes and from $V(\text{constant}) \ge 0$ (`FoldChess.chess_one`). To avoid being restricted to powers of 2, we use the argument of Fröhlich–Israel–Lieb–Simon, which takes the maximum of a ratio over a finite family of ranges and uses that folding preserves the maximum. In a model with a one-dimensional transfer kernel, the conclusion is not violated for reflection-positive kernels, and is violated for kernels that are not positive Reading . (ii) Apply it once in each of the four directions (`FoldChess.chess_four`). (iii) Derive the folding inequality from reflection positivity for planes through lattice sites in Cauchy–Schwarz form (20-01, every real β), and from the invariance of the measure under reflections and translations (`FoldChess.chess_data`; the concrete discharge is `FoldWilson.data`).

04

Rarity uniform in the volume — $\beta \ge 81$

Theorem of rarity uniform in the volume Lean: `FoldCap.rarity_explicit`. Set

$$q_1 = \frac{e^{\beta(48\cdot(1/20)^4 - 1)}}{h(1/20)^4} = \frac{e^{-0.88\,\beta}}{h(1/20)^4} \quad (3)$$

Then $q_1 < 1$ for every $\beta \ge 81$. Moreover, for every $m \ge 2$ ($N = 2m$) and every set X of unoriented plaquettes of $(\mathbb{Z}/N)^4$,

$$\int \prod_{p \in X} \mathbf{1}\big[\operatorname{Re} U_p < 0\big]\, d\mu_W \;\le\; Z \cdot q_1^{\lfloor |X|/6 \rfloor}. \quad (4)$$

The order of quantifiers is "for each β, there is a q_1 determined by β alone, and for all N and all X". q_1 depends neither on N nor on X.

Meaning. When the coupling is weak (β large), configurations in which many plaquettes are strongly twisted at once are exponentially unlikely in the number of twisted plaquettes, at a rate that does not depend on the size of the lattice. There is content from $|X| \ge 6$ onward (for $|X| < 6$ the right-hand side is Z). The event is not

empty. There is a configuration with $\mathrm{Re}\, U_p = -1$ on every plaquette (a fully frustrated $\mathbb{Z}_2$ configuration), and positive weight sits on an open set near it, so the left-hand side is positive for every X Proved here .

An intermediate form (also in Lean): for a general threshold $0 < y \le 2$, there exist $r > 0$ and β_0 (in the proof $r = y/20$) such that for $\beta \ge \beta_0$, $q1 = e^{\beta(48r^2 - y)}/h(r)^4 < 1$ and the form (4) holds (`FoldMix.rarity_uniform_all_mix`, $m \ge 2$). For plaquettes of a single orientation only, the rate is 1 per plaquette (`FoldMix.rarity_orient`).

05

The route of the proof of rarity

1. **Reflection positivity in four directions.** In directions 1 and 2 we use reflections in planes through lattice sites (every real β); in directions 3 and 4, reflections in planes through link midpoints (gauge-invariant functions, $\beta \ge 0$, $m \ge 2$) (20-01). Mixing in the midpoint planes makes the blocks lattice sites, and the carried plaquettes are distinct across all N^4 blocks. The indicator of a plaquette is gauge invariant ($\mathrm{Re}\, U_p$ is invariant). This builds the vessel of § 03 (`FoldMix.dataMix`). $m \ge 2$ is the condition of this stage.
2. **To a uniform pattern.** Place the block function "the plaquette of orientation $(1, 2)$ at the origin is large" on the blocks of a set S of lattice sites, place 1 elsewhere, and use (2): the probability is bounded by the $|S|/N^4$-th power of the weight of the event "all N^4 plaquettes of orientation $(1, 2)$ are large".
3. **The weight of the uniform pattern (energy).** On that event, $\mathrm{Re}\, U_p < 1 - y$ on N^4 plaquettes and $\mathrm{Re}\, U_p \le 1$ on the others, so the action is $\le 6N^4 - yN^4$ and the weight is $\le e^{\beta(6N^4 - yN^4)}$ (in the form of `FoldMix.action_le_mix`).
4. **A lower bound on the partition function (entropy).** If every link is placed in the cap $|U - 1| < r$, each plaquette is a product of four, so $|U_p - 1| \le 4r$ and hence $\mathrm{Re}\, U_p \ge 1 - 8r^2$. The Haar weight of this set of configurations is $h(r)^{4N^4}$, so $Z \ge e^{6N^4\beta(1-8r^2)} h(r)^{4N^4}$ (`FoldEntropy.Zw_ge`).
5. **Take the ratio.** Dividing 3 by 4 and taking the $1/N^4$-th power gives $q = e^{\beta(48r^2 - y)}/h(r)^4$ per plaquette.
6. **Align the orientations.** The measure is unchanged by involutions that map the six orientations to $(1, 2)$. A general X has at least $\lfloor |X|/6 \rfloor$ plaquettes of one of the six orientations (pigeonhole), and the indicator functions take values in $[0, 1]$, so we can drop to that subset (`FoldMix.rarity_all_mix`).

7. **An explicit threshold.** With $y = 1$ and $r = 1/20$, $48r^2 - 1 = -0.88$. The volume of the cap is bounded by $h(1/20) \geq 83^{-4}$ using a covering by the lattice $(1/41)\mathbb{Z}^4$ (`FoldCap.ballQ_ge`), and from $e^{71} > 2.718^{71} > 83^{16}$, $q_1 < 1$ for $\beta \geq 81$ (`FoldCap.q_lt_one_explicit`).

Detour: the bookkeeping behind $48r^2$ and $h(r)^4$

You may skip this. Per lattice site there are 6 plaquettes (the number of pairs of directions $\mu < \nu$) and 4 links. In the lower bound on the partition function, each plaquette loses at most $1 - \mathrm{Re}\, U_p \leq 8r^2$, so the loss of action per lattice site is $6 \cdot 8r^2 = 48r^2$, and the Haar weight is $h(r)^4$. On the side of the uniform-pattern event, one plaquette per lattice site loses at least y. Taking the ratio leaves $e^{\beta(48r^2 - y)}/h(r)^4$ per lattice site Proved here . Making r smaller makes $48r^2$ smaller, but $h(r)$ also shrinks at the rate r^3, so to get $q < 1$, β must be taken to be several times $\log(1/h(r))$. This is why the threshold is large.

06

The form with rate $1/4$ — including $m = 1$

Theorem of rarity with rate $1/4$ Lean: `FoldQuarter.rarity_uniform4`. For $0 < y \leq 2$, there exist $r > 0$ and β_0 such that for every $\beta \geq \beta_0$, $q_4 = e^{\beta(48r^2 - y/4)}/h(r)^4 < 1$, and for every $N = 2m$ ($m \geq 1$, in particular including $N = 2$) and every set X of plaquettes,

$$\int \prod_{p \in X} \mathbf{1}\left[1 - \mathrm{Re}\, U_p > y\right] d\mu_W \;\leq\; Z \cdot q_4^{\lfloor |X|/24 \rfloor}. \tag{5}$$

There are two differences. Since it is built only with reflections in planes through lattice sites, without those through link midpoints, the condition $m \geq 2$ is not needed; in exchange, the distinct plaquettes in a uniform pattern number $N^4/4$, and the rate becomes $1/4$. A general X is split into 24 classes (6 orientations $\times$ 4 parities of the outer directions) and pigeonhole is used.

The Lean statement asserts only the existence of r and β_0. In the proof, $r = y/1000$, and β_0 is taken as an expression containing $h(r)$; its numerical value is not in Lean. Putting in numbers with the formula for the volume of the cap gives $\beta_0 \approx 357$ at $y = 1$ Reading . The only explicit threshold held as a number inside Lean is the $\beta \geq 81$ of § 04 ($m \geq 2$).

07

Strength — in numbers

Quantity	Value	Label
True value of the volume $h(1/20)$ of the cap (formula $(\vartheta - \sin\vartheta\cos\vartheta)/\pi$, $\vartheta = 2\arcsin(1/40)$)	$2.652 \cdot 10^{-5}$	Reading
Lean's lower bound 83^{-4}	$2.107 \cdot 10^{-8}$	Reading
Ratio (true value / lower bound)	1259	Reading
Loss in the upper bound on q_1 (1259^4)	$\approx 2.5 \cdot 10^{12}$	Reading
q_1 at $\beta = 81$: Lean's upper bound / with the true h	0.56 / $2.2 \cdot 10^{-13}$	Reading
Threshold for $q_1 < 1$: with Lean's lower bound / with the true h	80.34 / 47.9	Reading
Threshold of the form with rate $1/4$ ($y = 1$, $r = y/1000$ as in the proof)	357	Reading

The upper bound should be far from the true probability, but there is no record comparing it numerically with the true probability at β above the threshold. What can be said goes only as far as the form of the rate and the threshold. A direct estimate without the chessboard diverges as $N \to \infty$ with X fixed, so it is not uniform in the volume. At small volumes the direct estimate can be the stronger one. The value of this chapter's estimate is that it is uniform in the volume, and in exchange the constants are weak.

08

What is not said

- **Coarse scales.** These are theorems at one scale (about the Wilson measure on the fine lattice itself). They do not reach estimates of large fields for the measure at a coarse scale pushed forward by the block map (19-04). Estimates of large fields across steps are on the side of the premises (28-04).
- **Lattice spacing.** They are uniform in the volume, but β is the coupling on a fixed lattice, and has nothing to do with the limit in which the lattice spacing is sent to 0.
- **Range of β.** § 04 covers only $\beta \geq 81$, and § 06 only couplings at or above a β_0 that is known only to exist. Nothing is said about the physical band (β between 2 and 3). 81 is a value of convenience for the proof.
- **Parity of the side.** Only even sides. § 04 needs $m \geq 2$; $m = 1$ ($N = 2$) is covered only by the form with rate $1/4$ of § 06.
- **What the ratio is taken against.** The probability is a ratio to Z. It is not a ratio to a partition function restricted to small-field events, nor to a twisted partition function.

- **Backgrounds.** Only the Wilson measure without twist (19-02).

What this chapter can say

Can say	Cannot say
Chessboard estimate: for $N = 2m$ ($m \geq 1$), every real β, and a family of functions of the links of a unit cube, $\lvert V(F)\rvert^{N^4} \leq \prod_w V(\text{constant})$. Lean: `FoldWilson.wilson_chessboard_even`	Odd sides. Measures with a twist
$\beta \geq 81$, $m \geq 2$: $P(X\text{: all its plaquettes have } \mathrm{Re}\, U_p < 0) \leq q_1^{\lfloor \lvert X\rvert/6 \rfloor}$, with $q_1 < 1$ independent of N and X. Lean: `FoldCap.rarity_explicit`	$\beta < 81$, $m = 1$. The measure at a coarse scale. The threshold 81 is a value fixed by a crude lower bound on the volume of the cap (with the true h about 48 Reading)
$0 < y \leq 2$, at or above some existing β_0, all $m \geq 1$: the form with rate $1/4$, $\leq q_4^{\lfloor \lvert X\rvert/24 \rfloor}$. Lean: `FoldQuarter.rarity_uniform4`	The value of β_0 (about 357 at $y = 1$ is Reading)
$Z \geq e^{6N^4\beta(1-8r^2)} h(r)^{4N^4}$ ($\beta \geq 0$). Lean: `FoldEntropy.Zw_ge`	—
$h(1/20) \geq 83^{-4}$. Lean: `FoldCap.ballQ_ge`	A Lean form of the true value of the volume of the cap

Sources and reproduction

Item	Kind	Source / tools
The Chessboard Estimate and the chessboard Peierls argument	Cited	The standard argument of J. Fröhlich, R. Israel, E. H. Lieb, B. Simon (1978). Confirmed from the abstract of M. Biskup, Reflection positivity and phase transitions in lattice spin models, arXiv:math-ph/0610025 (the text of the original paper not obtained). Same as 26-05
Reflection positivity (planes through lattice sites, planes through link midpoints)	Cited	The sources of 20-01 (K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471)
The theorems of § 03– § 06	—	Lean declarations of this book's study (names in the text and the scope table; no sorry, the three standard axioms, confirmed by re-checking with the kernel). The list is in 22-02
The true value of the volume of the cap, the floating-point value of the lower bound, the ratio 1259, the values of q_1, the thresholds 80.34, 47.9 and 357	Reading	The numerical record of this book's study (the formula for the volume of the cap cross-checked by Monte Carlo with $2 \cdot 10^6$ points at $r = 0.3, 0.5, 1$; the thresholds from the formula)

Item	Kind	Source / tools
That the one-directional core of the chessboard fails for kernels that are not reflection positive (a control)	Reading	The numerical record of this book's study (a model with a one-dimensional transfer kernel, $n = 4, 6, 10$; 0 violations for positive-definite kernels, violations for kernels that are not positive definite)

Terms used in this chapter — large plaquette, cap, chessboard estimate, uniform pattern, reflection positivity, plane through lattice sites, plane through link midpoints, Peierls argument, partition function, pigeonhole. Their meanings are in A-02 Glossary.

Next chapter to read: 22-02 What this part says, and what it does not — the theorems of Part 22 laid out in one table, checking the scope and what is not said.

What this part says, and what it does not

> This chapter is a guide. § 03 and § 04 are a map of open questions.

What this chapter shows — We lay out the theorems of Part 22 in one table and check their scope and what they do not say. We then describe what it means that this rarity is "a statement at one scale", how it relates to the published construction's handling of large fields by recounting them within each step of renormalisation, and the distance between statements of this type and the questions that remain (as an inference).

Prerequisite chapters — 22-00 Entrance, 22-01 Rarity of large plaquettes — the chessboard estimate. The published construction's handling of large fields is in 11-03 and 11-04.

Said first — This chapter adds no new theorem. The comparisons and assessments of § 03 and § 04 come from reading the statements in the source of the published construction Source and from inferences drawn from them Inferred . We do not say that the rarity of this part yields uniqueness of the infinite-volume limit or the mass gap.

The order of this chapter

1. List of theorems — scope and what is not said — Table 1
2. A statement at one scale — nothing is said about scales after renormalisation
3. Large fields in the published construction — recounting step by step — an inequality on the side of the action, and rarity on the side of the measure
4. Distance to the questions that remain — an estimate of the type that controls large fields (an inference)
5. What the part as a whole does not say

Labels — Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment (details in "Labels" in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are the labels of the whole book; Reading is a numerical reading of this book, and Lean a Lean theorem of this book. Source: `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0).

01

List of theorems — scope and what is not said

Theorem	Scope	What is not said	Lean
The Chessboard Estimate	$N = 2m$ ($m \geq 1$), all real β, families of functions of the links of a unit cube	Odd sides, measures with a twist, measures at a coarse scale	`FoldWilson.wilson_chessboard_even`
Rarity uniform in the volume	$\beta \geq 81$, $m \geq 2$, any set X of plaquettes: probability $\leq q_1^{\lfloor \lvert X\rvert/6\rfloor}$, with $q_1 < 1$ independent of N and X	$\beta < 81$, $m = 1$, coarse scales. 81 is a value fixed by a coarse lower bound on the volume of the cap (with the true volume it would be about 48 Reading)	`FoldCap.rarity_explicit`
The form with rate $1/4$	$0 < y \leq 2$, at or above some β_0, all $m \geq 1$: probability $\leq q_4^{\lfloor \lvert X\rvert/24\rfloor}$	The value of β_0 (about 357 for $y = 1$ is Reading)	`FoldQuarter.rarity_uniform4`
Lower bound on the partition function	$\beta \geq 0$: $Z \geq e^{6N^4\beta(1-8r^2)} h(r)^{4N^4}$	—	`FoldEntropy.Zw_ge`
Lower bound on the volume of the cap	For $K \geq 2$ and $2/K < r$, $h(r) \geq 1/(2K+1)^4$. With $K = 41$, $r = 1/20$, $h(1/20) \geq 83^{-4}$	A Lean form of the true value of the volume of the cap	`FoldCap.ballQ_ge`

Table 1. The theorems of Part 22. Each is a formalisation in this book's Lean of the type of the chessboard Peierls argument from reflection positivity Lean . Notation as in 22-01.

Probabilities are ratios to the partition function Z. They are not ratios to a partition function restricted to small-field events, nor to a twisted partition function. The upper bounds should be far from the true probabilities, but there is no record of a numerical

comparison with the true probabilities at β above the threshold. What can be said goes only as far as the form of the rate and the threshold (22-01 § 07).

02

A statement at one scale

The theorems of this part are statements about the Wilson measure itself on a fine lattice at a fixed lattice spacing. β is the coupling of that lattice, and no operation changing the lattice spacing is involved.

- **Nothing is said about scales after renormalisation.** Whether large coarse plaquettes are rare for the measure pushed forward to a coarse lattice by the block map (19-04) does not follow from the theorems of this part. The push-forward averages and composes the deviations of plaquettes, so there is no guarantee that the rarity of fine plaquettes becomes, as it stands, the rarity of coarse plaquettes Proved here .
- **Rarity uniform across steps is on the side of the premises.** Rarity of the same form at every step of renormalisation (with constants independent of the step and the lattice spacing) is placed as a premise in this book's study and is not shown (28-04, the large-field rarity condition).
- **Uniform in the volume, not uniform in the lattice spacing.** In the continuum limit β moves with the lattice spacing, so a statement at fixed β is not, as it stands, an estimate for any step of a continuum-limit sequence Proved here .

03

Large fields in the published construction — recounting step by step

The published construction too has to deal with large fields (places where the deviation of a plaquette exceeds a threshold). Its handling is of a different kind from the rarity of this part.

- **The form of the density.** The effective density at each step is carried in the form of a product of the exponential of an action truncated to be quadratic up to the threshold and linear beyond it, and a gas of large-field activities (forced activities) Source (11-01).
- **Recounting step by step.** The "seeds" of large fields are recounted at each step from the field of that step. It is not of the type in which a region that has once become a large field is fixed from then on Source . A persistent defect has its price paid again at each step Inferred .

- **Energy per seed.** The key of one step is a configuration-wise inequality: the difference of actions inside the halo is bounded below in proportion to the number of seeds (`Next.Fluctuations.grants`, `YM/Next/FLUCTUATIONS/Grants.lean:24`; with assumptions such as a gauge slice (`SliceSet`), regions (`Plot`) and `Funds`) Source . In the record of how declarations are used, this inequality is reached from one of the headline theorems (`SharpGram.habitat`) Source . How to read it is in 11-04. Where this inequality enters the dependency chain of the published theorems has not been checked.

The difference from the rarity of this part is as shown in Table 2 Inferred .

	Large fields in the published construction	Rarity in this part
Kind of statement	A configuration-wise inequality for the action (for all configurations, with conditions)	A statement about the measure (probabilities of events)
Scale	Each step of renormalisation, recounted step by step	One scale of the fine lattice
Coupling	Inside the weak-coupling band at every step	$\beta \geq 81$ (the coupling of one lattice)
Volume	The ledger of one step is a pointwise sum and does not depend on the volume. Estimates at the level of the density contain an exponential in the volume Source	The rate is uniform in the volume
Tools	Decomposition of the action, regions, partitions of unity, polymer sums	Reflection positivity, the chessboard, Peierls

Table 2. Two ways of handling large fields.

Neither follows from the other Inferred . The inequality of the published construction is a statement about the action within a step and says nothing about the rarity of a measure. The rarity of this part is a statement about the measure at one scale and says nothing about estimates across steps. What the published construction uses as estimates uniform in the volume are such things as the chessboard-root estimate on the side of the Wilson measure (22-00 § 03), and there too the chessboard is not used for the probability of large plaquettes Inferred (within the range of a search by name).

04

Distance to the questions that remain

This section is an inference Inferred .

One of the open questions in the assessment is the convergence and uniqueness of the infinite-volume limit (16-02). The published construction takes the infinite-volume

limit as a subsequence by compactness, and includes neither estimates of differences when the volume is changed nor material for uniqueness Source .

Arguments that show uniqueness usually take as a component some "lack of disorder" uniform in the volume. For uniqueness in the strong-coupling window (20-03), this was the Dobrushin condition (the smallness of the influence that one link receives from its surroundings). The rarity of this part provides, as a statement about the measure, a lack of disorder of the form "configurations with many large plaquettes are rare, at a rate independent of the volume" on the weak-coupling side. It is an estimate of the type that controls large fields and, like the Dobrushin condition in the strong-coupling window, has the form of a lack of disorder uniform in the volume. Its object, however, is the measure at one lattice spacing, which is different from the object of the assessment's question (the infinite-volume limit of the continuum theory).

For the following three reasons, the rarity of this part does not yield uniqueness.

1. Rarity alone says nothing about long-range correlations on the side of small fields (plaquettes that are not large). Uniqueness needs decay on that side, or an argument that replaces it.
2. It is a statement about one lattice with $\beta \geq 81$, a different object from the one for which the published construction takes infinite volume (correlation functions with the continuum limit taken first).
3. There is no rarity at renormalised scales (§ 02), so it is not in a form usable along a continuum-limit sequence.

Whether a statement of the same type can become a component of an argument for the mass gap (uniform decay) likewise remains open.

05

What the part as a whole does not say

- **Coarse scales.** Rarity of the pushed-forward measure and rarity uniform across steps are not said.
- **The range of β.** $\beta \geq 81$ (explicit), and at or above a β_0 whose existence only is given. Nothing is said about the physical band (β around 2 to 3).
- **Sides and background.** Only even sides and measures without a twist.
- **What the ratio is taken against.** Probabilities are ratios to Z.
- **Uniqueness, the mass gap, the continuum limit.** None of these follows from the theorems of this part.
- **Not a new result.** It is a formalisation of the chessboard Peierls argument for the SU(2) Wilson measure.

The scope of what this chapter says

Said	Not said
Lean The five items of Table 1 (the Chessboard Estimate, rarity uniform in the volume, the form with rate 1/4, the lower bound on the partition function, the lower bound on the volume of the cap). Details in 22-01	Coarse scales, $\beta < 81$, odd sides, measures with a twist
Source The published construction recounts large fields at each step and handles them by a configuration-wise inequality on the energy per seed	The path by which that inequality enters the chain of the published theorems (not checked)
Inferred Neither handling yields the other. The rarity of this part is an estimate of the type that controls large fields (in the form of a lack of disorder uniform in the volume)	A road to uniqueness or the mass gap

Sources and reproduction

Item	Kind	Source / tool
The theorems of Table 1	Lean	Lean declarations of this book's study (names in Table 1). The Chessboard Estimate is the standard argument of Fröhlich–Israel–Lieb–Simon (1978). Details of the sources in 22-01
The threshold values 48 and 357	Reading	The numerical record of this book's study (from the formula for the volume of the cap; 22-01 § 07)
The published construction's handling of large fields	Source	`yang-mills-group/OpenAI-YM-20261028`: `YM/Next/FLUCTUATIONS/Grants.lean:24` `grants`. Readings of the form of the density and of recounting are in 11-01, 11-03, 11-04
Open question (convergence and uniqueness)	Assessment	The assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), § 8, item 1

Terms used in this chapter — large plaquette, rarity, one scale, push-forward, truncated action, forced activity, seed, recounting, uniform in the volume, uniqueness. Their meanings are in A-02 Glossary.

Next chapter to read: 23-00 Entrance — the staircase, and the random walk, the ruler and the clock — from here on is the record of the study heading toward the band. To see the parts of firm theorems together, go to Table 1 of 16-04.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time entrance, staircase, five quantities · Kiichi, Shiori and Rin

Entrance to Part 23 — the staircase, and the random walk, the ruler and the clock

> This chapter is a guide (the entrance to this part).

This part is a record of the study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — Part 23 is the part that examines a road to the mass gap with the diffusive-relaxation-time assumption (the time over which fluctuations are smoothed out grows as a power of the square of the correlation length) as its axis. The diffusive-relaxation-time assumption is one sentence that restates a quantity of the problem statement (the gap in Euclidean time) as a quantity that can be computed on the lattice and that machine checking can reach (the relaxation time of Langevin dynamics). This part divides that sentence into a staircase of four steps, and lays out how far the claim has been built on the first step (the path measure of the double well in 0+1 dimensions). In this chapter, after looking at the picture of the staircase and the map of the part, we set up quantities such as the correlation length ξ and the Poincaré constant C_P one by one, with the metaphors of the random walk, the ruler and the clock.

Prerequisite chapters — 2-10 (relaxation time and the dynamical exponent), 4-11 (definition of the mass gap and the correlation length), 2-07 (the path integral and imaginary time).

Said first — This chapter is an entrance that lines up quantities through metaphors, and proves nothing. The relaxation-time assumption is an unproved assumption. The precise definitions are placed in 23-01 (the side of the four-dimensional lattice) and 23-04 (the side of the double well).

The order of this chapter

1. A map of the part — the list of eighteen chapters, and how to read the labels
2. The four steps of the staircase — from the double well to the mass gap in the limit (Figure 1)
3. The ruler — the correlation length ξ — in how many steps correlations die out

4. The smearing time — the Poincaré constant C_P — L^2 steps to even out a length L (Table 3). Detour: why the random walk goes as a square
5. The scale and the clock — the remaining three quantities — the number of rescalings, the graduation of the clock, the clock that runs slow
6. The relaxation-time assumption in one sentence — smearing time $\asymp$ ruler^2. "Some f" and "every f"
7. How to walk through Part 23 — in which chapter each metaphor becomes a precise sentence

---

01

A map of the part — the list of eighteen chapters, and how to read the labels

This part is a study with the relaxation-time assumption (the time over which fluctuations are smoothed out grows as a power of the square of the correlation length) as its axis. The relaxation-time assumption is not the mass gap itself but a sentence that restates it as a quantity of a dynamics measurable on the lattice (23-01 § 05), and this part builds that sentence on the first step of a four-step staircase — the path measure of the double well in 0+1 dimensions. We split the claim of the first step into four conditions, and for each condition lay out in one table the rows where a proof is given, rows citing original sources, computer-assisted rows, rows of numerical reading, and assumption rows.

The eighteen chapters of Part 23

0. **23-00 Entrance — the staircase, and the random walk, the ruler and the clock** [Y2] (this chapter)
1. 23-01 The line of construction and the diffusive-relaxation-time assumption [Y2]–[Y3]
2. 23-02 Roads tried on the four-dimensional lattice, and walls [Y3]
3. 23-03 The four steps of the staircase — why we start from 0+1 dimensions [Y2]
4. 23-04 The path measure of the double well, and kinks [Y2]–[Y3]
5. 23-05 The statement of the first step — the four double-well conditions and the upper-bound proposition [Y3]
6. 23-06 The in-phase local Poincaré condition [Res]
7. 23-07 The map of the saddle-tube condition — seven routes and the seam of the ring [Y3]

8. 23-08 What computer-assisted means — the conditions on the chain inside the window [Y2]–[Y3]
9. 23-09 The sector framework and the three strands of the window log-Sobolev route [Res]
10. 23-10 The list of detour plans — types of breakage [Y3]–[Res]
11. 23-11 Box reduction and the gap inside the box — the constant of the core [Res]
12. 23-12 The integrated table and the dependency graph [Y3]
13. 23-13 Lean statements and the limits of machine checking [Y2]–[Y3]
14. 23-14 Reconciliation — the numbers of the record and the known numbers [Y3]
15. 23-15 What remains and its cost — the distance to a theorem [Y3]
16. 23-16 What is not claimed [Y2]
17. 23-17 Insights and small questions [Y2]–[Y3]

The chapters run: why this assumption (23-00 to 23-02), the statement of the first step (23-03 to 23-05), the four conditions (23-06 to 23-11), the overall tables (23-12, 23-13), and reconciliation and what remains (23-14 to 23-17). The marks [Y2], [Y3] and [Res] are reader levels: [Y2] is second-year, [Y3] third-year and above (using measures, operators and Poincaré inequalities), and [Res] is research level. This chapter is [Y2]. Viewpoints other than the relaxation-time assumption, and new views that take the relaxation-time assumption as a premise, are placed in Part 24 The first road, second half.

The labels of this part

In this part the strength of a claim is divided by five labels (the labels of 0-01 § 03, refined for the material of this part). Each sentence gets one label; a sentence needing two is split.

Table 1 The five labels of this part (Lean is a column outside the labels)

Label	Meaning	What it does not guarantee
Proved here	Elementary. The proof is placed in the text or at the end of the chapter	That it contains no error (it is placed in a form the reader can check)
Cited	A known theorem cited from the original source (or a secondary source whose statement was checked)	That it is a result of this book
Computer-assisted	Computer-assisted. Checked as an inequality by interval arithmetic (premises such as rounding conventions are written on the same row)	That it holds outside the range computed

Label	Meaning	What it does not guarantee
Reading	A reading. Floating-point numbers, Monte Carlo, fits. The kind (a point value, a lower bound over a family, etc.) is added	That it is a proof of an upper or lower bound
Assumption	An assumption. A row put down without a proof. It is given a name	That it holds

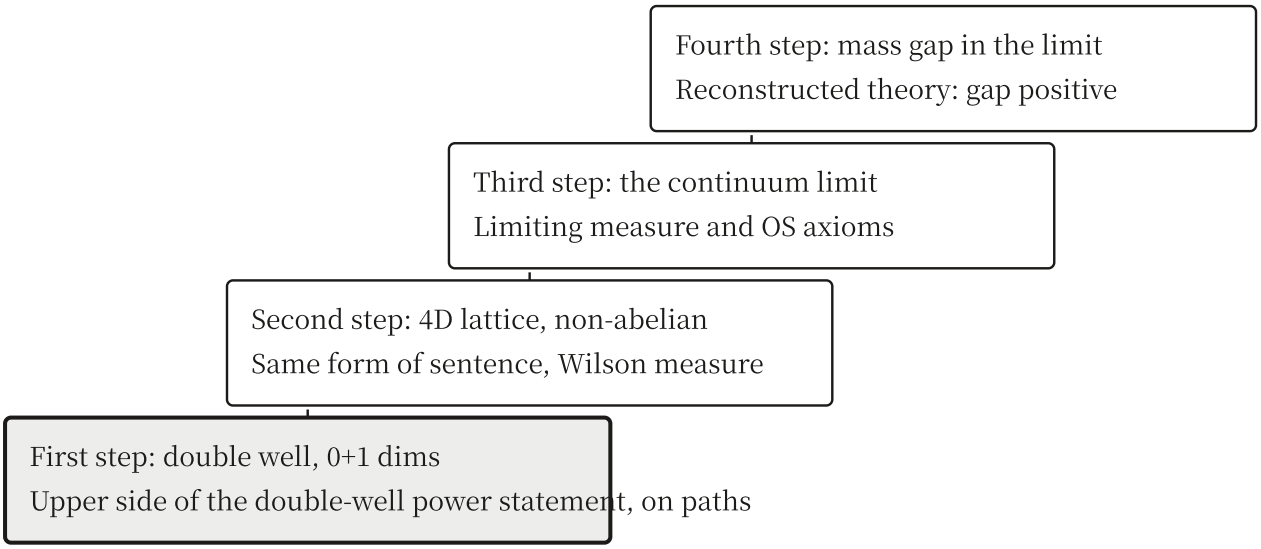
Lean is a column separate from the five labels. A Lean theorem is the "algebra from assumptions to conclusion" checked by machine, and the theorem name and the assumptions of that theorem are written on the same row. Whether the assumptions are true is outside the check (23-13).

02

The four steps of the staircase — from the double well to the mass gap in the limit

The relaxation-time assumption is the sentence "in some family of measures, the time over which fluctuations are smoothed out grows as a power of the square of the correlation length", and it can be stated independently of dimension. So we line up sentences of the same form in four steps, from easy places to hard ones.

Figure 1 The four steps of the staircase. This part deals with the inside of the first step; the arrows from the second step upwards stand for "a sentence of the same form", not for theorems of implication



Written out in words, Figure 1 is as follows. From the bottom: the first step = the double well in 0+1 dimensions (the path measure in imaginary time of a quantum-mechanical model); the second step = the non-abelian gauge theory on the four-dimensional lattice (the Wilson measure); the third step = the continuum limit with the lattice spacing taken to 0, and that limit satisfying the Osterwalder–Schrader axioms; the fourth step = the mass gap of the limiting theory.

Table 2 The objects of the four steps, and what can be called "closed" at each step

Step	Object	Form in which it can be called closed	State in this book

Step	Object	Form in which it can be called closed	State in this book
First step	The path measure of the double well in 0+1 dimensions (parameter $\hbar$)	Show the upper side of the double-well power statement (an upper bound on C_P), with a table of finitely many assumption rows	Split into the four double-well conditions, and up to the algebra on top of them (the Double-Well Relaxation-Time Upper-Bound Proposition). Every condition leaves Assumption rows
Second step	The Wilson measure on the four-dimensional lattice (parameter β)	A sentence of the same form, as $\beta \to \infty$	The relaxation-time assumption remains an assumption. Only estimates from below (23-02)
Third step	The limiting measure as the lattice spacing $a \to 0$	The limit exists and satisfies the OS axioms	Not dealt with. That a theory satisfying the Wightman axioms can be built as a limit of four-dimensional Wilson lattice gauge theory was proved in Lean by the construction published on 2026-11-10, according to the committee's assessment (Part 6). A probability measure for the continuum theory and uniqueness of the infinite-volume limit are not shown (6-04)
Fourth step	The reconstructed quantum field theory	The gap of the Hamiltonian is positive (4-11)	Not dealt with. The published construction does not show it either (6-04)

Even if the first step is closed, there is no guarantee that it lifts to the second step and beyond. What has a prospect of lifting is "the form of the sentence" and "the form of the tools"; constants and ranges of parameters do not lift. The reasons are given from both sides in 23-03 § 06.

03

The ruler — the correlation length ξ

Fluctuations are similar between nearby places and become unrelated when far apart. The length over which they become unrelated is the correlation length ξ. As a metaphor, it is the one ruler the measure carries.

As a formula, the correlation of two points (the expectation of the product with the mean subtracted) decreases exponentially with distance r, and the length of that decrease is ξ.

$$\langle O(0)\,O(r)\rangle - \langle O\rangle^2 \;\approx\; C\,e^{-r/\xi}, \qquad \xi = \frac{1}{m} \quad (1)$$

m is the gap of the transfer matrix (or transfer operator), that is, the difference between the lowest eigenvalue and the next, and r is the distance in the imaginary-time direction. That the decay of correlations and the gap are the same quantity is in 4-11 § 02, and the restatement as a correlation length is in 4-11 § 07 Cited .

Example 1 (free field). For the free field on a one-dimensional lattice (a Gaussian chain with neighbours joined by springs, mass m_0), correlations decrease exactly as $e^{-r/\xi}$, and ξ is determined by $m_0^2 = 2(\cosh(1/\xi) - 1)$. For $m_0 = 1$, $\xi \approx 1.04$; for $m_0 = 0.1$, $\xi \approx 10.0$ (18-03 § 05).

Example 2 (double well). In the quantum-mechanical double well (2-05 § 07), the gap is the tunnelling splitting Δ, determined by passing through between the two valleys, and it becomes exponentially small the higher the hill. Seen through paths in imaginary time, the correlation length is $\xi = \hbar/\Delta$, and as $\hbar$ is made smaller it grows at the rate $e^{S_0/\hbar}$ (S_0 is the action of the path that moves from valley to valley). This property, that the ruler grows exponentially, is one of the reasons for choosing the model of the first step (23-03 § 04, 23-04 § 03).

The ξ measurable on the lattice is a number in units of one graduation of the lattice (5-09 § 01). The physical length is ξ times the lattice spacing a, $a\xi$, and in the continuum limit $a\xi$ must remain finite while $a \to 0$. That is, ξ in lattice units must grow to infinity (5-09 § 05). The $\hbar \to 0$ of the double well is the smallest specimen of this regime, "the ruler becomes infinitely long compared with the graduations".

04

The smearing time — the Poincaré constant C_P

Drop a drop of ink on paper, and the ink gradually smears and spreads, and in the end becomes uniformly pale. What the smearing really is, is a random walk of the ink particles. Each particle moves one step left or right at random at each tick. After t steps the spread of the particles (the root-mean-square distance) is $\sqrt{t}$, and evening out a strip of length L takes on the order of L^2 steps.

The Poincaré constant C_P is this "longest time until evening out is finished", defined in general for fluctuations. As a metaphor, it is the reading of the one clock the measure carries: the smearing time.

We write it as a formula. Given a probability measure μ and the "size of the change from place to place" $\mathcal{E}(f)$ of a function f (the Dirichlet form),

$$\mathrm{Var}_\mu(f) \;\le\; C_P\,\mathcal{E}(f) \quad (\text{for every } f), \qquad C_P = \sup_f \frac{\mathrm{Var}_\mu(f)}{\mathcal{E}(f)} \;(2)$$

the best constant is C_P (1-10 § 07). If there is a function with large variance but small change, that is, "a large, slowly varying fluctuation", C_P becomes large. For the diffusion dynamics (Langevin dynamics) with μ as its equilibrium, $1/C_P$ is the gap of that dynamics, and the slowest fluctuation decreases by a factor $1/e$ every time C_P Cited .

Example: the random walk on a single road

Consider a single road of L cells in a row (reflecting at both ends). μ is the uniform distribution, and $\mathcal{E}(f) = \frac{1}{L}\sum_i (f_{i+1} - f_i)^2$. Then the C_P of (2) is the reciprocal of the eigenvalue next after 0 of the Laplacian matrix of the road, and can be written in closed form.

$$C_P(L) = \frac{1}{2\,(1-\cos(\pi/L))} \;\approx\; \frac{L^2}{\pi^2} \qquad (L \to \infty) \;(3)$$

Proved here Proof: $f_i = \cos(\pi(i-\frac{1}{2})/L)$ $(i = 1, \dots, L)$ is an eigenvector of the Laplacian matrix of the road reflecting at both ends, with eigenvalue $2(1-\cos(\pi/L))$ (use $\cos(A+B)+\cos(A-B) = 2\cos A \cos B$ on the sum of the two neighbouring terms, and check that at both ends the mirror conditions $f_0 = f_1$, $f_{L+1} = f_L$ hold). The other eigenvectors are $\cos(\pi k(i-\frac{1}{2})/L)$ $(k = 0, \dots, L-1)$, with eigenvalues increasing with k. Since μ is uniform, Var and $\mathcal{E}$ are divided by the same $1/L$, and the supremum of the ratio is the reciprocal of the eigenvalue next after 0. ∎

Proved here Table 3 C_P of the random walk on a single road (values of the closed formula (3)). C_P/L^2 approaches $1/\pi^2 \approx 0.10132$

L	Gap $2(1-\cos(\pi/L))$	C_P	C_P/L^2
16	0.0384294	26.02	0.10165
32	0.00963055	103.8	0.10140
64	0.00240909	415.1	0.10134
256	0.000150596	6640	0.10132

Doubling L makes the time until evening out is finished four times as long.

The C_P used in this part replaces this single road by "the space of configurations of whole paths". For the double well, the measure $e^{-S/\hbar}$ on the whole of the paths $x(\tau)$ in imaginary time is μ, $\mathcal{E}$ is the expected square of the change when a path is moved slightly, and Langevin dynamics diffuses whole paths along an "artificial time" s (23-04

§ 01). On the four-dimensional lattice, μ is the Wilson measure and what moves are the angles of the links (23-01, 7-09).

For the free field, the slowest fluctuation is in the direction of shifting the whole path uniformly, and its rate is the square of the mass, m_0^2. So $C_P = 1/m_0^2$, and from $m_0^2 = 2(\cosh(1/\xi) - 1)$ of Example 1, C_P/ξ^2 at $\xi = 1, 4, 16, 256$ is 0.921, 0.995, 0.9997, 0.999999, approaching 1 Proved here (values of the closed formula). The smearing time is the square of the ruler: that is the free field's answer.

Precisely, the C_P meant here is the Poincaré constant of "the distribution of configurations over the whole of spacetime (in 0+1 dimensions, imaginary time)" (the path measure). The Poincaré constant of the distribution at a single time (the $|\psi_0|^2$ of the ground state) is a different quantity, and it is tied to the gap of the Hamiltonian. The distinction among the three C_P is placed in 23-01. Known examples in which the power z (the dynamical exponent) with which C_P grows in ξ departs from 2 are in 2-10.

Detour: why the random walk goes as a square

A subsection you may skip. The position X_t after adding t independent steps ± 1 has mean 0, and its variance is t times the variance 1 of one step. This is because for sums of independent things, variances add (1-08).

$$\mathbb{E}[X_t^2] = \sum_{i=1}^{t} \mathbb{E}[\varepsilon_i^2] + \sum_{i \neq j} \mathbb{E}[\varepsilon i]\,\mathbb{E}[\varepsilon j] = t \quad (4)$$

The spread is $\sqrt{t}$, so reaching distance L takes $t \approx L^2$. A wave advances one step per tick, but a random walk goes back and forth, so reaching far is delayed by a square. The "square" of the relaxation-time assumption is this square of the random walk.

05

The scale and the clock — the remaining three quantities

Besides the ruler ξ and the smearing time C_P, there are three quantities that have been used in the record for the four-dimensional lattice. They are not the main players on the first step (the double well), but the reason for putting the relaxation-time assumption as a "power statement" lies here, so we only line up the metaphors.

The ruler	The smearing time	The number of rescalings	The graduation of the clock
The correlation length ξ	The Poincaré constant C_P		

the reciprocal of the gap m (§ 03)	L^2 steps to even out a length L (§ 04)	The bookkeeping length ξ_{RG} the number of steps that coarsen the map	The normalisation of $\mathcal{E}$ in what unit the same fluctuation is measured
`The clock that runs slow` Renormalisation of stochastic time the clock itself stretches with scale			

The number of rescalings. In four-dimensional lattice Yang–Mills there is no way to obtain ξ by formula as a function of the coupling β. So we coarsen the scale of the map one step at a time, and make separately a length that counts how many times it takes to reach a picture of fixed coarseness, the bookkeeping length ξ_{RG}. In the one-loop count, raising β just a little makes the bookkeeping length grow exponentially (4-07). That the bookkeeping length and ξ agree up to a constant factor is an assumption. The definitions and tables have been moved to 23-01 and 23-02.

The graduation of the clock. The C_P of (2) changes value depending on the unit in which $\mathcal{E}$ is measured. Multiplying $\mathcal{E}$ by c multiplies C_P by $1/c$. The same happens for the double well. Between the $\mathcal{E}$ that measures paths by the canonical field $x/\sqrt{\hbar}$ and the $\mathcal{E}_x = \mathcal{E}/\hbar$ that measures them by the position x as it is, $C_P^{(x)} = \hbar\, C_P$ Proved here Lean `ratioR_x_normalisation` (no assumptions; if $C_P^{(x)} = \hbar C_P$, the ratio is also multiplied by $\hbar$). The difference of graduation shows in "the width of a constant factor", but the factor of a power of $\hbar$ is small compared with $\ln \xi$, so it does not show in "the power".

The clock that runs slow. Langevin time itself stretches and shrinks when the scale is changed. In the perturbation theory of stochastic quantisation, the graduation of stochastic time also has to be renormalised (7-09) Cited (at the level of perturbation theory). On the four-dimensional lattice its effect is a factor of a fractional power of β, and this too does not show in the power. How to compare the four clocks is placed in a detour of 23-01.

Precisely, the three of them — the number of rescalings, the graduation of the clock and the clock that runs slow — move sentences stated in "the width of a constant

factor" (band statements), and do not move sentences stated in "a power". That is why the relaxation-time assumption in this part is a power statement.

06

The diffusive-relaxation-time assumption in one sentence — smearing time ≍ ruler²

With the quantities in place, we put the relaxation-time assumption into one sentence as a metaphor: "the smearing time grows with the same power as the square of the ruler". When a family of measures is lined up by one parameter, and moving the parameter makes the ruler grow without bound,

$$(\xi^2): \quad \frac{\ln C_P}{\ln \xi} \longrightarrow 2 \qquad (\xi \to \infty) \tag{5}$$

On the four-dimensional lattice the parameter is β with $\beta \to \infty$; for the double well the parameter is $\hbar$ with $\hbar \to 0$. For the free field it holds with $C_P/\xi^2 \to 1$, as in § 04. The question is whether it holds on the four-dimensional lattice with interaction, and the first step is to build the upper side on the double well, which is easier than that.

For the double well, the "width of a constant factor" sentence stronger than (5) ($C_P \le K\xi^2$) is read as being on the side that breaks. This is because the lower bound on the path C_P grows from 1 times to 6.5 times ξ^2 as $\hbar$ is lowered Reading (lower bound over a family, step $\varepsilon = 0.25$, $\hbar \in [0.07, 0.4]$, 23-04 § 04). In the count at the level of physics that counts kinks, the growth is first order in $S_0/\hbar \approx \ln \xi$ (the numbers in this range alone cannot distinguish it from exponential growth). If it is first order, the power statement (5) is kept. That is why the goal of the first step is not the band statement but the upper side of the power statement, $C_P \le K\xi^2 S_0/\hbar$ (23-03 § 05).

"Some f" and "every f"

The C_P of (2) is a supremum, so bounding it from below and from above differ in difficulty. To bound it from below (C_P is large), it is enough to show one test function with large $\mathrm{Var}(f)/\mathcal{E}(f)$. To bound it from above (C_P is not too large), the inequality must be shown for every f.

$$C_P \ge c\,\xi^2 \iff \exists f: \ \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} \ge c\,\xi^2, \qquad C_P \le C\,\xi^2 \iff \forall f: \ \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} \le C\,\xi^2 \tag{6}$$

As a metaphor, that smearing is slow can be said by showing one stain, but that it is not too slow must be said for every stain. What lies on the side of the mass gap is the upper side (C_P not too large), and test functions and numerics reach only the lower

side. The algebra that rereads the upper side as a lower bound on the mass, and the definitions of the upper and lower halves (the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption), are placed in 23-01.

Precisely, the lower bound $CP \geq \xi^2$ for the double well is a proposition of the form that a family of test functions local in time (functions of a single time summed over time) reaches up to equality (the One-Dimensional One-Sided Bridge Proposition in 23-04 § 04). On the upper side, even for the first step there are only conditional propositions.

07

How to walk through Part 23 — in which chapter each metaphor becomes a precise sentence

Each metaphor of this chapter is replaced once by a precise sentence somewhere in Part 23. Table 4 matches metaphors, symbols and the chapters that make them precise.

Table 4 Metaphors and the corresponding precise sentences

Metaphor	Symbol	Precise definition or sentence	State
The staircase	Four steps	23-03	The convention for how this part is built
The ruler	$\xi = 1/m$ (for the double well $\xi = \hbar/\Delta$)	4-11 § 07, 23-04 § 02	Cited Definition
The smearing time	CP (the path measure)	23-01, 23-04 § 01	Cited Definition. The value on the four-dimensional lattice is unknown
The number of rescalings	ξ_{RG}	23-01, 23-02	Convention (that it agrees with ξ is Assumption)
The graduation of the clock, the clock that runs slow	Normalisation of $\mathcal{E}$, stochastic time	23-01	Convention, level of perturbation theory
Smearing time $\asymp$ ruler^2	The relaxation-time assumption (power statement)	23-01, 23-03 § 05	Assumption
The upper side of the first step	$CP \leq K\xi^2 S_0/\hbar$ (the First-Step Upper-Bound Proposition)	23-05	Algebra on top of four conditions. The conditions leave Assumption rows

To follow the line first: 23-03 → 23-04 → 23-05 → 23-12 (the integrated table) → 23-15 (the distance to a theorem). To see the relation with the four-dimensional lattice first:

23-01 → 23-02. To see what is not claimed first, 23-16 is the shortcut. The Lean statements and assumptions are in 23-13, and how to read Lean is in A guide to Lean.

What this chapter can say

Said	Not said
Cited That the correlation length and the gap are the same quantity ((1)); the definition of the Poincaré constant, and that the gap of the diffusion dynamics is its reciprocal ((2))	That ξ diverges as $\beta \to \infty$ in four-dimensional lattice Yang–Mills; the value of C_P
Proved here The closed form (3) of C_P for the random walk on a single road; $C_P/\xi^2 \to 1$ for the one-dimensional free field; the mean-square distance $= t$ of the random walk ((4))	That the same ratio agrees in power in a model with interaction (the relaxation-time assumption)
Proved here That multiplying the normalisation by $\hbar$ also multiplies C_P by $\hbar$ (the graduation of the clock in § 05) Lean `ratioR_x_normalisation`	That the bookkeeping length ξ_{RG} of the four-dimensional lattice and the correlation length ξ agree up to a constant factor
Reading That the lower bound on the path C_P of the double well grows from 1 times to 6.5 times ξ^2 (lower bound over a family, $\varepsilon = 0.25$, $\hbar \in [0.07, 0.4]$)	An upper bound on C_P for the double well. The upper side of the first step is conditional on the four conditions (23-05)
The way of building it as a four-step staircase (a convention)	Implication from the first step to the second step and beyond. About the fourth step (the mass gap in the limit), this chapter shows nothing

Sources and reproduction

Item	Kind	Source / tools
Correlation length and gap, correlations in imaginary time	Cited	The table of sources of 4-11.
Poincaré constant, gap of diffusion	Cited	The table of sources of 1-10.
C_P of the random walk on a single road ((3), Table 3)	Proved here	The proof in § 04 of the text. The values in the table are values of the closed formula (3).
Tunnelling splitting of the double well, and S_0	Cited	(9)(10) of 2-05 § 07 and its table of sources.
Reading of the lower bound on the path C_P (§ 06)	Reading	23-04 § 04 and its table of sources.

Item	Kind	Source / tools
Rereading the normalisation (the graduation of the clock in § 05)	Proved here, Lean	Theorem `ratioR_x_normalisation` (no assumptions). The statement is in 23-13.
Bookkeeping length, renormalisation of stochastic time	Convention, Cited (perturbation theory)	The tables of sources of 4-06, 4-07 and 7-09. The detailed formulas are in 23-01.
Figure 1 (the staircase)	Convention	A schematic drawing of how this part is built.

Words used in this chapter — correlation length, Poincaré constant, Dirichlet form, path measure, staircase (four steps), the relaxation-time assumption (power statement), the five labels (proved here, cited, computer-assisted, reading, assumption), tunnelling splitting. Their meanings are in A-02 Glossary.

Next chapter to read: 23-01 The line of construction and the diffusive-relaxation-time assumption — splits the problem into eight statements, and defines the diffusive-relaxation-time assumption and its upper and lower halves. To go straight to the first step, 23-03 The four steps of the staircase.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time line of construction, the relaxation-time assumption, the one-sided bridge · Kiichi, Shiori and Rin

The line of construction and the diffusive-relaxation-time assumption — eight statements, the direction of the lattice version of the mass gap, and the relaxation-time assumption in six stages

> This chapter is this book's own investigation. § 02 and § 08 are a map of open questions; § 03 is established knowledge.

What this chapter shows — We line up the Clay problem as a single line, "lattice → infinite volume → continuum limit → OS axioms → mass gap", and split it into the eight statements of construction. The mass gap on the lattice is not "the gap in lattice units is uniformly positive" but "the gap in physical units stays positive". The diffusive-relaxation-time assumption, the axis of this part (the time C_P over which fluctuations are smoothed out grows with the same power as the square of the correlation length), is explained in six stages, each with an example that can be checked by hand: what we want to measure; why we restate it as a quantity of a dynamics; the statement (band statement → power statement → upper and lower halves [the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption]); what the relaxation-time assumption gives; what it does not give; and analogous cases. On a lattice with continuous time the one-sided bridge $C_P \geq \kappa \xi^2$ can be shown, and together with the upper-bound assumption it gives a lower bound on the gap of the lattice in physical units. The converse does not hold in general; under the assumption of the bridge, the upper-bound assumption is a stronger sentence than the lattice mass gap in physical units.

Prerequisite chapters — 0-03 (the problem statement), 5-09 (what to hold fixed when taking the limit), 4-11 (the gap and the correlation length), 1-10 (the Poincaré constant), 23-00 (the five quantities and the labels of the five grades).

Said first — This chapter does not solve the problem. Along the line of this chapter (for each β, first make the volume infinite, then β→∞), what is in hand in four dimensions is only the theorems of the reflection-positivity preservation statement and the OS reconstruction statement, and the infinite-volume uniqueness statement at small β (for the relation to the published construction, see the note in § 02). The relaxation-time assumption, the upper-bound assumption, the lower-bound assumption and the three chains are all assumptions. The one-sided bridge is a proposition on a finite spatial lattice with continuous time; towards the Wilson measure with discretised time and towards infinite volume, assumption rows remain. What was checked in Lean is only the algebra from assumptions to conclusions.

The order of this chapter

1. The conventions of this chapter — symbols, and the unit of the ruler
2. From the problem to the line of construction — eight statements (Table 1) and how they connect
3. The direction of the lattice version of the mass gap — a uniform lattice gap is on the opposite side (Table 2)

4. Restating with the Poincaré constant — three C_P (Table 3) and the free field (Table 4)
5. The relaxation-time assumption — (1) what we want to measure, (2) why we restate it, (3) the band statement, the power statement, the upper and lower halves (Table 5)
6. What the relaxation-time assumption gives and what it does not — (4) the one-sided bridge (Table 6), (5) a counterexample to the converse (the Free-Particle-on-a-Circle Counterexample) and the strong-coupling side (Table 7), (6) analogous cases, classification and the dependency graph (Table 8)
7. Detour: the number of rescalings and four clocks — logarithms that do not show in the power (Tables 9 and 10)
8. What of the line is in hand — on to 23-02 and 23-03

01

The conventions of this chapter — symbols, and the unit of the ruler

This chapter is about the side of the four-dimensional lattice. The group is $SU(2)$, the Wilson action is $S_W = \beta \sum_P (1 - \frac{1}{2} \operatorname{tr} U_P)$ (5-01), and $\beta = 4/g^2$ is the inverse of the coupling, not an inverse temperature. All quantities are counted in lattice units (lattice spacing 1).

$m_{\mathrm{lat}}(\beta)$ is the gap of the transfer matrix on gauge-invariant states (the mass in lattice units), $\xi = 1/m_{\mathrm{lat}}$ the correlation length, and $a(\beta)$ the lattice spacing measured in physical units. The mass in physical units is

$$\xi = \frac{1}{m_{\mathrm{lat}}}, \qquad \Delta(\beta) = \frac{m_{\mathrm{lat}}(\beta)}{a(\beta)} \quad (1)$$

There is more than one way to fix $a(\beta)$. If the string tension σ is taken as the physical unit, $a\sqrt{\sigma}$ is the lattice spacing; in the four-dimensional record of this part, a is set to the reciprocal of a length fixed by counting the steps that coarsen the lattice, the bookkeeping length ξ_{RG} (§ 07). Which one is taken goes into the assumptions of the one-scale chain and the bookkeeping-length chain among the three chains of § 05.

Example: if $m_{\mathrm{lat}} = 0.514$ at $\beta = 2.60$ (a Monte Carlo value of Lucini–Teper), then $\xi = 1.95$ lattice spacings Reading . Δ is a quantity fixed only when this is divided by a; the numbers on the lattice alone do not fix it (5-09).

02

From the problem to the line of construction — eight statements

The problem statement (0-03) asks that a non-trivial quantum Yang–Mills theory exist on $\mathbb{R}^4$, and that the theory have a mass gap $\Delta > 0$. On the road entering from the lattice, we line this up as a single line.

Lattice	Infinite volume	Continuum limit	OS axioms	Mass gap
The Wilson measure on a finite box	Make the box infinite the infinite-volume uniqueness statement, the	$\beta \to \infty$ $(a \to 0)$ the Wilson-loop limit statement, the rotation-recovery statement	Hilbert space and Hamiltonian	The gap in physical units is positive

a probability measure for each β and each volume	reflection-positivity preservation statement		the OS reconstruction statement, the non-triviality statement	the physical-unit gap statement

The order is to make the volume infinite first for the measure at each β, and then take $\beta \to \infty$ (the order of Chatterjee 2018; there is also the order of taking $a \to 0$ first in a finite box, 16-07). Table 1 makes each stage of the line a statement, and the demand of the problem statement becomes the following overall picture.

$$\textbf{the demand of the problem statement} \impliedby (\mathrm{S1}) \wedge (\mathrm{S2}) \wedge (\mathrm{S3}) \wedge (\mathrm{S4}) \wedge (\mathrm{S6}) \wedge (\mathrm{S7}) \wedge (\mathrm{S8}) \quad (2)$$

(2) is an overall picture, not a theorem. The relaxation-time band at the bookkeeping length is not in (2); it is a restatement of the physical-unit gap statement (§ 04).

Table 1 The eight statements of construction and their current status in four dimensions (in formulas they are written with the labels (S1)–(S8))

Statement	Content	Current status in four dimensions	Chapter
Infinite-volume uniqueness statement ((S1) in formulas)	For each β, the infinite-volume limit measure is uniquely determined	Cited Limit points exist by compactness. At small β, existence of the limit (Osterwalder–Seiler 1978) and uniqueness (Shen–Zhu–Zhu 2023). At large β, 'in general unknown' (Chatterjee § 3)	5-04
Reflection-positivity preservation statement ((S2) in formulas)	A positive self-adjoint transfer operator. Preserved under weak limits	Cited Osterwalder–Seiler 1978, Lüscher 1977	5-03
Wilson-loop limit statement ((S3) in formulas)	As $a \to 0$, renormalised Wilson loops have a non-trivial limit	Unsolved (the continuum string-tension problem (Chatterjee's Problem 5.2))	5-09, 7-07
Physical-unit gap statement ((S4) in formulas)	$\Delta(\beta) \to \Delta_\infty \in (0, \infty)$	Unsolved (the half on the lattice side is the lattice correlation-length problem (Chatterjee's Problem 5.1))	4-11, 5-09
Relaxation-time band at the bookkeeping length (slowness of fluctuations) ((S5) in formulas)	$a^2 C_P \asymp 1$	Assumption The formulation of this part. There is no proof	§ 04, § 05
Rotation-recovery statement ((S6) in formulas)	In the limit, invariance under rotations and translations of $\mathbb{R}^4$ returns	Unsolved along this line (shown in two dimensions; for the side of the published construction, see the note below)	16-07
OS reconstruction statement ((S7) in formulas)	Build the Hilbert space and the Hamiltonian from the axioms	Cited A theorem (Osterwalder–Schrader). Along this line it remains that the limit satisfies the axioms (for the side of the published construction, see the note below)	4-10
Non-triviality statement ((S8) in formulas)	The limit is not Gaussian, and at short distances agrees with the predictions of asymptotic freedom	Unsolved. For four-dimensional ϕ^4 cut off by a lattice, the limit is Gaussian Cited	0-03, 7-08

Note (relation to the published construction). The "current status in four dimensions" of Table 1 is the status along the line of this chapter (for each β, first make the volume infinite, then β→∞). The construction published on 2026-11-10, with a different assembly (stacking steps of renormalisation to take the ultraviolet limit, and taking infinite volume along a subsequence), shows that a quantum field theory satisfying the Wightman axioms can be built as a limit of four-dimensional Wilson lattice gauge theory, and the committee's assessment is that it is proved in Lean (Part 6). In that construction,

Euclidean invariance (recovery of rotations) and the OS axioms and reconstruction are shown for the theory of each phase (6-03). What is not shown is the convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory, a uniform mass gap, Wilson loops and the area law, and one-particle states (6-04). The physical-unit gap statement and the Wilson-loop limit statement are open under either assembly.

The eight are not independent. Written with arrows meaning "the source is a prerequisite of the target", they are as follows (this book's arrangement; the arrow labelled with the band form of the diffusive-relaxation-time assumption holds only under the band statement of § 05).

$$(\mathrm{S1}) \to (\mathrm{S3}) \to (\mathrm{S6}),\ (\mathrm{S8}); \qquad (\mathrm{S1}),(\mathrm{S2}) \to (\mathrm{S4}) \xleftarrow{(\xi')\text{band}} (\mathrm{S5}); \qquad (\mathrm{S2}),(\mathrm{S6}) \to (\mathrm{S7}) \quad (3)$$

Along the line of this chapter, what is in hand in four dimensions is the reflection-positivity preservation statement and the OS reconstruction statement (the part that is the reconstruction theorem). One is tempted to say "once the limit can be taken, the rest is automatic", but the footnote of the problem statement does not accept weak existence by compactness alone as a solution unless the properties of the limit ('such as the existence of a mass gap and the axioms') are shown separately (Jaffe–Witten) Cited .

$$\underbrace{\mu_{\beta_j} \to \mu \ \ (\text{subsequence})}_{\text{compactness only}} \implies (\mathrm{S2})\ \text{survives};\ \ (\mathrm{S4})\cdot(\mathrm{S6})\cdot(\mathrm{S8})\ \text{nothing can be said} \quad (4)$$

Example. Three arrows enter the physical-unit gap statement, and only the one from the relaxation-time band at the bookkeeping length is an arrow under the band statement. Only the infinite-volume uniqueness statement enters the Wilson-loop limit statement because the order is to make the volume infinite first.

03

The direction of the lattice version of the mass gap — a uniform lattice gap is on the opposite side

If the "mass gap" on the lattice is read as "the gap in lattice units $m_{\mathrm{lat}}(\beta)$ is at least a fixed value for every β", in the continuum limit it means the opposite. As in (1), the mass in physical units is the mass in lattice units divided by the lattice spacing.

$$\inf_{\beta} m_{\mathrm{lat}}(\beta) \geq c > 0 \implies \Delta(\beta) \geq \frac{c}{a(\beta)} \to \infty \qquad (a \to 0) \quad (5)$$

Proved here Just divide (1) by $a(\beta)$. As a metaphor, if the graduations of a ruler are made finer and finer but the thing measured is always a few graduations long, the actual length of the thing shrinks to 0. Precisely, under (5), ξ is bounded, and the correlation length in physical units $a\xi$ goes to 0. What the physical-unit gap statement asks for is the opposite: that m_{lat} goes to 0 at the same rate as a.

$$(\mathrm{S4}):\quad m_{\mathrm{lat}}(\beta) \sim \Delta_{\infty}\, a(\beta), \qquad \text{in particular}\ \ \xi(\beta) \to \infty \quad (\beta \to \infty) \quad (6)$$

This direction is exactly the lattice correlation-length problem of Chatterjee (2018): 'Show that for any β > 0, there exists some ξ(β) ∈ (0, ∞) such that ... Moreover, prove that $\lim_{\beta\to\infty} \xi(\beta) = \infty$.' Cited How to fix the scale is in 5-09.

Example: Table 2 takes the string tension σ as the physical unit and regards $a\sqrt{\sigma}$ as the lattice spacing. Over $\beta = 2.25 \to 2.60$, a becomes 1/3.16 and m_{lat} 1/2.70, while $\Delta/\sqrt{\sigma}$ stays at 3.3–3.8. The last column is the case where m_{lat} stops at its value at $\beta = 2.25$; it increases in inverse proportion to a.

Reading Table 2 Monte Carlo values from Table 1 of Lucini–Teper (2001) ($SU(2)$, lattices 8^4 to 24^4), and their ratios (errors propagated as independent)

β	$a\sqrt{\sigma}$	m_{lat}	$\Delta/\sqrt{\sigma}$	Case with m_{lat} held
2.25	0.4231(25)	1.390(30)	3.285(74)	3.29
2.30	0.3108(17)	1.090(33)	3.507(108)	4.47
2.40	0.2634(14)	0.953(19)	3.618(75)	5.28
2.475	0.2016(13)	0.754(10)	3.740(55)	6.89
2.55	0.15896(63)	0.586(10)	3.686(65)	8.74
2.60	0.13395(62)	0.514(8)	3.837(62)	10.38

$\Delta/\sqrt{\sigma}$ mostly increases and approaches the extrapolated value 3.844(61) of the original from below (it drops once, within errors, over $\beta = 2.475 \to 2.55$) Reading . The window is $\xi \lesssim 2$ lattice spacings, and it is not evidence that the limit exists (16-08).

04

Restating with the Poincaré constant — the statement of the slowness of fluctuations, and three C_P

We restate the physical-unit gap statement with the Poincaré constant C_P of the lattice measure μ_β. It is the supremum of the ratio of the variance of an observable to its sensitivity to the motion of the links (the Dirichlet form $\mathcal{E}$), and it is the time over which the slowest fluctuation is smoothed out in Langevin dynamics, which shakes the links while moving them towards lower action (1-10, 23-00 § 04).

$$C_P = \sup_f \frac{\mathrm{Var}_{\mu_\beta}(f)}{\mathcal{E}(f)}, \quad \mathcal{E}(f) = \mathbb{E}_{\mu_\beta}\Big[\sum_\ell |\nabla_\ell f|^2\Big], \qquad \text{(S5)}: \; a(\beta)^2\, C_P(\beta) \asymp 1 \quad (7)$$

∇_ℓ is the derivative in the direction of link ℓ (measured in the angle of the link), and $A \asymp B$ means that the ratio is bounded above and below by positive constants independent of β. Langevin time has the dimension of length squared (diffusion), so $a^2 C_P$ is the slowest relaxation measured in physical units, and the relaxation-time band at the bookkeeping length says it goes neither to 0 nor to infinity.

Three quantities are called by the same symbol C_P, so we distinguish them (Table 3). The C_P of this part is the path C_P, the Poincaré constant of the path measure (the distribution of all links in spacetime).

Table 3 Three C_P (the free-field column is for mass m with continuous time)

	Measure	Quantity it is tied to	Free field
Vacuum C_P	The vacuum distribution at a single time $\lvert\psi_0\rvert^2$	Gap of the Hamiltonian $\Delta = 1/(2C_P^{(\alpha)})$ Cited (ground-state transformation, 2-05, 1-10)	$1/(2m)$
Path C_P	The distribution of all fields in spacetime (paths)	The Langevin gap $\lambda_1 = 1/C_P$. Tied to m only through the relaxation chain	$1/m^2$
Path C_P measured in a graduation	The same measure as the path C_P	The graduation of $\mathcal{E}$. Multiplying $\mathcal{E}$ by c multiplies C_P by $1/c$	In angles $C_P^{(\vartheta)} = \frac{q^2}{4}\xi^2$, in fields $C_P^{(A)} = \xi^2$

For the free field $C_P^{(\beta)} = 4\,(C_P^{(\alpha)})^2$ Proved here (just square the free-field column of Table 3; on a time lattice of spacing 1 the right-hand side is multiplied by $1 + m^2/4$). The relaxation-time assumption is a sentence about the path C_P, so it can be reread as "the C_P of the paths agrees with the square of the C_P of the vacuum". In a gauge theory, however, the measure at a single time lives on the space of gauge orbits, and this rereading is an analogy from the form of the free field.

For the free field (with the graduation measured in field variables), C_P is the largest eigenvalue of the covariance $(-\Delta_{\text{lat}} + m_{\text{lat}}^2)^{-1}$ (1-10)Cited .

$$C_P = \frac{1}{m_{\text{lat}}^2} \quad \Longrightarrow \quad a^2 C_P = \frac{1}{\Delta^2} \ (8)$$

For the free field, the physical-unit gap statement and the relaxation-time band at the bookkeeping length say the same thing, and a uniform lattice gap $m_{\text{lat}} \geq c$ is $C_P \leq 1/c^2$, that is, the side of $a^2 C_P \to 0$. Table 4 is an example with $\Delta = 1$ held fixed and a made smaller.

Proved here Table 4 The free field on the four-dimensional lattice ($C_P = 1/m_{\text{lat}}^2$; ξ from $m_{\text{lat}}^2 = 2(\cosh(1/\xi) - 1)$. Values of the closed formulas)

a	m_{lat}	C_P	ξ	$a^2 C_P$	$a\xi$
1	1	1	1.0390	1	1.0390
1/4	0.25	16	4.0104	1	1.0026
1/16	0.0625	256	16.003	1	1.0002
2^{-k}	1 (held)	1	1.0390	4^{-k}	$1.039 \cdot 2^{-k}$

Each halving of a makes C_P four times and ξ twice as large, with $a^2 C_P = 1$ and $a\xi \to 1$. If the mass in lattice units is held at 1, $\Delta = 2^k$ diverges while $C_P = 1$ stays. Whether C_P and ξ^2 grow with the same power in a theory with interaction is the relaxation-time assumption of the next section.

Detour: two times

A subsection you may skip. For the free field the gap of the transfer matrix (Euclidean time) is $\approx m_{\text{lat}}$, and the gap $1/C_P$ of Langevin dynamics (the "fifth time") is m_{lat}^2. As a metaphor, it is the difference between the distance a wave reaches and the time ink takes to smear; precisely, Langevin dynamics is diffusion, and evening out a length ξ takes on the order of ξ^2 (18-03, the detour of 23-00).

05

The diffusive-relaxation-time assumption — what we want to measure, why we restate it, what it says

In one sentence, the relaxation-time assumption is the sentence "the time over which fluctuations are smoothed out grows with the same power as the square of the correlation length". In this section and the next we explain this sentence in six stages: (1) what we want to measure, (2) why we restate it as a quantity of a dynamics, (3) the statement, step by step, (4) what the relaxation-time assumption gives, (5) what the relaxation-time assumption does not give, (6) analogous cases. Each stage has a small example that can be checked by hand.

Its place in this part. The relaxation-time assumption is one restatement chosen in order to get a handle on the problem of the mass gap. It is neither the only road to the mass gap nor a road settled to be right. Viewpoints other than the relaxation-time assumption, and new ways of thinking that take the relaxation-time assumption as a premise, are dealt with in Part 24.

Why this part puts the relaxation-time assumption at the axis. The quantity the problem statement asks for is the gap in the direction of Euclidean time (the gap of the transfer matrix measured in physical units). The relaxation-time assumption replaces it with a quantity — the relaxation time C_P of Langevin dynamics and a power of the correlation length — that is fixed by the lattice measure alone, for which

tools exist to assemble inequalities "for every f" from local parts, and whose algebra of assembly can be checked by machine. As we see in § 06, this replacement can return to the side of the mass gap through the one-sided bridge; moreover, under the assumption of the one-sided bridge, the upper-bound assumption, the upper half of the relaxation-time assumption, is a stronger sentence than the gap in physical units on the lattice staying positive. Saying a stronger sentence in a language with more tools: this trade is the standpoint of this part, and all the chapters of Part 23 are built with the relaxation-time assumption at the axis. That is what the title of this part, a study taking the relaxation-time assumption as a handle, means.

(1) What we want to measure — the ruler and the smearing time

The part of the demand of the problem statement that can be stated on the lattice is (6): "the gap in physical units $\Delta = m_{\mathrm{lat}}/a$ stays positive as $\beta \to \infty$". Seen on the lattice, two things must happen at once: the ruler $\xi = 1/m_{\mathrm{lat}}$ grows without bound compared with the graduations of the lattice, and the way it grows matches the inverse of the lattice spacing ($a\xi$ goes neither to 0 nor to infinity). In terms of the five quantities of 23-00 § 05, the problem statement is a sentence asking about the relation between the ruler and the number of rescalings.

The ruler	The smearing time	The number of rescalings	The graduation of the clock	The clock that runs slow
The correlation length $\xi = 1/m$ lat the reciprocal of the gap in Euclidean time	The Poincaré constant C_P the reciprocal of the gap of Langevin dynamics	The bookkeeping length $\xi_{\mathrm{RG}} = 1/a$ the reciprocal of the lattice spacing (§ 07)	The normalisation of $\mathcal{E}$ the path C_P measured in a graduation, Table 3	Renormalisation of stochastic time the detour of § 07

The relaxation-time assumption brings the smearing time into this question. The time C_P over which the slowest fluctuation in the measure is smoothed out is a diffusion time, so evening out a length ξ should take on the order of ξ^2 (the detour of 23-00). So we make "the smearing time balances the square of the ruler" into one sentence, and state the relation between the ruler and the number of rescalings through the smearing time. That is the idea of the relaxation-time assumption.

Example 1 (counting the random walk on a single road by hand). On the single road of 23-00 § 04 (L cells, reflecting at both ends, uniform distribution, $\mathcal{E}(f) = \frac{1}{L}\sum_i (f_{i+1} - f_i)^2$), what corresponds to the ruler is the length of the road L. For $L = 2$, $f = (1, -1)$ is the slowest fluctuation, with $\mathrm{Var}(f) = 1$ and $\mathcal{E}(f) = \frac{1}{2} \cdot 2^2 = 2$, and the ratio is $\frac{1}{2}$. For $L = 3$, with $f = (1, 0, -1)$, $\mathrm{Var}(f) = \frac{2}{3}$ and $\mathcal{E}(f) = \frac{1}{3}(1 + 1) = \frac{2}{3}$, and the ratio is 1. Both agree with the values of $C_P(L) = 1/(2(1 - \cos(\pi/L)))$ from (3) of 23-00, and for $L = 4$ it is $1/(2 - \sqrt{2}) \approx 1.707$ Proved here . Doubling L makes C_P roughly four times as large, and for large L, $C_P \approx L^2/\pi^2$ (Table 3 of 23-00). This checks "the smearing time is the square of the ruler" on the smallest road.

Precisely, the L of the single road is a length given from outside, not a ruler the measure carries itself (a correlation length). The simplest example where the two agree is the free field: from $C_P = 1/m_0^2$ and $m_0^2 = 2(\cosh(1/\xi) - 1)$, $C_P/\xi^2 \to 1$ (23-00 § 04, Table 4 of this chapter). How the smearing time and the ruler balance in a measure with interaction is the question of the relaxation-time assumption.

(2) Why restate it as a quantity of a dynamics — measurable, with tools, within reach of checking

The quantity on the side of the ruler, m_{lat}, is the gap of the transfer matrix, a quantity of an operator on the Hilbert space of gauge-invariant states. Placing one trial state gives an upper bound on m_{lat} (a lower bound on ξ), but a lower bound needs all states. An upper bound on C_P also needs "every f", so the difficulty remains in the same direction. There are still three reasons to restate it.

(a) It can be measured from the lattice measure alone. As in (7), C_P is fixed by expectation values under μ_β alone: the variance and the Dirichlet form. It can be defined directly from the measure, without going through the Hilbert space or reconstruction; one test function gives a lower bound, and readings of lower bounds can also be taken from the autocorrelations of Langevin dynamics or Markov chains (23-02 § 04).

(b) There are tools to assemble "every f". For Poincaré inequalities, tools that assemble an upper bound on the whole from local parts are available in a form independent of dimension. With the conventions measure $\mu \propto e^{-S}$ and $\mathcal{E}(f) = \mathbb{E}|\nabla f|^2$,

$$\mathrm{Hess}\, S \geq \rho \;\Rightarrow\; C_P(\mu) \leq \frac{1}{\rho}; \qquad \mu' \propto e^{-W}\mu \;\Rightarrow\; C_P(\mu') \leq e^{\operatorname{osc} W}\, C_P(\mu); \qquad C_P(\mu_1 \otimes \mu_2) = \max\bigl(C_P(\mu_1),$$

The two on the left are Cited (the Bakry–Émery convexity criterion and the Holley–Stroock bounded perturbation; $\operatorname{osc} W = \sup W - \inf W$. Bibliographic data in the table of sources of 23-06), and the one on the right is Proved here (use the Poincaré inequality of each component on each term of the Efron–Stein inequality $\mathrm{Var}(f) \leq \mathbb{E}\,\mathrm{Var}_1(f) + \mathbb{E}\,\mathrm{Var}_2(f)$ to get $\leq \max$; equality for a function of one coordinate only). Adding to these the decomposition of the variance (the sum of the conditional variance and the variance of the conditional expectation, 23-05 § 01), these four are the parts from which the first step is assembled. The logarithmic Sobolev inequality (LSI) has tools of the same type, among them convexity criteria split by scale (23-02 § 05).

(c) Machine checking can reach it. The stage of combining the inequalities of the parts is algebra of real inequalities. The algebra can be checked in Lean, and numerical inequalities inside a finite window can be confirmed by interval arithmetic (computer-assisted). What is checked is the assembly from assumptions to conclusions, not the assumptions themselves (23-13).

Example 2 (where the tools work, for a Gaussian measure on two sites). Consider the measure $\mu \propto e^{-S}$ on $\mathbb{R}^2$, with $S(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}m_0^2(x_1^2 + x_2^2)$. The Hessian is $\begin{pmatrix} 1+m_0^2 & -1 \\ -1 & 1+m_0^2 \end{pmatrix}$, with eigenvalues m_0^2 (direction $(1,1)$) and $m_0^2 + 2$ (direction $(1,-1)$). The criterion on the left of (9) gives $C_P \leq 1/m_0^2$. On the other hand, for $f = x_1 + x_2$, since the covariance is the inverse of the Hessian, $\mathrm{Var}(f) = 2/m_0^2$ and $\mathcal{E}(f) = |\nabla f|^2 = 2$, and the ratio is $1/m_0^2$. Upper and lower bounds agree, and $C_P = 1/m_0^2$ Proved here. For the single road (Example 1) we found C_P by diagonalising a matrix; in this example (9) gives the answer without diagonalisation. For the double well $V(x) = (x^2-1)^2/8$, at the top $V''(0) = -\frac{1}{2} < 0$, and the convexity criterion of (9) cannot be used on the whole. So one combines the decomposition of the variance, splitting valley by valley, with convexity and perturbation after the split (23-04 § 06, 23-05 § 04). That an upper bound can be assembled from parts for a measure with interaction that cannot be diagonalised — this is the first gain of the restatement.

(3) The statement of the relaxation-time assumption, step by step — the band statement, the power statement, the upper and lower halves

First step: the band statement. Written plainly, it becomes the sentence "C_P lies within the width of a constant factor of the square of the ruler". Comparing with the physical unit a and a width of constant

factors independent of β (a band),

$$\exists\,\beta_0,\ \exists\, 0<c\le C:\qquad c\le a(\beta)^2\,C_P(\beta)\le C\qquad(\beta\ge\beta_0)\ \text{(10)}$$

In numbers, it is the sentence that when $\xi_{\mathrm{RG}}=1/a=10$, C_P lies between $100c$ and $100C$, and if $\xi_{\mathrm{RG}}=1000$, between 10^6c and 10^6C. Langevin time has the dimension of length squared (diffusion), so a^2C_P is the slowest relaxation measured in physical units, and (10) says it goes neither to 0 nor to infinity.

Example 3 (the single road satisfies the band statement). Taking L as the ruler for the single road of Example 1 and setting $x=\pi/L$, $C_P(L)/L^2=\frac{1}{\pi^2}\cdot\frac{(x/2)^2}{\sin^2(x/2)}$. Since $t/\sin t$ increases on $(0,\pi/2]$, for $L\ge 2$

$$\frac{1}{\pi^2}\;<\;\frac{C_P(L)}{L^2}\;\le\;\frac{1}{8}\ \text{(11)}$$

(the equality on the right is $C_P=\frac12$ at $L=2$), and the band statement holds with $c=1/\pi^2\approx 0.101$, $C=1/8$ Proved here .

Inserting the correlation length, the band statement splits into three factors.

$$a^2C_P=\underbrace{(a\xi)^2}_{\text{chains (ii)(iii)}}\cdot\underbrace{\frac{C_P}{\xi^2}}_{\text{chain (i)}},\qquad (a\xi)^2=\frac{1}{m_{\mathrm{phys}}^2},\qquad \frac{C_P}{\xi^2}=\frac{m_{\mathrm{fat}}^2}{\lambda_1}\ \text{(12)}$$

The first factor is determined by the mass in physical units $m_{\mathrm{phys}}=m_{\mathrm{lat}}/a$, and the second is the factor of the relaxation chain, "the Langevin gap λ_1 agrees with the square of the mass". Further splitting the first factor into "the square of the mass agrees with the string tension" (the one-scale chain) and "the string tension agrees with the square of the lattice spacing" (the bookkeeping-length chain), the three ratios are the three chains (17-12). All three are assumptions Assumption . What the upper and lower halves of each chain need is seen in 23-02 § 05.

Second step: the power statement. The band statement is moved by the logarithmic factors brought in by the graduation of $\mathcal{E}$, the order of the bookkeeping and the renormalisation of stochastic time (the detour of § 07). Lowering it to a level that ignores not only constant factors but also factors of powers of $\ln\xi$ gives the relaxation-time assumption.

$$(\xi^2):\qquad \lim_{\beta\to\infty}\frac{\ln C_P(\beta)}{\ln\xi(\beta)}=2\ \text{(13)}$$

This sentence has the same form as (2) of the statement of the first step, 23-03 § 05, $\lim_{\hbar\to0}\ln C_P(\hbar)/\ln\xi(\hbar)=2$. Only the family of measures differs: here the Wilson measure μ_β with $\beta\to\infty$, on the first step the path measure of the double well with $\hbar\to0$. So in this part (13) is treated as a dimension-independent power statement, called "the diffusive-relaxation-time assumption" at the stage of the four-dimensional lattice and "the double-well relaxation-time power statement" on the first step. Being a power statement, it does not change when $\mathcal{E}$ is multiplied by a power of β ($\ln\beta/\ln\xi\to0$) Proved here Lean `exponent_invariant_of_slowly_varying` (assumptions: $X_n,\ell_n>0$, $\ln X_n/\ln a_n\to p$, $\ln\ell_n/\ln a_n\to0$. Conclusion: $\ln(X_n\ell_n)/\ln a_n\to p$. Algebra of sums of limits).

Example 4 (the difference between band and power, in numbers). Consider the family $C_P=\xi^2\ln\xi$. The ratio $C_P/\xi^2=\ln\xi$ keeps increasing, $2.30,\ 6.91,\ 13.8$ at $\xi=10,10^3,10^6$, so the band statement breaks. On the other hand $\ln C_P/\ln\xi=2+\ln\ln\xi/\ln\xi$ approaches 2 slowly, $2.362,\ 2.280,\ 2.190$, and the power statement holds Proved here (divide the values $0.834,\ 1.933,\ 2.626$ of $\ln\ln\xi$ by $\ln\xi=2.303,\ 6.908,\ 13.82$). Conversely the single road of Example 3 satisfies the band statement, but $\ln C_P/\ln L=2-(\ln\pi^2$

$+ o(1))/\ln L$ is $\ln 6640/\ln 256 \approx 1.587$ at $L = 256$, far from 2. The power statement says only "2 in the limit", and allows a slow approach by the amount of the constants. The band statement implies the power statement ($\ln c/\ln \xi \to 0$), and the converse fails, as $C_P = \xi^2 \ln \xi$ shows Proved here .

(10) is a sentence that combines the relaxation-time assumption with the one-scale chain and the bookkeeping-length chain, and in this book it is not called the relaxation-time assumption. The names are fixed as in Table 5.

Table 5 Readings and names of "C_P agrees with the square of a length" (eight detailed readings in A-06)

Level, length compared	Sentence	Name in this book	In the free approximation
Power, ξ	$\ln C_P/\ln \xi \to 2$	The diffusive-relaxation-time assumption	Holds
Power, bookkeeping length	$\ln C_P/\ln \xi_{\mathrm{RG}} \to 2$	The relaxation-time assumption with the one-scale chain and the bookkeeping-length chain (power)	Holds
Band, ξ	$C_P \asymp \xi^2$	$(\xi^2)_{\mathrm{band}}[\mathcal{E}_\vartheta]$, $(\xi^2)_{\mathrm{band}}[\mathcal{E}_A]$ (naming the graduation)	Equality in the field graduation; in the angle graduation the lower half breaks with ratio $g^2/4$
Band, bookkeeping length	$a^2 C_P \asymp 1$ ((10))	Band statement with the one-scale chain and the bookkeeping-length chain = the relaxation-time band at the bookkeeping length	Determined by the order of the bookkeeping (§ 07)

The two rows at the power level are moved neither by the choice of graduation nor by the logarithmic factors of § 07. That is why this part places the relaxation-time assumption as a power statement.

Third step: upper and lower halves. Separating the right and left of the band statement (10) gives the upper-bound assumption and the lower-bound assumption.

$$(\xi^2)\uparrow:\ a^2 C_P \le C\ \ (C_P \lesssim \xi_{\mathrm{RG}}^2), \qquad (\xi^2)\downarrow:\ a^2 C_P \ge c\ \ (C_P \gtrsim \xi_{\mathrm{RG}}^2) \quad (14)$$

C_P is the supremum of (7), and $\xi_{\mathrm{RG}} = 1/a$ is a length fixed separately from the measure, so the two have opposite quantifiers over the test function f.

$$(\xi^2)\downarrow \iff \forall \beta \ge \beta_0\ \exists f:\ \mathrm{Var}(f) \ge c\,\xi_{\mathrm{RG}}^2\,\mathcal{E}(f), \qquad (\xi^2)\uparrow \iff \forall \beta \ge \beta_0\ \forall f:\ \mathrm{Var}(f) \le C\,\xi_{\mathrm{RG}}^2\,\mathcal{E}(f) \quad (15)$$

Proved here This just writes out the definition of the supremum. The lower-bound assumption needs only one test function to be shown; the upper-bound assumption is a genuine Poincaré inequality over all functions. "Easy from above, hard from below" for eigenvalues (1-10) becomes, for their reciprocal C_P, "easy from below, hard from above". This asymmetry determines the whole of the roads and walls of 23-02.

Example 5 (families where only one holds). In the family $C_P = \xi_{\mathrm{RG}}^2 \ln \xi_{\mathrm{RG}}$, for $\xi_{\mathrm{RG}} \ge e$ we have $a^2 C_P \ge 1$, so the lower-bound assumption holds with $c = 1$, and since $a^2 C_P = \ln \xi_{\mathrm{RG}} \to \infty$, the upper-bound assumption breaks (the power statement holds, as in Example 4). Conversely, in a family where C_P is bounded independently of β (the form of the theorems on the strong-coupling side, Table 7 in § 06), the upper-bound assumption holds and the lower-bound assumption breaks Proved here . The band statement is the sentence that neither of these two ways of breaking happens.

Note — "the relaxation-time assumption" and "the four double-well conditions"

In this part "the relaxation-time assumption" refers only to the power statement (13). The upper-bound assumption and the lower-bound assumption are the halves of the band statement, with

names separate from the relaxation-time assumption. The four conditions into which the upper side of the double-well power statement is split on the first step are written "the four double-well conditions", always with the word "conditions" (23-03 § 05, 23-05). The in-phase local Poincaré condition is not "the first half of the relaxation-time assumption" but a different statement, "local Poincaré within a phase" for the model of the first step. The letters in Lean theorem names (H1 and so on) are not changed.

Note — topological sectors, and the order of making the volume infinite

The supremum of (7) runs over every f, so the (smoothed) topological charge Q, which can no longer move between sectors in the continuum limit, is also among the test functions. On four-dimensional lattices with periodic boundaries, the autocorrelation time of Q is reported to grow clearly faster than ξ^2 Reading (Del Debbio–Panagopoulos–Vicari 2002 with local heat bath and overrelaxation, Schaefer–Sommer–Virotta 2011 with HMC-type algorithms, Bonanno–Bonati–D'Elia 2021 with local algorithms; as a power the exponent is about 5, or else exponential in $1/a$, and all three write that power and exponential cannot be distinguished by the data). The groups of the three original sources are all $SU(3)$ or larger, and whether the exponent exceeds 2 for $SU(2)$, where this book's relaxation-time assumption is stated, is not settled in the primary sources found. There is one primary source found that measures the dependence of the autocorrelation of the topological charge on the lattice spacing for $SU(2)$ (Abe–Morikawa–Suzuki 2025; HMC, periodic boundaries, small boxes with the physical side held fixed, three lattice spacings). There the integrated autocorrelation time is reported to grow close to a^{-2} Reading . So the following reading is a reading on the original sources for $SU(3)$ and larger. For reversible dynamics, the integrated autocorrelation time of any observable does not exceed C_P ($\tau_{\mathrm{int}} \le C_P$ Cited , 23-02 § 05), so under the conjecture that HMC and Langevin are in the same universality class Assumption , the upper-bound assumption can be read as false in a periodic box. Open boundaries remove this obstacle only inside a finite box with the physical side held fixed (in Lüscher–Schaefer 2011, the autocorrelation time of Q^2 with open boundaries grows roughly in proportion to the length in the time direction Reading). In this book's order, which makes the volume infinite first, this is expected not to help Assumption . If the relaxation-time assumption is to be kept, candidate directions are to rewrite it as a statement sector by sector, or as a local statement with support limited to a few correlation lengths; that study is dealt with in Part 24 (24-03, 24-04).

06

What the diffusive-relaxation-time assumption gives and what it does not — the one-sided bridge, a counterexample, analogous cases

The remaining three of the six stages: (4) what the relaxation-time assumption gives, (5) what the relaxation-time assumption does not give, (6) analogous cases. (4) and (5) start from the fact that the quantity of the problem statement (the gap in Euclidean time) and the quantity of the relaxation-time assumption (the Langevin relaxation time) are different quantities, and fix precisely the relation between the relaxation-time assumption and the mass gap through the one-sided bridge joining the two and an example in which the converse of the bridge fails.

(4) What the relaxation-time assumption gives — rereading as a mass, and the one-sided bridge

The version with equality. If the relaxation chain is put as the equality $C_P = 1/m_{\mathrm{lat}}^2$, the two halves of (14) are reread as statements about the mass in physical units $m_{\mathrm{phys}} = m_{\mathrm{lat}}\,\xi_{\mathrm{RG}}$.

$$(\xi^2)\uparrow \implies m_{\mathrm{phys}} \;\geq\; C^{-1/2}, \qquad (\xi^2)\downarrow \implies m_{\mathrm{phys}} \;\leq\; c^{-1/2} \quad (16)$$

Proved here Proof: multiply both sides of $1/m_{\mathrm{lat}}^2 \leq C\xi_{\mathrm{RG}}^2$ in the upper-bound assumption by m_{lat}^2/C to get $1/C \leq (m_{\mathrm{lat}}\xi_{\mathrm{RG}})^2$. For the lower-bound assumption, the same with the inequality reversed. ∎ The level of proof is algebra, and two Lean theorems check this algebra.

```
theorem hup_is_gap_positivity {CP xi mlat K : ℝ}
    (hmlat : 0 < mlat) (hCP : CP = 1 / mlat ^ 2) (h : CP ≤ K * xi ^ 2) :
    1 ≤ K * (mlat * xi) ^ 2

theorem hdown_is_mass_finiteness {CP xi mlat c : ℝ}
    (hmlat : 0 < mlat) (hCP : CP = 1 / mlat ^ 2) (h : c * xi ^ 2 ≤ CP) :
    c * (mlat * xi) ^ 2 ≤ 1
```

Lean `LatticeGauge.hup_is_gap_positivity` (assumptions: $m_{\mathrm{lat}} > 0$, $C_P = 1/m_{\mathrm{lat}}^2$, $C_P \leq K\xi^2$. Conclusion: $1 \leq K(m_{\mathrm{lat}}\xi)^2$) and `LatticeGauge.hdown_is_mass_finiteness` (assumptions: $m_{\mathrm{lat}} > 0$, $C_P = 1/m_{\mathrm{lat}}^2$, $c\,\xi^2 \leq C_P$. Conclusion: $c\,(m_{\mathrm{lat}}\xi)^2 \leq 1$). Inside Lean all four letters are named real numbers; that it is the Poincaré constant of the Wilson measure, and that C_P is a supremum (running over every f), are outside Lean (23-13). The upper-bound assumption is "the mass does not collapse to 0", that is, a lower bound on the side of the mass gap; the lower-bound assumption is the side "the mass does not escape to infinity".

The equality $C_P = 1/m_{\mathrm{lat}}^2$ is the value for the free field, and is too strong an assumption to put down for a theory with interaction. However, the proof that derives a lower bound on the mass from the upper-bound assumption uses only one side of the equality, $C_P \geq \kappa/m_{\mathrm{lat}}^2$. And on a lattice with continuous time, this one side can be shown as a proposition with its assumptions named.

The one-sided bridge (continuous time, finite spatial lattice, $T = \infty$)

Take the space to be a finite lattice, the space of configurations at a single time to be Q ($\mathbb{R}^n$, or a compact connected manifold), the metric of the kinetic term M, $H = -\frac{\hbar^2}{2}\Delta_M + V$, $A = (H - E_0)/\hbar$. Assume the following. (A1) The ground-state assumption: the spectrum of H is discrete, the ground state ψ_0 is non-degenerate, smooth and positive, and $\Delta = E_1 - E_0 > 0$. (A2) The stationary path-measure assumption: the path measure is the stationary measure at $T = \infty$ (the distribution at a single time is ψ_0^2, the transition is $e^{-\tau A}$). (A3) The metric comparison assumption: the metric of the Langevin Dirichlet form $\mathcal{E}_G(f) = \hbar \int \mathbb{E}\langle d_{q(\tau)}f, G\, d_{q(\tau)}f\rangle d\tau$ satisfies $G \leq \lambda_G M^{-1}$, and $\kappa := 1/\lambda_G$. Then, for $\xi = \hbar/\Delta$, $F = \psi 1/\psi 0$ and $f_T = \int_0^T F(q(\tau))\,d\tau$, for every $T > 0$

$$\frac{\mathrm{Var}(f_T)}{\mathcal{E}_G(f_T)} \;\geq\; \kappa\xi^2\Big[1 - \frac{\xi}{T}\big(1 - e^{-T/\xi}\big)\Big], \qquad \text{hence} \qquad C_P \;\geq\; \kappa\xi^2 = \frac{\kappa\hbar^2}{\Delta^2} \quad (17)$$

holds. With the canonical choice $G = M^{-1}$ ($\kappa = 1$), the supremum as $T \to \infty$ over the family of test functions local in time is exactly ξ^2.

Proved here Proof [Y3]. Set $m = \Delta/\hbar = 1/\xi$ and $\varphi = (F - \langle F\rangle)\psi_0$. Since $\langle F\rangle = \int \psi_0\psi_1 = 0$, $\varphi = \psi_1$ and $A\varphi = m\varphi$.

(i) The variance. Under the stationary path-measure assumption (A2), the correlations of the measure are $\mathrm{Cov}(F(q(s)), F(q(t))) = \langle \varphi, e^{-|s-t|A}\varphi\rangle = \|\varphi\|^2 e^{-m|s-t|}$ Cited (the formula writing correlations with the transfer operator, of the type of (12) in 2-07 § 08 and 23-04 § 04). Integrating over the square $[0,T]^2$ gives $\mathrm{Var}(f_T) = \|\varphi\|^2\big(2T/m - 2(1-e^{-mT})/m^2\big)$ (close $\iint e^{-m|s-t|} = 2\int_0^T (T-u)e^{-mu}du$ with one integration by parts).

(ii) The Dirichlet form. Differentiating f_T with respect to the path $q(\tau)$ gives $dF(q(\tau))$ for $\tau \in [0,T]$ and 0 otherwise. From stationarity and the metric comparison assumption (A3), $\mathcal{E}_G(f_T) = \hbar T \int \psi_0^2 \langle dF, G\, dF\rangle \le \hbar T \lambda_G \int \psi_0^2 |dF|_M^2 - 1$.

(iii) The ground-state transformation identity. For a smooth real function F,

$$\big\langle F\psi_0, (H - E_0)F\psi_0\big\rangle = \frac{\hbar^2}{2}\int \psi_0^2\, |dF|_M^2 - 1 \quad (18)$$

Proved here. Using $(V - E_0)\psi_0 = \frac{\hbar^2}{2}\Delta_M\psi_0$ gives $(H-E_0)(F\psi_0) = -\frac{\hbar^2}{2}\big(\psi_0\Delta_M F + 2\langle dF, d\psi_0\rangle_M - 1\big) = -\frac{\hbar^2}{2\psi_0}\,\mathrm{div}_M(\psi_0^2\,\nabla_M F)$, and multiplying by $F\psi_0$ and integrating by parts once gives the right-hand side (boundary terms vanish because Q is compact without boundary, or the functions decay at infinity). ∎ It has the same form as the one-dimensional calculation of 1-10 § 08 and 2-05 § 08. Since $F\psi_0 = \psi_1$, the left-hand side is $\Delta\|\varphi\|^2$, and $\int \psi_0^2 |dF|_M^2 - 1 = (2m/\hbar)\|\varphi\|^2$; hence $\mathcal{E}_G(f_T) \le 2\lambda_G m T \|\varphi\|^2$.

(iv) The ratio. Dividing (i) by (ii)(iii) gives $\mathrm{Var}/\mathcal{E}_G \ge \kappa\big(1/m^2 - (1-e^{-mT})/(Tm^3)\big)$, which is the left of (17). C_P is a supremum, so it is at least the right-hand side, and as $T\to\infty$, $C_P \ge \kappa\xi^2$. For the supremum in the canonical case, any f local in time gives $\mathrm{Var}/T \to 2\langle\varphi, A^{-1}\varphi\rangle$ and $\mathcal{E}/T = 2\langle\varphi, A\varphi\rangle$, and using $1/a \le a/m^2$ for the eigenvalues $a \ge m$ of A it is at most ξ^2, with equality when φ lies in the eigenspace of eigenvalue m (the same as (13) of 23-04 § 04). ∎

Lean `LatticeGauge.HupBridge.cont_ratio_closed` (assumptions: $0 < m$, $0 < T$. Conclusion: the ratio of (i) divided by (ii)(iii) equals $1/m^2 - (1-e^{-mT})/(Tm^3)$) and `one_sided_bridge` (assumptions: a lower bound on the variance, an upper bound on the Dirichlet form, $\mathrm{Var}/\mathcal{E} \le C_P$, positive numbers. Conclusion: $\kappa/m^2 \le C_P$) check only the algebra of numbers in (i)–(iv). The path measure, the formula for correlations, (18), and that C_P is a supremum are outside Lean.

Table 6 The assumptions of the one-sided bridge, and where they are used

Symbol	Assumption	Where it is used
The ground-state assumption (A1)	Discrete spectrum, $\Delta > 0$, $\psi_0 > 0$ smooth and non-degenerate, ψ_1 exists	$F = \psi_1/\psi_0$ can be defined and is smooth
The stationary path-measure assumption (A2)	The stationary path measure at $T = \infty$ (the distribution at a single time is ψ_0^2, the transition is $e^{-\tau A}$)	The formula for correlations in (i)
The metric comparison assumption (A3)	$G \le \lambda_G M^{-1}$ ($\kappa = 1/\lambda_G$)	The comparison in (ii)
Continuous time	The kinetic term of the paths has the form $\frac{1}{2}\lvert\dot q\rvert_M^2$	(18) is used with the same metric as the $\mathcal{E}$ of the path measure
Finite spatial lattice	Q is finite-dimensional	The ground-state assumption and the integration by parts in (18)

κ does not depend on the length of time T. T enters only the factor $1-(\xi/T)(1-e^{-T/\xi})\in(0,1)$ of (17), which approaches 1 as $T/\xi\to\infty$. Setting $G=M^{-1}=1$ for a one-dimensional well gives the One-Dimensional One-Sided Bridge Proposition (23-04 § 04), and the one-sided bridge is its multidimensional version with a metric. In the language of the chains, the one-sided bridge shows one side of the relaxation chain, $\lambda_1\le m_{\mathrm{lat}}^2/\kappa$ (the half on the side of the lower-bound assumption), on a lattice with continuous time. Whether a claim of the same form exists in the literature has not been checked (it may be known).

Why the power is two. The variance is the time integral of correlations, $T\cdot(2/m)\|\varphi\|^2$, that is, it grows to first order in $1/\Delta$. The Dirichlet form, for the function $F=\psi_1/\psi_0$ of the slowest excitation, whose gradient is small, is $T\cdot 2m\|\varphi\|^2$ by (18), that is, it becomes small to first order in Δ. The ratio is $1/\Delta^2$. If one overlooks the smallness of the gradient and estimates the Dirichlet form as a quantity independent of Δ, one drops to a lower bound of first order, c/Δ, and the next (19) does not follow.

Note (two meanings of the symbol Δ). In this chapter Δ is used with two meanings. The $\Delta(\beta)=m_{\mathrm{lat}}/a$ of (1) is the mass in physical units. The $\Delta=E_1-E_0$ of the one-sided bridge and the examples of this section is the gap of the Hamiltonian with space made a finite lattice, tied to the correlation length in lattice units by $\xi=\hbar/\Delta$ (with $\hbar=1$ it corresponds to m_{lat}, and the $\Delta(\beta)$ of (1) is that divided by the lattice spacing a). In Part 24, Δ is restricted to the meaning of the one-sided bridge, E_1-E_0, and the mass in physical units is written m_{phys} (24-09 § 02).

Example 6 (checking the bridge by hand on a two-site lattice). Take the same two sites as in Example 2 as the spatial lattice, with continuous time ($\hbar=1$, $M=G=1$, $V=\frac12 x^{\mathsf T}Kx$, where K is the Hessian of Example 2). The normal modes are $u=(x_1+x_2)/\sqrt2$ (angular frequency m_0) and $v=(x_1-x_2)/\sqrt2$ (angular frequency $\sqrt{m_0^2+2}$); the first excitation raises the oscillation of u by one, so $\Delta=m_0$, $\xi=1/m_0$ and $F=\psi_1/\psi_0\propto u$. On the paths, $u(\tau)$ is a stationary Gaussian process with $\mathbb{E}[u(s)u(t)]=e^{-m_0|s-t|}/(2m_0)$. For $f_T=\int_0^T u\,d\tau$, $\mathrm{Var}(f_T)=\frac{1}{2m_0}\big(\frac{2T}{m_0}-\frac{2(1-e^{-m_0T})}{m_0^2}\big)$; since $|\nabla u|^2=\frac12+\frac12=1$, $\mathcal{E}(f_T)=T$, and the ratio is $1/m_0^2-(1-e^{-m_0T})/(Tm_0^3)$, exactly the right-hand side of (17). In numbers, with $m_0=\frac12$ ($\xi=2$) and $T=20$ the ratio is $4-0.99995/2.5\approx 3.600$, and with $T=200$ it is $4-1/25=3.96$, approaching $\xi^2=4$ from below. (18) can also be checked: the left-hand side is $\Delta\,\mathbb{E}[u^2]=m_0\cdot\frac{1}{2m_0}=\frac12$ and the right-hand side is $\frac12\,\mathbb{E}|\nabla u|^2=\frac12$ Proved here . In this example the path measure is Gaussian and the largest eigenvalue of the covariance $(-\partial_\tau^2+K)^{-1}$ is $1/m_0^2$, so $C_P=\xi^2$, and the bridge holds with equality.

Combining the upper-bound assumption with the bridge. Setting the $C_P\le C\xi_{\mathrm{RG}}^2$ of the upper-bound assumption beside the $C_P\ge\kappa/m_{\mathrm{lat}}^2$ of the one-sided bridge gives

$$\frac{\kappa}{m_{\mathrm{lat}}^2}\ \le\ C_P\ \le\ C\,\xi_{\mathrm{RG}}^2\qquad\Longrightarrow\qquad m_{\mathrm{phys}}^2=(m_{\mathrm{lat}}\,\xi_{\mathrm{RG}})^2\ \ge\ \frac{\kappa}{C}\ \text{(19)}$$

Proved here . For example, with $\kappa=1$ and $C=4$, the gap in physical units stays at least $\frac12$ for every $\beta\ge\beta_0$. It is the form with the equality assumption of (16) weakened to one side, and in Lean

```
theorem hup_is_gap_positivity_of_lower {CP xi mlat K κ : ℝ}
    (hmlat : 0 < mlat) (hCP : κ / mlat ^ 2 ≤ CP) (h : CP ≤ K * xi ^ 2) :
    κ ≤ K * (mlat * xi) ^ 2
```

Lean `LatticeGauge.HupBridge.hup_is_gap_positivity_of_lower` (assumptions: $0 < m_{\mathrm{lat}}$, $\kappa/m_{\mathrm{lat}}^2 \le C_P$, $C_P \le K\xi^2$. Conclusion: $\kappa \le K(m_{\mathrm{lat}}\xi)^2$). Setting $\kappa = 1$ recovers the theorem on the left of (16) (`hup_is_gap_positivity_recovered`). The right of (16), on the side of the lower-bound assumption, uses $C_P \le 1/m_{\mathrm{lat}}^2$ (the hard side), so it cannot be weakened with the one-sided bridge.

We list the conditions for (19) to become a sentence on the side of the mass gap. First, κ must be uniform in β. In the field graduation $\mathcal{E}_A$, κ is a number independent of the coupling (its value depends on the convention for normalising the metric of the group), and the condition is met. In the link-angle graduation $\mathcal{E}_\vartheta$, $\kappa \propto g^2 \to 0$, and (19) gives $m_{\mathrm{phys}}^2 \gtrsim g^2/C$, that is, it loses by a logarithm (the path C_P measured in a graduation, Table 3). Second, the treatment of time. The one-sided bridge is a sentence on a lattice with continuous time (a Hamiltonian lattice with only space made a lattice); for the Wilson measure, which also discretises time, there is no counterpart of the ground-state transformation identity (18). On a time lattice of step 1, with $q = e^{-\Delta}$ the second eigenvalue of the transfer operator $\hat{T}$ (divided by its largest eigenvalue), putting down an assumption comparing the increment between neighbouring times with the gradient,

$$(\mathrm{D}): \quad \mathbb{E}_{\psi_0^2}|dF|_G^2 \le c_D \,\langle \varphi, (1-\hat{T})\varphi\rangle \qquad \Longrightarrow \qquad C_P \;\ge\; \frac{1+q}{c_D(1-q)^2} \;\ge\; \frac{2}{c_D\,\Delta^2} \tag{20}$$

gives a lower bound of the same type (the Discrete-Time One-Sided Bridge Proposition) Proved here (on top of (D). The variance over a window of N sites is $\|\varphi\|^2 \sum_{s,t\le N} q^{|s-t|}$, the Dirichlet form is at most $N\,c_D(1-q)\|\varphi\|^2$, and the ratio as $N \to \infty$ is the middle expression. The right inequality, with $y = \Delta/2$, is the same as $\sinh^2 y \le y^2 \cosh y\, e^y$, which follows from $\tanh y \le y$ and $\cosh y \le e^y$) Lean `pairSum_closed` (closed form of the window sum), `disc_limit_ge` (a weak version with coefficient 1; assumptions: $0 < \Delta$, $0 < c_D$). When $\hat{T} = e^{-(H-E_0)}$ holds exactly, (18) gives $c_D = 2\Delta/(1-e^{-\Delta}) \to 2$, returning to the one-sided bridge with continuous time. The transfer operator of the Wilson measure is not of this form, so (D) remains an assumption, together with a c_D uniform in β Assumption. Third, the volume. The one-sided bridge is a sentence about a finite spatial box, and the stage connecting the C_P of the infinite-volume measure with the limit of the gaps of the boxes has not been examined Assumption.

Why the question becomes one-sided on the first step. Using the correlation length fixed by the gap, $\xi = 1/m_{\mathrm{lat}}$ ($\hbar = 1$), the bridge $\kappa\xi^2 \le C_P$ gives the lower half of the power statement automatically.

$$\frac{\ln C_P}{\ln \xi} \;\ge\; 2 + \frac{\ln \kappa}{\ln \xi} \qquad (\xi > 1), \qquad \text{so} \quad \liminf \frac{\ln C_P}{\ln \xi} \ge 2 \tag{21}$$

Proved here (take ln and divide by $\ln \xi > 0$; if κ is uniform the second term goes to 0) Lean `power_lower_half_of_bridge` (assumptions: $0 < \kappa$, $1 < \xi$, $\kappa\xi^2 \le C_P$. Conclusion: the left of (21)). Under the assumption of the bridge, the content of the relaxation-time assumption is only the upper half $\limsup \ln C_P/\ln \xi \le 2$. For the double well, the One-Dimensional One-Sided Bridge Proposition is exactly this bridge ($\kappa = 1$, one-dimensional), so the power statement on the first step becomes the one-sided question "can the upper side be shown", and its target is the Double-Well Relaxation-Time Upper-Bound Proposition of 23-05, $C_P \le K\xi^2 S_0/\hbar$.

Algebra on the side of the chains. Assuming the six band inequalities of the three chains puts $a^2 C_P = a^2/\lambda_1$ in a positive band.

$$c_1 m^2 \le \lambda_1 \le C_1 m^2, \;\; c_2\sigma \le m^2 \le C_2\sigma, \;\; c_3 a^2 \le \sigma \le C_3 a^2 \;\Longrightarrow\; c_1c_2c_3\,a^2 \le \lambda_1 \le C_1C_2C_3\,a^2 \tag{22}$$

Proved here Lean `chain_band`, `sq_times_poincare_band` (assumptions: the six band inequalities and positivity of the coefficients. Conclusion: (22), and that a^2/λ_1 lies in $[1/C, 1/c]$). The chains themselves are assumptions, and the converse (chains from the band statement) does not follow. The one-sided bridge gives the right inequality of the relaxation chain, $\lambda_1 \le C_1 m^2$ ($C_1 = 1/\kappa$), on a lattice with continuous time.

(5) What the relaxation-time assumption does not give — a different quantity, the converse, the limit

That it is a different quantity. C_P is the Poincaré constant of the path measure, that is, the reciprocal of the gap of Langevin dynamics running in an artificial time, while the mass of the problem statement is the gap of the transfer matrix (the Hamiltonian) (the path C_P and the vacuum C_P of Table 3). They are different quantities, and the bridge of (4) only puts a one-directional inequality between them. Assuming the relaxation-time assumption does not make the two the same.

The converse does not hold in general. (19) was "the upper-bound assumption ⇒ the gap in physical units is positive". Its converse, "the gap in physical units is positive ⇒ the upper-bound assumption", fails in the following example.

> The Free-Particle-on-a-Circle Counterexample
>
> Let $Q = S^1$ (a circle of circumference 2π), $M = 1$, $V = 0$, $H = -\frac{\hbar^2}{2}\frac{d^2}{dx^2}$. The gap is $\Delta = \hbar^2/2 > 0$ and $\xi = \hbar/\Delta = 2/\hbar$, but the canonical C_P of the path measure at $T = \infty$ is $+\infty$.

Proved here Proof [Y3]. The eigenfunctions are e^{ikx} with eigenvalues $\hbar^2 k^2/2$, so $\Delta = \hbar^2/2$. The continuous lift $\tilde{x}(\tau)$ of a path $x(\tau)$ to the real line is determined up to a constant $2\pi k$. Take w with support $[0, L]$ vanishing at the ends, and set

$$f = \int w(\tau)\, d\tilde{x}(\tau) = -\int w'(\tau)\, \tilde{x}(\tau)\, d\tau, \qquad \mathrm{Var}(f) = \hbar \int w^2\, d\tau, \qquad \mathcal{E}(f) = \hbar \int w'^2\, d\tau \quad (23)$$

Since $\int w' = 0$, the indeterminacy of the constant of the lift disappears, and f is a function of the path. The variance follows from $\tilde{x}$ being a Brownian motion of strength $\hbar$ and the Itô isometry Cited, and the Dirichlet form from $\delta f/\delta x(\tau) = -w'(\tau)$. Stretching as $w(\tau) = w_0(\tau/L)$, the ratio is $L^2 \int w_0^2 / \int w_0'^2$, which grows without bound as $L \to \infty$. ∎ Lean `rotor_ratio_unbounded`, `rotor_gap_pos` (assumptions: $0 < r$ / $0 < \hbar$. Conclusions: for every B there is $L > 0$ with $B < L^2 r$ / $0 < \hbar^2/2$. Only the arithmetic of the skeleton of the Free-Particle-on-a-Circle Counterexample).

Example 7 (the random walk of the winding number). For $w_0(\tau) = \sin(\pi\tau)$ ($0 \le \tau \le 1$), $\int w_0^2 = \frac{1}{2}$ and $\int w_0'^2 = \pi^2/2$, so the ratio is L^2/π^2. It is the same number as the $C_P \approx L^2/\pi^2$ of the single road of Example 1, and not by chance. The winding number counts how many times the particle on the circle has gone round, and it performs a random walk on an endless single road. With $\hbar = 0.1$, $\xi = 20$ and $\xi^2 = 400$, but at $L = 100$ the ratio is about 1013, and at $L = 1000$ about 1.01×10^5, exceeding ξ^2 without limit. In the family $\hbar \to 0$, setting $a := 1/\xi$, $m_{\mathrm{phys}} = m\,\xi = 1$ is positive for every $\hbar$, yet since $C_P = \infty$, the upper-bound assumption breaks for every $\hbar$.

To name the type: relaxation becomes slower than the gap when a quantity not local in time and not reflected in the gap (a winding number, a topological charge, a number of kinks) is carried by diffusion. It is the same mechanism by which conserving dynamics has $C_P = \infty$ (the detour of 2-10); in gauge theory the freezing of the topological charge in a periodic box is a candidate of this type (23-02 § 05). Note the scope. For the four-dimensional Wilson measure on a finite periodic box the configuration

space is a compact connected finite-dimensional space, so $C_P < \infty$, and the Free-Particle-on-a-Circle Counterexample is not a counterexample in that class. What the Free-Particle-on-a-Circle Counterexample shows is a logical gap (a positive gap does not even imply finiteness of C_P); whether the converse is false within the class of Wilson measures is not settled.

In summary, under the assumption of the bridge, the upper-bound assumption implies a lower bound on the gap of the lattice in physical units, and the converse does not hold in general. **Under the assumption of the bridge, the upper-bound assumption is a stronger sentence than the lattice mass gap in physical units.** The road of the relaxation-time assumption is a road that restates a stronger sentence in a language where the tools of (2) can be used.

The relaxation-time assumption is not non-triviality

For a free massive field, as in Table 4, $a^2 C_P = 1$, and in the field graduation the relaxation-time assumption, the band statement and the lower-bound assumption all hold with equality Proved here . So the lower-bound assumption is a condition separate from the "non-trivial" of the problem statement (that there is interaction, 0-03), and the relaxation-time assumption as a whole does not include non-triviality either. What the relaxation-time assumption says is only the form "for a family of measures heading to a limit with a mass gap, how the time of fluctuations grows".

The strong-coupling side is the opposite side

Many theorems of lattice gauge theory are on the side of small β (strong coupling), and what they show is a lattice gap uniform in the volume, or Poincaré and logarithmic Sobolev inequalities, that is, the form $C_P = O(1)$. Shen–Zhu–Zhu (2023) show both inequalities with Langevin dynamics, and write 'the infinite volume measure has a strictly positive mass gap' Cited . Converting the condition to Wilson's β gives $\beta < N^2/(16(d-1)) = 1/12$ $(N = 2, d = 4)$.

Table 7 The β where theorems exist and the β on the side of the limit ($SU(2)$, $d = 4$. ξ is a rough guide from the leading term of the strong-coupling expansion $m_0 = -4 \ln u$, $u = I_2(\beta)/I_1(\beta)$)

Region	β	ξ	What is said
Condition of Shen–Zhu–Zhu	< 0.0833	0.065	Cited Uniqueness, both inequalities, exponential decay
Curvature after one-link integration	< 0.2722	0.093	The lower bound on curvature $-27\beta^2$ is Lean `StaplePerturbation.hess_gauge` (assumptions: the period L is even with $L \geq 3$, links are unit quaternions, perturbations are pure imaginary). The criterion is Cited (Bakry–Émery). The stage proceeding to the gap is Assumption (5-11)
Numerical window (Table 2)	2.25–2.60	0.72–1.95	Reading $\Delta/\sqrt{\sigma} \approx 3.3$ to 3.8
Two-loop extrapolation	3.24, 4.14, 5.02	10, 100, 1000	Reading A rough guide assuming asymptotic scaling

The ξ column of Table 7 is Reading (a rough guide from the leading term). The strong-coupling theorems are important results giving the infinite-volume uniqueness statement at small β, but if the range of β is bounded, improving the constants does not bring the physical-unit gap statement or the relaxation-time band at the bookkeeping length any closer. If they extended as they are uniformly in β, the upper-bound assumption would be satisfied and the lower-bound assumption broken, falling into the trivial limit $m_{\text{phys}} \to \infty$ (the direction is the same as (5); the second half of Example 5).

The limit and lifting. Even if the relaxation-time assumption closed on the four-dimensional lattice and the assumptions of the bridge were in place, what would be obtained goes only as far as a lower bound

on the lattice gap in physical units at each $\beta \geq \beta_0$. That $\Delta(\beta)$ has a limit (the rest of the physical-unit gap statement), the existence of the limiting measure (the infinite-volume uniqueness statement, the Wilson-loop limit statement), the recovery of rotations with the OS axioms and reconstruction (the rotation-recovery statement, the OS reconstruction statement), and the non-triviality statement lie outside the relaxation-time assumption. And even if the relaxation-time assumption closed on the first step (the double well), there is no theorem of implication to the relaxation-time assumption on the four-dimensional lattice; what has a prospect of lifting is only the form of the sentence and the form of the tools (23-03 § 06).

(6) Analogous cases — models where a question of the same type became a theorem

"Does the slowness of relaxation of a dynamics grow in the same form as a power of the correlation length" is the subject of theorems outside gauge theory.

Example 8 (the theorem of Bauerschmidt–Bodineau in one sentence). For the hierarchical version of the four-dimensional n-component $|\varphi|^4$ model (a model in which the lattice is replaced by nested boxes so that one step of renormalisation can be written down), in the family approaching the critical point from the high-temperature side, the spectral gap of Glauber-type dynamics goes to 0 with the same power as the gap of the free-field dynamics (with logarithmic corrections for $|\varphi|^4$) Cited (abstract and § 1– § 2 of Bauerschmidt–Bodineau, arXiv:1809.02075, checked). The gap of the free-field dynamics is the square of the mass, that is, it vanishes as a power of ξ^{-2}, so this is a sentence of the same type as the diffusive-relaxation-time assumption: "CP grows as a power of ξ^2, up to logarithmic factors" (the correspondence between mass and correlation length depends on definitions inside the model, so here we keep to a comparison of types).

Three things can be read from this example. First, the question of the type of the relaxation-time assumption stands as mathematics, and there is an example where it was answered for a four-dimensional model with interaction. Second, the answer comes with logarithmic corrections, and holds in the form of a power statement, not a band statement. This is the same situation as the reason for placing the relaxation-time assumption at the power level (the detour of § 07). Third, the method is an induction of Brascamp–Lieb type over discrete scales: the covariance is split scale by scale, and the spectral gap of the measure at each scale is handed on from the spectral gap at the next scale. The conclusion is in the form of a spectral gap (a Poincaré inequality); it uses as an assumption that the potential is uniformly close to convex at each scale, and takes results of equilibrium renormalisation as input. It is a result for scalar fields taking values in a flat space, not for gauge fields. How to transfer it to fields taking values in a group is dealt with, as a question outside this part, in 24-06 § 05 of Part 24.

Classification of the rows of the diffusive-relaxation-time assumption, and the dependency graph

We put the statements of (3)–(6) into one table in the format of the five grades (23-00 § 01). The numbers of the rows are the numbers of the nodes of the dependency graph below. The scope column says what the row is about (the range of β, the volume of space, the treatment of time).

Table 8 The relaxation-time assumption and the rows around it ("—" is an assumption row not shown in any range)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
[01]	Upper-bound assumption: $a^2 C_P \leq C$	Assumption	(14)(15)	—	—
[02]	Lower-bound assumption: $a^2 C_P \geq c$	Assumption	(14)(15)	—	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
[03]	The relaxation-time assumption (power statement): $\ln C_P / \ln \xi \to 2$	Assumption	(13)	—	—
[04]	Bands of the three chains (the relaxation chain, the one-scale chain, the bookkeeping-length chain)	Assumption	(12), 17-12	—	—
[05]	Equality in the relaxation chain, $C_P = 1/m_{\mathrm{lat}}^2$	Assumption	The assumption of (16) (the free-field value)	—	—
[06]	The ground-state assumption (A1): discreteness of the spectrum, simplicity and positivity of the ground state	Assumption	Assumption of the one-sided bridge (the statement in the original sources has not been checked in this part)	Finite spatial lattice	—
[07]	The stationary path-measure assumption (A2): the formula for correlations	Cited	(i), 2-07 § 08	$T = \infty$	—
[08]	The κ of the metric comparison assumption is uniform in β	Assumption	The first condition after (19) (a number independent of the coupling for $\mathcal{E}_A$, $\propto g^2$ for $\mathcal{E}_\vartheta$)	—	—
[09]	(D) Comparison of increments in discrete time with gradients, c_D uniform in β	Assumption	(20)	—	—
[10]	From the bridge in a box to the bridge in infinite volume	Assumption	The third condition after (19) (not examined)	—	—
[11]	The ground-state transformation identity (18)	Proved here	Proof of (iii)	Finite spatial lattice	—
[12]	The one-sided bridge: $C_P \geq \kappa \xi^2$	Proved here	(17), (i)–(iv) (conditional: (A1)–(A3), continuous time)	Finite spatial lattice, continuous time, $T = \infty$	`cont_ratio_closed`, `one_sided_bridge` (assumptions: positive numbers, a lower bound on the variance, an upper bound on the Dirichlet form)
[13]	The Discrete-Time One-Sided Bridge Proposition: $C_P \geq 2/(c_D \Delta^2)$	Proved here	(20) (conditional: the time-step comparison assumption (D))	Finite spatial lattice, time with step 1	`pairSum_closed`, `disc_limit_ge` (assumptions: $0 < \Delta$, $0 < c_D$)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
[14]	The Free-Particle-on-a-Circle Counterexample: $C_P = \infty$ with $\Delta > 0$ (negation of the converse)	Proved here	(23), Example 7	Circle, every $\hbar$	`rotor_ratio_unbounded`, `rotor_gap_pos` (arithmetic of the skeleton only)
[15]	(16) Rereading in the version with equality	Proved here	Proof of (16)	$\beta \geq \beta_0$	`hup_is_gap_positivity`, `hdown_is_mass_finiteness` (assumptions: $m_{\mathrm{lat}} > 0$, $C_P = 1/m_{\mathrm{lat}}^2$, one side of (14))
[16]	(19) $m_{\mathrm{phys}}^2 \geq \kappa / C$ from the upper-bound assumption and the bridge	Proved here	(19)	$\beta \geq \beta_0$	`hup_is_gap_positivity_of_lower` (assumptions: $0 < m_{\mathrm{lat}}$, $\kappa / m_{\mathrm{lat}}^2 \leq C_P$, $C_P \leq K \xi^2$)
[17]	(21) The lower half of the power statement is automatic	Proved here	(21)	$\xi > 1$	`power_lower_half_of_bridge` (assumptions: $0 < \kappa$, $1 < \xi$, $\kappa \xi^2 \leq C_P$)
[18]	(22) Algebra of the bands of the chains	Proved here	(22)	$\beta \geq \beta_0$	`chain_band`, `sq_times_poincare_band` (assumptions: the six band inequalities and positivity of the coefficients)
—	The gap of the dynamics of hierarchical $\lvert\varphi\rvert_4^4$ has the same power as the free field (with logarithms)	Cited	Example 8 (an analogous case; not used in the rows of the relaxation-time assumption)	Hierarchical model, criticality and its vicinity	—

The "Proved here" rows are all either algebra taking assumption rows as input, or propositions with their assumptions named. The one-sided bridge is a proposition whose proof is given in this section, but it is not a theorem that applies as it is to the four-dimensional Wilson measure (for discretised time the time-step comparison assumption remains, for infinite volume the stage from the bridge in a box to the bridge in infinite volume, and for the graduation the uniformity of κ).

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[01] ASM  relaxation-time upper-bound assumption: a²C_P ≤ C              —
[02] ASM  relaxation-time lower-bound assumption: a²C_P ≥ c              —
[03] ASM  diffusive-relaxation-time assumption: ln C_P / ln ξ → 2        —
[04] ASM  bands of the three chains                                      —
[05] ASM  equality in the relaxation chain, C_P = 1/m_lat²               —
[06] ASM  ground-state assumption: discrete spectrum,
          simple positive ground state                                   finite spa
[07] CIT  correlations of the stationary path measure                    T=∞
[08] ASM  κ uniform in β                                                 —
[09] ASM  time-step comparison assumption (c_D uniform)                  —
[10] ASM  from the box bridge to the infinite-volume bridge              —
[11] PRV  ground-state transformation identity (18) ⟸ [06]                finite sp
[12] PRV  one-sided bridge: C_P ≥ κξ² ⟸ [06][07][11]                      continuou
          Lean cont_ratio_closed, one_sided_bridge (algebra of numbers only)
[13] PRV  Discrete-Time One-Sided Bridge Proposition:
          C_P ≥ 2/(c_D Δ²) ⟸ [06][07][09]                                time with
          Lean pairSum_closed, disc_limit_ge
[14] PRV  Free-Particle-on-a-Circle Counterexample: Δ > 0 and C_P = ∞  every ħ
          Lean rotor_ratio_unbounded, rotor_gap_pos (arithmetic of the skeleton onl
[15] PRV  (16) upper-bound assumption ⇒ m_phys ≥ C^(-1/2),
          lower-bound assumption ⇒ m_phys ≤ c^(-1/2) ⟸ [01][02][05]
          Lean hup_is_gap_positivity, hdown_is_mass_finiteness
[16] PRV  (19) m_phys² ≥ κ/C ⟸ [01][08][10][12] (with discretised time, [13] inste
          Lean hup_is_gap_positivity_of_lower
[17] PRV  (21) liminf ln C_P / ln ξ ≥ 2 ⟸ [08][12] (or [13])
          Lean power_lower_half_of_bridge
[18] PRV  (22) a²C_P lies in a positive band ⟸ [04]
          Lean chain_band, sq_times_poincare_band
```

How to check. In every row with "⟸", check by eye that the numbers on the right are smaller than the number on the left ([11] from [06], [12] from [06][07][11], [13] from [06][07][09], [15] from [01][02][05], [16] from [01][08][10][12] (or [13]), [17] from [08][12] (or [13]), [18] from [04]). Every row keeps the rule, so there are no cycles. The ASM leaves are the nine [01]–[06] and [08]–[10]; the CIT leaf is [07]. The relaxation-time assumption [03] is a premise of no row. With the bridge, the lower half of the relaxation-time assumption follows automatically by [17], and the content of [03] is only the upper half: that is the content of (4). The Free-Particle-on-a-Circle Counterexample [14] is a counterexample with no premises, and shows that the converse of [16] does not hold in general.

Precisely, this graph is the dependency of the assembly of the relation between the relaxation-time assumption and the mass; it does not include the rows supporting the relaxation-time assumption itself (for the first step, the graph of 23-05 § 07). The one that joins everything together is the integrated graph of 23-12.

07

Detour: the number of rescalings and four clocks — logarithms that do not show in the power

A section you may skip. We look, through formulas for the three metaphors of 23-00 § 05 (the number of rescalings, the graduation of the clock, the clock that runs slow), at why the relaxation-time assumption is placed as a power statement rather than a band statement.

The number of rescalings — the bookkeeping length ξ_{RG}

In four-dimensional lattice Yang–Mills there is no way to obtain ξ by formula as a function of β. So we coarsen the scale of the map one step at a time, and make separately a length that counts how many times it takes to reach a picture of fixed coarseness. In one step that coarsens the lattice by a factor b (one step of the renormalisation group, 7-05), we count the coupling as decreasing as follows.

$$\beta \longmapsto \beta - \kappa\,\big(\ln b - \delta(b)\big), \qquad \kappa = \frac{11}{3\pi^2} \approx 0.3715 \tag{24}$$

κ is the one-loop coefficient (asymptotic freedom, 4-06) rewritten in the lattice β Cited , and $\delta(b)$ is what one step of bookkeeping misses (23-02 § 02). With $k(\beta)$ the number of steps it takes to come down to a fixed value β_*, set

$$\xi_{\mathrm{RG}} = b^{\,k(\beta)}, \qquad a := \frac{1}{\xi_{\mathrm{RG}}}, \qquad \xi_{\mathrm{RG}} = e^{(\beta-\beta_*)/\kappa} \quad (\delta = 0) \tag{25}$$

If $\delta = 0$, the number of steps is $(\beta - \beta_*)/(\kappa \ln b)$, and the bookkeeping length does not depend on b. It has the same form as the one-loop formula of dimensional transmutation (4-07).

Proved here Table 9 The number of rescalings (one loop, $\delta = 0$, $b = 2$, with $\beta_* = 2.0$ as an example. Values of the closed formula (25))

β	Number of steps k (number of halvings)	$\log_{10} \xi_{\mathrm{RG}}$
2.5	1.94	0.58
4	7.77	2.34
6	15.53	4.68
10	31.07	9.35
20	69.90	21.04

Each increase of β by 1 adds about 3.9 halvings. The choice of β_* affects only a constant factor. The bookkeeping length is not a length fixed directly by the measure but a length fixed by the convention of bookkeeping, and that it agrees with ξ up to a constant factor is the assumption of the one-scale chain and the bookkeeping-length chain Assumption . If the bookkeeping is truncated at one loop, the difference from the two-loop running (for pure $SU(N)$, $b_1/(2b_0^2) = 51/121$) remains as a factor $\beta^{51/121}$ Cited (at the level of perturbation theory, 4-06).

The graduation of the clock and the clock that runs slow

Writing an $SU(2)$ link as $U = e^{i\vartheta\cdot\sigma}$, the relation to the field of the continuum theory is $A = \sqrt{\beta}\,\vartheta$. $\mathcal{E}$ is a square of derivatives, so

$$\mathcal{E}_A = \frac{\mathcal{E}_\vartheta}{\beta}, \qquad C_P^{(A)} = \beta\, C_P^{(\vartheta)}, \qquad \text{in the free approximation } C_P^{(A)} = \xi^2,\ C_P^{(\vartheta)} = \frac{g^2}{4}\,\xi^2 \tag{26}$$

Proved here . Further, Langevin time itself stretches and shrinks with scale. When stochastic quantisation (7-09) is renormalised in perturbation theory, besides the field and the coupling, the graduation of stochastic time is also renormalised multiplicatively Cited (at the level of perturbation theory). Transferring its one-loop coefficient into the language of graduations, it can be read as

$$C_P^{(A)} \;\propto\; \xi^2\,\beta^{13/22}, \qquad C_P^{(\vartheta)} \;\propto\; \xi^2\,\beta^{-9/22} \tag{27}$$

can be read Assumption (a reading that presupposes that it is the leading one-loop logarithm, that the leading logarithm of the lattice agrees with that of dimensional regularisation, the multiplicative renormalisability of the five-dimensional theory, and the relaxation-time assumption itself). The power $\beta^{-9/22}$ in the angle graduation is the same power as the factor $g_0^{9/11}$ that Lüscher–Schaefer (2011) write for the autocorrelation time. It comes from the same one-loop coefficient, so it is not an independent check.

Table 10 lines up C_P/ξ^2 for four clocks: two graduations, with and without the slowing. ξ is replaced by the bookkeeping length ξ_{RG} of (25) ($\beta_* = 0$), and the constants are set to 1 at $\beta = 2.5$. The power is $d\ln C_P/d\ln\xi = 2 + \varepsilon\kappa/\beta$.

Proved here Table 10 Four clocks (schematic. Values of the closed formulas $\xi = e^{\beta/\kappa}$ and (26)(27))

β	$\log_{10}\xi$	Field A ($\varepsilon = 0$)	Angle ϑ ($\varepsilon = -1$)	Field A, slowed ($\varepsilon = 13/22$)	Angle ϑ, slowed ($\varepsilon = -9/22$)
2.5	2.92	1	1	1	1
4	4.68	1	0.625	1.32	0.8251
10	11.69	1	0.25	2.269	0.5672
100	116.90	1	0.025	8.844	0.2211
1000	1168.99	1	0.0025	34.48	0.0862
Power ($\beta = 2.5$)	–	2.0000	1.8514	2.0878	1.9392
Power ($\beta = 100$)	–	2.0000	1.9963	2.0022	1.9985

For every clock the power approaches 2, and the constant factor does not settle. Precisely, Table 10 is a schematic representing the interaction only through one-loop logarithmic factors; it is not a computation of the C_P of four-dimensional lattice Yang–Mills. The three logarithms of graduation, order of bookkeeping and stochastic time move the band statement, and do not move the power statement (13).

08

What of the line is in hand — on to 23-02 and 23-03

We line up the statements of this chapter as a single arrow, in the order of the line.

$$\underbrace{\text{one step of bookkeeping}}_{\text{23-02}} \xrightarrow{(\xi^{\prime})\wedge\sum\delta<\infty} \text{(S5)} \xrightarrow{(\xi^{\prime})\text{band}} \text{(S4)} \xrightarrow{\text{(S3)(S6)(S7)(S8)}} \text{the demand of the}$$

The first arrow is the sentence at the power level with the finiteness of the sum of the shortfalls of bookkeeping $\sum\delta$ added (23-02 § 02); the next arrow needs the band statement. Of (28), what is in hand is only part of the first box (the algebra of one step of bookkeeping), and none of the arrows is proved. The ultraviolet side has been crossed by a different road (a construction stacking steps of renormalisation) in the construction published on 2026-11-10. That a theory satisfying the Wightman axioms can be built as a limit of four-dimensional Wilson lattice gauge theory is proved in Lean, according to the committee's assessment (Part 6). Even so, a uniform mass gap is not shown (6-04). Where the setters name the need for new ideas is also the infrared: 'New ideas are needed to prove the existence of a mass gap that is uniform in the volume of space-time.' (Jaffe–Witten § 6.5)Cited .

What follows if the relaxation-time assumption closes. As in § 06, if the upper-bound assumption closes on the four-dimensional lattice and the assumptions of the one-sided bridge are in place (a lattice with continuous time, or, on a lattice with discretised time, the time-step comparison assumption ((D) in the paper "one-sided bridge"); κ uniform in β; the stage from the box to infinite volume), it follows that at each $\beta \geq \beta_0$ the lattice gap in physical units stays at least $(\kappa/C)^{1/2}$ ((19)). From the relaxation-time assumption alone (a power statement, not the band statement), no gap in physical units follows. C_P is the reciprocal of the gap of Langevin dynamics running in an artificial time, a different quantity from the gap of the Hamiltonian of the transfer matrix (the mass) (the vacuum C_P and the path C_P of Table 3). The bridge is a one-directional inequality between them, and the converse does not hold in general (the Free-Particle-on-a-Circle Counterexample). Even with the relaxation-time assumption and the bridge in place, it does not follow that $\Delta(\beta)$ has a limit, nor the existence of the limit (the infinite-volume uniqueness statement, the Wilson-loop limit statement), the OS axioms and reconstruction (the

rotation-recovery statement, the OS reconstruction statement), or the non-triviality statement. And even if the relaxation-time assumption closed on the first step (the double well), there is no theorem of implication to the relaxation-time assumption on the four-dimensional lattice; what has a prospect of lifting is only the form of the sentence and the form of the tools (23-03 § 06).

The roads that pushed the arrows of (28) on the four-dimensional lattice as it is, and where each stopped, are gathered in 23-02. As seen there, on the side of the lower-bound assumption (some f) there are several roads that reach, and on the side of the upper-bound assumption (every f) no tool that reaches has been found. The relaxation-time assumption is a sentence independent of dimension, so the first step of the staircase is to move the same sentence to the smallest model where the relaxation-time assumption is expected to hold, and build the assembly of the upper side there first (23-03).

What this chapter can say

Said	Not said
Cited The split into eight statements and their current status in four dimensions (Table 1), the footnote of the problem statement and § 6.5, the direction of the lattice correlation-length problem, the strong-coupling theorems (Table 7), the relation of the vacuum C_P among the three C_P to the gap, the convexity and perturbation criteria of (9), the gap of the dynamics of hierarchical $\lvert\varphi\rvert^4_4$ (Example 8, as an analogous case)	The infinite-volume uniqueness statement at large β; the Wilson-loop limit statement, the physical-unit gap statement, the rotation-recovery statement and the non-triviality statement (along the line of this chapter; for the relation to the published construction, see the note in § 02). This chapter shows nothing about them
Proved here That a uniform lattice gap means a divergent physical mass ((5)), that the physical-unit gap statement and the relaxation-time band at the bookkeeping length say the same thing for the free field ((8), Table 4), the relation between band and power ((11), Example 4), the asymmetry of quantifiers ((15)), the hand calculations of Examples 1–7, the conversion of graduations (26), the values of the closed formulas in Tables 9 and 10	Joining the physical-unit gap statement and the relaxation-time band at the bookkeeping length in a theory with interaction (the relaxation-time assumption, the band statement)
Proved here The one-sided bridge ($C_P \geq \kappa\xi^2$; conditions: (A1)–(A3), continuous time, finite spatial lattice, $T = \infty$) and the Discrete-Time One-Sided Bridge Proposition (condition: (D)), the ground-state transformation identity (18), the Free-Particle-on-a-Circle Counterexample (on the circle $\Delta > 0$ and $C_P = \infty$)	Applying the bridge to the four-dimensional Wilson measure ((D) for discretised time, uniformity of κ, the stage to infinite volume). Whether "positive gap ⇒ the upper-bound assumption" is false within the class of Wilson measures
Proved here The algebra rereading the upper-bound assumption and the lower-bound assumption as lower and upper bounds on the mass ((16)) Lean `hup_is_gap_positivity`, `hdown_is_mass_finiteness` (assumptions: $m_{\mathrm{lat}} > 0$, $C_P = 1/m^2_{\mathrm{lat}}$, one side of (14)). The version weakened to one side, (19), `hup_is_gap_positivity_of_lower` (assumptions: $0 < m_{\mathrm{lat}}$, $\kappa/m^2_{\mathrm{lat}} \leq C_P$, $C_P \leq K\xi^2$). The lower half of the power statement, (21), `power_lower_half_of_bridge` (assumptions: $0 < \kappa$, $1 < \xi$, $\kappa\xi^2 \leq C_P$). The algebra of the bands of the chains, (22), `chain_band` (assumptions: the six band inequalities)	The relaxation-time assumption, the upper-bound assumption, the lower-bound assumption and the three chains themselves. That $C_P = 1/m^2_{\mathrm{lat}}$ holds in a theory with interaction
Reading That $\Delta/\sqrt{\sigma}$ is roughly constant in the Monte Carlo values of the original source (Table 2); the rough guide to ξ in Table 7	That the extrapolated value is the value of the constructed theory
Assumption The exponents of stochastic time $13/22$ and $-9/22$ (a one-loop reading with premises), that the bookkeeping length and ξ agree up to a constant factor, the assumption rows of Table 8	That those exponents apply to the C_P of four-dimensional lattice Yang–Mills. About the mass gap and the continuum limit, this chapter shows nothing

Sources and reproduction

Item	Kind	Source / tools
Problem statement, footnote, § 6.5	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000). Text checked (the table of sources of 0-03).
The lattice correlation-length problem and the continuum string-tension problem (Chatterjee's Problems 5.1 and 5.2), current status of the infinite-volume uniqueness statement	Cited	S. Chatterjee, arXiv:1803.01950 § 3, § 5 (Springer Proc. Math. Stat. 283, 2019). Text checked.
Uniform estimates at strong coupling and conversion of the condition	Cited	H. Shen, R. Zhu, X. Zhu, Commun. Math. Phys. 400 (2023) 805–851, arXiv:2204.12737 (abstract, § 1 and formula (1.2) checked). K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (bibliographic data only).
Transfer operator, OS reconstruction, triviality of ϕ^4_4	Cited	M. Lüscher, Commun. Math. Phys. 54 (1977) 283. K. Osterwalder, R. Schrader, Commun. Math. Phys. 31 (1973) 83, 42 (1975) 281. M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163. Bibliographic data only (tables of sources of each chapter).
The convexity criterion and bounded perturbation of (9)	Cited	D. Bakry, M. Émery, LNM 1123 (1985). The type of the Holley–Stroock perturbation lemma checked in M. Ledoux's restatement. Bibliographic data in the table of sources of 23-06.
Values in Table 2	Reading	B. Lucini, M. Teper, JHEP 0106 (2001) 050, hep-lat/0103027, Tables 1 and 5 (text checked). Ratios by division and propagation of errors.
Tables 4, 9 and 10, Examples 1–7, (8)(11)(26)	Proved here	The closed formulas $C_P = 1/m^2$, $m^2 = 2(\cosh(1/\xi) - 1)$, $C_P(L) = 1/(2(1 - \cos(\pi/L)))$, the values of (25)–(27), and the hand calculations of the text.
The one-sided bridge, the Discrete-Time One-Sided Bridge Proposition, the Free-Particle-on-a-Circle Counterexample, and (18)	Proved here	The proofs in § 06 of the text. The formula for correlations in the one-sided bridge is 2-07 § 08; the Itô isometry is cited as a known fact. Whether a claim of the same form exists in the literature has not been checked.
The gap of the dynamics of hierarchical $\lvert\varphi\rvert^4_4$	Cited	R. Bauerschmidt, T. Bodineau, Commun. Math. Phys. 373 (2020) 1167–1206, arXiv:1809.02075 (abstract and § 1– § 2 checked. The method is an induction of Brascamp–Lieb type over discrete scales; the conclusion is a spectral gap).
One-loop and two-loop coefficients	Cited	D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343. H. D. Politzer, ibid. 1346 (bibliographic data only; the table of sources of 4-06).
Renormalisation of stochastic time, and (27)	Cited, Assumption	L. Baulieu, D. Zwanziger, Nucl. Phys. B 581 (2000) 604, arXiv:hep-th/9909006. M. Lüscher, S. Schaefer, JHEP 04 (2011) 104, arXiv:1103.1810. J. Zinn-Justin, D. Zwanziger, Nucl. Phys. B 295 (1988) 297 (abstract only). The premises of the reading of (27) are in the text of § 07.
The algebra of (15)(16)(19)(21)(22) and the Lean theorems	Proved here, Lean	Theorems `hup_is_gap_positivity`, `hdown_is_mass_finiteness`, `chain_band`, `sq_times_poincare_band` (bundle `LatticeGaugeOneScale`); `exponent_invariant_of_slowly_varying` (bundle `LatticeGaugePoincareNormalisation`); `hup_is_gap_positivity_of_lower`, `hup_is_gap_positivity_recovered`, `power_lower_half_of_bridge`, `cont_ratio_closed`, `one_sided_bridge`, `pairSum_closed`, `disc_limit_ge`, `rotor_ratio_unbounded`, `rotor_gap_pos` (bundle `LatticeGaugeHupBridge`). Statements and assumptions are in 23-13.

Item	Kind	Source / tools
The note of § 05 (topological sectors and the order of volume)	Reading, Cited, Assumption	L. Del Debbio, H. Panagopoulos, E. Vicari, arXiv:hep-th/0204125 § 2. S. Schaefer, R. Sommer, F. Virotta, Nucl. Phys. B 845 (2011) 93, arXiv:1009.5228 § 4.1, § 5. C. Bonanno, C. Bonati, M. D'Elia, arXiv:2012.14000v2 § 1, Fig. 4. M. Abe, O. Morikawa, H. Suzuki, PTEP 2025 (2025) 063B03, arXiv:2501.00286 § 3.2, Fig. 6 ($SU(2)$, HMC). M. Lüscher, S. Schaefer, JHEP 07 (2011) 036, arXiv:1105.4749 § 3, § 5.3. $\tau_{\mathrm{int}} \leq C_P$ is the Lean theorem of 23-02 § 05.
The eight statements of construction, the relaxation-time assumption, the band statement, the three chains, (A1), (D)	Assumption, convention	The formulation of this part. The origin of the three chains is 17-12. The list of readings is in A-06.

Words used in this chapter — the eight statements of construction, the lattice version of the mass gap, the Poincaré constant (three C_P), the diffusive-relaxation-time assumption (power statement), the band statement, the upper-bound assumption and the lower-bound assumption, the three chains, the one-sided bridge (the version with continuous time and the version with discretised time), the ground-state transformation identity, the Free-Particle-on-a-Circle Counterexample (winding number), the bookkeeping length ξ_{RG}, graduation (normalisation of $\mathcal{E}$), renormalisation of stochastic time, the four double-well conditions (name only). Their meanings are in A-02 Glossary.

To the next chapter — 23-02 Roads tried on the four-dimensional lattice, and walls: puts on one sheet the seventeen roads and twenty-one walls that pushed the arrows of (28) on the four-dimensional lattice as it is, and sees that they reach from below but not from above, and that this is why we go down to the first step. To go straight to the first step, 23-03 The four steps of the staircase.

Next chapter to read: 23-02 Roads tried on the four-dimensional lattice, and walls.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time seventeen roads, twenty-one walls (five types) · Kiichi, Shiori and Rin

Roads tried on the four-dimensional lattice, and walls — they reach from below, and not from above

> This chapter is this book's own investigation. § 01 is guide material; § 03, § 06 and § 08 are a map of open questions.

What this chapter shows — We put on one sheet the seventeen roads tried on the four-dimensional lattice towards the diffusive-relaxation-time assumption and the arrows of the line of construction, and the twenty-one places where they stopped (the walls). The roads through test functions, gauge fixing and spectroscopy all end in lower bounds on C_P or upper bounds on the mass; tools that reach the upper side (inequalities for every f) existed only on the strong-coupling side, where curvature works, and in a form moved into a single assumption. What was closed in Lean is the algebra from assumptions to conclusions. At the end of this chapter we say, through the correspondence between tools and walls, why we step down to the first step (the double well in 0+1 dimensions).

Prerequisite chapters — 23-01 (the eight statements, the diffusive-relaxation-time assumption, the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption, the three chains), 1-10 (the variational principle, the Poincaré constant), 4-11 (the gap and finite windows), 5-02 (the strong-coupling expansion), 7-05 (steps of the renormalisation group). Notation as in 23-01 ($SU(2)$, $d = 4$, $\beta = 4/g^2$, lattice units).

Said first — This chapter does not solve the problem. "Stopped" for a road means "we checked that this tool, in this family, does not reach this arrow"; it does not say that the proposition not reached is false. The seventeen roads and twenty-one walls of this chapter are the series on the side of the four-dimensional lattice, separate from the series of the first step (the four double-well conditions, routes, strands, detour plans). Many of the numbers are readings inside models, not computations for four-dimensional $SU(2)$.

The order of this chapter

1. How to read the map — the formula of arrows, and the names of the roads and walls
2. One step of bookkeeping — complete families, one-link integration, the deficit of bookkeeping, the shortfall and the form of the answer (Table 1)
3. The seventeen roads tried — one row each (Table 2)
4. Reaching from below — test functions, gauge fixing and spectroscopy end in upper bounds

5. Not reaching from above — curvature, tubes, one-dimensional chains of scales, boundaries
6. Twenty-one walls — five types (Table 3)
7. What remained — algebra closed in Lean, and numerical readings (Table 4)
8. Why we step down to the first step — which tool stopped at which wall (Table 5)

01

How to read the map — the formula of arrows, and the names of the roads and walls

The roads and walls of this chapter are attempts to push one of the arrows of the line of construction in 23-01 § 08. Rewritten with the band statement split into its upper and lower halves, it has the following form.

$$\underbrace{\text{one step of bookkeeping}}_{\S 02} \xrightarrow{\sum_k \delta(bk)<\infty} \text{(S5)} \xrightarrow{(\xi^{\angle})\downarrow \wedge (\xi^{\angle})\uparrow} \text{(S4)} \xrightarrow{\text{properties of the limit}} \text{t}$$

$\delta(b)$ is the shortfall of one step (§ 02); in the formula, (S5) (the relaxation-time band at the bookkeeping length) is $a^2 C_P \asymp 1$, (S4) (the physical-unit gap statement) is that the gap in physical units stays positive, and the properties of the limit are the infinite-volume uniqueness statement, the reflection-positivity preservation statement, the Wilson-loop limit statement, the rotation-recovery statement, the OS reconstruction statement and the non-triviality statement (23-01 Table 1). The lower-bound assumption is $C_P \gtrsim \xi_{\mathrm{RG}}^2$ (one test function suffices), and the upper-bound assumption is $C_P \lesssim \xi_{\mathrm{RG}}^2$ (an inequality over all test functions). None of the arrows of (1) is proved.

A road is the pair of one question and one set of tools applied to it, and is named after the content of the question. The road of comparison with the defect of a submission and the road of a one-dimensional chain of scales take up the original question again. A wall is a one-sentence statement of why a certain road, with certain tools, stopped short of a certain arrow; the walls are divided into five types (principle, structure, tools, assumptions, measurement and literature) (§ 06).

Example: the road of lower-bound test functions (test functions giving the lower-bound assumption) pushed the lower-bound assumption of (1), and stopped short of the upper-bound assumption at the one-sided test-function wall (test functions bound C_P only from below). The road of strong-coupling curvature is not on (1); it sits outside to the left. Uniform estimates at strong coupling are $C_P = O(1)$, the opposite side of the relaxation-time band at the bookkeeping length (23-01 § 06).

02

One step of bookkeeping — from complete families to the sum of shortfalls

One step is the map that coarsens the lattice spacing from a to ba, integrates out some of the fine variables, and makes the action of the coarse lattice (7-05). The bookkeeping here takes three moves: (i) choose links that can be integrated in closed form, (ii) write the remaining action with characters, (iii) read the coefficient of the fundamental representation of the coarse plaquette as β'. It is of the "decimation" type, keeping some of the original links (Migdal–Kadanoff is an approximation of the same type). The area law, dimension counting and the one-loop coefficient are known rewritings, not counted as results.

Complete families. If the links to be integrated never lie on the same plaquette, then with the rest held fixed the weight of each link separates independently and can be integrated. A family that removes exactly one link from every plaquette is called a complete family.

$$F \subset E \text{ is complete} \iff \text{for every plaquette } P,\ |P \cap F| = 1; \qquad f_x \le 2,\ \frac{1}{|V|}\sum_x f_x = \frac{d}{2} \implies d \le 4$$

Proved here An edge lies on $2(d-1)$ plaquettes, so counting the pairs $(P, \ell \in P \cap F)$ in two ways gives the density of the family $|F|/|E| = 1/4$. Two perpendicular edges at the same vertex are two adjacent edges of one plaquette, so the number f_x of edges of the family at a vertex x is at most the two along one axis, and comparing with the average $d/2$ gives $d \le 4$. The classification including existence is Table 1; the statements are Lean `PerfectFamily.perfectK_one_iff` (assumption: $2 \le d$. Conclusion: the box $(\mathbb{Z}/L)^d$ of period L has a complete family of edges $\iff$ $d \le 4$ and L is even) and `no_perfectZ_of_five` (assumption: $d \ge 5$. There is none on $\mathbb{Z}^d$). No prior record was found in what we searched, but it may be known. This 4 is a combinatorial number of the lattice; its agreement with the four dimensions of spacetime is a coincidence with no relation.

Proved here Table 1 Existence of complete families of edges on $(\mathbb{Z}/L)^d$ (the statement of `perfectK_one_iff` written out. In brackets, the number of edges in one family = 1/4 of the number of edges)

Period	$d=2$	$d=3$	$d=4$	$d=5$
$L=2$	Yes (2)	Yes (6)	Yes (16)	No
$L=3$	No	No	No	No
$L=4$	Yes (8)	Yes (48)	Yes (256)	No

One-link integration. The 6 plaquettes on which an edge ℓ of the family lies contain no other edges of the family, so U_ℓ pairs only with the sum M of staples (the products of the three edges of a plaquette other than ℓ), and the integral is determined by $|M|$ alone.

$$\int_{SU(2)} e^{\frac{\beta}{2}\operatorname{tr}(UM)}\,dU = \frac{2I_1(\beta|M|)}{\beta|M|} =: F(\beta|M|), \qquad \frac{\beta^2 t}{8} - \frac{\beta^4 t^2}{64} \le \ln F(\beta\sqrt{t}) \le \frac{\beta^2 t}{8} \quad (3)$$

The left is the standard formula of the $SU(2)$ heat-bath method Cited (5-07); the two-sided estimate on the right is Lean `log_F0_le`, `log_F0_ge` (assumption: only that the argument is non-negative). The conditional density of the remaining edges is a product of six sources and

does not close into the one-link type, so this trick works only once Assumption (there is a paper argument, but this book gives no proof).

The deficit of bookkeeping. After integrating a complete family, not a single "intact" plaquette with all 4 edges remaining is left, and the coupling can only be reborn from broken plaquettes. For the fraction e of remaining edges and the fraction q of intact plaquettes, counting pairs gives

$$e - q = \frac{1}{4|\mathcal{P}|} \sum_{P \text{ broken}} |P \cap E_K| \;\geq 0 \quad (4)$$

Proved here (double counting) Lean `Incidence.intact_frac_le` (assumption: the incidence is regular. Conclusion: $q \leq e$). Stacking steps does not reach it: in a sequence of complete steps the remaining edges are $(2^k+1)/2^{k+1} > 1/2$, `no_blocking` (assumption: a sequence of complete steps), and in any sequence removing at most $1/4$ at each step, reducing the edges to $1/16$ leaves at most $1/65536$ of the 4-loops, `blocking_shortfall` (assumption: fraction $\leq 1/4$). The reading of this subsection is that the wall of bookkeeping was neither curvature nor packing, but the way of bookkeeping that counts only intact plaquettes.

Discs and a weak-coupling step. Integrating the interior of a disc-shaped set of A plaquettes raises the coefficients of every channel of the character expansion to the power A: $u(\beta') = u(\beta)^A$, and for a 2×2 disc, $\beta' = \beta^4/64 + O(\beta^6)$ Cited (a rewriting of the strong-coupling area law and two-dimensional decimation, 5-02, 7-02). It is a semigroup in the area, with $\beta' < \beta$ everywhere Lean `discStep_semigroup`, `step_lowers_coupling` (assumption: surfaces of disc shape). In the Gaussian approximation at weak coupling, the b^2 plaquettes of a disc act in series and the b^{d-2} transverse layers in parallel, giving

$$\beta'_{\text{disc}} = \frac{\beta}{b^2}, \qquad \beta'_{\text{tree}} = \beta\, b^{d-4}, \qquad \kappa_{\text{disj}}(b) = \frac{1}{b^2} + \frac{2}{b^2+4b} < 1 \quad (5)$$

βb^{d-4} is the dimension count of the coupling, marginal at $d = 4$ Cited (4-04). The contribution of surfaces sharing no edges does not reach 1 Lean `kappaDisj_lt_one` (assumption: $b \geq 2$), and the bookkeeping of surfaces has the multiplicative form $\beta' = \kappa(b)\beta$, not the subtractive form of one loop $\beta' = \beta - \kappa \ln b$.

The one-loop slope and the shortfall. On top of the tree-level $\beta' = \beta$ sits the one-loop correction from the fast modes (momenta between $\pi/(ba)$ and π/a). Measured by the integral over a lattice window, it takes the form

$$\beta' = \beta - \kappa L_{\text{eff}}(b), \qquad L_{\text{eff}}(b) = \ln b + C - \delta(b), \qquad \delta(b) = \frac{c_2}{b^2} + O(b^{-4}) \quad (6)$$

$\kappa = 4Nb_0 = 11/(3\pi^2) = 0.3715110$ is the known one-loop coefficient Cited (4-06). The slope read from the difference of windows is 1.000028 at $b = 32 \to 64$; multiplied by κ it is 0.371521, agreeing with $11/(3\pi^2)$ to a relative 2.8×10^{-5} Reading (an agreement inside the model of the one-loop kernel; not a non-perturbative check). The coefficients of the shortfall are $c_2 = 0.0197(5)$ (weight w) and $0.4200(5)$ (weight w^2), and at $b = 2$, $\delta(2) = 0.1335$ and 0.2284 Reading

(fits in the same model). On the other side, the effective slope $\Delta\beta/\ln(a_1/a_2)$ read from the actual lattice $a\sqrt{\sigma}$ (Table 2 of 23-01) is 0.304 over $\beta = 2.25 \to 2.60$, reaching neither the one-loop 0.3715 nor the two-loop 0.397 Reading . Assuming the step-by-step form $\xi(\beta - \kappa \ln b) = \xi(\beta)/b$ for every $b \geq 1$ gives $\xi(\beta) = \xi(0)e^{\beta/\kappa}$ Proved here Lean `xi_eq_exp` (assumption: that functional equation; neither continuity nor monotonicity is assumed). The physical content lies on the side of the assumed functional equation (the one-loop running put down as an equation).

The form of the answer. Stacking steps with b fixed, the shortfall accumulates by the same amount each step, and if $\delta(b) \neq 0$ the sum diverges. For a sequence with growing step scales $b_k = k + 2$, $\sum_k c_2/(k+2)^2 < \infty$ and it stays bounded Proved here Lean `fixed_step_bounded_iff` (assumption: a constant error $\varepsilon_j = d \geq 0$. Conclusion: the partial sums are bounded $\iff d = 0$). For example, with the provisional value $c_2 = 1$, fixing $b = 2$ gives a total of 250 after 1000 steps, while with $b_j = j + 2$ the total stays within $\pi^2/6 - 1 = 0.645$ (values of closed formulas). What ties this shortfall to the first arrow of (1) is the algebra of two powers.

$$C_P \asymp \xi^{\,2-\eta}, \qquad a\,\xi \propto a^{\theta} \quad\Longrightarrow\quad a^2 C_P \propto a^{p}, \qquad p = \eta + (2-\eta)\,\theta \;(7)$$

θ is the exponent of accumulation of the shortfall (with b fixed, $\delta(b)/\ln b$), and η is the dynamical power η_{dyn}. If $0 \leq \eta \leq 2$ and $\theta \geq 0$, then $p = 0 \iff \eta = 0 \wedge \theta = 0$ Proved here Lean `drain_exponent`, `drainExp_eq_zero_iff` (assumptions: the power form of (7) and the signs). In words, "under the relaxation-time assumption ($\eta = 0$), the physical-unit gap statement $\Longleftrightarrow \sum_k \delta(b_k) < \infty$", and this is "the form of the answer". Besides the relaxation-time assumption, this statement stands on the power form of (7), a model in which the errors of the steps are additive and bounded, and δ being the value of the model of the one-loop kernel Assumption . The algebra that if the partial sums of the errors of the steps are bounded then $a_k^2 C_P^{(k)}$ stays in a positive band is also closed Lean `S4_two_sided` (assumptions: partial sums $|\sum_{j<k} \varepsilon_j| \leq M$, positivity of coefficients. Reading this as $a_k^2 C_P^{(k)}$ is on the side of the text).

A road that changes the kernel to make $\delta(2) = 0$ exists in the one-parameter family $c_1(q) = \alpha + (1-\alpha)\cos q$ at $\alpha_0 = 1.4400$, but the neighbouring weight is negative Reading . For kernels with positive weights, $c_2 > 0$ Lean `c2_ge_of_nonneg_weights` (assumption: c_2 is an increasing affine function $A + \kappa S$ of the kernel quantity $S \geq 0$, with $A, \kappa > 0$, a measured dependence. Not an unconditional claim). Negative weights can collide with reflection positivity, so within the families examined, removing the shortfall and the condition for returning to a quantum theory are not compatible. Outside the families it is undecided.

03

The seventeen roads tried — one row each

Table 2 lines up the seventeen roads one row each. The names in the "Where it stopped" column are the walls of § 06, and the "What remained" column is algebra closed in Lean (assumptions in brackets) or labelled statements.

Table 2 The seventeen roads tried on the four-dimensional lattice

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of strong-coupling curvature	Widen the β where the curvature method works (outside (1), strong-coupling side)	Lower bound on the Hessian form after one-link integration of a complete family, and the Bakry–Émery criterion	Up to $\beta < \sqrt{2/27} = 0.2722$. The correlation length is about 1/10 of a lattice spacing, on the opposite side from the limit (upper-bound tool wall)	Lean `hess_gauge` (assumptions: the period L is even with $L \geq 3$, unit quaternions, pure imaginary perturbations. Conclusion: $\partial_\tau^2 \Phi \geq -27\beta^2 \sum \lvert X_e \rvert^2$). The stage proceeding to the gap is Assumption
Road of one-step bookkeeping	Bookkeep the continuum limit as a sequence of steps (one-step bookkeeping)	Complete families, character expansion, disc steps, one-loop windows	One-link integration works only once (single-step integral wall). The bookkeeping of surfaces is multiplicative and does not produce the one-loop slope	`discStep_semigroup`, `step_lowers_coupling`, `kappaDisj_lt_one`, `intact_frac_le`, `no_blocking`, `blocking_shortfall`
Road of the sum of shortfalls	Is the sum of the shortfalls $\delta(b)$ finite (the summability-of-shortfalls assumption)	Window integrals of the model of the one-loop kernel	With b fixed the sum diverges. The kernel with $\delta(2) = 0$ has a negative neighbouring weight (positive-weight kernel wall). The difference of c_2 with the choice of gauge went only as far as reading it positive in two gauges (gauge dependence of the shortfall wall)	`fixed_step_bounded_iff`, `c2_ge_of_nonneg_weights` (assumption: the measured dependence). Reading c_2

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of the strong-coupling series	Can the dynamical power η be read from the strong-coupling series (the η of the relaxation-time assumption)	High-order strong-coupling expansion, Dlog-Padé	Near the pole the series takes the renewal form $G = A/(1 - K(p))$, which forces $\eta = 0$. At an essential singularity under asymptotic freedom the exponent cannot be read (renewal-series wall)	Lean `renewal_chi_mul_massSq`, `renewal_residue_pos` (assumption: coefficients of the renewal form $A, c_0 > 0$. Conclusion: $\chi m_0^2 = A/c_0 > 0$). That it takes the renewal form is Assumption
Road of the dimension of complete families	In which dimensions do complete families of edges and faces exist (one-step bookkeeping)	Double counting, exhaustive search for exact covers	For edges, only for $d \le 4$ with an even period. For k-cells there is an upper limit $d \le 3k + 1$, and $d = 9$ for $k = 3$ is undecided	`perfectK_one_iff`, `no_perfectZ_of_five`, `no_perfect_k4_d9`, `no_perfect_k5_d8` (assumption: period 2 only)
Road of the dimensional threshold	Why four dimensions is the borderline (the summability-of-shortfalls assumption)	The recursion for the drift of the steps $x_{k+1} = b^{d-4} x_k + \varepsilon_k$	Contracts for $d < 4$, accumulates for $d = 4$. The same as the standard dimension count; not a reproduction of the constructions in lower dimensions	Lean `dev_le_of_contracting` (assumptions: $0 \le \lambda < 1$, $\lvert\varepsilon_j\rvert \le E$. Conclusion: drift $\le E/(1 - \lambda)$), `dev_eq_of_marginal` (with $\lambda = 1$ and constant E, drift $= nE$)
Road of reflection positivity	Use reflection positivity for an estimate from below (the reflection-positivity preservation statement, the OS reconstruction statement)	Representation by the transfer matrix	For the overlap $\tilde{Z}$ with the lightest state, an upper bound $\tilde{Z} \le \chi m$ came out where a lower bound was wanted. The question moved to the road of lower-bound test functions	Lean `overlap_le_susc_mul_gap` (assumptions: $m > 0$, $\tilde{Z} \ge 0$, $\tilde{Z} \coth(m/2) \le \chi$)

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of an area law in the bookkeeping	Does the form of the area law appear inside the bookkeeping (the Wilson-loop limit statement)	Coefficient $4(\beta/4)^{A}$	A rewriting of the strong-coupling area law, which holds even for abelian groups. Not counted as evidence of confinement. The content went to the road of the lower half at one scale	Cited (Wilson 1974, Osterwalder–Seiler 1978)
Road of multi-link integrals	Is the closed form of k-link integration analytic even in the continuum limit (one-step bookkeeping)	Closed form of a single sum	The two-link correction is of size β^{3} and not an obstacle. Analyticity has not been shown	Assumption (a calculation inside the strong-coupling expansion)
Road of comparison with the defect of a submission	Is the "small defect" of a submission that claims a construction the same quantity as δ (the summability-of-shortfalls assumption)	Comparison of types	The sum of defects is a quantity of the contracting sector; δ is the shortfall of the coefficient of the marginal sector. The words sit in the same place, but the quantities differ	We do not judge whether the submission is right or wrong (the policy of 16-01)
Road of multiscale curvature	Stack the curvature method scale by scale to make a tool for the upper-bound assumption (the upper-bound assumption)	Multiscale Bakry–Émery, curvature on the Gribov region, one step of gradient flow	Cannot be written beyond the cross terms of two scales (upper-bound tool wall). Details in § 05	Lean `comp_class_eq_of_equivariant_bijection` (assumption: the flow is a bijection commuting with the gauge action), `autocov_sum_le_on_invariant` (assumption: Poincaré on an invariant subspace)

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of volume and boundary	Is the estimate of C_P uniform in volume and boundary condition (the infinite-volume uniqueness statement)	Bare link angles, comparison of boundary conditions	With bare link angles the ratio is 1 and the worry about zero modes disappeared. Instead the boundary condition matters, and the upper-bound assumption can break in a periodic box (periodic-boundary conjecture wall)	Assumption (a reading under the conjecture that HMC and Langevin are in the same universality class)
Road of lower-bound test functions	Is there a test function giving the lower-bound assumption (the lower-bound assumption)	Three families: plaquettes, bare link angles, gauge-fixed long-wavelength modes	Local operators have dimension 4, so $\eta \geq 2$ (local-operator dimension wall); link angles give ratio 1; gauge fixing runs into single-valuedness (gauge-fixing single-valuedness wall)	`profileQuotient_isGreatest`, `fisherEta_eq_zero_iff_canonical`, `eta_ge_two_of_bounded_susc` (assumption: susceptibility uniformly bounded), `prop_at_gap_scale` (assumption: pure-power propagator), `jump_energy_ge`
Road of gauge-fixed long wavelengths	Can gauge-fixed long-wavelength modes be made single-valued (the lower-bound assumption)	Weighted averages over orbits (the schemes of Parrinello–Jona-Lasinio and Zwanziger)	Sharp means multivalued, blunt means the long wavelengths vanish (gauge-fixing single-valuedness wall). The numbers are tests on compact $U(1)$	`orbitAvg_gauge_invariant`, `orbitAvg_continuous` (assumptions: orbits of a finite group, positive continuous weights)

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of spectroscopy	Bound the gap with smoothed operators and spectroscopy (the lower-bound assumption and upper bounds)	Wilson flow, APE, variational method, effective mass	Ends in upper bounds (spectroscopic upper-bound wall). A lower bound needs a lower bound on the overlap. The cost of measuring the numerator is $\geq cL^{d+2}$ (measurement-cost wall)	`meff_antitone`, `meff_ge_of_corr`, `gap_ge_of_overlap` (assumption: a lower bound q on the overlap), `light_admixture_invisible`, `cost_ge`, `sqQuotient_between`
Road of the lower half at one scale	Build the lower half at one scale, $m^2 \gtrsim \sigma$ (the one-scale chain)	Entropy of tubes, deficit $D = m - 4\sigma$, pillar model, Wulff shape	Moved into the tube transfer-operator inequality (the tube inequality), but there is no type of proof for it (tube-inequality wall)	`mass_sq_ge_of_tube` (assumption: a norm estimate that is a consequence of the tube inequality), `mass_bound_localized`, `linear_regime_breaks_onescale`, `casimir_only_no_bound`, `linear_deficit_breaks_inv`
Road of a one-dimensional chain of scales	Sandwich the upper-bound assumption with a one-dimensional chain of scales (the upper-bound assumption)	The radial type of holography, two-weight Hardy inequality, sandwiching autocorrelations	Sandwiched only under the assumption of projection onto a one-dimensional chain of scales. The projection assumption is not weaker than the relaxation-time assumption, and in a periodic box it breaks through the topological charge (projection-assumption wall)	`conditional_scale_bound` (assumptions: the projection assumption, shortfalls non-negative with bounded partial sums, geometric-series weights), `const_defect_scale_unbounded`, `tau_int_le`, `rayleigh_le_tau`

Seen arrow by arrow, there is a bias. Stopped roads gather at the lower-bound assumption and outside (1). Only two roads aimed at the upper-bound assumption, the road of multiscale curvature and the road of a one-dimensional chain of scales; the road of multiscale curvature stopped at the cross terms, and the road of a one-dimensional chain of scales sandwiched only under the projection assumption. No road of this chapter reached the rotation-recovery statement or the non-triviality statement (on the side of the published construction, the committee's assessment is that the recovery of rotations is shown for the theory phase by

phase; 6-03). The Wilson-loop limit statement and the rotation-recovery statement are claims of a kind that do not come out however many times the map of one step is composed (limit-property wall).

04

Reaching from below — test functions, gauge fixing and spectroscopy end in upper bounds

The road of reflection positivity, the road of lower-bound test functions, the road of gauge-fixed long wavelengths and the road of spectroscopy all aimed at estimates from below, and stopped at a place of the same form. C_P is a supremum over test functions, and the gap is an infimum over states, so what one test function or one operator gives is in either case only one side.

$$C_P = \sup_f \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} \;\geq\; \frac{\mathrm{Var}(f_0)}{\mathcal{E}(f_0)}, \qquad E_{\mathrm{low}} = \inf_{\psi \perp \Omega} \frac{\langle \psi, H\psi\rangle}{\langle \psi,\psi\rangle} \;\leq\; m_{\mathrm{eff}}(k) \quad (8)$$

Cited The variational principle (1-10). Ω is the vacuum, H the Hamiltonian of the transfer matrix, and $m_{\mathrm{eff}}(k) = \ln(C(k)/C(k+1))$ the effective mass at time k. The lower-bound assumption is a statement about "some f", the side test functions can reach; the upper-bound assumption is about "every f", the side they cannot.

Local operators (the road of lower-bound test functions). In the family made by translating one operator O_x and summing with weights, the freedom of the weights collapses to a single best momentum Lean `profileQuotient_isGreatest` (assumptions: finitely many momenta, the Fourier transform of the denominator is positive). The numerator is a susceptibility, and estimating it by the integral up to radius ξ,

$$\frac{\mathrm{Var}(f_O)}{\mathcal{E}(f_O)} \asymp \xi^{\,2-\eta_s}, \qquad \eta_s = \min\bigl(2\Delta_O - (d-2),\, 2\bigr), \qquad \Delta_O \geq 4 \Rightarrow \eta_s = 2 \quad (9)$$

(assuming a power form of the correlations). η_s is the static power of each operator, a different quantity from the dynamical η_{dyn} (18-05). The lowest dimension of gauge-invariant local operators is the 4 of $\mathrm{tr}\, F^2$ Cited , so the ratio does not grow with ξ Lean `eta_ge_two_of_bounded_susc` (assumption: the susceptibility is uniformly bounded; the boundedness comes from the physical assumptions of the form of correlations and $\Delta = 4$). For bare link angles, under the measure without gauge fixing each link angle is uniform and independent, and the ratio is 1 independently of β and the volume Proved here (a direct calculation from the invariance of the Haar measure; the same mechanism as Elitzur's theorem).

Gauge fixing (the road of gauge-fixed long wavelengths). Gauge-fixed long-wavelength modes are non-local and avoid the wall of dimension, but the choice of representative must be single-valued. Taking a weighted average at temperature T over the orbit makes it single-valued and continuous Lean `orbitAvg_gauge_invariant`, `orbitAvg_continuous`. In the smallest example, the $U(1)$ ring (L edges), the amplitude of the averaged mode vanishes exponentially

with L as $(I_1/I_0)^{L-1}$ for blunt fixing, and jumps at the total angle $\Phi = \pi$ for sharp fixing Proved here (closed formulas from Fourier series). The same functional serves both as "a tool for choosing among copies" and "a tool for evening out long wavelengths", so the two demands sit at the two ends of a single knob. The numbers measured for four-dimensional compact $U(1)$ (the scatter between copies being 3.25 standard deviations, and so on) are Reading ; they were not measured for $SU(2)$.

Spectroscopy (the road of reflection positivity, the road of spectroscopy). Smoothing in the three spatial directions only stays on one time slice, and correlations keep the form $C(k) = \sum_n w_n e^{-E_n k}$ $(w_n \geq 0)$ Cited (5-03). With this form the effective mass decreases monotonically Lean `meff_antitone` (assumption: a representation with non-negative weights), and at every k it is an upper bound on the lowest energy `meff_ge_of_corr`. A lower bound needs a different input.

$$E_{\text{low}} \;\geq\; \frac{1}{k}\Big[\ln q + \ln \frac{C(0)}{C(k)}\Big] \qquad (w_{\text{low}} \geq q\, C(0) \text{ assumed}) \quad (10)$$

Proved here Lean `gap_ge_of_overlap` (assumptions: a representation with non-negative weights and a lower bound q on the overlap). The price of the lower bound is exactly one lower bound on the overlap; without q, a light state of weight ε changes the correlation by only ε at every k `light_admixture_invisible` (a concrete two-level example). That no lower bound on the gap follows from finitely many measurements is a restatement of the known framework of the equivalence of a gap and exponential clustering Cited (4-11). On the numerical side, the effective mass of a smoothed operator at $\beta = 2.6$, 12^4 was $\leq 0.501 \pm 0.071$, touching the $am(0^{++}) = 0.514(8)$ of the original source from below Reading (Monte Carlo, reading of an upper bound). A four-dimensional flow that also smooths in the time direction does not stay on one slice, and breaks the upper-bound property with an effective mass that increases monotonically Proved here (closed formulas for the free field). The cost of measuring is $\geq cL^{d+2}$, from the relaxation time $\propto L^2$ of the softest mode in a box of side L `cost_ge` (assumptions: a lower bound on the relaxation time, and the number of sweeps needed for relative precision).

The three roads reached the same wall. Each needs "the overlap of gauge-invariant operators with the lightest fluctuation is bounded from below", which includes "no other light state is hidden", a statement about all states. It has the same form as the upper-bound assumption.

05

Not reaching from above — curvature, tubes, one-dimensional chains of scales, boundaries

The upper-bound assumption is an upper bound on CP. The types of tools that give upper bounds are limited; there are two representative ones.

$$\nabla^2 U \geq K > 0 \implies CP \leq \frac{1}{K} \quad \text{(Bakry–Émery)}, \qquad B \leq CP \leq 4B \quad \text{(one dimension, Muckenhoupt}$$

Cited The left comes from a lower bound on the curvature of the measure e^{-U} (1-10); the right sandwiches C_P within a factor of 4 by the maximum B of "the mass before a point $\times$ the resistance beyond it", which can be computed only in one dimension.

Curvature scale by scale (the road of strong-coupling curvature, the road of multiscale curvature). The known places where inequalities of the type of the upper-bound assumption stand as theorems are only curvature, diameter, convexity, strong coupling, and the multiscale Bakry–Émery uniform in the lattice regularisation for the high-temperature phase of continuum scalar fields in two and three dimensions Cited (Theorem 1.1 of Bauerschmidt–Dagallier; the criterion is Theorem 2.5 of Bauerschmidt–Bodineau). For four-dimensional gauge fields, the curvature after one-link integration is positive only at strong coupling (the road of strong-coupling curvature). Examining the curvature on the Gribov region in a finite-dimensional model, it was positive only on the strong-coupling side, and the non-convexity came from the zero-momentum mode Reading (on samples with $L = 2, 3, 4$). The ultraviolet fluctuations remaining after removing the zero modes cannot be overcome by one step of gradient flow: the flow is a bijection commuting with the gauge action, so it does not shrink the space of functions Lean `comp_class_eq_of_equivariant_bijection`; the only gain is in changing the metric, and the curvature of the pushed-forward measure became more deeply negative with flow time Reading (within the range measured: open box $\ell = 3$, typical points at $\beta = 2.3, 4$, $t \le 0.5$).

The deficit of tubes (the road of the lower half at one scale). We moved the lower half of the one-scale chain, $m^2 \gtrsim \sigma$, into an inequality between the transfer operator $\mathcal{T}$ of closed tubes and the perimeter operator $\hat{\ell}$.

$$T(c,1): \quad \mathcal{T} \;\le\; \exp\Big(-\big(\sigma\hat{\ell} + c/\hat{\ell}\big)\Big) \qquad \Longrightarrow \qquad m \;\ge\; \min_{\ell>0}\Big(\sigma\ell + \frac{c}{\ell}\Big) = 2\sqrt{c\sigma} \quad (12)$$

Proved here (the right is the AM–GM inequality) Lean `mass_sq_ge_of_tube` (assumption: the norm of the transfer operator is at most $e^{-2\sqrt{c\sigma}}$. Conclusion: $m^2 \ge 4c\sigma$). The power of $1/\ell$ is the only power that makes the coefficient c dimensionless. The content is on the side of the tube inequality, which itself has no type of proof. In the region where the strong-coupling surface expansion converges, the lower half at one scale is itself false `linear_regime_breaks_onescale` (assumption: a sandwich in the region of convergence); in counting with the pillar model the deficit can have either sign, the Wulff shape is a rhombus, and the tube inequality stands only after the anisotropy is subtracted `linear_deficit_breaks_inv` (negative control). The final form is three assumptions, "localised T on short loops, a Casimir floor on large loops, and a seam between them" `mass_bound_localized`; the floor alone gives no lower bound `casimir_only_no_bound`.

A one-dimensional chain of scales (the road of a one-dimensional chain of scales). In holographic models the gap is an eigenvalue problem in a single radial variable, and on that variable the band statement $C_P \asymp \xi^2$ becomes an identity needing no proof Proved here (ground-state transformation $H = P^\dagger P$ and the variational principle). To bring this onto the lattice, we put a one-dimensional chain on the scale $s = \ln \ell$ and sandwiched C_P with the discrete version of the right of (11).

$$(\mathrm{P}) \ \wedge \ \sum_k \delta(b_k) \le S_\delta \quad \Longrightarrow \quad e^{-2T} C_P \text{ is bounded independently of } T = \ln(1/a) \tag{13}$$

Lean `conditional_scale_bound` (assumptions: the projection assumption, shortfalls non-negative with bounded partial sums, geometric-series weights; the Hardy sandwich for finite chains is a reproof of a known result). As a negative control, with a constant shortfall the boundedness breaks `const_defect_scale_unbounded`. The content is on the side of the projection assumption, and the projection assumption splits into four, "the variance can be split scale by scale", "the conditional Poincaré constants of each stage are uniform", "the conductance across stages is bounded", "the state space is a single line of scales", of which the last three have not been shown Assumption . The projection assumption is not weaker than the relaxation-time assumption. The one-dimensional chain did not show the upper-bound assumption; it restated the upper-bound assumption as the projection assumption.

Boundaries and the topological charge (the road of volume and boundary, the road of a one-dimensional chain of scales). In a periodic box the state space splits into sheets by topological charge Q, connected only at the ultraviolet end. For reversible dynamics, for every observable

$$R(f) = \frac{\operatorname{Var} f}{\mathcal{E}(f)} \ \le \ \tau_{\mathrm{int}}(f) \ \le \ C_P \tag{14}$$

holds Cited Lean `tau_int_le`, `rayleigh_le_tau` (finite reversible chains; a reproof of known facts), and every measured τ_{int} is a lower bound on C_P. The original source reports for HMC-type dynamics an effective dynamical exponent of the square of the topological charge of "about 5" Reading (Schaefer–Sommer–Virotta 2011); combining this with the conjecture that HMC and Langevin are in the same universality class Assumption and (14), in a periodic box $C_P \gtrsim a^{-5}$ and the upper-bound assumption breaks. It is a conditional reading on top of a conjecture. "About 5" is a value for $SU(3)$; the same measurement for $SU(2)$ was found in one source (HMC, periodic boundaries, small boxes, three points), reported to be close to a^{-2} Reading (Abe–Morikawa–Suzuki 2025; the note in 23-01 § 05). Open boundaries remove this barrier only inside a finite box with the physical side held fixed (a count using the charge of a subregion as test function Assumption ; in the original source too, the autocorrelation time of Q^2 with open boundaries grows roughly in proportion to the length in the time direction Reading). In this book's order, which makes the volume infinite first, this is expected not to help, and the candidate direction is to rewrite the statement of the relaxation-time assumption itself as a sentence sector by sector of topology, or as a local sentence with support limited to a few correlation lengths (the note in 23-01 § 05; how to rewrite it is dealt with in Part 24).

Thus every road on the upper side stopped by naming the place to which the difficulty had moved.

$$m^2 \gtrsim \sigma \ \Longleftarrow \ T_{\mathrm{loc}}(c) \wedge \text{floor} \wedge \text{seam}, \qquad C_P \lesssim \xi^2 \ \Longleftarrow \ (\mathrm{P}) \wedge \textstyle\sum \delta < \infty \tag{15}$$

The arrows go only one way, from right to left, and every right-hand side is an assumption.

06

Twenty-one walls — five types

A wall is a one-sentence statement of why a certain road, with certain tools, stopped short of a certain arrow of (1). There are five types: walls of principle come from the direction of an inequality and do not depend on the details of the tool; walls of structure are where demands are not compatible within the chosen family, bookkeeping or series (outside the family it is undecided); walls of tools are where the type of inequality needed is known but there is no tool to show it; walls of assumptions are the footing of conditional statements; walls of measurement and literature are what was not settled by numbers, boxes or primary sources. No wall is a proof that it "cannot be crossed".

Table 3 Twenty-one walls ("To move" is what follows from the definition)

No.	Wall	One line	To move	Arrow
1	One-sided test-function wall	Test functions bound C_P only from below Cited	An inequality of a type other than test functions	Upper-bound assumption
2	Spectroscopic upper-bound wall	Spectroscopy ends in upper bounds on the gap. A light state with small weight is invisible in a finite window Cited	A lower bound on the overlap with the lightest state	Physical-unit gap statement
3	Local-operator dimension wall	Local gauge-invariant operators have dimension 4 or more, so $\eta_S = 2$ (algebra under the assumption that the susceptibility is bounded)	Smooth out up to radius ξ ($\to$ measurement-cost wall)	Lower-bound assumption
4	Gauge-fixing single-valuedness wall	Gauge fixing: sharp means multivalued, blunt means the long wavelengths vanish Reading ($U(1)$)	A construction separating selection from smoothing (no proposal)	Lower-bound assumption
5	Measurement-cost wall	The numerator of smoothed operators is the zero-momentum susceptibility, and the cost of measuring it is $\geq cL^{d+2}$	Bound the numerator by estimates rather than measurement	Lower-bound assumption
6	Positive-weight kernel wall	The kernel with $\delta(2) = 0$ has a negative neighbouring weight. For kernels with positive weights $c_2 > 0$ (under the measured dependence)	Kernels outside the family	Summability of shortfalls
7	Renewal-series wall	The strong-coupling series has the renewal form, forces $\eta = 0$, and does not let the exponent be read	Information from outside the series	The η of the relaxation-time assumption
8	Single-step integral wall	One-link integration of a complete family works for one step only	A different bookkeeping that stacks steps	One-step bookkeeping
9	Limit-property wall	The Wilson-loop limit statement and the rotation-recovery statement do not come out of compositions of the map of one step	Another tool to show the properties of the limit	Limit
10	Upper-bound tool wall	There is no tool of the type of the upper-bound assumption. The proposal of stacking curvature scale by scale stops at the cross terms	Estimates of the cross terms	Upper-bound assumption

No.	Wall	One line	To move	Arrow
11	Tube-inequality wall	$m^2 \gtrsim \sigma$ moved to the tube inequality, but the tube inequality has no type of proof	A gauge-invariant definition of the transfer operator on loop space	One-scale chain
12	Projection-assumption wall	The projection assumption is not weaker than the relaxation-time assumption. The lower-bound assumption does not drop to one dimension	Derive the projection assumption from a weaker statement	Upper-bound assumption
13	Checking-tool wall	The Riemannian geometry of the one-link argument, and the generalised eigenvalue problem of the matrix variation of CP, lack tools for machine checking	Building up the library	One-step bookkeeping
14	Wall of the assumption itself	The relaxation-time assumption and the three chains (the relaxation chain, the one-scale chain, the bookkeeping-length chain) are not proved Assumption	The upper-bound assumption itself, the projection assumption, or one of the lower bounds of the three chains	The η of the relaxation-time assumption, chains
15	Equivalence-scaffold wall	The scaffold of the equivalence for the physical-unit gap statement: the power form, $\eta \geq 0$, additive errors of the steps, the δ of the one-loop kernel Assumption	Remove each assumption	Summability of shortfalls
16	Periodic-boundary conjecture wall	The breaking of the upper-bound assumption with periodic boundaries depends on the conjecture that HMC and Langevin are in the same universality class Assumption	Dynamical exponent of local dynamics for $SU(2)$	Upper-bound assumption
17	Fitting wall	That the breaking of $m/\sqrt{\sigma}$ is $O(a^2)$ is a fit to tables of numbers Reading	Estimates of the distortion of the steps	One-scale chain
18	Box-size wall	The boxes are small (8^4 is in the small-box phase for $\beta \gtrsim 2.45$, and 12^3 has $L\sqrt{\sigma} < 3$)	Large boxes with matched physical volume	Lower-bound assumption
19	Gauge dependence of the shortfall wall	The difference of c_2 with the choice of gauge went only as far as reading it positive in both of two gauges inside the model of the one-loop kernel Reading	A gauge-independent definition of δ	Summability of shortfalls
20	Tube-model sign wall	In the tube model, the sign of the $1/P$ coefficient at finite u is undecided	An effective model of the zero modes of shape	One-scale chain
21	Primary-source wall	Gaps in primary sources (the dynamical exponent of local dynamics for $SU(2)$; the texts of Bałaban, Magnen–Rivasseau–Sénéor and Osterwalder–Seiler)	Obtain them	Physical-unit gap statement

Walls of principle. The one-sided test-function wall and the spectroscopic upper-bound wall are the two one-sided halves of (8), and they do not move however many measurements are stacked. For example, for the standard normal distribution ($C_P = 1$, with equality at $f = x$), the maximum ratio over a family of odd polynomials not containing x approaches 1 as the family is enlarged, but the numbers inside the family alone do not tell whether 1 is the final value Reading (lower bound over a family). One knows it is 1 because of an input from outside the family ($C_P \le 1$ from the convexity $V'' = 1$), and an input of this kind is what the upper-bound assumption needs.

Walls of structure. Incompatibilities inside the chosen family; they say nothing about outside it. The local-operator dimension wall, the gauge-fixing single-valuedness wall and the measurement-cost wall are the ways the three roads of § 04 stopped; the positive-weight kernel wall is the tension between reflection positivity and removing the shortfall, the same tension, in the opposite direction, as what the problem statement says about continuum regularisations, "once reflection positivity is lost there are few ways to recover it". The renewal-series wall, the single-step integral wall and the limit-property wall are limits of the form of bookkeeping and series.

Walls of tools. They gather in the column of the upper-bound assumption. The upper-bound tool wall, the tube-inequality wall and the projection-assumption wall are the three places of § 05 to which the difficulty moved; the checking-tool wall alone is not a mathematical obstacle but a lack of tools for checking. The $\beta < 0.2722$ where the curvature theorem exists and the $\beta = 2.60$ used by spectroscopy (correlation length 1.95 lattice spacings) are about 9.6 times apart, and an estimate crossing that interval uniformly is the content of the upper-bound tool wall.

Walls of assumptions. The band statement $a^2 C_P \asymp 1$ splits into the product of three ratios $(a^2/\sigma)(\sigma/m^2)(m^2/\lambda_1)$, and the central equivalence also stands on footing other than the relaxation-time assumption. Each assumption corresponds to a concept of Parts 1–5 and Parts 7, 17 and 18: the relaxation-time assumption, the one-scale chain and the bookkeeping-length chain are candidates for restating "a single scale" (4-07) in terms of the Poincaré constant; the relaxation chain is the dynamical exponent $z = 2$ (18-03); the one-scale chain is a single scale (17-12); the bookkeeping-length chain is the specialisation of asymptotic scaling (5-05) to the bookkeeping length. These are analogies, not mathematical equivalences.

Walls of measurement and literature. They move by computation or by obtaining sources, but even if they move, the logic of (1) does not change. The representative is the size of the box: $12 \times 0.15896 = 1.91$ ($\beta = 2.55$) and $12 \times 0.13395 = 1.61$ ($\beta = 2.60$), both small boxes that do not contain 3 physical lengths $1/\sqrt{\sigma}$ Reading .

The walls in the column of the upper-bound assumption are the one-sided test-function wall, the upper-bound tool wall, the projection-assumption wall and the periodic-boundary conjecture wall, with no wall of measurement and literature. That is, short of the upper-bound assumption there is no wall that moves by measuring. Where the setters wrote that "new ideas are needed" is also the side of the gap uniform in the volume (Jaffe–Witten § 6.5) Cited , the same place where the walls that measurement does not move gather.

07

What remained — algebra closed in Lean, and numerical readings

Most of what remained from the seventeen roads is finite inequalities and algebra with the assumptions made explicit as arguments. Table 4 lines up the theorems cited in the text by what their statements are about. Inside Lean every theorem is a sentence about real numbers, sequences, or sums over finite sets; that they are quantities of the Wilson measure is a reading attached by the text. There are no theorems about expectation values, correlations or gaps of the Wilson measure (23-13).

Table 4 The main theorems closed in Lean on the side of the four-dimensional lattice (character: reproof = a known fact proved again / arrangement = algebra of definitions and assumptions / conditional = only the arrow from unproved assumptions to conclusions / negative control = an example that breaks when an assumption is removed / lattice theorem = a theorem about families of edges or effective actions, assuming only the shape of the lattice)

Theorem	Assumptions	Statement	Character
`log_F0_le`, `log_F0_ge`	Argument non-negative	Two-sided estimate (3) of the logarithm of one-link integration	Reproof
`Incidence.intact_frac_le`	Incidence regular	Fraction of intact faces $\leq$ fraction of remaining edges (4)	Arrangement
`no_blocking`, `blocking_shortfall`	Sequence of complete steps / fraction removed at each step $\leq 1/4$	Stacking steps does not bring intact faces back	Arrangement
`discStep_semigroup`, `step_lowers_coupling`	Surfaces of disc shape	Disc steps form a semigroup in the area, with $\beta' < \beta$	Reproof
`kappaDisj_lt_one`	$b \geq 2$	The contribution of surfaces sharing no edges is less than 1 (5)	Arrangement
`xi_eq_exp`	$\xi(\beta - t) = \xi(\beta)e^{-t/\kappa}$ for all steps	$\xi(\beta) = \xi(0)e^{\beta/\kappa}$	Arrangement
`fixed_step_bounded_iff`, `S4_two_sided`	Constant error / partial sums bounded	Partial sums bounded $\iff d = 0$ / stays in a positive band	Arrangement, conditional
`drain_exponent`, `drainExp_eq_zero_iff`	Power form of (7), $0 \leq \eta \leq 2, \theta \geq 0$	$p = 0 \iff \eta = \theta = 0$	Arrangement
`c2_ge_of_nonneg_weights`	$c2 = A + \kappa S$ (measured dependence)	$c2 > 0$	Conditional
`PerfectFamily.perfectK_one_iff`, `no_perfectZ_of_five`	$2 \leq d$ / $d \geq 5$	Classification of complete families of edges (Table 1)	Lattice theorem

Theorem	Assumptions	Statement	Character
`StaplePerturbation.hess_gauge`	Period even and ≥ 3, unit quaternions, pure imaginary perturbations	Hessian form of the effective action $\geq -27\beta^2$	Lattice theorem
`renewal_chi_mul_massSq`	Coefficients of the renewal form positive	$\chi m_0^2 = A/c_0$	Arrangement
`dev_le_of_contracting`, `dev_eq_of_marginal`	$0 \leq \lambda < 1$ / $\lambda = 1$	Drift bounded / proportional to the number of steps	Arrangement
`overlap_le_susc_mul_gap`	$m > 0$, $\tilde{Z} \coth(m/2) \leq \chi$	$\tilde{Z} \leq \chi m$	Arrangement
`profileQuotient_isGreatest`	Finitely many momenta, denominator positive	The supremum of the ratio over a rank-one family is the maximum of the componentwise ratios	Reproof
`eta_ge_two_of_bounded_susc`	Susceptibility uniformly bounded	$\eta_s \geq 2$	Conditional
`orbitAvg_gauge_invariant`	Action of a finite group	Weighted averages over orbits are gauge-invariant	Reproof
`meff_antitone`, `meff_ge_of_corr`	Representation with non-negative weights	The effective mass decreases monotonically and is an upper bound on the lowest energy	Reproof
`gap_ge_of_overlap`, `light_admixture_invisible`	Lower bound q on the overlap / two-level example	(10) / a light state is invisible in the window	Conditional, negative control
`cost_ge`	Lower bound on the relaxation time, number of sweeps	Cost $\geq cL^{d+2}$	Conditional
`mass_sq_ge_of_tube`, `mass_bound_localized`	Consequence of the tube inequality / localised T, floor, seam	$m^2 \geq 4c\sigma$ (12)	Conditional
`linear_regime_breaks_onescale`, `casimir_only_no_bound`, `linear_deficit_breaks_inv`	Region of convergence / floor only / raw perimeter	Under each assumption the lower half of a single scale is false	Negative control
`conditional_scale_bound`, `const_defect_scale_unbounded`	The projection assumption, bounded shortfall / constant shortfall	(13) / boundedness breaks	Conditional, negative control
`tau_int_le`, `rayleigh_le_tau`	Finite reversible chains	(14)	Reproof

Theorem	Assumptions	Statement	Character
`comp_class_eq_of_equivariant_bijection`	Bijection, commuting with the action	The class of functions of the flowed field is the same as the class of invariant functions	Arrangement

The content of the conditional rows lies on the side of the assumptions, and the theorems do not take a single step towards showing those assumptions. Only the two rows of lattice theorems (the classification of complete families and the lower bound on the Hessian form of the effective action) are theorems on the lattice that assume only the shape of the lattice. The former is a combinatorial fact about the lattice; the latter is an estimate that works on the strong-coupling side, where the correlation length is about 1/10 of a lattice spacing. Both concern places far from the continuum limit.

What remained on the numerical side is readings Reading : the agreement of the slope in the model of the one-loop kernel with the known coefficient (relative 2.8×10^{-5}, inside the model), the values of c_2 (two weights and two gauges), the effective slope 0.304 read from the lattice $a\sqrt{\sigma}$ of the original source, the upper bound 0.501 ± 0.071 on the effective mass of smoothed operators ($\beta = 2.6$, 12^4), the scatter of gauge-fixed copies for compact $U(1)$, the strong-coupling range where the curvature of the Gribov region is positive (samples of a finite-dimensional model), and the curvature of the pushed-forward measure of the gradient flow (at the typical points measured). None of them is used in a proof of an upper or lower bound.

08

Why we step down to the first step — which tool stopped at which wall

Summing up § 04– § 06: there is no tool to attack the upper side of the relaxation-time assumption (the upper-bound assumption, every f) while staying on the four-dimensional lattice. The tools of the lower side (test functions, spectroscopy) stopped at walls of principle, and the tools of the upper side (curvature, tubes, one-dimensional chains) all stopped where they had moved the difficulty into a single assumption. Where each stopped is as in the following table, tool by tool. The right column is the prospect of what changes when the same tool is brought to the model of the first step (the path measure of the double well in 0+1 dimensions, 23-04).

Table 5 Correspondence of tools and walls, and what changes on the first step (the right column is a prospect; the content on the first step is in each chapter)

Tool	Wall where it stopped on the four-dimensional lattice	What changes on the first step
Test functions, spectroscopy (lower side)	One-sided test-function wall, spectroscopic upper-bound wall, local-operator dimension wall, gauge-fixing single-valuedness wall, measurement-cost wall	On the lower side, a family of test functions local in time reaches equality in $C_P \geq \xi^2$ (the One-Dimensional One-Sided Bridge Proposition, 23-04 § 04). That the upper side needs different tools does not change on the first step either

Tool	Wall where it stopped on the four-dimensional lattice	What changes on the first step
Curvature (Bakry–Émery)	Upper-bound tool wall (positive only at strong coupling; cross terms)	The measure as a whole has two valleys and is not convex, but inside one phase without kinks local Poincaré can be asked through an assembly of sectors (the in-phase local Poincaré condition, 23-06)
One-dimensional chain of scales (the projection assumption)	Projection-assumption wall (the projection assumption is not weaker than the relaxation-time assumption)	The slow degree of freedom is the position of the kinks, and the form of the projection to one dimension is visible from the model. The projection assumption splits into, and is named as, transport along the family of saddles (the saddle-tube condition) and reduction to the box dynamics (the box-reduction condition) (23-07, 23-11)
Transfer operator of tubes (the tube inequality)	Tube-inequality wall (there is no type of proof for the lower bound on the mass)	The side of the mass is not needed: the correlation length $\xi = \hbar/\Delta$ is obtained accurately and independently from the tunnelling splitting, and what it is compared with is clear (23-04)
One-step bookkeeping (the summability-of-shortfalls assumption)	Positive-weight kernel wall, single-step integral wall, equivalence-scaffold wall	No need to make a bookkeeping length: the correlation length grows in $\hbar$ at the rate $e^{S_0/\hbar}$, and the length to compare with is fixed directly by the measure
Boundaries and the topological charge	Periodic-boundary conjecture wall (the breaking in a periodic box rests on a conjecture)	Stated in a regime where the length of the paths T is made infinite first, so the question is set up in a form where the topological sheets of a periodic box do not appear
Boxes and primary sources	Walls of measurement and literature (box size, gauge dependence of the shortfall, tube-model sign, primary sources)	With numerics of the transfer operator, semiclassical formulas and interval arithmetic, many of the quantities compared are obtained to high precision or as inequalities (23-08)

Some things do not change. That the upper-bound assumption is an inequality over "every f", and that tools of the lower side cannot reach it in principle (the one-sided test-function wall), are the same on the first step. So what the first step does is to build the assembly of the upper side first, in the form of propositions with their assumption rows named. The relaxation-time assumption is a sentence independent of dimension (23-01 § 05), so we build the same sentence in the smallest model where the ruler grows exponentially in the parameter, the slow motion is the motion of "changing phase", and the quantities compared are known exactly.

Even if the first step is closed, there is no guarantee that it lifts to the four-dimensional lattice. What has a prospect of lifting is the form of the sentence and the form of the tools; constants and ranges of parameters do not lift (23-03 § 06). The table of walls in this chapter is kept as a table for reading which walls the tools of the first step might hit again on the second step.

The relaxation-time assumption is one restatement chosen as a handle, and the roads and walls of this chapter too are within the scope of a study with the relaxation-time assumption at its axis. Viewpoints other than the relaxation-time assumption are dealt with in Part 24.

What this chapter can say

Said	Not said
The arrangement of seventeen roads and twenty-one walls (Tables 2 and 3), the correspondence of tools and walls (Table 5)	None of the arrows of (1) is proved. "Stopped" does not mean that anything was shown about the mass gap or the continuum limit, nor that the propositions not reached are false
Cited The variational principle and finite windows, the lowest dimension of local operators, reflection positivity, the strong-coupling area law, the one-loop coefficient, dimension counting, the Bakry–Émery and Muckenhoupt inequalities, $R \le \tau_{\mathrm{int}} \le C_P$	Counting these as results
Proved here The count (2) giving $d \le 4$ for complete families, the identity (4) of the deficit of bookkeeping, the algebra of powers in (7), the algebra of (10) and (12), the ratio 1 for bare link angles, the closed formulas for gauge fixing on a ring. Lean The theorems of Table 4 (assumptions on the same row)	Proofs of the relaxation-time assumption, the three chains, the tube inequality, the projection assumption, boundedness of the susceptibility, lower bounds on the overlap, the measured dependence of c_2, the model of errors of the steps
Reading The agreement of κ in the model of the one-loop kernel and c_2, the effective slope of the lattice, upper bounds from spectroscopy, gauge fixing for $U(1)$, the curvature of the Gribov region and of the gradient flow, the effective exponent of the topological charge	That the numbers of the models are non-perturbative quantities of four-dimensional $SU(2)$. Using readings as proofs of upper or lower bounds
Assumption That one-link integration works only once, the renewal form, the breaking of the upper-bound assumption in a periodic box (on top of a conjecture), what changes on the first step (the prospects in the right column of Table 5)	That results of the first step lift to the second step and beyond. About the mass gap and the continuum limit, this chapter shows nothing

Sources and reproduction

Item	Kind	Source / tools
One-link integration, the heat-bath method, the strong-coupling area law, two-dimensional decimation	Cited	Source tables of 5-07, 5-02 and 7-02 (K. G. Wilson, Phys. Rev. D 10 (1974) 2445; K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440; A. A. Migdal, Zh. Eksp. Teor. Fiz. 69 (1975) 810. Bibliographic data only).
One-loop coefficient b_0	Cited	D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343. H. D. Politzer, ibid. 1346 (bibliographic data only).
Lattice $a\sqrt{\sigma}$ and $am(0^{++})$	Reading	B. Lucini, M. Teper, JHEP 0106 (2001) 050, hep-lat/0103027, Table 1 (text checked).

Item	Kind	Source / tools
Variational principle, finite windows, Bakry–Émery	Cited	The tables of sources of 1-10 and 4-11. D. Bakry, M. Émery, Lecture Notes in Math. 1123 (1985) 177–206 (bibliographic data only).
What multiscale Bakry–Émery has reached	Cited	Theorem 1.1 of R. Bauerschmidt, B. Dagallier, arXiv:2202.02295 (text checked). Theorem 2.5 of R. Bauerschmidt, T. Bodineau, Comm. Pure Appl. Math. 74 (2021).
Two-weight Hardy inequality	Cited	B. Muckenhoupt, Studia Math. 44 (1972) 31–38 (bibliographic data only).
Local gauge symmetry, Gribov, stochastic gauge fixing, averages over orbits	Cited	S. Elitzur, Phys. Rev. D 12 (1975) 3978. I. M. Singer, Commun. Math. Phys. 60 (1978) 7. C. Parrinello, G. Jona-Lasinio, Phys. Lett. B 251 (1990) 175. D. Zwanziger, Nucl. Phys. B 345 (1990) 461. D. Mehta, M. Schröck, Phys. Rev. D 89 (2014) 094512 (abstract checked; the others bibliographic data only).
Wilson flow, APE	Cited	M. Lüscher, JHEP 08 (2010) 071, arXiv:1006.4518 (abstract checked). M. Albanese et al., Phys. Lett. B 192 (1987) 163 (bibliographic data only).
Effective exponent of the topological charge, HMC and Langevin	Reading, Assumption	S. Schaefer, R. Sommer, F. Virotta, Nucl. Phys. B 845 (2011) 93, arXiv:1009.5228. M. Lüscher, S. Schaefer, JHEP 07 (2011) 036, arXiv:1105.4749 (abstracts of both checked). M. Abe, O. Morikawa, H. Suzuki, PTEP 2025 (2025) 063B03, arXiv:2501.00286 § 3.2, Figure 6 (HMC for $SU(2)$; text checked).
Holographic models	Cited (inside the model)	U. Gürsoy, E. Kiritsis, F. Nitti, JHEP 0802 (2008) 019, arXiv:0707.1349. C. Csáki, H. Ooguri, Y. Oz, J. Terning, JHEP 9901 (1999) 017, hep-th/9806021 (abstracts checked). 17-09.
Problem statement § 6.5	Cited	A. Jaffe, E. Witten, the Clay problem statement (the table of sources of 0-03).
Theorems and assumptions of Table 4	Lean	Bundles `LatticeGaugeCoarseGraining`, `LatticeGaugeRunning`, `LatticeGaugeMultiStep`, `LatticeGaugeSpectralHinge`, `PerfectFamilyEdges`, `LatticeGaugeOneLink`, `LatticeGaugeOverlapExponent`, `LatticeGaugeTubeEntropy`, `LatticeGaugeWulffCorrection`, `LatticeGaugeHardyLadder`, `LatticeGaugeAutocorrelationBound` and others. The `#print axioms` at the end of each bundle is a subset of the three standard axioms. Statements and limits in 23-13.
Numbers of the models (c_2, slopes of windows, gauge fixing, curvature, flow)	Reading	Computations of this part's record for the four-dimensional lattice (the model of the one-loop kernel, finite-dimensional models, Monte Carlo in small boxes). Values and conditions in each row of the text.

Words used in this chapter — one-step bookkeeping, complete family, one-link integration, shortfall $\delta(b)$, the form of the answer, the seventeen roads tried on the four-dimensional lattice, the twenty-one walls (five types), renewal form, lower bound on the overlap, the tube inequality, the projection assumption, topological charge, negative control. Their meanings are in A-02 Glossary.

To the next chapter — 23-03 The four steps of the staircase — why we start from 0+1 dimensions: divides the relaxation-time assumption into four steps, and fixes the conditions for choosing the double well for the first step and the form of the statement of the first step

(the double-well relaxation-time power statement and the Double-Well Relaxation-Time Upper-Bound Proposition).

Next chapter to read: 23-03 The four steps of the staircase — why we start from 0+1 dimensions.

The four steps of the staircase — why we start from 0+1 dimensions

> This chapter is this book's own investigation. § 03 and § 06 are a map of open questions; § 07 is guide material.

What this chapter shows — the diffusive-relaxation-time assumption is a sentence of a dimension-independent form: "the relaxation time of a path measure grows as a power of the square of the correlation length of that measure". So we divide the diffusive-relaxation-time assumption into four steps (the double well in 0+1 dimensions / the four-dimensional lattice, non-abelian / the continuum limit and the OS axioms / the mass gap in the limit), and give the reasons for choosing the double well for the first step. The statement of the first step is "the double-well relaxation-time power statement" (below, the double-well power statement), $\ln C_P / \ln \xi \to 2$, and putting its upper side in the form $C_P \le K \xi^2 S_0/\hbar$ (the First-Step Upper-Bound Proposition) is the centre of the assembly of the first step. The First-Step Upper-Bound Proposition is a conditional proposition on top of four conditions, and is not proved. We give both the prospects of lifting from the first step to the second and beyond, and the reasons it does not lift.

Prerequisite chapters — 23-00 (the entrance, the five labels), 2-07 (the path integral and imaginary time), 2-05 § 07 (the tunnelling splitting of the double well). Definitions on the side of the four-dimensional lattice are in 23-01.

Said first — The relaxation-time assumption is an assumption both on the four-dimensional lattice and for the double well. The First-Step Upper-Bound Proposition is algebra on top of the four double-well conditions, and every one of the four conditions leaves assumption rows. This chapter fixes the form of the statement and proves nothing new.

The order of this chapter

1. The relaxation-time assumption is a sentence of a dimension-independent form — a family of measures, the correlation length, the relaxation time
2. Definition of the four steps — the form in which each step can be called "closed" (Table 1)

3. Why we do not attack it on the four-dimensional lattice as it is — there are no tools for the upper side
4. Conditions on the model chosen for the first step — a ruler that grows exponentially, two phases, quantities that can be compared (Table 2)
5. The power statement and the form of the First-Step Upper-Bound Proposition — the lower bound ξ^2 and the target for the upper bound
6. What lifts and what does not (Table 3)
7. A map inside the first step — a guide to 23-04 to 23-11

01

The diffusive-relaxation-time assumption is a sentence of a dimension-independent form — a family of measures, the correlation length, the relaxation time

Three things appear in the relaxation-time assumption: a family of probability measures μ_s lined up by one parameter, the correlation length $\xi(s)$ of each μ_s, and the Poincaré constant $C_P(s)$ of each μ_s (the relaxation time of the Langevin dynamics having that measure as equilibrium). When moving the parameter gives $\xi(s) \to \infty$, the relaxation-time assumption is the one sentence

$$(\xi^2): \quad \lim \frac{\ln C_P(s)}{\ln \xi(s)} = 2 \quad (1)$$

This sentence does not say on what space the measure lives. On the four-dimensional lattice μ_s is the Wilson measure with $s = \beta \to \infty$ (23-01); for the double well in 0+1 dimensions μ_s is the measure on paths in imaginary time $e^{-S/\hbar}$ with $s = 1/\hbar \to \infty$ (23-04). Sentences of the same form can be lined up with the object changed.

Example: for the free field (a Gaussian chain of mass m_0), $C_P = 1/m_0^2$ and $\xi \approx 1/m_0$, and (1) holds as $m_0 \to 0$ (23-00 § 04) Proved here . With interaction, new mechanisms that slow relaxation enter (crossing barriers, moving between configurations in different phases), and (1) is no longer trivial.

Precisely, "the relaxation-time assumption" in this part refers only to the sentence (1) at the power level. A sentence stated within the width of a constant factor (a band statement) is called by another name, not the relaxation-time assumption (23-01). Also, the four conditions of the first step are written "the four double-well conditions", always with the word "conditions" so as not to be confused with the relaxation-time assumption. The letters in Lean theorem names (H1 and so on) are not changed.

02

Definition of the four steps — the form in which each step can be called "closed"

We place four steps, from easy places to hard ones.

- **The first step**: the path measure of the double well in 0+1 dimensions. Parameter $\hbar \to 0$.
- **The second step**: non-abelian gauge theory on the four-dimensional lattice (the Wilson measure of SU(2) or SU(3)). Parameter $\beta \to \infty$.
- **The third step**: the continuum limit as the lattice spacing $a \to 0$. That the limiting measure exists and satisfies the Osterwalder–Schrader axioms (4-10).
- **The fourth step**: the mass gap of the limiting theory (4-11).

Table 1 Object, target, tools and state in this book, step by step

Step	Object	Form in which it can be called "closed"	Main tools	State in this book
First step	The path measure of the double well	State the upper side of the double-well power statement with a table of finitely many named assumption rows, and fill each row with a proof, an original source, or computer assistance	Splitting of the variance, local Poincaré, reduction to the box, interval arithmetic	Up to the First-Step Upper-Bound Proposition (algebra on top of the four double-well conditions). Every one of the four conditions leaves Assumption rows (23-05, 23-12)
Second step	The Wilson measure on the four-dimensional lattice	The sentence (1) of the same form, as $\beta \to \infty$	Prospect of transferring the types made on the first step (§ 06)	The relaxation-time assumption is Assumption. Only estimates from below have been reached (23-02)

Step	Object	Form in which it can be called "closed"	Main tools	State in this book
Third step	The measure of the continuum limit	Existence of the limit and the OS axioms	The sequence of renormalisation group steps, constructive field theory (Part 7)	Not dealt with. That a theory satisfying the Wightman axioms can be made as the limit of four-dimensional Wilson lattice gauge theory is, in the committee's assessment, proved in Lean by the construction published on 2026-11-10 (Part 6). The uniqueness of the infinite-volume limit and the continuum probability measure have not been shown (6-04)
Fourth step	The reconstructed quantum field theory	The gap of the Hamiltonian is positive	The OS reconstruction (4-10)	Not dealt with. The published construction does not show it either (6-04)

"Closed" for the first step means putting the proposition of the upper side into a form where "which rows are proofs and which are assumptions" can be read on one table. Only when every row is filled with a proof, an original source or computer assistance, and the stage of the First-Step Upper-Bound Proposition goes through as a single proof, does the first step become a theorem. At present it is not so (23-12).

To go up from the second step, one more bridge is needed between each pair of steps. Even if the sentence (1) of the second step holds, that alone does not give the existence of the continuum limit (the third step), and the gap in the limit (the fourth step) is a form that can be reread only after adding to the upper side of (1) the assumptions joining the correlation length to physical length (the chains of 23-01). This part has no theorem of implication for any bridge between steps.

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03
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Why we do not attack it on the four-dimensional lattice as it is — there are no tools for the upper side

In the record for the four-dimensional lattice (23-02), every road through test functions, gauge fixing and spectroscopy ended in lower bounds on C_P or upper bounds on the mass, and no tool reaching the upper side (C_P not too large, over every f) has been found. There are estimates on the strong-coupling side (C_P bounded) and the Bakry–Émery type that works for convex measures, but neither reaches the regime $\xi \to \infty$. In an example applying a known volume-uniform criterion to the path lattice of the double well, the upper bound obtained is of the type $e^{c\xi}$ (doubly exponential in $\hbar$), read as far from a power statement Cited (criterion: Theorem 2.1 of Bauerschmidt–Dagallier) Reading ($c \approx 1.4$ to 1.8; numbers for $\varepsilon = 0.25$, $\hbar \in [0.1, 0.4]$). So we first make an "assembly from above" on the smallest test bed where the relaxation-time assumption is expected to hold. The detailed roads and walls are in 23-02.

04

Conditions on the model chosen for the first step — a ruler that grows exponentially, two phases, quantities that can be compared

There are three conditions we ask of the model of the first step.

1. **The ruler grows exponentially in the parameter.** The correlation length of the four-dimensional lattice grows exponentially in β in the one-loop count (4-07). To make a test bed that does not confuse powers with exponentials, a ruler growing at the same rate is needed.
2. **The slow motion is the motion of "changing phase".** What slows relaxation is a rare event of moving between two (or more) phases. There are two scales: fluctuations inside the two phases are fast, and only the motion of changing phase is slow.
3. **The quantities compared are known exactly.** The correlation length can be found to high precision by an independent method. If what it is compared with is blurred, the growth of C_P/ξ^2 will be misread.

The path measure of the double well in 0+1 dimensions, $V(x) = (x^2-1)^2/8$, satisfies all three. The ruler grows exponentially as $\xi = \hbar/\Delta \propto \hbar^{1/2} e^{S_0/\hbar}$ (Δ is the tunnelling splitting, $S_0 = 2/3$ the action of the path moving from valley to valley); the two phases are "the intervals where the path is in the right valley" and "those where it is in the left valley", and the event of changing phase is a kink (a short transition with action S_0). Δ is found by two routes, the semiclassical formula ((10) of 2-05 § 07) and numerics of the transfer operator (23-04).

The action S_0 of one kink, and the correlation length $\xi \propto e^{S_0/\hbar}$. This pair has, in the smallest dimension, the structure in which "the action of an instanton" and "a correlation length growing exponentially" sit side by side on the four-dimensional lattice. This is an analogy, however, not a theorem of correspondence.

Table 2 How the double well and the four-dimensional lattice line up (the right column is an analogy, not a theorem of correspondence)

Item	Double well (first step)	Four-dimensional lattice (second step)
Measure	The path measure $e^{-S/\hbar}$ (the line of imaginary time)	The Wilson measure (four-dimensional lattice)
Parameter and limit	$\hbar \to 0$	$\beta \to \infty$
Ruler	$\xi \propto \hbar^{1/2} e^{S_0/\hbar}$ Cited (leading semiclassical term)	$\propto e^{3\pi^2 \beta/11}$ from one-loop bookkeeping (SU(2); that it agrees with ξ is Assumption)
Slow degrees of freedom	The positions of kinks (action $S_0 = 2/3$)	The positions, sizes and orientations of instantons, the density of topological charge (analogy)
Lower side	$CP \geq \xi^2$ (One-Dimensional One-Sided Bridge Proposition, 23-04 § 04)	Lower bounds from test functions (23-02)
Upper side	Conditional on four conditions (the First-Step Upper-Bound Proposition)	None

The first step is an easy model, but not too easy. No known theorem giving the upper side of the double-well power statement was found in what we searched. There are theorems of Eyring–Kramers type that hold the length of the interval fixed and take $\hbar \to 0$, but the regime that makes the length of the paths T infinite first ($T \gg \xi$) is outside the scope of those theorems Cited (Brooks–Di Gesù 2026 state explicitly in the text that the regime of infinite length and the family of translations are outside their scope).

05

The power statement and the form of the First-Step Upper-Bound Proposition — the lower bound ξ² and the target for the upper bound

The statement of the first step, for the path measure of the double well (taking $T \to \infty$ first), is

$$\text{power form of Assumption } (\xi^2): \quad \lim_{\hbar \to 0} \frac{\ln C_P(\hbar)}{\ln \xi(\hbar)} = 2 \quad (2)$$

Setting $R := CP/\xi^2$, $\ln CP / \ln \xi = 2 + \ln R / \ln \xi$, so (2) is the same as "$\ln R$ is small compared with $\ln \xi$".

The label "power assumption (ξ^2)" on the left of (2) is what the text calls the double-well power statement (the double-well relaxation-time power statement).

The lower side. $R \geq 1$, that is $C_P \geq \xi^2$, holds for every well in 0+1 dimensions (the One-Dimensional One-Sided Bridge Proposition; the proof is in 23-04 § 04).

The target for the upper side. The position of a kink moves with diffusion coefficient $D = \hbar / S_0$, and the time to cross a length ξ by diffusion is $\xi^2 / D = \xi^2 S_0 / \hbar$ (23-04 § 03). So we set the target for the upper side as

$$C_P \;\leq\; K\,\xi^2\,\frac{S_0}{\hbar} \qquad (K \text{ does not depend on } \hbar\,) \quad (3)$$

In the count at the level of physics (counting kinks as a dilute gas) it is read as $C_P \approx \xi^2 S_0/\hbar$, and (3) is that with a margin of a constant factor Assumption (the count is heuristic, not a proof).

From (3) to (2). From (3) and $R \geq 1$, $1 \leq R \leq K S_0/\hbar$. When $\hbar$ is small, $\ln \xi \geq s/\hbar$ (for some $s > 0$), so

$$0 \;\leq\; \frac{\ln R}{\ln \xi} \;\leq\; \frac{\hbar\,\big(\ln(K S_0) + \ln(1/\hbar)\big)}{s} \;\longrightarrow\; 0 \qquad (\hbar \to 0) \quad (4)$$

and (2) follows Proved here Lean `power_H_of_kink_ratio` (assumptions: $0 < K$, $0 < s$, $0 < \delta$, and for $0 < \hbar < \delta$, $1 \leq R \leq K/\hbar$ and $s/\hbar \leq \ln \xi$. The K of Lean is the $K S_0$ of the text). $\ln \xi \geq s/\hbar$ is a row coming from the leading semiclassical term $\ln \xi \approx S_0/\hbar$, and in the lattice model it is Reading. The values are in 23-04 § 07.

The form of the First-Step Upper-Bound Proposition

(3) is not proved. The assembly of the first step splits (3) into the following four conditions, and puts it in the form of a conditional proposition: on top of them, (3) follows.

The First-Step Upper-Bound Proposition (conditional; statement only)

For the path measure of the double well in 0+1 dimensions, assuming the four double-well conditions, taking $T \to \infty$ first, as $\hbar \to 0$, $C_P \leq K \xi^2 S_0/\hbar$ (K depends only on the constants of the conditions). Hence the double-well power statement (2) follows.

- The in-phase local Poincaré condition (local Poincaré within a phase without kinks)
- The saddle-tube condition (transport along the family of flat saddles)
- The box-reduction condition (reduction to the box dynamics)

- The gap-inside-the-box condition

The content of the four conditions, the algebra of the splitting of the variance for the First-Step Upper-Bound Proposition, and how what has been shown is separated from the assumption rows are in 23-05. The current status of each condition is in 23-06 to 23-11, and the table putting all rows on one sheet is 23-12.

Precisely, every one of the four conditions leaves assumption rows at present. And there is not yet a single $\hbar$ at which all rows of all conditions hold at once (23-12). The First-Step Upper-Bound Proposition is a proposition naming "what must be shown for the first step to close", not a proposition saying that the first step has closed.

06

What lifts and what does not

Even if (3) closes on the first step, the sentence of the second step (the four-dimensional lattice) does not follow. This section states, with reasons, what has a prospect of lifting and what does not. Both are descriptions of prospects, not theorems of implication.

Table 3 From the first step to the second (prospects)

	Prospect of lifting	Reason it does not lift
Form	The sentences (1) and (2) do not depend on dimension. The two-scale split "fast inside a phase / slow when changing phase" can also be written in the same form for any measure with slow degrees of freedom	The slow degrees of freedom of the four-dimensional lattice (collective coordinates of instantons, the density of topological charge) increase in number in proportion to the volume. Even on the first step, when the path is much longer than ξ, a split taking only one slow variable cannot be used (the band of 23-04 § 05). In four dimensions this situation is expected to be stronger still
Tools	The splitting of the variance, the lemma of two projections, the Efron–Stein inequality and the stage of sums over sectors are algebraic stages that do not use the dimension of the space of the measure, and can be written as they are	Many stages of the first step depend on properties peculiar to one-dimensional nearest-neighbour measures (the transfer operator, the boundary between phases being a point, kinks being countable as a dilute gas). Gauge symmetry and the geometry of the orbit space (3-04) are absent from the first step
Constants	—	The constants of the first step were fixed by computer assistance and numerical readings at particular nodes and windows of $\hbar$, and give no information about the β of the four-dimensional lattice. The correspondence between $\hbar$ and β is also an analogy

We still start from the first step because the procedure itself of assembling the proposition of the upper side in a form where "which rows are assumptions" is visible is also needed on the second step. The procedure of naming the assumption rows, checking the direction of dependence (23-12), and separating what machine checking guarantees (23-13) does not depend on the model. Taking the procedure through on the first step means that when a proposition of the same form is set up on the second step, we can name which rows are newly needed.

Precisely, setting up "a sentence of the same form" on the second step is not evidence that the sentence of the second step holds. Even if the relaxation-time assumption is read as holding for the model of the first step (counts at the level of physics and readings of lower bounds), whether it holds on the second step is a separate question (23-15).

07

A map inside the first step — a guide to 23-04 to 23-11

Chapters of the first step

4. 23-04 The path measure of the double well, and kinks — definition of the setting, kinks, the lower bound $CP \geq \xi^2$ [Y2]–[Y3]
5. 23-05 The statement of the first step — the four double-well conditions and the upper-bound proposition — the splitting of the variance, and the assembly of the upper-bound proposition [Y3]
6. 23-06 The in-phase local Poincaré condition — local Poincaré in a phase without kinks [Res]
7. 23-07 The map of the saddle-tube condition — seven routes and the seam of the ring [Y3]
8. 23-08 What computer-assisted means — the conditions on the chain inside the window — checking inequalities by interval arithmetic [Y2]–[Y3]
9. 23-09 The sector framework and the three strands of the window log-Sobolev route — the common wall of the plateau at the top [Res]
10. 23-10 The list of detour plans — types of breakage [Y3]–[Res]
11. 23-11 Box reduction and the gap inside the box — the constant of the core — the box-reduction condition, the gap-inside-the-box condition, and the constant of the core [Res]

The [Res] chapters place at the start a summary section that can be read at the [Y3] level, so one can read only the summaries and go on to 23-12.

What this chapter can say

Said	Not said
The way of building it by writing the relaxation-time assumption in the dimension-independent form (1) and dividing it into four steps (a convention)	Any implication between steps. The relaxation-time assumption on the second step, the continuum limit, the mass gap in the limit
Proved here The algebra (4) by which the double-well power statement (2) follows from (3), the lower bound $R \geq 1$ and $\ln \xi \geq s/\hbar$ Lean `power_H_of_kink_ratio` (assumptions are the rows of § 05)	(3) itself. The First-Step Upper-Bound Proposition is conditional on the four double-well conditions, and every condition leaves assumption rows
Cited That Eyring–Kramers-type theorems with the interval held fixed place the regime taking $T \to \infty$ first outside their scope	That there is no known theorem giving the upper side in this regime (only as far as: none was found in what we searched)
Prospects of lifting and reasons it does not lift (Table 3, prospects)	That a proposition of the same form can be set up, or holds, on the second step

Sources and reproduction

Item	Kind	Source / tools
Definitions of the Poincaré constant and the correlation length	Cited	The table of sources of 23-00.
Tunnelling splitting of the double well, and S_0	Cited	(9)(10) of 2-05 § 07 and its table of sources.
Eyring–Kramers-type theorems, and their scope	Cited	G. Di Gesù, D. Le Peutrec, arXiv:1506.04434 (Theorem 1.1); B. Brooks, G. Di Gesù, arXiv:2606.16083 (the statement of scope on p.4 of the text). Both checked in the text.
A volume-uniform criterion, and its values on the path lattice of the double well	Cited, Reading	R. Bauerschmidt, B. Dagallier, arXiv:2202.02295 (Theorem 2.1, checked in the text). Values from numerics of the transfer operator (step $\varepsilon = 0.25$, $T = \infty$).
The algebra (4) of the double-well power statement	Proved here, Lean	§ 05 of the text. Theorem `power_H_of_kink_ratio` (assumptions in § 05). The statement is in 23-13.
The count $C_P \approx \xi^2 S_0/\hbar$	Assumption	A count at the level of physics using a dilute gas of kinks and the diffusion coefficient of the projection (23-04 § 03).
Statement of the First-Step Upper-Bound Proposition	With a table of assumptions	The content is in 23-05.

Words used in this chapter — staircase (four steps), the double-well power statement, the four double-well conditions, the First-Step Upper-Bound Proposition, kink, tunnelling splitting, phase (phase without kinks), local Poincaré, box dynamics (box reduction). Their meanings are in A-02 Glossary.

Next chapter to read: 23-04 The path measure of the double well, and kinks — defines the setting of the first step, and derives the lower bound $C_P \geq \xi^2$ and the form of the target for the upper bound.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time double well, kinks, lower bound · Kiichi, Shiori and Rin

The path measure of the double well, and kinks

In this chapter, § 01– § 03 are established knowledge and § 04 onward are this book's own investigation.

What this chapter shows — The setting of the first step is the measure $e^{-S/\hbar}$ on paths in imaginary time for the double well $V(x) = (x^2-1)^2/8$. The correlation length $\xi = \hbar/\Delta$ (Δ is the tunnelling splitting) grows at the rate $\hbar^{1/2}e^{S_0/\hbar}$, where $S_0 = 2/3$ is the action of the path moving from valley to valley (the kink). The Poincaré constant C_P of whole paths is at least ξ^2 (the One-Dimensional One-Sided Bridge Proposition), and the target for the upper side is a constant times the diffusion time of kinks, $\xi^2 S_0/\hbar$. At three values of $\hbar$, $\ln \xi$ is 218 / 661 / 2213 (readings).

Prerequisite chapters — 2-07 (the path integral, imaginary time, the transfer matrix), 2-05 § 07 (the double well), 1-10 § 07– § 09 (Poincaré inequalities), 23-03 (the staircase and the double-well relaxation-time power statement).

Said first — § 01– § 03 and § 07 are written at [Y2], § 04– § 06 at [Y3]. No upper bound on C_P is shown here. The One-Dimensional One-Sided Bridge Proposition is a lower bound, and the target for the upper side (15) is the conclusion of the conditional Double-Well Relaxation-Time Upper-Bound Proposition (23-05). All the numbers in this chapter are numbers of a 0+1-dimensional model, not of four-dimensional gauge theory.

The order of this chapter

1. Setting — the action, the path measure, the lattice ε=½, the Dirichlet form, C_P
2. Two times — the correlation length ξ=ħ/Δ and the Langevin gap 1/C_P
3. Kinks — action S_0=2/3, the position degree of freedom, diffusion coefficient D=ħ/S_0, how ξ grows
4. The lower bound C_P ≥ ξ² — the One-Dimensional One-Sided Bridge Proposition, and readings beyond it
5. The target for the upper side, and the band of long paths — why the naive reduction cannot be used
6. The plateau at the top — the curvature falls ¼ short of convexity

7. Numerical examples — $\ln \xi$ at three values of $\hbar$ (Table 2). Detour: the tunnelling splitting by Herring's formula

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01

Setting — the action, the path measure, the lattice, the Dirichlet form

A particle of mass 1 sits in a potential with two valleys.

$$V(x) = \frac{(x^2-1)^2}{8}, \qquad V'(x) = \frac{x(x^2-1)}{2}, \qquad V''(x) = \frac{3x^2-1}{2} \quad (1)$$

The valleys are at $x = \pm 1$, the curvature at the bottom of a valley is $V''(\pm 1) = 1$ (the angular frequency of oscillation inside a valley is 1), and the top of the hill is at $x = 0$ with $V''(0) = -\frac{1}{2}$. This is (9) of 2-05 § 07 with $a = 1$.

To a path $x(\tau)$ in imaginary time τ ($0 \le \tau \le T$) we attach the Euclidean action

$$S[x] = \int_0^T \left[\frac{1}{2}\dot{x}(\tau)^2 + V\big(x(\tau)\big) \right] d\tau \quad (2)$$

and make a probability measure on paths with weight $e^{-S[x]/\hbar}$. This is the measure of (7) of 2-07 § 05 (mass $m = 1$); closing the path into a loop gives the partition function of (9) of 2-07 § 06, that is, a classical chain in which the length T of the loop plays the role of inverse temperature. In this part we take $T \to \infty$ (temperature 0, the side of the ground state) **first**, and then $\hbar \to 0$. The quantity with the order reversed (holding T fixed and taking $\hbar \to 0$) is a different quantity. For short paths ($T < 2\pi\sqrt{2}$), from the known theorem that the mean time to move between valleys grows at the rate $e^{T/(8\hbar)}$ Cited (Theorem 2.6 of Berglund–Gentz, of Eyring–Kramers type with the interval held fixed), C_P/ξ^2 is read as falling exponentially to 0 Reading (the rereading from mean times to eigenvalues is by reduction to two states).

The lattice. The numbers in the later chapters of this part are numbers on a lattice that discretises imaginary time with step $\varepsilon = \frac{1}{2}$. With x_i the value at site $i \in \mathbb{Z}$,

$$S_{\mathrm{lat}} = \sum_i \frac{(x_{i+1} - x_i)^2}{2\varepsilon} + \varepsilon \sum_i V(x_i), \qquad \varepsilon = \frac{1}{2} \quad (3)$$

and the measure at $T = \infty$ is a one-dimensional nearest-neighbour Gibbs measure. The integral kernel of the transfer operator ((13) of 2-07 § 08) is positive, so the measure is expected to be uniquely determined Assumption (using the integral-operator version of the Perron–Frobenius theorem of 2-07 § 08 on $\mathbb{R}$. A statement of this form in

an original source has not been checked in this part, so it is put as an assumption row). Some chapters also cite numbers with step $\varepsilon = 0.25$, and say so each time.

The Dirichlet form and C_P. For a function f of the path, we set the "size of the change when it is moved" to be

$$\mathcal{E}(f) = \hbar \int \mathbb{E}\Big|\frac{\delta f}{\delta x(\tau)}\Big|^2 d\tau, \qquad \text{on the lattice} \quad \mathcal{E}(f) = \frac{\hbar}{\varepsilon}\sum_i \mathbb{E}\,|\partial_i f|^2 \tag{4}$$

(the canonical Dirichlet form), and set the Poincaré constant as $C_P = \sup_f \mathrm{Var}(f)/\mathcal{E}(f)$ ((2) of 23-00 § 04). The $\hbar$ attached to $\mathcal{E}$ is a normalisation matching the form differentiated with respect to the canonical field $x/\sqrt{\hbar}$ rather than the position x (the scale of the clock in 23-00 § 05). The Langevin dynamics with this $\mathcal{E}$ is, on the lattice,

$$dx_i = -\frac{1}{\varepsilon}\frac{\partial S_{\mathrm{lat}}}{\partial x_i}\,dt + \sqrt{\frac{2\hbar}{\varepsilon}}\,dW_i \tag{5}$$

(W_i are independent Brownian motions); it has $e^{-S_{\mathrm{lat}}/\hbar}$ as equilibrium, and its gap is $1/C_P$ Cited (1-10 § 07). Along an artificial time t the whole path sways, and the slowest fluctuation decreases by a factor $1/e$ every time C_P.

Example (free field). With $V = \frac{1}{2}x^2$ the measure is Gaussian, $C_P = 1$, the correlation length is also $\xi = 1$, and the ratio C_P/ξ^2 is exactly 1 Proved here Lean `ratioR_free_eq_one` (assumptions: $C_P = 1/m^2$, $m \neq 0$). On the lattice it is the same in the zero-momentum direction.

Precisely, the measures of (2)–(5) are path measures of a 0+1-dimensional quantum-mechanical model, and no gauge field enters anywhere. The relation to four-dimensional gauge theory is only the analogy of 23-03 § 04.

02

Two times — the correlation length ξ=ħ/Δ and the Langevin gap 1/C_P

This model has two "times" of different character.

The first is a length in imaginary time, the correlation length ξ. With $\Delta = E_1 - E_0$ (the tunnelling splitting) the difference between the lowest two levels of the quantum-mechanical Hamiltonian $H = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V$, the correlation of two points on the path decreases as

$$\langle x(0)\,x(\tau)\rangle - \langle x\rangle^2 \;\approx\; C\,e^{-\tau\Delta/\hbar}, \qquad m := \frac{\Delta}{\hbar}, \qquad \xi := \frac{1}{m} = \frac{\hbar}{\Delta} \tag{6}$$

(the ratio of eigenvalues of the transfer operator; the same line of argument as 2-07 § 07) Cited . ξ is a length in the imaginary-time direction; counted in lattice sites it is $\xi/\varepsilon = 2\xi$.

The second is an artificial time, the smearing time C_P. It is the time for the whole path to approach equilibrium under the dynamics (5), a time in a direction different from imaginary time.

The quantity comparing the two is

$$R := \frac{C_P}{\xi^2} = C_P\, m^2 \tag{7}$$

For the free field $R = 1$, and the double-well relaxation-time power statement (the power statement) is the same as "$\ln R$ is small compared with $\ln \xi$" (23-03 § 05).

Precisely, the ξ of (6) is the definition for the model with continuous time; for the lattice model it is fixed by $\xi = \varepsilon/\ln(\lambda_0/\lambda_1)$ from the top two eigenvalues $\lambda_0 > \lambda_1$ of the transfer operator. As $\hbar \to 0$ the difference between the lattice value and the continuum value enters the exponent as the difference of actions $S_0^{\text{lat}} \neq S_0$, so lattice values are used when writing numbers (§ 03, § 07).

03

Kinks — action S₀=2/3, the position degree of freedom, diffusion coefficient D=ħ/S₀

When $\hbar$ is small, the weight $e^{-S/\hbar}$ concentrates on paths with small action. What makes the action 0 is the path staying in the right valley all along, $x \equiv 1$, and the path staying in the left valley all along, $x \equiv -1$. The path with the next-smallest action, "moving from valley to valley", is the **kink**.

The stationarity equation of (2) is $\ddot{x} = V'(x)$ (motion in the potential $-V$, 2-07 § 05), and solutions that settle into valleys as $\tau \to \pm\infty$ satisfy conservation of energy $\frac{1}{2}\dot{x}^2 = V(x)$. From (1), $\dot{x} = \sqrt{2V(x)} = (1 - x^2)/2$, and solving this gives

$$x_{\text{cl}}(\tau) = \tanh\frac{\tau - \tau_0}{2}, \qquad S_0 = \int \dot{x}_{\text{cl}}^2\, d\tau = \int_{-1}^{1} \sqrt{2V(x)}\, dx = \int_{-1}^{1} \frac{1 - x^2}{2}\, dx = \frac{2}{3} \tag{8}$$

Proved here (putting $\frac{1}{2}\dot{x}^2 = V$ into (2) makes the integrand $\dot{x}^2$, and $dx = \dot{x}\, d\tau$ turns it into an integral over x). The width of the kink is about 2 in τ, and its action is the number $2/3$, independent of $\hbar$. This is the $S_0 = 2a^2/3$ ($a = 1$) of 2-05 § 07, in the language of physics the action of an instanton (3-05).

The position degree of freedom. The τ_0 of (8) is arbitrary, and shifting the kink in the imaginary-time direction does not change the action. The second derivative of the action (the fluctuation operator) $-\partial_\tau^2 + V''(x_{\mathrm{cl}})$ has $\dot{x}_{\mathrm{cl}}$ as an eigenfunction with eigenvalue 0 Proved here (differentiating the stationarity equation $\ddot{x}_{\mathrm{cl}} = V'(x_{\mathrm{cl}})$ with respect to τ gives $\dddot{x}_{\mathrm{cl}} = V''(x_{\mathrm{cl}})\dot{x}_{\mathrm{cl}}$). This flat direction (the zero mode) is one slow degree of freedom, the position of the kink. The next eigenvalue is $3/4$, and above it is the continuous part $[1, \infty)$, as is known Assumption (an operator of Pöschl–Teller type; the statement in the original source has not been checked in this part).

The diffusion coefficient. Projecting the noise of (5) onto the direction of the position of the kink, the position X diffuses. The direction that shifts the position by one is $\dot{x}_{\mathrm{cl}}$, and the square of its length is $\int \dot{x}_{\mathrm{cl}}^2 d\tau = S_0$, so the noise strength $\hbar$ is divided by S_0, giving

$$\mathbb{E}\big[(X_t - X_0)^2\big] \approx 2Dt, \qquad D = \frac{\hbar}{S_0} \quad (9)$$

follows. Cited The same projection count reproduces the coefficient of the theorem that a single interface in the stochastic Allen–Cahn equation on an interval converges to Brownian motion with diffusion coefficient $3/4$ (the Single-Interface Diffusion Theorem (Theorem 2.1 of Bertini–Brassesco–Buttà)). That is, at the level of "the motion of a single kink", (9) is a restatement of a known theorem. Reading In Langevin numerics following one kink on a ring with step $\varepsilon = 0.25$, $D/(\hbar/S_0)$ was 1.00 ± 0.03 ($\hbar = 0.02$), 1.04 (0.07) and 1.14 ± 0.05 (0.1).

Precisely, (9) is a sentence about the process of the position of one kink, not about the gap $1/C_P$ of whole paths. Whether, on a long path with many kinks, the gap is determined by the diffusion of kinks is exactly the question of the upper side (the First-Step Upper-Bound Proposition).

How the ruler grows

The tunnelling splitting Δ is determined, in terms of paths in imaginary time, by "the weight of inserting one kink", $e^{-S_0/\hbar}$. The leading semiclassical term is ((10) of 2-05 § 07 written with $\hbar$)

$$\frac{1}{\xi} = \frac{\Delta}{\hbar} \approx \frac{4}{\sqrt{\pi\hbar}}\, e^{-S_0/\hbar}, \qquad \xi^2 \approx \frac{\pi\hbar}{16}\, e^{2S_0/\hbar}, \qquad \ln \xi = \frac{S_0}{\hbar} + \frac{1}{2}\ln\hbar + O(1) \quad (10)$$

Cited (the formula of the leading term is a copy from the source of 2-05. When Δ is found by two independent numerical methods, the ratio to the leading term approaches 1, from 0.47 at $\hbar = 0.5$ to 0.92 at $\hbar = 0.05$ Reading), and the correlation length grows exponentially as $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$. In the lattice model, instead of S_0 the

action of the lattice kink $S_0^{\mathrm{lat}} = 0.665266$ ($\varepsilon = \frac{1}{2}$; it differs from $2/3$ by 0.2%) enters the exponent Reading . At $\hbar = 10^{-3}$ this difference moves $\ln \xi$ by 1.4, so lattice values are used when writing numbers.

04

The lower bound C_P ≥ ξ² — the one-dimensional one-sided bridge, and readings beyond it

C_P is a supremum, so one test function bounds it from below (23-00 § 06). The plainest test function is a function $h(x)$ of a single time, summed over time,

$$f = \int_0^T h\big(x(\tau)\big)\, d\tau \quad (11)$$

(a test function local in time). In this family the lower bound reaches exactly the square of the ruler.

The One-Dimensional One-Sided Bridge Proposition (continuous time, $T \to \infty$, any one-dimensional well)

The supremum of the limit as $T \to \infty$ of the Rayleigh quotient $\mathrm{Var}(f)/\mathcal{E}(f)$ of f of the form (11) is exactly $\xi^2 = 1/m^2$, attained at $h = \psi_1/\psi_0$ (ψ_0, ψ_1 are the lowest two eigenfunctions of H). Hence $C_P \geq \xi^2$, that is, $R \geq 1$.

Proved here Proof [Y3]. Set $\varphi := (h - \langle h \rangle)\psi_0$ and $A := (H - E_0)/\hbar$. A is the generator of imaginary time, and $A \geq m$ on functions orthogonal to ψ_0. Writing the correlations with the transfer operator (2-07 § 08),

$$\frac{\mathrm{Var}(f)}{T} \longrightarrow \int_{-\infty}^{\infty} \langle \varphi, e^{-|\tau|A}\varphi \rangle\, d\tau = 2\,\langle \varphi, A^{-1}\varphi \rangle \quad (12)$$

On the other hand $\delta f/\delta x(\tau) = h'(x(\tau))$, so $\mathcal{E}(f)/T = \hbar \int \psi_0^2 h'^2\, dx$, and from the ground-state transformation identity $\langle \varphi, (H - E_0)\varphi \rangle = \frac{\hbar^2}{2} \int \psi_0^2 h'^2\, dx$ (2-05 § 08, 1-10 § 08), $\mathcal{E}(f)/T = 2\langle \varphi, A\varphi \rangle$. With the eigendecomposition of A, $\varphi = \sum_k c_k e_k$ (eigenvalues $a_k \geq m$), $1/a \leq a/m^2$ gives

$$\frac{\sum_k |c_k|^2/a_k}{\sum_k |c_k|^2 a_k} \;\leq\; \frac{1}{m^2} = \xi^2 \quad (13)$$

with equality when φ lies in the eigenspace of eigenvalue m, that is, when $h - \langle h \rangle \propto \psi_1/\psi_0$. ∎ The transfer-operator formula (12) and the ground-state transformation identity

are cited as known facts Cited . Lean `local_rayleigh_ratio_le_one` (assumptions: $0 < m$, $a_k \geq m$, $c_k \geq 0$, $\sum c_k a_k > 0$) checks only the skeleton of finite sums of (13); the eigendecomposition, (12) and the ground-state transformation are outside Lean.

The One-Dimensional One-Sided Bridge Proposition holds not only for the double well but for every one-dimensional well. Whether a claim of the same form exists in the literature has not been checked (it may be known).

Readings beyond it. The One-Dimensional One-Sided Bridge Proposition also says that "for test functions local in time, $R = 1$ is the ceiling". Seeing $R > 1$ needs test functions that span times (for example, blurring the path in time and then looking at the sign, or counting the collective position of kinks). Evaluating the Rayleigh quotients of such test functions with samples drawn directly from the transfer operator (not samples approached by running a Markov chain), the lower bound on R grew from 1.06 at $\hbar = 0.4$ to 6.5 at $\hbar = 0.07$ Reading (lower bound over a family, step $\varepsilon = 0.25$, a window of the measure at $T = \infty$, variation between seeds ± 10–15%). It also agreed, for $\hbar = 0.3$ to 0.15, with values from measuring the decay of the slowest observable in Langevin dynamics Reading (both are lower bounds, not upper bounds).

The type of growth, in the count at the level of physics that counts kinks (a dilute gas of kinks, the diffusion coefficient of (9)), is $R \approx S_0/\hbar \approx \ln \xi$, that is,

$$C_P \;\approx\; \xi^2 \,\frac{S_0}{\hbar} \;\approx\; \frac{\pi}{24}\, e^{4/(3\hbar)} \qquad (14)$$

Assumption (the count is heuristic, not a proof). The numbers in this range alone cannot distinguish whether R grows as $1/\hbar$ or as $e^{c/\hbar}$. In the former case the double-well power statement holds; in the latter it breaks. This is why, even for the double well, the sentence stated within the width of a constant factor, $C_P \leq K\xi^2$, is read as being on the side that breaks.

05

The target for the upper side, and the band of long paths

Matching the form of (14), we put the target for the upper side as

$$C_P \;\leq\; K\,\frac{\xi^2}{D} = K\,\xi^2\,\frac{S_0}{\hbar} \qquad (K \text{ does not depend on } \hbar\,) \qquad (15)$$

Together with the One-Dimensional One-Sided Bridge Proposition this gives $1 \leq R \leq K S_0/\hbar$, and the double-well power statement follows ((4) of 23-03 § 05). (15) is the conclusion of the First-Step Upper-Bound Proposition on top of the four double-well conditions, and is not proved (23-05).

The naive road to showing (15) is to choose one slow variable (for example the mean position of the path, the magnetisation) and show that "functions orthogonal to it relax fast". For short paths ($T \ll \xi$, at most one pair of kinks) this form gives the right count. For long paths ($T \gg \xi$), however, this road cannot be used.

The reason can be seen in the Glauber dynamics of the one-dimensional Ising chain. The space spanned by functions of a single spin is closed under the generator, and the Fourier component of wave number k, $S_k = \sum_j e^{ikj}\sigma_j$, is an eigenfunction with eigenvalue $\alpha(1-\gamma\cos k)$ (α is the rate, $\gamma = \tanh 2K$). The magnetisation is $k=0$, and its ratio to the slowest component orthogonal to the magnetisation, $k = 2\pi/N$, is

$$\frac{1-\gamma\cos(2\pi/N)}{1-\gamma} = 1 + \frac{\gamma\,(1-\cos(2\pi/N))}{1-\gamma} \;\approx\; 1 + \Big(\frac{2\pi\xi}{N}\Big)^2 \qquad (16)$$

which approaches 1 when the chain is much longer than the correlation length ($N/\xi \to \infty$) Proved here (rewriting the ratio) Lean `glauber_band_ratio` (assumption: $\gamma \neq 1$). For the formula of the eigenvalues itself, we saw to 8 digits that the two eigenvalues appear as they are in the diagonalisation of all states of chains with $N = 8, 10$ Reading (the statement in Glauber's original paper has not been checked in this part). The slowest fluctuation is not an isolated single mode but the lower edge of a continuous band. On long paths of the double well too, long-wave fluctuations of the density of kinks make the same band, and the Poincaré constant of the conditional measure with the magnetisation fixed stays of the same size as ξ^2/D.

So on long paths one has to take a number of slow variables proportional to the volume, or cut the problem into boxes of length $C\xi$. The latter is the box-reduction condition (reduction to the box dynamics) and the gap-inside-the-box condition (23-11).

06

The plateau at the top — the curvature falls ¼ short of convexity

If the measure $e^{-S/\hbar}$ had a convex action (a log-concave density), the Bakry–Émery criterion would bound C_P by the reciprocal of the curvature of the action, independently of length and of $\hbar$ Cited (1-10). The path measure of the double well is not convex. We look on the lattice (3) at where convexity breaks.

The second derivative of the action (3) (the Hessian) is a matrix with $2/\varepsilon + \varepsilon V''(x_i)$ on the diagonal and $-1/\varepsilon$ between neighbours. For $\varepsilon = \frac{1}{2}$ the diagonal is $4 + \frac{1}{2}V''(x_i)$ and the neighbours -2. For uniform convexity over a long interval it suffices that the diagonal be at least the size 4 of the sum of the two neighbours (diagonal dominance).

Inside a valley ($|x_i|$ close to 1), $V'' \approx 1$ and the diagonal is 4.5, which is enough, but at the top of the hill $x_i = 0$, $V''(0) = -\frac{1}{2}$, so

$$\underbrace{4 + \tfrac{1}{2} \cdot \left(-\tfrac{1}{2}\right)}_{\text{curvature at one site at the top}} = \frac{15}{4} = 4 - \frac{1}{4} \quad (17)$$

which falls short by $\frac{1}{4}$ Proved here . For the path with every site at the top, $x \equiv 0$, the smallest eigenvalue of the Hessian (in the uniform direction) is exactly $\varepsilon V''(0) = -\frac{1}{4}$.

Between a pair of kinks, the interval where the path stays near the top makes a flat direction (a plateau) along which the action hardly changes. Conversely, confining the path to the convex core deep in a valley, $\{x_i \geq a\}$ ($a > 1/\sqrt{3}$), Bakry–Émery gives $C_P \leq 2/(3a^2 - 1)$ uniformly in length and $\hbar$ Proved here (lower bound on curvature) Cited (of Bakry–Émery type) Lean `convex_core_curvature` (assumptions: $1/3 < a^2$, $0 \leq a \leq x$). But the action of an excursion that leaves the core and comes back, $2/3 - a + a^3/3$, is smaller than the action of a kink $2/3$ (0.1535 at $a = 1/\sqrt{3}$) Lean `excursion_action_lt_kink` (assumption: $0 < a < 1$), and an interval of length ξ contains exponentially many excursions. The convex core alone does not give local Poincaré within a phase (the in-phase local Poincaré condition).

This shortfall of $\frac{1}{4}$ comes back, in changed form, to the same place on every road that builds the upper side. In later chapters it is called the common wall (23-09).

07

Numerical examples — ln ξ at three values of ħ

The later chapters of this part use three values of $\hbar$ as representatives. The order is always from large to small, $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$. The reason for the choice is that $\ln \xi$ changes widely, from about 218 to about 2213, making them representatives of the regime where the correlation length grows exponentially in $\hbar$. Table 2 gives values on the lattice $\varepsilon = \frac{1}{2}$.

Reading Table 2 $\ln \xi$ and $\ln(\xi^2 S_0/\hbar)$ at three values of $\hbar$ (lattice $\varepsilon = \frac{1}{2}$, $T = \infty$. ξ is a length in imaginary time. Floating-point values)

Quantity	$\hbar = 3 \cdot 10^{-3}$	$\hbar = 10^{-3}$	$\hbar = 3 \cdot 10^{-4}$
$\ln \xi$	218.08	661.04	2212.72
$S_0^{\mathrm{lat}}/\hbar$	221.76	665.27	2217.55
$\ln \xi - S_0^{\mathrm{lat}}/\hbar - \frac{1}{2} \ln \hbar$	−0.774	−0.777	−0.778
$\ln(\xi^2 S_0/\hbar)$ (scale of the target for the upper side)	441.6	1328.6	4433.1

$\ln \xi$ is the value obtained for the difference of the top two eigenvalues of the transfer operator by Herring's formula, which involves no subtraction (the next detour). At intermediate $\hbar$ (0.2, 0.05, 0.04), the road of finding the eigenvalues of the matrix directly and the road of Herring's formula agreed to six digits Reading . That the remainder in the third row approaches a number independent of $\hbar$ is consistent with the form $\ln \xi = S_0/\hbar + \frac{1}{2}\ln \hbar + O(1)$ of (10), and the difference from the continuum semiclassical value $\frac{1}{2}\ln(\pi/16) \approx -0.814$ is read as a lattice correction.

The fourth row is the scale of the right-hand side of (15), and its difference from $2\ln \xi$ is $\ln(S_0/\hbar)$. The ratio of the two columns is close to 2, and approaches 2 as $\hbar$ is lowered. The "2" of the double-well power statement is a restatement of how small this difference is.

Precisely, Table 2 gives floating-point values, not values enclosed by interval arithmetic. ξ is of size e^{218} to e^{2213}, not a length that can be measured directly by Monte Carlo but a quantity read from the eigenvalues of the transfer operator. The three values of $\hbar$ are representative points, with a role different from the "nodes" and "windows" of later chapters (the distinction is in 23-12).

Detour: the tunnelling splitting by Herring's formula

A subsection you may skip. At $\hbar = 10^{-3}$, $\Delta/\hbar$ is of size e^{-661}, and if the two eigenvalues λ_0, λ_1 are found separately and subtracted, every digit of floating point is lost. So we use a formula that writes the difference directly.

The integral kernel $K(x,y)$ of the transfer operator is symmetric and unchanged under $x \to -x$, $y \to -y$. Making two kernels on the half-line $x, y > 0$, $A(x,y) = K(x,y)$ and $C(x,y) = K(x,-y)$, the even eigenfunctions are eigenfunctions of $A + C$ and the odd eigenfunctions of $A - C$ (eigenvalues λ_e and λ_o). Since A and C are both symmetric, for an even eigenfunction u and an odd eigenfunction v,

$$v \cdot (A+C)u - u \cdot (A-C)v = (\lambda_e - \lambda_o)\, u \cdot v \quad \Longrightarrow \quad \lambda_e - \lambda_o = \frac{2\, u \cdot Cv}{u \cdot v} \tag{18}$$

Proved here (on the left $v \cdot Au = u \cdot Av$ cancels, and two copies of $v \cdot Cu = u \cdot Cv$ remain). The right-hand side involves no subtraction. The kernel of $A - C$ is positive for $x, y > 0$ (since $(x+y)^2 > (x-y)^2$, $K(x,y) > K(x,-y)$), so the odd eigenfunction can also be taken positive on the half-line, and can be computed in the logarithmic domain keeping relative precision out to the tails. This is the transfer-operator version of Herring's formula of quantum mechanics (the formula writing the splitting through overlaps of wave functions).

Classification of the statements of this chapter

We put the statements of this chapter into one table in the format of the five grades of this part (Table 1 of 23-00 § 01). The columns are the same as in the tables of propositions in later chapters.

Table 3 The statements of this chapter (one grade per row; the machine-checking column is outside the grades)

Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
Shape of the kink $\tanh(\tau/2)$ and action $S_0 = 2/3$	Proved here	(8) of § 03	Independent of $\hbar$	—
The fluctuation operator has $\dot{x}_{\mathrm{cl}}$ with eigenvalue 0 (the position degree of freedom)	Proved here	§ 03	Independent of $\hbar$	—
Next eigenvalue of the fluctuation operator $3/4$, continuous part $[1, \infty)$	Assumption	§ 03 (Pöschl–Teller type; original source unchecked)	Independent of $\hbar$	—
Diffusion coefficient of the position of one kink $D = \hbar/S_0$ (at the level of the process)	Cited	(9) of § 03. A copy of the coefficient of the Single-Interface Diffusion Theorem of Bertini–Brassesco–Buttà	$\hbar \to 0$	—
$D/(\hbar/S_0) = 1.00$ to 1.14	Reading	§ 03 (Langevin point values, $\varepsilon = 0.25$)	$0.02, 0.07, 0.1$	—
Leading semiclassical term (10) of ξ	Cited	§ 03, 2-05 § 07	$\hbar \to 0$	—
Lattice $\ln \xi$ (Table 2) and $S_0^{\mathrm{lat}} = 0.665266$	Reading	§ 07 (transfer operator, Herring's formula, floating point)	Three points	—

Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
The One-Dimensional One-Sided Bridge Proposition: the ceiling of the family local in time is ξ^2, hence $C_P \geq \xi^2$	Proved here	Proof in § 04 ((12) and the ground-state transformation cited). Continuous time, $T \to \infty$	Every $\hbar$	`local_rayleigh_ratio_le_one` (assumptions: $0 < m$, $a_k \geq m$, $c_k \geq 0$, $\sum c_k a_k > 0$. Skeleton of finite sums only)
The lower bound on R grows from 1.06 to 6.5	Reading	§ 04 (lower bound over a family, $\varepsilon = 0.25$)	$[0.07, 0.4]$	—
$C_P \approx \xi^2 S_0/\hbar$ ((14))	Assumption	§ 04 (count with a dilute gas; heuristic)	$\hbar \to 0$	—
The ratio (16) to the component orthogonal to the magnetisation approaches 1 (the band)	Proved here	§ 05 (rewriting the ratio)	Independent of $\hbar$	`glauber_band_ratio` (assumption: $\gamma \neq 1$)
The Glauber eigenvalues $\alpha(1 - \gamma \cos k)$ appear in the spectrum of all states	Reading	§ 05 (diagonalisation for $N = 8, 10, 8$ digits)	—	—
Curvature at one site at the top $15/4 = 4 - \frac{1}{4}$	Proved here	(17) of § 06	Independent of $\hbar$	—
$C_P \leq 2/(3a^2 - 1)$ on the convex core $\{x_i \geq a\}$	Proved here	§ 06 (the lower bound on curvature is proved here; the Bakry–Émery type is cited)	Independent of $\hbar$	`convex_core_curvature` (assumptions: $1/3 < a^2$, $0 \leq a \leq x$)
Target for the upper side (15), $C_P \leq K\xi^2 S_0/\hbar$	Assumption	Conclusion of the First-Step Upper-Bound Proposition (on top of the four double-well conditions). 23-05	$\hbar \to 0$	From (15) to the double-well power statement by `power_H_of_kink_ratio` (assumptions in 23-03 § 05)

What this chapter can say

Said	Not said
Proved here The kink action $S_0 = 2/3$ and the zero mode, the shortfall $\frac{1}{4}$ of curvature at the top, the transfer-operator version (18) of Herring's formula	Checking the rest of the spectrum of the fluctuation operator (the next eigenvalue) in the original source
Proved here The One-Dimensional One-Sided Bridge Proposition: $C_P \geq \xi^2$ (continuous time, $T \to \infty$; (12) and the ground-state transformation cited)	An upper bound on C_P. Whether R grows as $1/\hbar$ or exponentially
Cited The diffusion coefficient of one kink $D = \hbar/S_0$ (at the level of the process), the leading semiclassical term of ξ	That the gap of whole paths is determined by the diffusion of kinks (the question of the upper side)
Reading The growth of the lower bound on R (1.06→6.5), $\ln \xi$ at three values of $\hbar$ (Table 2), the numbers for D	Enclosing the values of Table 2 by interval arithmetic. Conversion between readings at $\varepsilon = 0.25$ and readings at $\varepsilon = \frac{1}{2}$
Assumption The target for the upper side (15) was put in the form of the conclusion of the First-Step Upper-Bound Proposition	(15) itself. The double-well power statement. About four-dimensional gauge theory, this chapter shows nothing

Sources and reproduction

Item	Kind	Source / tools
The path integral, the measure in imaginary time, the transfer matrix	Cited	The table of sources of 2-07.
Leading term of the tunnelling splitting of the double well, S_0	Cited	The table of sources of 2-05 (formula (2.8) of Zinn-Justin–Jentschura 2004). The transcription with $\hbar$ and the numbers for the ratio to the leading term are numbers of the record (two independent methods, $\hbar = 0.5$ to 0.05).
Poincaré constant, Langevin gap, the Bakry–Émery type	Cited	The table of sources of 1-10.
Diffusion coefficient of a single interface	Cited	L. Bertini, S. Brassesco, P. Buttà, arXiv:math-ph/0611076 (the Single-Interface Diffusion Theorem, checked in the text).
Transition times in the regime with the interval held fixed	Cited	N. Berglund, B. Gentz, arXiv:1202.0990 (Theorem 2.6, checked in the text). The rereading as eigenvalues is a reading.
Single-spin eigenvalues of Glauber dynamics	Assumption, Reading	R. J. Glauber, J. Math. Phys. 4 (1963) 294 (bibliographic data from memory, not checked in this part). Eigenvalues checked by diagonalisation for $N = 8, 10$.

Item	Kind	Source / tools
The One-Dimensional One-Sided Bridge Proposition, the finite sums of (13)	Proved here, Lean	§ 04 of the text. Theorem `local_rayleigh_ratio_le_one` (assumptions in the row of Table 3 for the One-Dimensional One-Sided Bridge Proposition).
$R = 1$ for the free field, the band ratio, the convex core, the action of excursions	Proved here, Lean	Theorems `ratioR_free_eq_one`, `glauber_band_ratio`, `convex_core_curvature`, `excursion_action_lt_kink` (assumptions in each row of the text). The list of statements is in 23-13.
Lower bound on R, numbers for D, Table 2	Reading	Floating-point computations of the record (Rayleigh quotients with samples drawn directly from the transfer operator, Langevin, the transfer operator and Herring's formula). Values not enclosed by interval arithmetic.

Words used in this chapter — double well, path measure, Dirichlet form, kink, zero mode, diffusion coefficient of kinks, tunnelling splitting, correlation length (of the double well), Herring's formula, plateau, convex core, phase (phase without kinks). Their meanings are in A-02 Glossary.

Next chapter to read: 23-05 The statement of the first step — the four double-well conditions and the upper-bound proposition — splits the upper side into four conditions by the algebra of the splitting of the variance, and states the First-Step Upper-Bound Proposition divided into the parts shown and the assumption rows.

The statement of the first step — the four double-well conditions and the upper-bound proposition

> This chapter is this book's own investigation. § 05 is established knowledge.

What this chapter shows — We split the upper side $CP \le K\xi^2 S_0/\hbar$ for the path measure of the double well into the four double-well conditions (the in-phase local Poincaré condition, the saddle-tube condition, the box-reduction condition and the gap-inside-the-box condition). The box-reduction condition reduces infinitely long paths to boxes of length $C\xi$, the gap-inside-the-box condition controls the gap inside a box, and the in-phase local Poincaré condition and the saddle-tube condition enter the gap-inside-the-box condition as the fast part inside the box. On top of this we state, as one line of argument, how $R = CP/\xi^2$ is squeezed into $1 \le R \le K'/\hbar$ and the double-well relaxation-time power statement follows, adding for each step "what it uses and which grade it has". At the end of the chapter the steps of the First-Step Upper-Bound Proposition are collected in a table of the five grades and a dependency graph.

Prerequisite chapters — 23-03 § 05 (the double-well power statement and the form of the First-Step Upper-Bound Proposition), 23-04 (the path measure, kinks, the One-Dimensional One-Sided Bridge Proposition), 1-10 § 07– § 09 (Poincaré inequalities), 23-00 § 01 (the labels of the five grades).

Said first — The level is [Y3]. The First-Step Upper-Bound Proposition is algebra on top of four conditions, and every one of the four conditions leaves assumption rows. This chapter does not say that the first step is closed. It names what would have to be shown to close it. This part is a study that chose the diffusive-relaxation-time assumption as its foothold, and the split into the four conditions is one way of building within it. Other views are treated in Part 24.

The order of this chapter

1. The splitting of the variance — the algebra of splitting into two valleys, and why that alone is not enough for long paths
2. The parts shown — the algebra of the splitting, the convex core, the reduced plateau, the diffusion coefficient of kinks

3. The statements of the four conditions — for which measure, which constant, uniform in what
4. The First-Step Upper-Bound Proposition as one line — box reduction → inside the box → $C_P \le K\xi^2 S_0/\hbar \to R \le K'/\hbar$ → the double-well power statement
5. The distance from known theorems — where they reach and where they do not (Table 1)
6. The table of the five grades — the rows of the steps of the First-Step Upper-Bound Proposition (Table 2)
7. The dependency graph — checking by the numbers that there is no circle
8. A $\hbar$ that fits all, and where the four conditions stand — a guide to 23-06–23-12

01

The splitting of the variance — the algebra of splitting into two valleys

The setting is as in 23-04 § 01. The lattice is $\varepsilon = \frac{1}{2}$, the action S_{lat}, the measure $\mu \propto e^{-S_{\mathrm{lat}}/\hbar}$, the Dirichlet form $\mathcal{E}(f) = (\hbar/\varepsilon)\sum_i \mathbb{E}|\partial_i f|^2$, $C_P = \sup_f \mathrm{Var}(f)/\mathcal{E}(f)$, the diffusion coefficient of kinks $D = \hbar/S_0$, and $S_0 = 2/3$. We take $T \to \infty$ first and then $\hbar \to 0$.

The first tool for building the upper side is the algebra of splitting the measure into two parts. Split the space of configurations in two (for example by the sign of the smoothed value of the path) and write $\mu = Z_+\mu_+ + Z_-\mu_-$ ($Z_\pm \ge 0$, $Z_+ + Z_- = 1$, and $\mu_\pm$ the measures restricted to each part and normalised). Then for every f

$$\mathrm{Var}_\mu f = Z_+\,\mathrm{Var}_{\mu_+} f + Z_-\,\mathrm{Var}_{\mu_-} f + Z_+Z_-\big(\mathbb{E}_{\mu_+} f - \mathbb{E}_{\mu_-} f\big)^2 \quad (1)$$

holds Proved here (the identity for the variance of a mixture. Put $m_\pm = \mathbb{E}_{\mu_\pm} f$ and $m = Z_+m_+ + Z_-m_-$, split $\mathbb{E}_\mu(f-m)^2$ on each part into $f - m_\pm$ and $m_\pm - m$, and use $Z_+(m_+ - m)^2 + Z_-(m_- - m)^2 = Z_+Z_-(m_+ - m_-)^2$).

Suppose that each of the three terms on the right-hand side has the following inequality.

$$\mathrm{Var}_{\mu_\pm} f \le C_{\mathrm{loc}}\,\mathcal{E}_\pm(f), \qquad \big(\mathbb{E}_{\mu_+} f - \mathbb{E}_{\mu_-} f\big)^2 \le C_{\mathrm{md}}\,\mathcal{E}(f), \qquad Z_+\mathcal{E}_+(f) + Z_-\mathcal{E}_-(f) \le$$

The first is local Poincaré inside a valley, the second is the "transport of the mean difference", which controls the difference of the means of the two valleys by the Dirichlet form, and the third is the additivity of the form. Since $Z_+Z_- \le \frac{1}{4}$,

$$C_P \;\le\; C_{\mathrm{loc}} + \frac{C_{\mathrm{md}}}{4} \quad (3)$$

holds Proved here Lean `split_poincare_le` (assumptions: $Z_+ + Z_- = 1$, $Z_\pm \ge 0$, the splitting formula (1), the two local Poincaré inequalities, the mean difference, additivity of the form, non-negative constants). The splitting formula (1) has the same form as the Variance-Splitting Lemma (Lemma 2.4 of Menz–Schlichting) Cited (the skeleton of the Poincaré version of finite-dimensional Eyring–Kramers).

Of the two constants in (3), C_{loc} later becomes the in-phase local Poincaré condition and C_{md} becomes the saddle-tube condition. But (3) cannot be applied to long paths as it stands.

For long paths, one coarse variable is not enough. (1) is a split of the form "take one slow variable (which valley the path is in) and treat the rest as fast". When the path is much longer than the correlation length ($T \gg \xi$), long-wave fluctuations of the density of kinks form a continuous band, and the Poincaré constant of the conditional measure with one valley fixed stays of the same size as ξ^2/D (the Glauber band of 23-04 § 05). That is, C_{loc} does not become $O(1)$. So instead of taking a number of slow variables proportional to the volume, we cut the problem into boxes of length $C\xi$. This is the box-reduction condition and the gap-inside-the-box condition, and the in-phase local Poincaré condition and the saddle-tube condition are used inside the box (§ 04).

02

The parts shown — the algebra of the splitting, the convex core, the reduced plateau, the diffusion coefficient

First we list the parts of the construction of the upper side that can be said without assumption rows.

(a) The algebra of the splitting. (1)–(3) are as in § 01 Proved here .

(b) The convex core. For the measure restricted to the convex region deep in a valley, $\{x_i \ge a\ \forall i\}$ ($a > 1/\sqrt{3}$), the Hessian of the action is at least $\varepsilon\kappa_a$ ($\kappa_a = (3a^2-1)/2$), so Bakry–Émery on a convex region gives $C_P \le 2/(3a^2-1)$ uniformly in the length, $\hbar$ and the boundary Proved here (the lower bound on the curvature) Cited (Bakry–Émery on a convex region, Theorem 1.2 of Kolesnikov–Milman). The proof and the Lean statement are in 23-04 § 06. But excursions out of the core have smaller action than a kink, and exponentially many of them fit in an interval of length ξ, so the convex core alone does not give the in-phase local Poincaré condition.

(c) The two-state value of the reduced plateau. On a short ring ($T \ll \xi$) the path that moves between valleys is one pair of kinks, and the separation ℓ of the pair becomes

the one slow coordinate. In the one-dimensional model that keeps only ℓ (ℓ moves with diffusion coefficient $2D$, with an atom of a valley at each end and the plateau weight $\pi(\ell)$ in between), we control the mean difference by the Cauchy–Schwarz inequality, write the difference of the values at the two ends of ℓ as $\int f'(\ell)\,d\ell$, and

$$\big(f(\text{right valley}) - f(\text{left valley})\big)^2 \;\le\; \int \frac{d\ell}{\pi(\ell)} \cdot \int \pi(\ell)\, f'(\ell)^2\, d\ell \quad (4)$$

gives a value of the form $C_{\mathrm{md}}/4 = J/(4D)$, with equality nearly attained by the sign test function Proved here ((4) is the Cauchy–Schwarz inequality) Lean `reduced_two_state_value` (assumptions: $0 < T$, $0 < D$, $J = 4\xi^2 I/T$, with I the integral of the plateau weight). The reduced model itself (that everything other than ℓ may be thrown away) is an assumption of the model, and this value is "a value on the reduced model" Assumption .

(d) The diffusion coefficient of kinks. The position of a single kink moves with diffusion coefficient $D = \hbar/S_0$. At the level of the process of a single interface, this is a copy of the coefficient of a known theorem Cited ((9) of 23-04 § 03). It is not a statement about the gap of the whole path.

These four are all. The remaining parts, namely that local Poincaré inside a valley is uniform in the length and in $\hbar$, that transport along the family of flat saddles is uniform in the dimension, the reduction of long paths to boxes, and the gap inside a box, lie outside the assumptions of known theorems (§ 05). We name them as four conditions.

03

The statements of the four conditions — for which measure, which constant, uniform in what

We write the four conditions making three things explicit: the measure, the constant and the uniformity. All are statements about the 0+1-dimensional double well $V = (x^2 - 1)^2/8$, the lattice $\varepsilon = \frac{1}{2}$ and the Dirichlet form $\mathcal{E}$. "Uniform" means that for some $\hbar_0 > 0$ with $\hbar \in (0, \hbar_0]$, the constant does not depend on $\hbar$ (does not grow as $\hbar \to 0$).

The in-phase local Poincaré condition (local Poincaré within a phase)

Measure: the measure of the phase without kinks. The measure $\mu^{(+)} = \mu(\cdot \mid x_i \ge 0\ \forall i)$ restricted to configurations in which the path stays on the side of one valley, for example to configurations with $x_i \ge 0$ at every site (reflected at the wall $x_i = 0$).
Constant: the Poincaré constant C_{loc} of $\mu^{(+)}$, that is, $\mathrm{Var}_{\mu^{(+)}} f \le C_{\mathrm{loc}}\, \mathcal{E}(f)$.
Uniformity: $C_{\mathrm{loc}} = O(1)$ uniformly in $\hbar$ and in the length of the path. For use

inside the box, the version for segments with end values (length $\leq C\xi$) is needed (the segment local Poincaré assumption), and it must also be uniform in the end values.

The saddle-tube condition (transport along the family of flat saddles)

Measure: the conditional measure of configurations in the tube around the family of kink–antikink pairs (the family of saddles, parametrised by the centre X and separation ℓ of the pair, along which the action is almost flat). **Constant**: the constant of transport of the mean difference along that family from valley to valley (of the type of C_{md} in (2)), and the Poincaré constant inside the tube. **Uniformity**: uniform in the number of sites of the path (the dimension) and in $\hbar$. For use inside the box it becomes two rows: Poincaré in the tube uniform in tube length $\leq C\xi$ (the saddle-tube Poincaré assumption), and Poincaré of the core inside the tube (the interval that stays near the top) uniform in the core length k (the core Poincaré uniformity assumption).

The box-reduction condition (reduction to the box dynamics)

Measure: the measure μ on the lattice line $\mathbb{Z}$ with $T = \infty$, and the conditional measure $\mu_B^{a,b}$ with the values $(a, b) \in \mathbb{R}^2$ on both neighbours of a box B of length L sites ($L\varepsilon = C\xi$) fixed. **Constants**: $C_{\mathrm{box}}(L) := \sup_{a,b\in\mathbb{R}} C_P(\mu_B^{a,b})$ and the constant c_M of mixing between boxes. **Statement**: $C_P(\mu) \leq C_{\mathrm{box}}(L)/c_M$. **Uniformity**: c_M depends only on lengths in units of ξ (C and the fraction of overlap of the boxes), not on $\hbar$ or the boundary values.

The gap-inside-the-box condition

Measure: the conditional measure $\mu_B^{a,b}$ of a box of length $C\xi$. The number of kinks in the box is $O(1)$ on average (the density of kinks is $1/(2\xi)$, so the expectation is $C/2$). **Constant**: the gap $1/C_P(\mu_B^{a,b})$. **Statement**: $1/C_P(\mu_B^{a,b}) \geq c\,D/(C\xi)^2$, that is, $C_{\mathrm{box}} \leq K_4(C\xi)^2/D$ ($K_4 = 1/c$). **Uniformity**: c is uniform in $\hbar$ and in all boundary values (a, b) (it may depend on C).

The four are not conditions of the same weight. The box-reduction condition is the step that "moves the problem of long paths to a problem of boxes", the gap-inside-the-box condition is the step that "splits the inside of a box into slow variables (the arrangement of kinks) and fast variables", and the in-phase local Poincaré condition

and the saddle-tube condition are statements on the side of those fast variables. The next section puts into one line the place where the in-phase local Poincaré condition and the saddle-tube condition enter the gap-inside-the-box condition.

To be precise, the four double-well conditions are different from the relaxation-time assumption (the power statement, 23-01 § 05). The relaxation-time assumption is a statement on the side of the conclusion, and the four double-well conditions are conditions put in place to derive it in the first step (23-03 § 05).

04

The First-Step Upper-Bound Proposition as one line

The First-Step Upper-Bound Proposition (conditional)

For the path measure of the 0+1-dimensional double well (lattice $\varepsilon = \frac{1}{2}$, taking $T \to \infty$ first), assume the four double-well conditions and the premise rows of Step 0 below. Then as $\hbar \to 0$

$$C_P \;\le\; K\,\xi^2\,\frac{S_0}{\hbar} = K\,\frac{\xi^2}{D} \qquad (K \text{ depends only on the constants of the conditions, not on}$$

holds, hence $1 \le R = C_P/\xi^2 \le KS_0/\hbar$, and the double-well power statement $\ln C_P / \ln \xi \to 2$ follows.

(5) is the same formula as (15) of 23-04 § 05 and (3) of 23-03 § 05. Below, each step is given with "what it uses" and its grade.

Step 0 (the step aligning the assumptions) The premise rows

That the measure μ with $T = \infty$ is uniquely determined (simplicity of the principal eigenvalue, from the positivity of the integral kernel of the transfer operator) Assumption (the original source for the version with a positive kernel on $\mathbb{R}$ has not been checked; the same row as in 23-04 § 01). The relation between $C_P(T)$ of the finite ring and C_P of the measure with $T = \infty$ (the step of the limit), and the regularity of the functions used in the steps below (conditional measures, smoothness of the kink coordinates) Assumption . These three are premises placed outside the First-Step Upper-Bound Proposition, and they are counted as assumption rows in the integrated table (23-12) too.

Step 1 (the box-reduction step) Box reduction — the box-reduction condition in the form of the Box-Reduction Proposition

Cover the line $\mathbb{Z}$ by boxes of length L sites. Neighbouring boxes overlap in l sites ($l \le L/3$), and the boxes are divided alternately into an even family E and an odd family O. Let $\mathcal{G}_E$ be the σ-algebra generated by the values at the sites outside the even family, define $\mathcal{G}_O$ in the same way, and let the maximal correlation of the two σ-algebras be

$$c_{\mathrm{alt}} := \sup\{\operatorname{corr}_\mu(F,G) : \ F \in L^2(\mathcal{G}_E),\ G \in L^2(\mathcal{G}_O)\} \quad (6)$$

.

The Two-Projection Lemma. If $c = c(\mathcal{G}_1,\mathcal{G}_2) < 1$, then for all $f \in L^2(\mu)$, $\operatorname{Var} f \le (1-c)^{-1}\big(\mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_1) + \mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_2)\big)$.

Proved here Proof. Put $\tilde f = f - \mathbb{E}f$ and $a_i = \mathbb{E}[f \mid \mathcal{G}_i] - \mathbb{E}f$. Then $s := \|a_1\|^2 + \|a_2\|^2 = \langle \tilde f, a_1 + a_2\rangle \le \|\tilde f\|\,\|a_1+a_2\|$, while $\|a_1+a_2\|^2 \le s + 2c\|a_1\|\|a_2\| \le (1+c)s$. Hence $s \le (1+c)\|\tilde f\|^2$, and adding $\mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_i) = \|\tilde f\|^2 - \|a_i\|^2$ gives at least $(1-c)\|\tilde f\|^2$. ∎ Lean `two_proj_var` (assumption: $A + B \le (1+c)V$. The geometry of the projections is outside Lean).

The product over boxes. Even boxes are $L - 2l \ge 1$ sites apart, so by the Markov property of the nearest-neighbour interaction they are mutually independent under $\mathcal{G}_E$. The variance of a function of independent components is at most the sum of the conditional variances component by component (of the Efron–Stein type), so

$$\mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_E) \ \le \ C_{\mathrm{box}}(L) \sum_{B \in E} \mathcal{E}_B(f) \quad (7)$$

holds Proved here (the Markov property and the variance inequality for independent products; the proof is placed in 23-11). The same holds for the odd family. Each site lies in the even family at most once and in the odd family at most once, and only the overlaps lie in both.

Averaging over shifts. Shift the arrangement of boxes by l at a time, in $n = \lfloor (L-l)/l \rfloor$ ways. Every shift has the same c_{alt}, and the overlap sites are disjoint between shifts. Applying the Two-Projection Lemma and (7) to each shift and averaging gives

$$C_P(\mu) \ \le \ \Big(1 + \frac{1}{n}\Big)\frac{C_{\mathrm{box}}(L)}{1 - c_{\mathrm{alt}}} \quad (8)$$

Proved here Lean `box_two_family`, `box_shift_average` (assumptions: the two-family inequality for each shift, $C_{\mathrm{box}} \ge 0$, the sum of the forms on the overlaps at most $\mathcal{E}$). This is the Box-Reduction Proposition, and c_M of the box-reduction condition

corresponds to $(1 - c_{\mathrm{alt}})/(1 + 1/n)$. This step uses only Poincaré inequalities, not the logarithmic Sobolev inequality.

One assumption row remains in (8): that the alternating maximal correlation $c_{\mathrm{alt}} < 1$ is uniform in $\hbar$ (the alternating maximal-correlation assumption for boxes). Its value is read as $c_{\mathrm{alt}} \approx 0.73$ at $C = 2$ and overlap $\theta = l/L = \frac{1}{4}$ Reading (a family value copied from the closed formula for the two-state telegraph chain; not a proof of an upper bound). The type of the box reduction has the same form as the Box-Dynamics Proposition for Glauber dynamics of discrete spins (Propositions 3.4 and 3.5 of Martinelli's lectures) Cited (discrete spins, in the form of a total-variation coupling; here rewritten in one dimension, in L^2, in the form of maximal correlation).

Step 2 (the step inside the box) Inside the box — the gap-inside-the-box condition in the form of the Gap-Inside-the-Box Proposition, and where the in-phase local Poincaré condition and the saddle-tube condition enter

Fix a box B ($L\varepsilon = C\xi$) and boundary values (a, b). Let the slow variable inside the box be Y = the arrangement of kinks (the number of kinks, and the position of each kink defined by projection onto the manifold of kinks). Splitting the variance by Y gives

$$\operatorname{Var} f = \mathbb{E} \operatorname{Var}(f \mid Y) + \operatorname{Var}\big(\mathbb{E}[f \mid Y]\big) \quad (9)$$

Proved here (the identity for conditional expectations).

First term = the fast side. The in-phase local Poincaré condition and the saddle-tube condition enter here. By the nearest-neighbour Markov property, the conditional measure with Y fixed splits into a product of the following parts. (i) Segments of the phase without kinks between kinks (length $\le C\xi$, with values at both ends). (ii) The tube around each kink or kink pair, and the core inside it (the interval that stays near the top). The Poincaré constant of a product measure is the maximum of the constants of the components (the variance inequality for independent products), so

$$\mathbb{E} \operatorname{Var}(f \mid Y) \;\le\; C_{\mathrm{loc}}\, \mathcal{E}(f), \qquad C_{\mathrm{loc}} = \max\big(C_{(\mathrm{H1}^{\mathrm{seg}})},\, C_{(\mathrm{H2}^{\mathrm{tube}})},\, C_{(\mathrm{T0}'_{\mathrm{P}})}\big) \quad (10)$$

Proved here (the Poincaré constant of a product measure is the maximum of the constants of the components. That it splits into a product depends on how Y is defined, and its regularity is included in the row of Step 0). The constant of (i) is the segment version of the in-phase local Poincaré condition, and the constants of (ii) are the tube and core rows of the saddle-tube condition Assumption (all three are assumption rows).

Second term = the slow side. Longitudinal transport along the family of saddles enters here. $\mathbb{E}[f \mid Y]$ is a function of Y alone, and its fluctuations relax by the diffusion

of kink positions and by the creation and annihilation of kink pairs. The Dirichlet form $\mathcal{E}_Y$ on Y has mobility per kink of the form $D = \hbar/S_0$ (the kink diffusion-coefficient row: the projection formula, 23-04 § 03). Here we put two rows.

- The kink-gas gap assumption: the Poincaré constant of the box for the reduced kink gas, $C_{\mathrm{kg}} \le (C\xi)^2/\big(c_{\mathrm{kg}}(C)(1-\eta_{\mathrm{KG}})D\big)$, with $\eta_{\mathrm{KG}} < 1$ uniform in $\hbar$ and the boundary values Assumption (the assumption that the kink gas falls into the same type as the Glauber dynamics of the one-dimensional Ising model).
- The cross-term uniformity assumption: the cross term $\mathcal{E}_Y(\mathbb{E}[f \mid Y]) \le (1+\chi)\,\mathcal{E}(f)$, with χ uniform in $\hbar$ and the boundary values Assumption (of the type of the cross constant of a two-scale inequality; a statement of this form is not taken from a known theorem).

c_{kg} of the kink-gas gap assumption comes from reading in the continuum limit the closed formula for the gap of the Glauber box with fixed ends, $(1-\gamma)+\gamma(1-\cos(\pi/(L+1)))$,

$$\frac{D}{\xi^2} + \frac{\pi^2 D}{L_\tau^2} = \big(C^2+\pi^2\big)\frac{D}{(C\xi)^2} \qquad (L_\tau = C\xi) \quad (11)$$

which gives $c_{\mathrm{kg}} = C^2+\pi^2$ Proved here (the algebra of (11)) Lean `kg_constant` (assumption: $C\xi \neq 0$). The first term D/ξ^2 is the creation and annihilation of pairs, and the second is the diffusion of kink positions. When a boundary value is near the top, half a kink can sit at the end of the box (corresponding to a free end), and in the sup including that case $c_{\mathrm{kg}} = c_{\mathrm{ff}}(C)$, read as 7.17 at $C = 2$ Reading (a family value obtained from lattice values of the gap of the free-end box of the one-dimensional Ising model).

Combining (9)(10) with the kink-gas gap assumption and the cross-term uniformity assumption gives

$$C_{\mathrm{box}} \;\le\; C_{\mathrm{loc}} + (1+\chi)\,C_{\mathrm{kg}} \;\le\; K_4\,\frac{(C\xi)^2}{D}, \qquad K_4 = \frac{1+\chi}{c_{\mathrm{kg}}(1-\eta_{\mathrm{KG}})} + C_{\mathrm{loc}}\,\frac{D}{(C\xi)^2} \quad (12)$$

Proved here Lean `H4_two_scale` (assumptions: the inequality of decomposition (9), first term $\le C_{\mathrm{loc}}\mathcal{E}$, the kink-gas gap assumption $\le C_{\mathrm{kg}}\mathcal{E}_Y$, the cross-term uniformity assumption, $C_{\mathrm{kg}} \ge 0$), `H4_gap_form` (assumptions: $D>0$, $C\xi \neq 0$, $c_{\mathrm{kg}}>0$, $\eta_{\mathrm{KG}}<1$, $1+\chi \ge 0$, the two inequalities above). This is the Gap-Inside-the-Box Proposition.

The second term of K_4 is C_{loc} multiplied by $D/(C\xi)^2$, and $D/\xi^2 = \hbar/(S_0\xi^2)$ is of size $e^{-441.6}/e^{-1328.6}/e^{-4433.1}$ at the three values of $\hbar$ ($\hbar = 3\cdot10^{-3}/10^{-3}/3\cdot10^{-4}$) Reading (Table 2 of 23-04 § 07). Unless C_{loc} grows exponentially in $\hbar$, this term vanishes as $\hbar \to$

0. In this sense the demand of the in-phase local Poincaré condition for $C_{\mathrm{loc}} = O(1)$ is a demand with room to spare.

A note on where the saddle-tube condition goes. In the two-valley split of § 01, the saddle-tube condition was the mean-difference constant C_{md} itself. In the box split, the direction along the family of saddles (the kink positions) was taken as the slow variable Y, so transport in that direction is handled by the kink-gas gap assumption and the cross-term uniformity assumption in the second term, and what remains in the saddle-tube condition is the Poincaré of the tube and the core with Y fixed. In this book the tube and core rows are counted as rows of the saddle-tube condition, and the kink-gas gap assumption and the cross-term uniformity assumption as rows of the gap-inside-the-box condition (the same counting as the integrated table, 23-12).

Step 3 (the composition step) Composition — $C_P \le K\xi^2 S_0/\hbar$

Putting (12) into C_{box} of (8) and using $L\varepsilon = C\xi$ and $D = \hbar/S_0$ gives

$$C_P(\mu) \;\le\; K\,\xi^2\,\frac{S_0}{\hbar}, \qquad K = \Big(1+\frac{1}{n}\Big)\frac{C^2 K_4}{1-c_{\mathrm{alt}}} \tag{13}$$

Proved here Lean `H3_H4_compose` (assumptions: the conclusion (8) of the Box-Reduction Proposition, the conclusion $C_{\mathrm{box}} \le K_4(C\xi)^2/D$ of the Gap-Inside-the-Box Proposition, $D = \hbar/S_0$, $n > 0$, $\hbar > 0$, $S_0 > 0$, $c_{\mathrm{alt}} < 1$). That K does not depend on $\hbar$ is the uniformity of c_{alt}, χ, η_{KG}, c_{kg} and C_{loc}, that is, the very content of the assumption rows.

Putting in the values of the reduced model ($\chi = \eta_{\mathrm{KG}} = 0$, $n = 3$, $\theta = \frac{1}{4}$) gives $K \approx 2.76$ at $C = 2$ (the sup including boundary values at the top), 1.43 (ends on the well side only), and $K \approx 1.78$ at $C = 4$ Reading (family values, not upper bounds on K) Lean `K_ff_C2`, `K_ff_C4` (assumptions: the measured values of c_{ff} and c_{alt}; arithmetic only). The value is the same at the three values of $\hbar$ because $\hbar$ does not enter the reduced model.

Step 4 (the step bracketing the ratio) Bracketing the ratio R — $1 \le R \le K'/\hbar$

Dividing (13) by ξ^2 gives $R = C_P/\xi^2 \le KS_0/\hbar =: K'/\hbar$ Proved here (division). From below, the One-Dimensional One-Sided Bridge Proposition (23-04 § 04) gives $R \ge 1$ Proved here (continuous time, $T \to \infty$; the proof is placed in 23-04). Together,

$$1 \;\le\; R \;\le\; \frac{K'}{\hbar}, \qquad K' = KS_0 \tag{14}$$

The One-Dimensional One-Sided Bridge Proposition is a lower bound that uses no conditions, and the left inequality of (14) does not depend on the four double-well

conditions.

Step 5 (the step reading the power) The double-well power statement — $\ln C_P / \ln \xi \to 2$

$\ln C_P / \ln \xi = 2 + \ln R / \ln \xi$, and (14) gives $0 \le \ln R \le \ln K' + \ln(1/\hbar)$. The remaining input is how the ruler grows, $\ln \xi \ge s/\hbar$ (for some $s > 0$ and small $\hbar$), which comes from $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$. The semiclassical leading term of the continuum model is a copy of a known formula Cited , and for the lattice model it is a reading that $\ln \xi - S_0^{\mathrm{lat}}/\hbar - \frac{1}{2}\ln \hbar$ settles to a constant, $-0.774/-0.777/-0.778$, at the three values of $\hbar$ Reading (23-04 § 07). On this input

$$0 \;\le\; \frac{\ln R}{\ln \xi} \;\le\; \frac{\hbar\big(\ln K' + \ln(1/\hbar)\big)}{s} \;\longrightarrow\; 0 \qquad (\hbar \to 0) \quad (15)$$

and the double-well power statement follows Proved here Lean `power_H_of_kink_ratio` (assumptions: $0 < K$, $0 < s$, $0 < \delta$, and $1 \le R \le K/\hbar$ and $s/\hbar \le \ln \xi$ for $0 < \hbar < \delta$. K in Lean is K' in the text).

Checking the single line

Shrinking Steps 1–5 to one line gives

$$\underbrace{(\mathrm{H1}^{\mathrm{seg}}),\ (\mathrm{H2}^{\mathrm{tube}}),\ (\mathrm{T0}'_{\mathrm{P}})}_{C_{\mathrm{loc}}} \xrightarrow{(9)(10),\ (\mathrm{KG}),\ (\mathrm{X})} \underbrace{C_{\mathrm{box}} \le K_4 \frac{(C\xi)^2}{D}}_{\text{Proposition H4}} \xrightarrow{\text{Proposition H3, (M}}$$

Of the inputs written above the arrows, the assumption rows are the segment local Poincaré assumption, the saddle-tube Poincaré assumption, the core Poincaré uniformity assumption, the kink-gas gap assumption, the cross-term uniformity assumption, the alternating maximal-correlation assumption for boxes, and the premises of Step 0; the rest are elementary algebra, known theorems or readings.

Here we also check that the statements of the assumption rows do not contain the quantities of the conclusion. The quantities of the conclusion are $C_P(\mu)$ of the whole measure with $T = \infty$ and the ratio R. The segment local Poincaré assumption, the saddle-tube Poincaré assumption and the core Poincaré uniformity assumption are constants of the measures of segments, tubes and cores, the kink-gas gap assumption is a constant of the reduced gas, the cross-term uniformity assumption is a comparison of forms, and the alternating maximal-correlation assumption for boxes is a quantity of correlation. None of them contains $C_P(\mu)$ or R. A form that puts the conclusion among the premises (a circle of dependence) does not appear in the way these steps are built (for how to check this, see 23-13).

To be precise, the conclusion of the First-Step Upper-Bound Proposition is the first-step statement that "the relaxation time C_P of Langevin dynamics grows as a power of ξ^2". C_P is a different quantity from the gap of the transfer operator (the tunnelling splitting), and even if every row of the First-Step Upper-Bound Proposition were in place, the statements from the second step on would not follow (23-01 § 08, 23-03 § 06).

05

The distance from known theorems

Within the range searched, no known theorem gives the upper bound (5) of the First-Step Upper-Bound Proposition. Table 1 summarises how far nearby theorems reach and where they stop.

Table 1 Where known theorems reach and where they do not (the grade is the grade of the row that cites the theorem)

Known theorem	Where it reaches	Where it does not	Grade
The Poincaré version of Eyring–Kramers (Bovier–Gayrard–Klein; the Variance-Splitting Lemma of Menz–Schlichting, Theorems 2.9 and 2.12)	For a Morse function at fixed dimension, from the splitting of the variance, local Poincaré and transport of the mean difference up to the asymptotics of the gap and the prefactor	The constants are not uniform in the dimension. Degenerate families of saddles such as translation of a kink are outside the assumptions	Cited
Eyring–Kramers for the stochastic Allen–Cahn equation on a fixed interval (Di Gesù–Le Peutrec; Brooks–Di Gesù)	Asymptotics of the gap with the length of the interval fixed below the bifurcation threshold	The regime with infinite interval length and the translation family are stated by the authors themselves in the text to be out of scope	Cited
Transition times of short paths (Berglund–Gentz Theorem 2.6)	The mean time to move between valleys for $T < 2\pi\sqrt{2}$ (uniform saddle)	Long paths ($T \gg \xi$)	Cited
Diffusion of a single interface (Bertini–Brassesco–Buttà Theorem 2.1)	That the position of a single kink approaches Brownian motion with diffusion coefficient D (at the level of the process)	Not a statement about the spectral gap of the whole path	Cited
A criterion uniform in the volume (Bauerschmidt–Dagallier Theorem 2.1)	Applies directly to the lattice of paths and gives an upper bound uniform in the volume	The value is $C_P \le e^{c\xi}$ with $c \approx 1.4 \to 1.8$ ($\varepsilon = 0.25$, $\hbar = 0.4 \to 0.1$), doubly exponential in $\hbar$. It does not reach the double-well power statement	Cited (theorem) Reading (value)

Known theorem	Where it reaches	Where it does not	Grade
The Box-Dynamics Proposition (Propositions 3.4 and 3.5 of Martinelli's lectures)	For discrete spins, the type that controls the whole gap by the box dynamics and the gap inside the box	Unbounded spins, sup over boundary values. Step 1 borrows only the type, rewritten in one dimension in the L^2 form	Cited (type)

As the last three rows of Table 1 show, the known tools sit at the two ends, "a single kink" and "an upper bound uniform in the volume but coarse", and what lies between them (an upper bound on long paths sharp up to the diffusion time ξ^2/D of the kink gas) is the place of the First-Step Upper-Bound Proposition. The Bauerschmidt–Dagallier bound becoming $e^{c\xi}$ is read as follows: the criterion looks at a single quantity, the susceptibility, so it cannot distinguish slow but local relaxation from slow and global relaxation, and ξ enters the exponent Reading .

06

The table of the five grades — the rows of the steps of the First-Step Upper-Bound Proposition

We put the steps of § 04 into one table in the format of the five grades (23-00 § 01). Each row has one grade, and the column of machine checking is outside the grades. The row numbers are the numbers of the nodes of the dependency graph in § 07 (the labels in the formulas, (H1^{seg}), (H2^{tube}), (T0′$_{\mathrm{P}}$), (KG) and (X), are the assumptions [02]–[06] respectively).

Table 2 The rows of the steps of the First-Step Upper-Bound Proposition ("—" in the range of $\hbar$ marks an assumption row not shown at any $\hbar$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[01]	Uniqueness of the measure with $T = \infty$, the step of the limit, regularity	Assumption	§ 04 Step 0 (the original source for the principal eigenvalue of a positive kernel on $\mathbb{R}$ is unchecked)	—	—
[02]	The segment local Poincaré assumption: Poincaré of segments without kinks uniform in the length, the end values and $\hbar$	Assumption	§ 03, (10) of § 04 Step 2. The node propositions for the ring version are in 23-06	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[03]	The saddle-tube Poincaré assumption: Poincaré of the tube uniform in length $\leq C\xi$	Assumption	§ 03, (10)	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[04]	The core Poincaré uniformity assumption: Poincaré of the core uniform in the core length	Assumption	§ 03, (10). The proposition with an assumption at the lower end of the window is in 23-09 and 23-11. For the core on the sector side (both ends in the well), it is supported for every $\hbar \in (0,1]$ by the Dobrushin criterion of the coarse chain: conditional on a proof in the paper (the Coarse-Chain Dobrushin Theorem (Theorem CC of the paper)) + Computer-assisted ($\hbar \in [10^{-5},1]$, 239 intervals, interval arithmetic, $\rho_{\mathrm{col}} \leq 0.400$) + a paper proof and Computer-assisted (the Small-$\hbar$ Coarse-Chain Proposition (Proposition PIN of the paper): $\hbar \leq 10^{-5}$, $\rho_{\mathrm{col}} \leq 0.874$) + Cited (the Poincaré Version of Helffer's Criterion in Bobkov–	Sector side: $(0,1]$	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
			Ledoux 2000 (Proposition 3.1 of Ledoux)), and the Coarse-Chain Brascamp–Lieb Proposition (Proposition BLc of the paper). There are no Reading rows. The core on the box side is not covered (the paper "Convex blocks and a coarse Dobrushin chain")		
[05]	The kink-gas gap assumption: the box gap of the reduced kink gas $\geq c_{\mathrm{kg}}(1-\eta_{\mathrm{KG}})D/(C\xi)^2$	Assumption	§ 04 Step 2 (the assumption that it falls into the Glauber type)	—	—
[06]	The cross-term uniformity assumption: cross term $\mathcal{E}_Y(\mathbb{E}[f \mid Y]) \leq (1+\chi)\mathcal{E}(f)$	Assumption	§ 04 Step 2	—	—
[07]	The alternating maximal-correlation assumption for boxes: $c_{\mathrm{alt}} < 1$ uniformly in $\hbar$	Assumption	§ 04 Step 1	—	—
[08]	Mobility of a single kink $D = \hbar/S_0$	Cited	§ 02 (d), 23-04 § 03 (a copy of the coefficient of a theorem at the level of the process)	$\hbar \to 0$	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[10]	The Two-Projection Lemma	Proved here	Proof in § 04 Step 1	Independent of $\hbar$	`two_proj_var` (assumption: $A + B \le (1 + c)V$)
[11]	Conditional independence of even boxes and the product variance inequality (7)	Proved here	§ 04 Step 1 (proof in 23-11)	Independent of $\hbar$	—
[14]	The Box-Reduction Proposition: (8) $C_P \le (1 + 1/n)C_{\mathrm{box}}/(1 - c_{\mathrm{alt}})$	Proved here	§ 04 Step 1 (the Two-Projection Lemma, the product variance of even boxes, averaging over shifts)	Independent of $\hbar$	`box_two_family`, `box_shift_average` (assumptions: two-family inequality, $C_{\mathrm{box}} \ge 0$, sum over overlaps $\le \mathcal{E}$)
[12]	Decomposition of the variance (9) and into a product (10)	Proved here	§ 04 Step 2 (the product decomposition depends on how Y is defined; regularity is in the row of common premises [01])	Independent of $\hbar$	—
[13]	$c_{\mathrm{kg}} = C^2 + \pi^2$ (the algebra of (11))	Proved here	§ 04 Step 2 (the closed Glauber formula is inside the kink-gas gap assumption)	Independent of $\hbar$	`kg_constant` (assumption: $C\xi \ /= 0$)
[15]	The Gap-Inside-the-Box Proposition: (12) $C_{\mathrm{box}} \le K_4 (C\xi)^2 / D$	Proved here	§ 04 Step 2	Independent of $\hbar$	`H4_two_scale`, `H4_gap_form` (assumptions are the rows of § 04 Step 2)
[16]	The First-Step Upper-Bound Proposition: (13) $C_P \le K \xi^2 S_0 / \hbar$	Proved here	§ 04 Step 3	$\hbar \to 0$	`H3_H4_compose` (assumptions: (8), (12), $D = \hbar / S_0$, $n, \hbar, S_0 > 0$, $c_{\mathrm{alt}} < 1$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
—	$K \approx 2.76/1.43$ ($C = 2$), 1.78 ($C = 4$)	Reading	§ 04 Step 3 (reduced model, $\chi = \eta$ KG $= 0$, family values)	Independent of $\hbar$	`K_ff_C2`, `K_ff_C4` (assumptions: measured values; arithmetic only)
[17]	The One-Dimensional One-Sided Bridge Proposition: $R \geq 1$	Proved here	23-04 § 04	Every $\hbar$	`local_rayleigh_ratio_le_one` (the skeleton of the finite sum only; the assumptions are in Table 3 of 23-04)
[18]	(14) $1 \leq R \leq K'/\hbar$	Proved here	§ 04 Step 4	$\hbar \to 0$	—
[09]	$\ln \xi \geq s/\hbar$ ($\xi \propto \hbar^{1/2} e^{S0/\hbar}$)	Reading	§ 04 Step 5 (the continuum leading term is Cited, the values of the lattice model are Reading)	Read at three points; $\hbar \to 0$ from the leading term	—
[19]	The double-well power statement: $\ln C_P / \ln \xi \to 2$	Proved here	(15) of § 04 Step 5	$\hbar \to 0$	`power_H_of_kink_ratio` (assumptions: $0 < K, s, \delta$, $1 \leq R \leq K/\hbar$, $s/\hbar \leq \ln \xi$)

The "Proved here" rows of Table 2 are all algebra that takes assumption rows as input. That "Proved here" rows line up does not mean that the conclusion of the First-Step Upper-Bound Proposition has been shown. The conclusion becomes unconditional only when every "Assumption" row and the "Reading" row (the row of the growth of the correlation length, [09]) have been replaced.

07

The dependency graph — checking by the numbers that there is no circle

We turn the rows of Table 2 into a list of numbered nodes. The format is common to this part (the integrated graph of 23-12 is the same), and **the numbers of the premises to the right of "⟸" are always placed smaller than the number of the conclusion on the left**. The leaves (assumptions, known results, readings) have small numbers, and the conclusion has the largest number.

```
[01] ASM  row of common premises: uniqueness of the T=∞ measure, limi
[02] ASM  segment local Poincaré assumption: Poincaré of segments wit
[03] ASM  saddle-tube Poincaré assumption: Poincaré of the tube (leng
[04] ASM  core Poincaré uniformity assumption (uniform in core length
[05] ASM  kink-gas gap assumption: box gap of the reduced kink gas
[06] ASM  cross-term uniformity assumption: cross term ℰ_Y ≤ (1+χ)ℰ
[07] ASM  alternating maximal-correlation assumption for boxes: c_alt
[08] CIT  kink diffusion-coefficient row: mobility of a single kink D
[09] RDG  row of the growth of the correlation length: ln ξ ≥ s/ħ (ξ
[10] PRV  Two-Projection Lemma
          Lean two_proj_var (assumption: A+B ≤ (1+c)V)
[11] PRV  conditional independence of even boxes, product variance (7
[12] PRV  decomposition of variance (9), into a product (10) ⟸ [01]
[13] PRV  c_kg = C²+π² (algebra of (11))
          Lean kg_constant (assumption: Cξ ≠ 0)
[14] PRV  Box-Reduction Proposition (8) ⟸ [07][10][11]
          Lean box_two_family, box_shift_average (assumptions: two-fa
[15] PRV  Gap-Inside-the-Box Proposition (12) ⟸ [02][03][04][05][06]
          Lean H4_two_scale, H4_gap_form (assumptions: decomposition,
[16] PRV  First-Step Upper-Bound Proposition (13) C_P ≤ Kξ²S₀/ħ ⟸ [0
          Lean H3_H4_compose (assumptions: (8), (12), D = ħ/S₀, n, ħ,
[17] PRV  One-Dimensional One-Sided Bridge Proposition R ≥ 1 ⟸ [01]
          Lean local_rayleigh_ratio_le_one (skeleton of the finite su
[18] PRV  (14) 1 ≤ R ≤ K'/ħ ⟸ [16][17]
[19] PRV  double-well power statement ln C_P / ln ξ → 2 ⟸ [09][18]
          Lean power_H_of_kink_ratio (assumptions: 0 < K, s, δ; 1 ≤ R
```

How to check. In every row with "⟸", look to see that the numbers on the right are smaller than the number on the left. [11] and [12] use only [01], [14] uses [07][10][11], [15] uses [02]–[06][08][12][13], [16] uses [08][14][15], [17] uses [01], [18] uses [16][17], and [19] uses [09][18]. Every row keeps the rule. If the rule is kept, following the premises always lowers the number and eventually reaches a leaf, so there is no circle.

Among the leaves, "Assumption" covers the seven [01]–[07], "Cited" is [08], and "Reading" is [09]. The conclusion [16] of the First-Step Upper-Bound Proposition depends on all of [01]–[08] (on [01] also through [11][12]), and the double-well power statement [19] depends further on [09]. Only the One-Dimensional One-Sided Bridge Proposition [17] depends on nothing but [01] and does not use the four double-well conditions.

To be precise, this graph is the dependency of the construction of the steps of the First-Step Upper-Bound Proposition, and it does not include the content of each assumption

row (for example the Six-Node In-Phase Poincaré Proposition, which supports [02] at the nodes). Those are in the graphs of the chapters on each condition, and the graph that joins them all into one is the integrated graph of 23-12.

08

A $\hbar$ that fits all, and where the four conditions stand

There is no $\hbar$ at which all rows are in place. The node propositions of the in-phase local Poincaré condition are at six points with $\hbar \geq 0.005$, the conditional propositions that support the core row are at the lower end of the window, of order $\hbar \leq 10^{-3}$, and the two do not meet. The kink-gas gap assumption, the cross-term uniformity assumption and the saddle-tube Poincaré assumption are assumption rows at every $\hbar$. The band figure and the way of fitting them together are placed in 23-12.

We list where each of the four conditions stands, one line each. The details are in each chapter.

Where the four conditions stand

1. The in-phase local Poincaré condition — the Six-Node In-Phase Poincaré Proposition for the ring version: margin ≥ 0.067 at the six nodes $\hbar \in \{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$, with neighbours and ends in $[0, 2.5]$ (computer-assisted). What remains is between the nodes, $\hbar < 0.005$, ends > 2.5, and the segment version (23-06)
2. The saddle-tube condition — a map split into seven routes. The condition on the chain outside the window is computer-assisted on the whole window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$, five steps of the sector framework are propositions, and for the one remaining proposition three strands stop at the curvature shortfall $\frac{1}{4}$ at one site at the top (23-07–23-10)
3. The box-reduction condition — the Box-Reduction Proposition (§ 04 Step 1). The alternating maximal-correlation assumption for boxes and the $\sup$ over boundary values remain (23-11)
4. The gap-inside-the-box condition — the Gap-Inside-the-Box Proposition (§ 04 Step 2). The cross-term uniformity assumption, the kink-gas gap assumption, the segment local Poincaré assumption and the core row remain, and the core constant is controlled in three steps (23-11)

This part is a study that chose the relaxation-time assumption as its foothold, and the First-Step Upper-Bound Proposition is one way of splitting to build the upper side of

the first step. Other views are treated in Part 24.

What this chapter can say

What is said	What is not said
Proved here The algebra of the splitting of the variance (3), the Two-Projection Lemma, the Box-Reduction Proposition (8), the Gap-Inside-the-Box Proposition (12), the composition (13), the bracketing of the ratio (14), the algebra to the double-well power statement (15). All are algebra on named assumption rows Lean (the assumptions are the rows of Table 2)	None of the assumption rows [01]–[07]. Hence neither (5) itself nor the double-well power statement for the first step
Proved here The One-Dimensional One-Sided Bridge Proposition: $R \geq 1$ (uses no conditions)	Whether R grows like $1/\hbar$ or exponentially
Cited The diffusion coefficient of a single kink; how far known theorems reach (Table 1)	A known theorem giving an upper bound on long paths sharp up to the diffusion time of the kink gas (only as far as "not found within the range searched")
Reading $K \approx 2.76$ and others (reduced model), $c_{\mathrm{alt}} \approx 0.73$, how $\ln \xi$ grows	Using these in proofs of upper or lower bounds
That the dependency graph has no circle (the rule of the numbers)	A $\hbar$ at which all rows are in place. Implications from the first step to the second and later steps

Sources and reproduction

Item	Kind	Source / tools
The splitting of the variance, the skeleton of the Poincaré version of Eyring–Kramers	Cited	G. Menz, A. Schlichting, Ann. Probab. 42 (2014), arXiv:1202.1510 (the Variance-Splitting Lemma, Theorems 2.9 and 2.12; checked in the text). A. Bovier, V. Gayrard, M. Klein, J. Eur. Math. Soc. 7 (2005) 69–99 (bibliographic data).
Bakry–Émery on a convex region	Cited	D. Bakry, M. Émery, LNM 1123 (1985); A. V. Kolesnikov, E. Milman, J. Geom. Anal. 27 (2017), arXiv:1310.2526 (Theorem 1.2; text checked).
Eyring–Kramers on a fixed interval	Cited	G. Di Gesù, D. Le Peutrec, arXiv:1506.04434 (Theorem 1.1); M. Brooks, G. Di Gesù, arXiv:2606.16083 (the explicit statement of what is out of scope; text checked).
Transition times of short paths, diffusion of a single interface	Cited	The table of sources of 23-04 (Berglund–Gentz, Bertini–Brassesco–Buttà).

Item	Kind	Source / tools
A criterion uniform in the volume	Cited, Reading	R. Bauerschmidt, B. Dagallier, Comm. Pure Appl. Math. 77, arXiv:2202.02295 (Theorem 2.1; text checked). The application to the lattice of paths and the value $e^{c\xi}$ are floating-point computations of the record (transfer matrix, $\varepsilon = 0.25$, 1.0049 on the harmonic positive control).
The type of box dynamics	Cited	F. Martinelli, LNM 1717 (1999) 93–191 (checked in § 3.6 and Propositions 3.4 and 3.5 of the preprint).
The lemma of two projections, the Efron–Stein type	Proved here	§ 04 Step 1 of the text (the original sources for the Friedrichs angle and Efron–Stein are bibliographic only).
The closed formula for the box gap of Glauber dynamics, values with free ends	Assumption, Reading	R. J. Glauber, J. Math. Phys. 4 (1963) 294 (bibliographic only). The fixed-end formula agrees to 6 digits with diagonalisation over all states of small chains; free ends are lattice values (floating point).
Values of the alternating maximal correlation	Reading	Family values copied from the closed formula for the two-state telegraph chain (floating point).
The algebra of the steps	Proved here, Lean	Theorems `split_poincare_le`, `reduced_two_state_value`, `two_proj_var`, `box_two_family`, `box_shift_average`, `kg_constant`, `H4_two_scale`, `H4_gap_form`, `H3_H4_compose`, `K_ff_C2`, `K_ff_C4`, `power_H_of_kink_ratio` (the assumptions are the rows of Table 2 and of the text). The list of statements is in 23-13.

Words used in this chapter — splitting of the variance, local Poincaré, transport of the mean difference, the four double-well conditions, the First-Step Upper-Bound Proposition, phase (phase without kinks), segment, tube, core, box dynamics (box reduction), maximal correlation, alternating maximal correlation, the Two-Projection Lemma, Efron–Stein inequality, reduced kink gas, cross term, dependency graph. Their meanings are in A-02 Glossary.

Next chapter to read: 23-06 The in-phase local Poincaré condition — turns the Poincaré constant of the measure of the phase without kinks into node propositions at six values of $\hbar$, by a two-scale construction of kept sites and bridges, and describes the method and cost of filling in between the nodes.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time the in-phase local Poincaré condition, the Six-Node In-Phase Poincaré Proposition, six nodes · Kiichi, Shiori and Rin

The in-phase local Poincaré condition

This chapter is this book's own investigation. The established knowledge it builds on is in 2-05 and 2-07.

What this chapter shows — The in-phase local Poincaré condition is the statement that the Poincaré constant of the measure restricted to the phase without kinks (configurations in which the path stays on the side of one valley) is $O(1)$ uniformly in $\hbar$ and in the length of the path. In this chapter we turn the version on the ring into the Six-Node In-Phase Poincaré Proposition, by a two-scale construction that splits the path into kept sites and bridges of 11 sites between them. The Six-Node In-Phase Poincaré Proposition holds at six nodes of $\hbar$, $\{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$, and its key inequality (margin on the coarse side ≥ 0.067) is computer-assisted. At the end we list the method and cost of filling in the $\hbar$ between the nodes, and what remains that needs other methods, such as $\hbar < 0.005$ and large end values.

Prerequisite chapters — 23-05 (the four conditions and the Double-Well Relaxation-Time Upper-Bound Proposition, the format of the table of the five grades and the dependency graph), 23-04 § 06 (the plateau at the top and the convex core), 1-10 § 07– § 09 (Poincaré inequalities). For how to read computer-assisted results, see 23-08.

Said first — § 01 is a summary readable at [Y3], and § 02 onwards is at the research level [Res]. You can read only the summary and go on to 23-07. The Six-Node In-Phase Poincaré Proposition is not a proof of the in-phase local Poincaré condition. The $\hbar$ outside the six nodes, end values > 2.5, and the segment version used inside the box remain as assumption rows.

The order of this chapter

1. Summary [Y3] — statement, construction, the Six-Node In-Phase Poincaré Proposition, what remains
2. The statement and the measure of the phase — what is hard
3. The two-scale construction — kept sites, bridges, nesting, the criterion on the coarse side, the algebra of the rate
4. The Six-Node In-Phase Poincaré Proposition — the margins at the six nodes (Table 1) and the constant

5. The table of the five grades — the rows of the Six-Node In-Phase Poincaré Proposition (Table 2)
6. The $\hbar$ between the nodes — method and cost (Tables 3 and 4)
7. The remaining rows — what needs other methods (Table 5)
8. The dependency graph

01

Summary [Y3]

Statement. On the lattice of paths of the double well ($\varepsilon = \frac{1}{2}$), consider the measure $\mu^{(+)}$ restricted to $x_i \geq 0$ at every site. The in-phase local Poincaré condition says "$C_P(\mu^{(+)})$ is at most a constant independent of $\hbar$ and of the length". Bakry–Émery on the convex core works only deep in a valley ($x_i \geq a > 1/\sqrt{3}$), and exponentially many excursions within the valley fit in an interval of length ξ, so this measure is not convex (23-04 § 06).

Construction. Keep one site in every 12 (the coarse side), and treat the 11 sites between them as bridges with both ends fixed (the fine side). Split the variance into "the conditional variance with the kept sites fixed" and "the variance of its expectation". The fine side is controlled by splitting each bridge further into three small bridges and two sites, and the coarse side is judged on the chain of kept sites by the gap at one site and the strength of the coupling between neighbours. The key to the criterion on the coarse side is the margin inequality

$$\underbrace{\hbar\rho}_{\text{gap at one site}} - 2 \underbrace{\hbar\kappa}_{\text{coupling to neighbours}} > 0 \quad (1)$$

, and interval arithmetic is used to check it for every combination of end values.

The Six-Node In-Phase Poincaré Proposition. At each of the six nodes of $\hbar$, the margin ≥ 0.067 is checked computer-assisted for all neighbour and end values in $[0, 2.5]$, and on top of the remaining rows (assumptions and known theorems) $C_P(\mu^{(+)}) \leq C$ with $C \leq 1.6 \times 10^{10}$ (from paper and interval-arithmetic inputs only). With numerical readings as input, $C \leq 658$. Neither depends on the length of the ring.

What remains. For the $\hbar$ between the nodes, there is a scheme that fills in subinterval by subinterval by making $\beta = 1/\hbar$ a third direction of tilting, and its cost is estimated at tens of hours on one core. $\hbar < 0.005$ needs infinitely many subintervals and so needs another method (asymptotics as $\hbar \to 0$). End values > 2.5, the sharpness of the constant, the scrap version used inside the box, and checking the bibliography of the cited theorems also remain.

02

The statement and the measure of the phase — what is hard

The measure of the phase. From the lattice action on a ring of $N = 12n$ sites ($n \geq 3$),

$$S = \sum_i \frac{(x_{i+1} - x_i)^2}{2\varepsilon} + \varepsilon \sum_i V(x_i), \qquad V(x) = \frac{(x^2-1)^2}{8}, \qquad \varepsilon = \frac{1}{2} \quad (2)$$

we form the measure $\mu^{(+)} \propto e^{-S/\hbar} \mathbf{1}\{x_i \geq 0 \ \forall i\}$ restricted to configurations with $x_i \geq 0$ at every site. The orthant is convex, so this measure is the equilibrium of Langevin dynamics reflected at the wall $x_i = 0$ (Neumann condition). The Dirichlet form is $\mathcal{E}(f) = (\hbar/\varepsilon) \sum_i \mathbb{E}|\partial_i f|^2$. The path stays on the side of the valley $x = 1$ and has no kinks. This is "the phase without kinks".

The in-phase local Poincaré condition (ring version)

There exist $\hbar_0 > 0$ and $C < \infty$ such that $\mathrm{Var}_{\mu^{(+)}} f \leq C\, \mathcal{E}(f)$ for all $\hbar \in (0, \hbar_0]$ and all n.

The version used inside the box (the segment local Poincaré assumption used in the step inside the box, 23-05 § 04) uses, instead of the ring, segments of length $\leq C\xi$ with values at both ends, and asks for uniformity in the end values as well.

What is hard. Three naive roads all lose a factor exponential in $\hbar$.

- The convex core: restricting to $x_i \geq a > 1/\sqrt{3}$ gives $C_P \leq 2/(3a^2 - 1)$ by Bakry–Émery, but the action of an excursion out of the core (0.1535 at $a = 1/\sqrt{3}$) is smaller than the kink action $2/3$, and $e^{(S_0 - c_a)/\hbar}$ excursions fit in an interval of length ξ Proved here (23-04 § 06).
- Global perturbation (of the Holley–Stroock type): the oscillation of the difference from a convex action is proportional to the length of the path, and the constant is of the type $e^{T\delta/\hbar}$ Cited (the Holley–Stroock perturbation lemma; type checked in the restatement as Proposition 6.1 of Ledoux).
- Perturbation site by site: the non-convexity per site is $\varepsilon/(8\hbar)$, but the coupling between neighbours $(x_{i+1} - x_i)^2/(2\varepsilon)$ is strong, so sites cannot be detached and treated one at a time Reading .

The numbers do not contradict the in-phase local Poincaré condition. Measuring the integrated autocorrelation time of the magnetisation for Langevin dynamics on the ring with a wall gave $0.86/1.40/1.50/1.39$ at $\hbar = 0.3/0.15/0.1/0.07$, sizes that do not depend on $\hbar$ Reading (the autocorrelation of a single observable gives only a lower bound on the Poincaré constant, and is not evidence for an upper bound).

03

The two-scale construction

Kept sites and bridges. Keep the sites $y_j := x_{12j}$, and treat the 11 sites $x_{12j+1}, \dots, x_{12j+11}$ between y_j and y_{j+1} as a bridge with both ends fixed at y_j, y_{j+1}. Splitting the variance by the kept sites y gives

$$\mathrm{Var}_\mu(+)\ f = \mathbb{E}\big[\mathrm{Var}(f \mid y)\big] + \mathrm{Var}\big(\mathbb{E}[f \mid y]\big) \quad (3)$$

(the variance-decomposition row: the decomposition of the variance is elementary; the regularity and density of the functions used are assumptions). Because the interaction is nearest-neighbour, the bridges are independent once y is fixed, and the marginal measure of y is a nearest-neighbour chain of the kept sites (the nearest-neighbour chain row: the Markov property).

The fine side: nesting inside the bridge. A bridge of 11 sites is not convex (the smallest eigenvalue of the Hessian of the bridge is $(1/\varepsilon)\lambda_{\min}(L_{11}) - \varepsilon/2 = -0.11$). So we keep sites 4 and 8 of the bridge as one more level and split it into three small bridges (3 sites) and two sites (x_4, x_8).

- A small bridge of 3 sites is log-concave: Hessian $\geq (1/\varepsilon)\lambda_{\min}(L_3) - \varepsilon/2 = 2(2-\sqrt{2}) - \frac{1}{4} \approx 0.92$. Bakry–Émery gives the gap $\rho_f = 0.922$ Proved here (lower bound on the curvature) Cited (Bakry–Émery on a convex region).
- The cross term between the small bridges and the two sites is $\kappa_f = 1.98$ by the Brascamp–Lieb inequality Cited .
- The gap of the marginal measure of the two sites (x_4, x_8) is $\lambda_c \geq 1.3 \times 10^{-4}$ (end values $[0,4]^2$, $\hbar \in (0, 0.2]$) Computer-assisted (the two-site gap row).

Combining the three by the nested two-scale algebra, the gap of the bridge is at least the smaller root of $t^2 - (\rho_f + \lambda_c + \kappa_f^2/\rho_f)t + \rho_f \lambda_c = 0$, and $\lambda_S \geq 2.3 \times 10^{-5}$ Proved here (arithmetic of the root).

The coarse side: the gap at one site and the coupling. The one-dimensional conditional measure of y_j with the neighbour values c, d fixed is $\propto e^{-U(t)/\hbar}$, with

$$U'(t) = \varepsilon V'(t) + \frac{2t - e(c,t) - e(d,t)}{\varepsilon}, \qquad e(a,b) := \mathbb{E}_{a,b}\, x_{11} \quad (4)$$

($\mathbb{E}_{a,b}$ is the expectation over the bridge with ends a, b). Write $\hbar\rho(c,d)$ for its Poincaré gap multiplied by $\hbar$, and let the strength of the coupling between the two ends of the bridge be

$$\hbar\kappa(a,b) := \frac{\mathrm{Cov}_{a,b}(x_1, x_{11})}{\varepsilon^2 \hbar} = \frac{1}{\varepsilon}\,\partial_a e(a,b) \ \geq\ 0 \quad (5)$$

(the second equality is the identity obtained by differentiating $\partial_a \log Z(a,b)$ in b Proved here ; non-negativity follows from e being increasing in a Cited (the FKG correlation inequality)). Using the Poincaré Version of Helffer's Criterion in the Gershgorin form on the ring, the gap of the chain on the coarse side is at least the margin

$$\lambda := \min_{c,d} \hbar\rho(c,d) - 2 \max_{a,b} \hbar\kappa(a,b) \quad (6)$$

Cited (the Poincaré Version of Helffer's Criterion, Proposition 3.1 of Ledoux) Proved here (the Gershgorin form) Lean `ring_gershgorin` (the inequality for quadratic forms on the ring; assumption: $\kappa \geq 0$). (1) means $\lambda > 0$.

The cross term and the algebra of the rate. The strength of the cross term linking the fine side and the coarse side is $\kappa^2 = 2(v/\hbar)\lambda_S/\varepsilon^2$, with $v/\hbar \leq 2\big(644/1455 + (60/97)^2/\lambda_c\big) \leq 5888$ (the crossing row) Proved here (arithmetic) Lean `T4_paper_numeric` (assumption: $\lambda_c \geq 1.3 \times 10^{-4}$) Cited (Brascamp–Lieb). Both λ and λ_S are in units of the gap for the flat Dirichlet form $\int |\nabla f|^2$ multiplied by $\hbar$. Then if $0 \leq t < \lambda$ satisfies

$$t\,\kappa^2 \;\leq\; \lambda_S(\lambda_S - t)(\lambda - t) \quad (7)$$

, we have $C_P(\mu^{(+)}) \leq \varepsilon/t$ Proved here Lean `H1_assembly` (algebra with the assumptions named: decomposition (3) [`hT5`], the fine side [`hFine`; ρ in Lean is λ_S in the text], the conclusion of the criterion on the coarse side [`hLedoux`], the margin $\lambda \leq \hbar\rho - 2\hbar\kappa$ [`hT1`], $0 < \lambda$, the cross term [`hT4`], the rate condition (7)).

The whole construction, as a tree of decompositions of the variance, is as follows.

```
Var f = E[Var(f|y)] + Var(E[f|y])                                  (varia
 ├ fine side: bridges of 11 sites with y fixed
 │    └ nesting {4,8}: small bridges BE ρ_f=0.922, cross term BL κ_f=1.9
 ├ coarse side: ring of kept sites (nearest-neighbour chain row) ── Poin
 │    └ margin row: gap at one site ħρ (Muckenhoupt, FKG band, tail) − 1
 └ crossing: κ² = 2(v/ħ)λ_S/ε² (crossing row ← two-site gap row)
 → rate condition tκ² ≤ λ_S(λ_S−t)(λ−t) (H1_assembly) → C_P ≤ ε/t
```

04

The Six-Node In-Phase Poincaré Proposition — the margins and the constant

The Six-Node In-Phase Poincaré Proposition (0+1 dimensions, ring version, with a table of assumptions)

Let $V = (x^2-1)^2/8$, $\varepsilon = \frac{1}{2}$, the ring $N = 12n$ ($n \geq 3$), the measure of the phase $\mu^{(+)}$ with the wall $x_i \geq 0$, the kept sites $y_j = x_{12j}$, and the nested sites inside the bridge $\{4,$

8}. Under the "Assumption" and "Cited" rows of Table 2, at each point $\hbar \in \{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$,

$$\mathrm{Var}_{\mu}(+)\, f \;\le\; C\,\mathcal{E}(f), \qquad C \le 1.6 \times 10^{10} \quad (8)$$

holds independently of n. With numerical readings as input, $C \le 658$.

The key row is the margin (6), and the values of Table 1 are obtained at the six nodes for every combination of neighbour and end values in $[0, 2.5]$.

Computer-assisted Table 1 The margin on the coarse side at each node (neighbour and end values $[0, 2.5]^2$; $\hbar\rho$ is a lower bound and $\hbar\kappa$ an upper bound, values rounded outwards)

$\hbar$	$\hbar\rho \ge$	$\hbar\kappa \le$	**Margin** $\hbar\rho - 2\hbar\kappa \ge$
0.2	0.7755	0.0358	0.7039
0.1	0.7140	0.0664	0.5812
0.05	0.8121	0.1222	0.5677
0.02	0.8846	0.2636	0.3574
0.01	0.8670	0.39996	0.0670
0.005	0.92182	0.41868	0.0844

The interval arithmetic is implemented by adding outward widths to float64 operations (rounding of non-negative sums is widened outwards by a relative 10^{-10}, interval arithmetic operations by a relative 10^{-12}, weights below e^{-700} are replaced from above, and the region $\varepsilon V/\hbar > 690$ is covered by a tail upper bound). It is not rounding with a direction specified for each operation, but an added width that presupposes correct IEEE rounding. This premise is written in the same row, row [01] of Table 2. The arithmetic that gives the margin ≥ 0.067 from the six pairs of values is Lean `T1_margin_six_nodes` (no assumptions, numbers), and the step from the interval output to the margin at one node is `T1_node_margin` (assumptions: the two inequalities of the interval output and $0.067 \le r - 2k$).

Tools that extend the intervals to the continuum between nodes. Table 1 gives values "for every combination of neighbour and end values", and to pass from computations on finitely many boxes to the continuum we use three lemmas.

- The $\hbar\rho$ side: Muckenhoupt's estimate, which squeezes the Poincaré constant of a one-dimensional measure by a Hardy-type quantity B as $B \le C_P \le 4B$ Cited ; a lemma that builds a band from the values at the nodes to the continuum of neighbours by the monotonicity (FKG) of $e(c, t)$ in the neighbour value Cited (FKG); and a lemma for the tail on the side of large t (the continuous version of Holley's comparison and a tilted Gaussian bridge) Assumption (the theorem number of the continuous version has not been checked in the original source).

- The $\hbar\kappa$ side: the Tilted Second-Order Taylor Lemma, which tilts the measure at the centre of a box of end values and controls the whole box by the second-order Taylor expansion of the cumulant generating function and a Hölder remainder Proved here (the Taylor remainder and Hölder's inequality). This formula is exact whatever constant is chosen for the centring.
- The smaller of two upper bounds is also an upper bound, and the reflection symmetry $\hbar\kappa(a,b) = \hbar\kappa(b,a)$ covers half Proved here Lean `min_of_two_upper`, `symm_upper` (assumptions: two upper bounds / symmetry and a one-sided upper bound).

The moves that made the margin positive at $\hbar = 0.01$ and 0.005. When $\hbar$ is small the upper bound on $\hbar\kappa$ swells, and in a coarse computation the margin was negative at $\hbar = 0.01$, with 0.634. What limits it differs from place to place, and three moves bring it down. (i) At the corner of the ends $(0,0)$, the centring constant of the Tilted Second-Order Taylor Lemma is retaken for each subrectangle as the median of that range (the formula is exact for any centre, so only the upper bound goes down). (ii) The second-order term acts as the square of the box width δ, so δ is halved. (iii) At the large end $(2.5, 2.5)$, the bridge jumps far into the well in one step, so the quadrature error is the limit, and the quadrature step is halved. At $\hbar = 0.005$, refining the quadrature on the $\hbar\rho$ side raises the lower bound from 0.6945 to 0.92182. The last two rows of Table 1 are the results of these moves Computer-assisted .

The constant. $C \le 1.6 \times 10^{10}$ in (8) is the ε/t that follows from $t = 3.2 \times 10^{-11}$ satisfying the rate condition (7) with the paper and interval-arithmetic inputs ($\lambda \ge 0.067$, $\lambda_S \ge 2.3 \times 10^{-5}$, $v/\hbar \le 5888$) Proved here (arithmetic) Lean `H1_rate_paper_six_nodes` (no assumptions, numbers). Taking the gap of the two sites to be the numerical $\lambda_C = 0.57$ gives $\lambda_S \ge 0.093$, $v/\hbar \le 1.0015$, then $t = 7.6 \times 10^{-4}$ and $C \le 658$ Reading (the input is a floating-point reading) Lean `H1_rate_numeric_six_nodes` (no assumptions, numbers). Most of the difference between the two values comes from the weakness of the paper value of λ_C.

05

The table of the five grades — the rows of the Six-Node In-Phase Poincaré Proposition

We put the inputs of the Six-Node In-Phase Poincaré Proposition into one table in the format of the five grades. Items whose paper proof is placed in this book are "Proved here", those whose proof is not are "Assumption", interval arithmetic is "Computer-assisted" with the premises of its implementation written in the same row, and cited theorems are split into "Cited" or "Assumption" according to the level to which their bibliography has been checked. The numbers in the row column are the numbers of the nodes of the dependency graph in § 08, and "—" marks a row not in the graph.

Table 2 The rows of the Six-Node In-Phase Poincaré Proposition ("six nodes" in the range of $\hbar$ means $\{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[01]	Margin row: margin $\hbar\rho - 2\hbar\kappa \geq 0.067$ (neighbours and ends $[0, 2.5]^2$)	Computer-assisted	§ 04 Table 1. float64 + outward widths (relative 10^{-10}, 10^{-12}), presupposing correct IEEE rounding; tails covered by upper bounds	Six nodes	`T1_margin_six_nodes` (numbers), `T1_node_margin` (assumption: the interval output)
[02]	The Tilted Second-Order Taylor Lemma (second-order Taylor of the tilt and Hölder remainder, any centre)	Proved here	§ 04 (Taylor remainder, Hölder's inequality)	Independent of $\hbar$	—
[03]	Squeezing the one-dimensional Poincaré constant, $B \leq C_P \leq 4B$ (Muckenhoupt)	Cited	Statement checked in the restatement as Theorems 1 and 2 of Barthe–Roberto	Independent of $\hbar$	—
[04]	Monotonicity of $e(c, t)$ and $\hbar\kappa \geq 0$ (the MTP_2 correlation inequality)	Cited	Proposition 2.1 of Karlin–Rinott (checked in the restatement as Proposition 3.5 of Fallat et al.)	Independent of $\hbar$	—
[05]	The continuous version of Holley's comparison used for the tail lemma	Assumption	§ 04 (the theorem number in the original source is unchecked)	Six nodes	—
[06]	The Poincaré Version of Helffer's Criterion (gap of the chain on the coarse side $\geq \lambda$)	Cited	The Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux; checked in the original text)	Independent of $\hbar$	—
[14]	The Gershgorin form on the ring	Proved here	§ 03 (6)	Independent of $\hbar$	`ring_gershgorin` (assumption: $\kappa \geq 0$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[11]	Nearest-neighbour chain row: the marginal of the kept sites is a nearest-neighbour chain	Proved here	§ 03 (the Markov property)	Independent of $\hbar$	—
[07]	Two-site gap row: $\lambda_c \geq 1.3 \times 10^{-4}$ (ends $[0,4]^2$)	Computer-assisted	§ 03. Interval arithmetic (multiprecision intervals and float64 + outward widths)	$(0, 0.2]$	—
[08]	The Poincaré–Miranda theorem used to cover the boxes of the two-site gap row	Assumption	The statement in the original source is unchecked (bibliographic only)	$(0, 0.2]$	—
[15]	Nesting: small bridge $\rho_f = 0.922$ (Bakry–Émery)	Proved here	§ 03 (lower bound on the curvature; Bakry–Émery is Cited via Theorem 1.2 of Kolesnikov–Milman)	Independent of $\hbar$	—
[09]	Nesting: cross term $\kappa_f = 1.98$ (also used in the crossing row; Brascamp–Lieb)	Cited	Theorem 4.1 of Brascamp–Lieb (checked in a restatement); convex regions by Kolesnikov–Milman	Independent of $\hbar$	—
[16]	Crossing row: arithmetic of $v/\hbar \leq 5888$ and $\lambda_S \geq 2.3 \times 10^{-5}$	Proved here	§ 03	$(0, 0.2]$	`T4_paper_numeric` (assumption: $\lambda_c \geq 1.3 \times 10^{-4}$)
[12]	Variance-decomposition row: decomposition of the variance (3)	Proved here	§ 03	Independent of $\hbar$	—
[10]	Variance-decomposition row: regularity and density of the functions used	Assumption	§ 03 (not checked in the original source)	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[17]	The algebra of the rate (7) $\Longrightarrow C_P \leq \varepsilon/t$	Proved here	§ 03	Independent of $\hbar$	`H1_assembly` (the assumptions are the seven named ones of § 03)
[18]	$C \leq 1.6 \times 10^{10}$	Proved here	§ 04 (arithmetic from the values of [01], [07] and [16])	Six nodes	`H1_rate_paper_six_nodes` (numbers)
—	$C \leq 658$	Reading	§ 04 ($\lambda_c = 0.57$ is floating point)	Six nodes	`H1_rate_numeric_six_nodes` (numbers)
—	Margin row (large-end side): neighbours and ends > 2.5	Assumption	§ 07	—	—
—	Margin row (between nodes): the $\hbar$ between the nodes	Assumption	§ 06 (method and cost)	Outside the nodes of $(0.005, 0.2)$	—
—	Margin row (small ħ): $\hbar < 0.005$	Assumption	§ 07	—	—
—	Two-site gap row (large-end side): ends of the two sites > 4	Assumption	§ 07	—	—
—	The segment local Poincaré assumption: from the ring to segments (with end values)	Assumption	§ 07 (the transfer to the chain version of the criterion is a reading)	—	—

The "Computer-assisted" rows [01] and [07] of Table 2 are inequalities at the computed nodes and within the computed ranges. They say nothing about $\hbar$ or end values outside those ranges, and that is what the five "Assumption" rows at the end of Table 2 (with "—" in the number column) take on.

06

The $\hbar$ between the nodes — method and cost

The Six-Node In-Phase Poincaré Proposition is a statement at six points. The in-phase local Poincaré condition is a statement on the continuum of $\hbar$, so the gaps between the points must be filled. This section describes the method for that, the result of trying the method on one subinterval, and an estimate of the cost for the whole range.

(a) β as a third direction of tilting. $\beta = 1/\hbar$ enters the measure $e^{-\beta S}$ only as an exponent of the weight, so we regard an interval $[\hbar_1, \hbar_2]$ of $\hbar$ as an interval $[\beta_{\mathrm{lo}}, \beta_{\mathrm{hi}}]$ of β, and add a third direction (the observable $-S$) to the two directions of tilting in the end values in the Tilted Second-Order Taylor Lemma. β enters the margin (6) in six places (the weight, the width of the tilt, the prefactor, the quadrature step, the tail, the monotonicity band), and for each it is determined at which end of the interval the worst case lies Proved here . A guide for the ratio of one step, $r = \beta_{\mathrm{hi}}/\beta_{\mathrm{lo}}$, is $r \approx 1.17$, from the fluctuation of the action $\mathrm{Var}(\beta S)$ (about half the degrees of freedom of the bridge, near $11/2$ Reading), and $\hbar \in [0.005, 0.2]$ can be covered by 24 subintervals Proved here $(0.2/1.17^{24} \le 0.005)$ Lean `nodes_reach_117` (no assumptions, numbers).

(b) The chord form breaks. With the end values (a, b) fixed, $f(\beta) = \hbar\kappa(a, b; 1/\beta)$ is smooth in β, and a lower bound on the second derivative $f'' \ge -C$ gives the chord upper bound

$$f(\beta) \;\le\; \max\bigl(K2(\beta_{\mathrm{lo}}), K2(\beta_{\mathrm{hi}})\bigr) + \frac{(\beta_{\mathrm{hi}} - \beta_{\mathrm{lo}})^2}{8} \sup|f''| \quad (9)$$

Proved here Lean `chord_beta_upper` (assumptions: $f \in C^2$, $f'' \ge -C$, $0 \le C$, $\beta_{\mathrm{lo}} \le \beta_{\mathrm{hi}}$, upper bounds at the ends). But f'' is a connected cumulant, and controlling it by a product of absolute values (Hölder) gives 5×10^3–1.8×10^6 times the true value Reading , so at $r = 1.17$ the remainder is 0.25–0.64, far beyond what is allowed Lean `holder_r117_exceeds` (numbers).

(c) A step-by-step scheme (a scheme that carries $\hbar$ as an interval). Compute the unnormalised integral $J_X(\beta) = \int X e^{-\beta H}\,dx$ ($X \ge 0$ is the weight of the product of end observables) at three points of β, and interpolate by a quadratic. The interpolation error is $|\omega(\tau)|/6 \cdot \sup|J_X'''|$, with $\max|\omega| = 2\Delta^3/(3\sqrt{3})$ Proved here (ω is the cubic that vanishes at the three nodes. Choosing K so that $g(t) = J_X(t) - p_X(t) - K\omega(t)$ vanishes at the three interpolation points and at τ, four points in all, Rolle's theorem used three times gives a zero of g''', and $K = J_X'''(\eta)/6$). The third derivative is controlled by $|y|^3 \le c_3(\lambda_+)e^{\lambda+y} + c_3(\lambda_-)e^{-\lambda-y}$ and the log-convexity of J_X, through the maximum at the ends of the integrals at four shifted values of β Proved here . Splitting the interval of β further into small pieces, on each piece the expectation is a linear-fractional function, and its value lies between the values at the two ends of the piece, so it can be enclosed by the interval hull Proved here Lean `mobius_between` (assumption: the denominator is positive at both ends of the piece), `lebesgue_le` (assumption: $|u| \le 1$), `cube_le_exp_split` (assumption: $\lambda_\pm > 0$). This scheme never uses monotonicity. The remainder enters as a third-order quantity of raw integrals that needs no cancellation, and the Hölder loss of (b) does not appear.

(d) A trial on one subinterval. Table 3 is the result of running (c) in interval arithmetic on $\hbar \in [0.008547, 0.01]$ ($r = 1.17$) and the rectangle of end values $[0, 0.18]^2$ Computer-assisted (the same implementation premises as Table 1).

Computer-assisted Table 3 Trial on one subinterval ($\hbar \in [0.008547, 0.01]$, rectangle $[0, 0.18]^2$, values of the largest box)

Quantity	Value	How to read it
Upper bound at the point $\hbar = 0.01$ (cross-check)	0.18996	Agrees with the computation of Table 1 to five digits
Upper bound on the subinterval (including the remainder)	0.23638	Swelling from the point upper bound $+0.0464$
Share of the remainder	0.00968	Within the uniform allowance 0.0335
Breakdown of the swelling	–	Growth of the node upper bound with β 48%, interpolation and hull of the pieces 31%, remainder 21%

The allowance. What is needed on the subinterval at $\hbar = 0.01$ is "point upper bound + swelling $\leq \hbar\rho/2 = 0.4335$" in every box. If the same amount is allowed to every box, the uniform allowance set by the worst box (upper bound 0.39996) is $0.4335 - 0.39996 \approx 0.0335$, and the swelling 0.0464 of Table 3 exceeds it. On the other hand, the allowance for the boxes in the rectangle of Table 3 is $0.4335 - 0.19 \approx 0.24$, and the swelling falls within it.

What limits it. The limit is not the remainder but the growth of the node upper bound itself with β (the quadrature width widens with β). The upper bound 0.39996 of the worst box is mostly quadrature width, and the floating-point value for the same box is 0.0076 Reading . Recomputing the band containing this box with finer quadrature brings the upper bound of the box down to 0.0751 Computer-assisted , but the maximum over the whole range only comes down to 0.39971. This is because about 32,000 boxes with upper bounds above 0.19 remain at the coarse setting in a frame along the four sides of the square of end values.

Cost. Table 4 is the estimate for the whole range $\hbar \in [0.005, 0.2]$ (computation time on one core; all are estimates, none has been run yet). The cost of one subinterval is about 3.4 times that of a point computation (measured in the trial of Table 3), and the smaller $\hbar$ is, the larger the cost of one point.

Table 4 Estimated cost of filling in between the nodes (the $\hbar\kappa$ side; the $\hbar\rho$ side is needed separately)

Road	Premise	r	Subintervals	Computation time
After recomputing the frame with finer quadrature	That the point upper bounds in the frame come down to 0.318 or below (not yet computed)	1.17	24	About 23–39 hours (+ 30–50 minutes to recompute the frame)
Point upper bounds as they are (a model in which the swelling acts additively)	The swelling of Table 3 does not depend on the box	1.12	33	About 22 hours

Road	Premise	r	Subintervals	Computation time
Point upper bounds as they are (a model in which the swelling acts multiplicatively)	The ratio 1.363 of Table 3 does not depend on the box	1.04	94	About 63 hours

Which of the two models is right cannot be decided from a trial on one rectangle. The road that recomputes the frame first passes at $r = 1.17$ regardless of the model if the upper bound in the frame falls below 0.318, so it also narrows the range of the cost estimate. The $\hbar\rho$ side (a one-dimensional measure, 23 degrees of freedom) is read by the same guide as $r \approx 1.12$ and 33 subintervals Reading .

To be precise, Table 3 is the result for one rectangle of one subinterval, and does not close that subinterval. To close a subinterval, every rectangle in it must be computed by the scheme of (c).

07

The remaining rows — what needs other methods

Table 5 lists the rows that remain in going from the Six-Node In-Phase Poincaré Proposition to the in-phase local Poincaré condition, and to the segment local Poincaré assumption used inside the box. The cost column is filled only where there is an estimate.

Table 5 The remaining rows of the in-phase local Poincaré condition

Row	Content	Current state	What is needed
Remaining item between nodes	The $\hbar \in (0.005, 0.2)$ between the nodes	The scheme of the method of § 06 exists, and has been tried on one rectangle of one subinterval	Recomputing the frame, and computing all subintervals and all rectangles (Table 4). The $\hbar\rho$ side by the same scheme too
Small-ħ remaining item	$\hbar < 0.005$	The computer-assisted scheme cannot be used: with the ratio r fixed, the number of subintervals $\ln(0.005/\hbar)/\ln r$ grows without bound as $\hbar \to 0$, and the cost of each one grows too	An argument uniform in the asymptotics $\hbar \to 0$ (another method)
Large-end-value remaining item	Neighbour and end values > 2.5	The road of covering by monotonicity is closed: FKG gives an order of likelihood ratios, and the gap is not monotone (the Gap Non-Monotonicity Proposition). Also $e(c, t)$ is not bounded in c (it grows at the rate $c^{3^{-11}}$)	A paper argument that the one-dimensional U becomes uniformly convex for large neighbours. The finite range $[2.5, 4]$ is expected to be covered by the same computation if $\hbar \geq 0.05$

Row	Content	Current state	What is needed
Sharp-constant remaining item	Sharpness of the constant	1.6×10^{10} with paper inputs, 658 with numerical inputs. The difference is carried by the weakness of the paper value of the gap of the two sites λ_c	Rebuilding the two-site gap row. The statement of the in-phase local Poincaré condition does not need the size of the constant (only that it does not depend on $\hbar$)
Segment-version remaining item	The segment version (the segment local Poincaré assumption)	The Six-Node In-Phase Poincaré Proposition is a statement on the ring. Inside the box, segments with values at both ends are needed	Transfer to the chain version of the criterion (fewer neighbours than the ring). At present at the level of a reading
Bibliographic remaining item	Bibliography of the cited theorems	Muckenhoupt and FKG have their statements checked in restatements in secondary sources, Ledoux in the original. The theorem number of Holley's continuous version and the original source of the Poincaré–Miranda theorem are unchecked	Obtaining the original sources. If they cannot be obtained, the rows stay "Assumption" ([05] and [08] of Table 2)

The Gap Non-Monotonicity Proposition is a small proposition of a negative type. The variance $p(1-p)$ of the measure $(1-p, p)$ on two points $\{0, 1\}$ is not monotone when p is raised in the order of likelihood ratios $(3/16 < 1/4 > 3/16)$ Proved here Lean `tilt_variance_not_monotone` (no assumptions, numbers). The same holds for the family obtained by tilting the symmetric double well linearly, and FKG alone does not give "the gap is monotone in the neighbour".

08

The dependency graph — the Six-Node In-Phase Poincaré Proposition

We turn the rows of Table 2 into a list of nodes in the same format as 23-05 § 07. The numbers of the premises are always placed smaller than the number of the conclusion.

```
[01] CMP  margin row: margin ≥ 0.067 (neighbours and ends [0,2.5]²)
[02] PRV  Tilted Second-Order Taylor Lemma (any centre)
[03] CIT  Muckenhoupt squeeze B ≤ C_P ≤ 4B
[04] CIT  MTP₂ correlation inequality (monotone e, ħκ ≥ 0)
[05] ASM  continuous version of Holley's comparison (tail lemma)
[06] CIT  Poincaré version of Helffer's criterion (Ledoux Prop. 3.1)
[07] CMP  two-site gap row: λ_c ≥ 1.3×10⁻⁴ (ends [0,4]²)
[08] ASM  Poincaré–Miranda theorem
[09] CIT  Brascamp–Lieb (cross term κ_f, crossing row)
[10] ASM  variance-decomposition row: regularity, density
[11] PRV  nearest-neighbour chain row: kept sites form a nearest-neighb
[12] PRV  variance-decomposition row: decomposition of the variance (3)
[13] PRV  continuum statement of the margin λ ⟸ [01][02][03][04][05]
          Lean T1_margin_six_nodes (numbers), T1_node_margin (assumptio
[14] PRV  coarse-side gap ≥ λ (Gershgorin) ⟸ [06][11][13]
          Lean ring_gershgorin (assumption: κ ≥ 0)
[15] PRV  nesting λ_S ≥ 2.3×10⁻⁵ ⟸ [07][08][09]
[16] PRV  crossing row: v/ħ ≤ 5888 ⟸ [07][09]
          Lean T4_paper_numeric (assumption: λ_c ≥ 1.3×10⁻⁴)
[17] PRV  algebra of the rate C_P ≤ ε/t ⟸ [12][14][15][16]
          Lean H1_assembly (assumptions: hT5, hFine, hLedoux, hT1, 0 < 
[18] PRV  Six-Node In-Phase Poincaré Proposition: C ≤ 1.6×10¹⁰ ⟸ [17]
          Lean H1_rate_paper_six_nodes (numbers)
```

In every row with "⟸" the numbers on the right are smaller than the one on the left, so there is no circle. Among the leaves, "Assumption" covers [05][08][10] and "Computer-assisted" covers [01][07]. $C \leq 658$ from numerical inputs (the row of Table 2 with numerical inputs) puts reading values into the algebra of [17], and does not enter the conclusion [18] of this graph. The five rows at the end of Table 2 are not premises of the Six-Node In-Phase Poincaré Proposition but rows for going from the Six-Node In-Phase Poincaré Proposition to the in-phase local Poincaré condition and the segment local Poincaré assumption, and in the integrated graph (23-12) they come between [18] and the node of the in-phase local Poincaré condition.

The construction of this chapter is one way of handling one condition within the first step, with the diffusive-relaxation-time assumption as the foothold. Other views are treated in Part 24.

What this chapter can say

What is said	What is not said
Computer-assisted At the six nodes, margin ≥ 0.067 for all neighbours and ends in $[0, 2.5]^2$ (Table 1, with the premises of the implementation). Gap of the two sites $\lambda_c \geq 1.3 \times 10^{-4}$	The $\hbar$ between the nodes, $\hbar < 0.005$, $\hbar > 0.2$. Neighbours and ends > 2.5
Proved here The algebra of the two-scale construction and the rate condition (7), the constant 1.6×10^{10} of the Six-Node In-Phase Poincaré Proposition (on the assumption rows of Table 2) Lean (the assumptions are in Table 2)	The in-phase local Poincaré condition itself (a statement uniform on the continuum of $\hbar$). The segment version (the segment local Poincaré assumption)
Cited The Muckenhoupt squeeze, the MTP_2 correlation inequality, the Poincaré Version of Helffer's Criterion, Bakry–Émery, Brascamp–Lieb	Checking the continuous version of Holley's comparison and the Poincaré–Miranda theorem in the original sources ([05] and [08] of Table 2 are Assumption)
Reading $C \leq 658$ with numerical inputs, the autocorrelation time of the phase with a wall, the guide for the step ratio between nodes	Using these as upper bounds
The scheme of the method for filling in between the nodes, the trial on one rectangle of one subinterval (Table 3), and the cost estimate (Table 4)	That even one subinterval has been closed. The times in Table 4 are estimates, not times actually run

Sources and reproduction

Item	Kind	Source / tools
Squeezing the one-dimensional Poincaré constant	Cited	B. Muckenhoupt, Studia Math. 44 (1972) 31–38 (bibliographic data). The statement checked in Theorems 1 and 2 of F. Barthe, C. Roberto, Potential Anal. 29 (2008) 167–193, arXiv:math/0612026.
The MTP_2 correlation inequality, Holley's comparison	Cited, Assumption	Proposition 2.1 of S. Karlin, Y. Rinott, J. Multivariate Anal. 10 (1980) 467–498 (checked in the restatement as Proposition 3.5 of S. Fallat et al., Ann. Statist. 45 (2017), arXiv:1510.01290). R. Holley, CMP 36 (1974) 227–231 (theorem number of the continuous version unchecked).
The Poincaré Version of Helffer's Criterion	Cited	The Poincaré Version of Helffer's Criterion in M. Ledoux, Sém. Probab. XXXV, LNM 1755 (2001) 167–194 (checked in the text). Original: B. Helffer, Ann. IHP Probab. Statist. 35 (1999) 483–508.
Bakry–Émery, Brascamp–Lieb	Cited	D. Bakry, M. Émery, LNM 1123 (1985); H. J. Brascamp, E. H. Lieb, J. Funct. Anal. 22 (1976) 366–389; convex regions: Theorem 1.2 of A. V. Kolesnikov, E. Milman, J. Geom. Anal. 27 (2017), arXiv:1310.2526.
The Poincaré–Miranda theorem	Assumption	W. Kulpa, Amer. Math. Monthly 104 (1997) 545–550 (bibliographic only; statement unchecked).

Item	Kind	Source / tools
Interval values of Tables 1 and 3	Computer-assisted	Interval arithmetic of the record (float64 + outward widths, the premise of row [01] of Table 2). Cross-checked that floating-point values from an independent route (trapezoidal rule, different steps) all lie below the upper bounds at random boxes and the largest box of each rectangle.
Costs of Table 4	Estimate	Estimated from the measurement in the trial of Table 3 (one subinterval is about 3.4 times a point) and a fit of the $\hbar$-dependence of the computation time per node.
The algebra and numbers of the construction	Proved here, Lean	Theorems `H1_assembly`, `ring_gershgorin`, `T1_margin_six_nodes`, `T1_node_margin`, `T4_paper_numeric`, `H1_rate_paper_six_nodes`, `H1_rate_numeric_six_nodes`, `min_of_two_upper`, `symm_upper`, `tilt_variance_not_monotone`, `chord_beta_upper`, `holder_r117_exceeds`, `nodes_reach_117`, `mobius_between`, `lebesgue_le`, `cube_le_exp_split` (the assumptions are in the text and the rows of Table 2). The list of statements is in 23-13.

Words used in this chapter — phase (phase without kinks), kept site, bridge, nesting, Helffer's criterion, margin, node, subinterval, allowance, swelling, frame, segment version (the segment local Poincaré assumption), computer-assisted, interval arithmetic. Their meanings are in A-02 Glossary.

Next chapter to read: 23-07 The map of the saddle-tube condition — seven routes and the seam of the ring — puts on one sheet the map that splits transport along the family of saddles (the saddle-tube condition) into seven routes, and the seam of the ring (the outer chain closes on the whole window, the inner chain has finite coefficients).

The map of the saddle-tube condition — seven routes and the seam of the ring

> This chapter is this book's own investigation. § 02 is a map of open questions; § 07 is guide material.

What this chapter shows — We put the saddle-tube condition (transport along the family of flat saddles) on one sheet, as a map split into seven routes. For each route we give, in one line, what it tries to show, its current grade (proved here, cited, computer-assisted, reading, assumption), what remains, and the chapter that treats it in detail. The second half deals with the place where a kink and an antikink meet on a periodic path (the seam of the ring). The seam is the plateau at the top itself. The outer chain becomes a computer-assisted inequality on the whole computed window of $\hbar$, and the inner chain remains with a finite coefficient above 1.

Prerequisite chapters — 23-05 § 03 (the statements of the four conditions), 23-05 § 04 The step inside the box (where the saddle-tube condition enters the box), 23-04 § 06 (the plateau), 23-00 § 01 (the labels of the five grades).

Said first — The level is [Y3]. This part places the diffusive-relaxation-time assumption (the time over which fluctuations are smoothed out grows as a power of the square of the correlation length, 23-01) at the axis of the study. The diffusive-relaxation-time assumption rephrases the quantities of the problem statement as quantities that can be computed on the lattice and pursued with machine checking, and the saddle-tube condition is one of the four pillars that support its first step. This chapter does not say that the saddle-tube condition is closed. It puts on one sheet which routes have come how far and what remains. Another view is treated in Part 24.

The order of this chapter

1. The statement of the saddle-tube condition and two places — the rows left in the saddle-tube condition by the box split, the tube of the bridge and the seam of the ring
2. The map of the seven routes — grade and what remains, one line each (Table 1)
3. The seam of the ring — the plateau at the top, the outer chain and the inner chain (Table 2)
4. The remaining numbers of the route of the window-constant numerics and their cost — estimates of the number of runs and the time
5. The table of the five grades — the rows of the saddle-tube condition (Table 4)
6. The dependency graph — checking by the numbers that there is no circle
7. A guide to 23-08–23-11

01

The statement of the saddle-tube condition and two places

The statement of the saddle-tube condition is as in 23-05 § 03. We copy it here.

The saddle-tube condition — transport along the family of flat saddles (copied from 23-05)

Measure: the conditional measure of configurations in the tube around the family of kink–antikink pairs (the family of saddles, parametrised by the centre X and separation ℓ of the pair, along which the action is almost flat). **Constant**: the constant of transport of the mean difference along that family from valley to valley, and the Poincaré constant inside the tube. **Uniformity**: uniform in the number of sites of the path (the dimension) and in $\hbar$. For use inside the box it becomes two rows: Poincaré in the tube uniform in tube length $\leq C\xi$ (the saddle-tube Poincaré assumption), and Poincaré of the core inside the tube (the interval that stays near the top) uniform in the core length k (the core Poincaré uniformity assumption).

What the box split leaves in the saddle-tube condition. In the box split of 23-05, the direction along the family of saddles (the kink positions) was taken as the slow variable, so longitudinal transport in that direction is handled by the rows of the gap-inside-the-box condition (the kink-gas gap assumption and the cross-term uniformity assumption) (the note in the step inside the box, 23-05 § 04). What remains in the saddle-tube condition are the **tube and core rows** after the arrangement of kinks is fixed, namely the saddle-tube Poincaré assumption and the core Poincaré uniformity assumption. The map of this chapter is a map of the tools that support these two rows.

Why it is hard. On the core $Q_k = (-x_c, 0)^k$ ($x_c = 0.89$; k sites of boxes, each running from the side of the valley -1 to just short of the top 0), the action is almost flat at the top and the curvature at one site is negative (the summit-curvature shortfall wall of Table 4 in § 05; for details see 23-09 § 06). Convexity tools (Bakry–Émery) do not apply to the whole core, and the core length k grows up to the correlation length, so that the constant is uniform in k must be shown by another means.

Two places — the tube of the bridge and the seam of the ring

The rows of the saddle-tube condition appear in two places where the shape of the path differs.

- **The tube of the bridge.** The tube around a kink, for configurations with one kink on an interval with both ends fixed in the two valleys (a bridge). The core is the interval inside the tube that stays near the top. The saddle-tube Poincaré assumption and the core Poincaré uniformity assumption inside the box have this shape.
- **The seam of the ring.** Configurations on a short periodic path (a ring, length $T \ll \xi$) with one kink–antikink pair, at the place where the separation ℓ of the pair shrinks and the two meet. The configuration just before the pair disappears is the plateau at the top itself (§ 03). The argument for the tube of the bridge uses the fact that the kinks are apart, so the seam needs a different treatment.

In both places, the tool that controls the core constant is of the same type. We attach a coarse mark "valley side or top side" to the sites in the core, and combine the logarithmic Sobolev constant AW of the marginal of the marks on a range of 17 sites around one site (the site window W) with the contraction ratio q of the chain extending outside the window. The contraction condition $q < 1$ of

this "chain outside the window" is the centre of the route of the chain outside the window and of the outer chain in the route of the seam of the ring, and the size of A_W is the centre of the route of the window-constant numerics, the route of the sector framework and the route of the window log-Sobolev inequality. The site window W is different from the range of $\hbar$ treated computer-assisted (the window of $\hbar$, $[3\cdot10^{-4},3\cdot10^{-3}]$). The window in "the chain outside the window" and "the stages inside the window" is the site window, while the window in "the whole window" and "the lower end of the window" is the window of $\hbar$. Where they could be confused we add "site".

02

The map of the seven routes

Seven routes were set up to close the core rows of the saddle-tube condition. The routes are called by the names of their content. The order of the rows of the table indicates neither order nor weight. "Grade" in Table 1 uses the symbols of the five grades of 23-00 § 01, and when two or more grades are mixed in one route they are listed from the main one.

Table 1 The seven routes (window = $\hbar\in[3\cdot10^{-4},3\cdot10^{-3}]$, three points = $\hbar=3\cdot10^{-3}/10^{-3}/3\cdot10^{-4}$)

Route	What it shows	Current grade	Range of $\hbar$	What remains	Chapter
The route of the chain outside the window (contraction of the chain outside the window)	In the core of the bridge, the Stein-kernel ratio $q_{\mathrm{free}}<1$ of the unmarked chain extending outside the site window	Computer-assisted $q_{\mathrm{free}}\le0.998854904$ (float64 + outward widths, presupposing correct IEEE rounding, $\hbar$ as an interval, 25 subintervals)	The whole window	Says nothing outside the window or as $\hbar\to0$	23-08
The route of the window-constant numerics (remaining numbers)	Constants of the site window: minimal weight π_*, rarity Λ, one-site Poincaré C_{LD}^{W}, gap of the mark dynamics gap_W, ratio of the inner chain of the window of the bridge	Reading (floating point at three points) Assumption (the paper reduction to a number of computations is not placed in the book)	Three points	The number of interval-arithmetic runs and the time (§ 04). Decided not to run them because of the floor of the product form	This chapter, § 04
The route of the sector framework	The five steps of the sector sum (sector sum, Gibbs decomposition, componentwise localisation, volume locality, sector composition) and the closing condition $b(A_W c_H+D_W+\ln2)<\kappa\ell_{\min}$	Proved here (the proof of the five steps is placed in 23-09). Inputs are Assumption , Reading	Steps independent of $\hbar$ / inputs at three points	Inputs (the marginal log-Sobolev input for the window, the bridge volume-locality input, E_m, the defect rows, regularity)	23-09 § 02

Route	What it shows	Current grade	Range of $\hbar$	What remains	Chapter
The route of the sector input (small assumptions)	The small assumptions of the sector input (eleven rows: the bridge volume-locality input, the core log-Sobolev uniformity assumption, the longitudinal term and regularity, plus seven rows whose content has not yet been written in one line)	Assumption (list only; not started)	—	Each row	This chapter, § 05
The route of the window log-Sobolev inequality (the one remaining proposition)	Window Log-Sobolev Threshold Proposition: the logarithmic Sobolev constant of the marginal on the site window lies within the needed value $A_{\max}(\hbar)$	Assumption . None of the three strands (componentwise, scalar, Feynman–Kac) is closed	Window (the Scalar-Strand Proposition is on $[3\cdot 10^{-4}, 10^{-3}]$)	The remaining step of each of the three strands	23-09 § 05
The route of the seam of the ring	The core on the ring: contraction of the outer chain and the coefficient of the inner chain	Outer: Computer-assisted (pin ≤ 0; the seam limit assumption and the random-end-value assumption are Assumption) / inner: Reading $q_{\text{in}} \approx 1.24$–$1.28$	Window	The seam limit assumption, the random-end-value assumption, the ratio conditioned on the plateau, the boundary stage, putting the inner chain into intervals	This chapter, § 03
The route of the range of $\hbar$	Outside the window and $\hbar \to 0$	Outside the scope of this part	—	Paper asymptotics	23-12, 23-15

Three notes on how to read Table 1.

(a) Only two rows are computer-assisted. They are the route of the chain outside the window and the outer chain in the route of the seam of the ring. Both are closed in the sense that "within the computed window of $\hbar$, the inequality holds given the table of outputs of interval arithmetic", and say nothing outside the window or as $\hbar \to 0$ (23-08).

(b) Even if the route of the chain outside the window closes, the saddle-tube condition does not close. The ratio q_{free} of the route of the chain outside the window enters only one factor $S_W = 1 + 2\big(16 + 1/(1-q_{\text{free}})\big)$ of the upper bound on the logarithmic Sobolev constant A_W of the site window ([12] of the dependency graph in § 06). In the current construction of the upper bound, the upper bound on A_W lies outside the needed value (by 4–5 orders of magnitude in the product form of the factors for the 17-site window, and by 29–74 times even in the form for the 3-site window with the needed side raised, 23-09 § 03), and the one remaining proposition (the route of the window log-Sobolev inequality) is being attacked by other strands.

(c) The route of the range of ħ was decided to be out of scope. The double-well relaxation-time power statement is a statement as $\hbar \to 0$, so in the end the four double-well conditions must be stated in the asymptotics $\hbar \to 0$, and computer assistance on a fixed window alone does not reach that. Outside the window, paper asymptotics are needed. In this part we first check the grades of the rows inside the window, and how to fit them together is described in 23-12 and 23-15.

03

The seam of the ring — the plateau at the top, the outer chain and the inner chain

A ring is a periodic path of length $T \ll \xi$ with one kink–antikink pair, and we write ℓ for the length of the arc on the negative side. The reduced plateau model of 23-05 § 02 (c) kept only this ℓ. The seam is where ℓ becomes small and the pair disappears.

The seam is the plateau at the top itself

The continuum minimal action $S(\ell)$ with the length of the negative arc fixed at ℓ tends, as $\ell \to 0^+$, not to 0 but to S_0 (the path touches the top and returns). For $\ell \le \pi\sqrt{2}$ the minimising configuration has no kinks: it is a plateau lying flat on the top 0, with action $S_0 + \ell/8$. For $\ell > \pi\sqrt{2}$ a dip appears and the action tends to $2S_0$ ($2S_0 - S \approx 8e^{-\ell}$) Assumption (a continuum variational argument and a cross-check with floating-point minima on the lattice; the proof is not placed in the book). On the lattice ($\varepsilon = \frac{1}{2}$) too, the minimum with m sites on the negative side lies on $S_{\mathrm{lat}}(1) + (m-1)/16$ for $m \le 9$ Reading (floating-point minima, difference $\le 5 \times 10^{-11}$). **The configuration at the seam is the core itself (the interval that stays near the top)**, and the same difficulty as for the core of the tube of the bridge appears here.

The ℓ direction is easy. One-dimensional transport in ℓ alone can be controlled, seam included. On each of the band, the collar and the plateau of the seam, the Muckenhoupt quantity, which squeezes the one-dimensional Poincaré constant within a factor 4, is at most $64S_0\hbar$, $L_0^2/(8D)$ and T^2/D, and the algebra passes machine checking Assumption (paper estimates on the one-dimensional model; the reduction to the model is itself an assumption). The hard part is the side of the coordinates: at the seam, the number of sign changes does not become a slow variable. So we fold by sectors that do not depend on the sign, namely runs of sites with $|z| < \theta$ ($\theta = \frac{1}{2}$) (**lumps**). With the rest fixed, the action of a lump of k sites is convex for $k \le 7$, from $V'' \ge -\frac{1}{2}$ Proved here (the smallest eigenvalue $8\sin^2\frac{\pi}{2(k+1)} - \frac{1}{4}$ of the tridiagonal matrix $\mathrm{tridiag}(-2, 4+w, -2)$, $w \ge -\frac{1}{4}$, is positive for $k \le 7$).

The outer chain and the inner chain

On a site window W of the ring (17 sites) we put marks: one lump (length $k \le 7$, possibly absent), the other sites of the window on the valley side, and outside the window the side of fixed sign. The construction of the logarithmic Sobolev constant of the window involves contraction ratios of two kinds of chain.

- **The outer chain**: the chain extending outside the site window. It moves in the region of fixed sign $[0, \infty)$, and mapped to $(-\infty, 0]$ by the reflection $x \mapsto -x$ it becomes a chain of the same type as the outer chain of the core of the bridge. The action is invariant under the reflection Proved here (V is even) Lean `chain_action_reflect` (no assumptions; the image of the measure and the reversal of the Stein kernel are outside Lean), but it is not the same chain as the chain of the bridge. The box of the bridge has a lower wall $-x_c$, and the outer chain of the ring has none.
- **The inner chain**: the stages inside the site window. From stage to stage the box alternates between "valley side" and "lump", and the lump box $(-\frac{1}{2}, \frac{1}{2})$ straddles 0.

The difference between the two chains can be read from the corner at the wall. The classical corner ratio at a wall at position b is $r(b) = \Delta/(\Delta - \varepsilon^2 V'(b))$. It is exactly 1 at a wall with $V'(b) = 0$ (the wall 0 of the outer chain), and exceeds 1 at a wall where the force pushes out of the box Proved here (the algebra of the sign of the ratio) Lean `corner_eq_one` (assumption: $\Delta \neq 0$), `corner_gt_one` (assumption: $0 < c < \Delta$). The formula for the ratio itself comes from a computation of the envelope, and its regularity is Assumption. At the upper wall $+\frac{1}{2}$ of the lump, $\varepsilon^2 V'(\frac{1}{2}) = -3/64$, so the force points towards the wall and the corner exceeds 1 (the classical value for an isolated wall is 1.2857).

Table 2 The outside and inside of the seam of the ring (triples in the order $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$)

Quantity	Value	Grade	Conditions
Outer chain (unconditioned form) q_{out}	≤ 0.998888530 at every $\hbar$ of the window (the maximum is at the lower end of the window; on the upper half $[10^{-3}, 3 \cdot 10^{-3}]$ it is ≤ 0.996606914)	Computer-assisted	All pins $p \le 0$, evaluation points $s \le 0$, end $z_0 = -1$. The seam limit assumption, the random-end-value assumption and regularity are Assumption. float64 + outward widths, presupposing correct IEEE rounding
Deep side of the outer chain (pin $p < -3$)	≤ 0.592898559	Computer-assisted	As above (uses the upper bound on the tail variance ≤ 0.357608022)
Boundary stage (pin $\in (0, \frac{1}{2})$)	≈ 1.004–1.045 (at the four points $\hbar = 3 \cdot 10^{-3}/10^{-3}/(3/9100)/3 \cdot 10^{-4}$: $1.0043/1.0282/1.0436/1.0446$)	Reading	A finite stage that appears only in configurations where the lump touches the end of the window. The outer-chain rows above do not cover this pin
Inner chain q_{in}	$1.2707/1.2765/1.2421$ (in the order of the triple; 1.2449 at the fourth point $\hbar = 3/9100 \approx 3.3 \cdot 10^{-4}$)	Reading	Floating-point transfer; it moves upwards as the grid is refined. The limit is set by the stage inside the lump with the pin at the edge of the far wall of the lump
Finite coefficient $\bar{q} = \max(q_{\text{in}}, 1)$	$\bar{q} \approx 1.3$, $\bar{q}^{16} \approx 27$–67 (applied to each of S_W and T_W; $\bar{q}^{32}$ for the product A_W)	Reading	$\bar{q} = 1$ does not hold
$S_W = 1 + 2\bar{q}^{16}(16 + 1/(1 - q_{\text{out}}))$	≤ 122745 (≤ 1833 if $\bar{q} = 1$)	Proved here (arithmetic)	Input $q_{\text{in}} \le 13/10$ is Reading, q_{out} is Computer-assisted. Lean `sw_inner_le` (assumptions: $q_{\text{in}} \le 13/10$, $q_{\text{out}} \le 0.998888530$)

In one line, Table 2 says: **the outside is closed (inside the window, conditional on pin ≤ 0), and the inside has a finite coefficient**. "Closed" here means that within the computed window of $\hbar$ the contraction of the outer chain became a computer-assisted inequality, not that the saddle-tube condition is closed on the ring. The result for the outer chain was first closed for the chain conditioned on the box (upper bound 0.998888525) and then extended to the unconditioned chain (the tail term put into the remainder of the interval arithmetic, deep pins covered by a closed-form upper bound). The extension removed the need for the four assumption rows that returned the conditioned chain to the original measure.

Why the inner chain exceeds 1, and the cost of putting it into intervals. What limits the inner chain is the stage with the pin at the edge of the far wall of the lump box, which is exactly the corner pushing outwards described above (close to the classical value 1.2857). The interval

machinery of the outer chain is built on the premises that the right wall is at 0 with $V'(0) = 0$, that there is one box, and that the pins and the grid coincide, so it cannot be used for the inner chain just by swapping the inputs. The estimated cost of putting the inner chain into interval arithmetic is 1–2 paper steps + 1–2 scripts + about 50 runs, about 42 hours (one core), and what it gives is only an upper bound on the finite coefficient $\bar{q} \approx 1.3$. This coefficient enters A_W as a factor $\bar{q}^{32} \approx 10^{3.6}$, so it is not a road to meeting the closing condition of the sector framework (23-09 § 02).

The rows that remain at the seam of the ring. The seam limit assumption (that the limit of stacking stages without end falls inside the enclosure of the fixed point at each $\hbar$), the random-end-value assumption (the case where the value at the end of the chain is given not by a point but by a distribution), the ratio conditioned on the plateau (the ratio conditioned on the length of the plateau), the boundary stage, and putting the inner chain into intervals. All remain Assumption or Reading .

04

The route of the window-constant numerics — the remaining numbers and their cost

In the current construction (the Product Bound for the Window Log-Sobolev Constant), the upper bound on the logarithmic Sobolev constant of the site window has the form of a product of factors $A = c_W \cdot C_{\mathrm{LD}}^{W} \cdot S_W \cdot T_W$. c_W is a constant of the marginal measure of the marks, built from the minimal weight π_* and the gap of the mark dynamics gap_W, and S_W and T_W are built from the contraction ratios of the chains. The route of the window-constant numerics is the paper reduction and estimate of the number of runs for moving the values of the factors from floating point at three points to interval arithmetic.

Table 3 The numbers of the route of the window-constant numerics, and the number of runs to move them to interval arithmetic (the numbers of runs and the times are estimates; not a single run has been made yet)

Quantity	Paper reduction (not placed in the book)	Estimated runs	Current value
Minimal weight π_*	The marginal measure of the marks is MTP_2, and the sequence giving the minimal weight is pushed to the extreme sequence (all on the valley side) (the preservation theorem of Karlin–Rinott Cited , statement checked in a restatement in a secondary source)	2–3 runs	Reading all on the valley side is the minimum at the three points
Rarity Λ	The sup over boundary values of a monotone function is attained by a single chain at the extreme (MTP_2 association)	1–2 runs	Reading
One-site Poincaré C_{LD}^{W}	The Muckenhoupt criterion quantity Cited (checked in a restatement in a secondary source), cell by cell of boundary values	About 20 runs	Reading of order ≤ 0.66
Gap gap_W	Since one mark screens the interaction of the interfaces, the congestion of the flow is controlled by the chain of the extreme sequence	4 runs	Reading 0.959–1.000
Ratio of the inner chain of the window of the bridge	The Stein-kernel ratio is not monotone in the sequence of marks (it fails for around 300 of the 665 pairs enumerated), so it is split by type of sequence and a band on the far side	About 13 runs	Reading < 1 at the three points

The estimate totals about 42 runs and about 21 hours (one core). **We decided not to run these 42.** The reason is the floor of the product form itself. From $S_W \geq 1 + 2/(1 - q_{\mathrm{free}})$, $T_W \geq |W|$ and c_W

$\geq \ln\bigl(1/\nu(\sigma_i = \text{valley side})\bigr) \approx E_U/\hbar$, even with the true values of the factors, $A \gtrsim 1.4 \times 10^4$–$1.4 \times 10^6$ Reading (estimate), which lies outside the needed value of the sector framework (23-09 § 03, tens to hundreds) by tens to tens of thousands of times. Even if the factors are confirmed in interval arithmetic, this form as it stands does not reach the closing condition. The paper reduction of the route of the window-constant numerics (screening, the extreme sequence) is kept as a tool that enters unchanged into upper bounds not of product form (the strands of the route of the window log-Sobolev inequality).

05

The table of the five grades — the rows of the saddle-tube condition

We put the rows of Tables 1–3 into one table in the format of the five grades. Each row has one grade, and the column of machine checking is outside the grades. The numbers of the rows are the numbers of the nodes of the dependency graph in § 06, and "—" marks a row not in the graph.

Table 4 The core rows of the saddle-tube condition ("—" in the range of $\hbar$ marks an assumption row not shown at any $\hbar$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[01]	Premises of interval arithmetic: correct rounding of IEEE 754 arithmetic, upper bounds on summation errors, upper bounds on the errors of our own exponential function	Assumption	23-08 § 03	—	—
[02]	The seam limit assumption, the random-end-value assumption, regularity	Assumption	§ 03	—	—
[03]	Window Log-Sobolev Threshold Proposition: $A_W < A_{\max}(\hbar)$	Assumption	The route of the window log-Sobolev inequality, 23-09 § 03	—	—
[04]	All the small assumptions of the route of the sector input	Assumption	Table 1	—	—
[05]	Product Bound for the Window Log-Sobolev Constant: $A_W \leq c_W C_{\mathrm{LD}}^{W} S_W \mathcal{T}_W$ (product form)	Assumption	§ 04 (the paper construction is not placed in the book)	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[06]	The constants of the route of the window-constant numerics, π_*, Λ, C_{ID}^{W}, gap_W, and the ratio of the inner chain of the window of the bridge	Reading	Table 3 (floating point at three points)	Three points	—
[07]	Inner chain of the ring $q_{\mathrm{in}} \approx 1.24$–$1.28$	Reading	Table 2 (floating-point transfer; not an upper bound)	Four points	—
[08]	Boundary stage $q \approx 1.004$–1.045	Reading	Table 2	Four points	—
[09]	Row of the contraction of the chain outside the window: $q_{\mathrm{free}} \le 0.998854904$	Computer-assisted	23-08 § 05 (the premise is the row of interval-arithmetic premises)	Window	`win_qfree_window_le_of_interval_table` (assumption: the table of outputs of interval arithmetic on 25 subintervals)
[10]	Outer chain of the ring $q_{\mathrm{out}} \le 0.998888530$ (pin ≤ 0)	Computer-assisted	23-08 § 06 (the premises are the row of interval-arithmetic premises and the row of the seam limit assumption)	Window	`unc_window_lt_one_of_table` (assumption: the table of outputs of interval arithmetic on 25 subintervals)
[11]	The five steps of the sector framework	Proved here	23-09 § 02 (the inputs are the row of small assumptions and the row of the seam limit assumption)	Independent of $\hbar$	`cell_sum_sup`, `comp_close` (the assumptions are in the table of 23-09)
[12]	$S_W \le 1 + 2(16 + 1/(1-q)) \approx 1779.6$ of the bridge	Proved here	Sum of a geometric series (23-08 § 02)	Window	—
[13]	$S_W \le 122745$ of the ring	Proved here	Table 2 (the input q_{in} is a reading)	Window	`sw_inner_le` (assumptions: $q_{\mathrm{in}} \le 13/10$, $q_{\mathrm{out}} \le 0.998888530$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[14]	Values of A_W in product form $\approx 2.5 \times 10^6/5.4 \times 10^6/4.7 \times 10^7$	Reading	§ 04 (one point of c_W is an estimate including extrapolation)	Three points	—
[15]	A positive lower bound on the gap of the core, uniform in k (in the form of sector composition)	Proved here	23-09 § 02 (on the window log-Sobolev threshold row and the sector-framework row)	—	`comp_close` (assumptions: the three inequalities of Barta, componentwise localisation and Bochner)
—	The curvature at one site at the top falls $\frac{1}{4}$ short of 4 (the common wall)	Proved here	23-09 § 06	Independent of $\hbar$	—

The two "Computer-assisted" rows of Table 4 ([09] and [10]) are both computer-assisted on the premise, the row of interval-arithmetic premises, and what Lean checked is "the construction with the table of outputs of interval arithmetic placed as an assumption" (23-08 § 08). The "Proved here" core lower-bound row is algebra on the assumption, the window log-Sobolev threshold row, and does not mean that a lower bound on the gap of the core has been shown.

06

The dependency graph — checking by the numbers that there is no circle

We turn the rows of Table 4 into numbered nodes. **The numbers of the premises to the right of "⟸" are always placed smaller than the number of the conclusion on the left** (the format common to this part, 23-05 § 07).

```
[01] Assumption        row of interval-arithmetic premises (correct IEEE roundi
[02] Assumption        seam limit assumption, random-end-value assumption, regu
[03] Assumption        Window Log-Sobolev Threshold Proposition: A_W < A_max(ħ)
[04] Assumption        small assumptions of the sector input
[05] Assumption        Product Bound for the Window Log-Sobolev Constant (A_W ≤
[06] Reading           constants of the route of the window-constant numerics
[07] Reading           inner chain of the ring q_in ≈ 1.24–1.28
[08] Reading           boundary stage q ≈ 1.004–1.045
[09] Computer-assisted row of the contraction of the chain outside the window:
                        Lean win_qfree_window_le_of_interval_table (assumption:
[10] Computer-assisted outer chain of the ring q_out ≤ 0.998888530 (pin ≤ 0) ⟸
                        Lean unc_window_lt_one_of_table (assumption: table of i
[11] Proved here       the five steps of the sector framework ⟸ [02][04]
                        Lean cell_sum_sup, comp_close (assumptions in the table
[12] Proved here       S_W of the bridge ≤ 1779.6 ⟸ [09]
[13] Proved here       S_W of the ring ≤ 122745 ⟸ [07][08][10]
                        Lean sw_inner_le (assumptions: q_in ≤ 13/10, q_out ≤ 0.
[14] Reading           A_W in product form ≈ 2.5·10⁶–4.7·10⁷ ⟸ [05][06][12]
[15] Proved here       positive lower bound on the core gap uniform in k ⟸ [0
                        Lean comp_close (assumptions: the three inequalities of
[16] Proved here       the form of the core Poincaré uniformity assumption of 2
```

How to check. [09] has only [01] on the right, [10] only [01][02], [11] only [02][04], [12] only [09], [13] only [07][08][10], [14] only [05][06][12], [15] only [03][11], and [16] only [02][15]. In every row the numbers on the right are smaller than the one on the left, so there is no circle.

What to read from this graph. (1) The road to the core row [16] passes through [15], and [15] depends on the assumption row [03] (the one remaining proposition of the route of the window log-Sobolev inequality). (2) The computer-assisted [09][10] enter only the product-form value [14] through [12][13], and [14] does not satisfy [03] (it lies outside the needed value). That is, **in the current graph the two computer-assisted rows are not on the road to [16]**. When an upper bound not of product form satisfies [03], [09][10] become part of the grounds of [03] if the ratio q enters that bound. (3) None of the reading rows [06][07][08] is used in the proof of an upper bound.

We also check that the statements of the assumption rows do not contain the quantities of the conclusion. The quantities of the conclusion are the gap of the measure of the core ([15]) and the constant of the core Poincaré uniformity assumption ([16]). [03] is the logarithmic Sobolev constant of the marginal of the marks on the site window, [04] constants of the sector input, [05] the construction of the product-form upper bound, and [02] limits and regularity. None of them contains the gap of the core itself.

07

A guide to 23-08–23-11

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Chapters to read after this one
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1. 23-08 What computer-assisted means — using the route of the chain outside the window and the outer chain in the route of the seam of the ring as examples, how interval arithmetic turns floating-point values into inequalities, its premises, and what it does not say
2. 23-09 The sector framework and the three strands of the window log-Sobolev route — the proof of the five steps of the route of the sector framework, the derivation of the needed value $A_{\max}$, the three strands and the common wall
3. 23-10 The list of detour plans — the detour plans tried in the route of the window log-Sobolev inequality, and the types of breakage that appeared repeatedly
4. 23-11 Box reduction and the gap inside the box — the constant of the core — the construction that controls the core constant in three steps from the side of the box

This part places the relaxation-time assumption at the axis of the study and treats the saddle-tube condition as one of the pillars of its first step. Another view is treated in Part 24.

What this chapter can say

What is said	What is not said
Computer-assisted $q_{\text{free}} \le 0.998854904$ at every $\hbar$ of the window $[3\cdot 10^{-4}, 3\cdot 10^{-3}]$, and $q_{\text{out}} \le 0.998888530$ for the outer chain of the ring (pin ≤ 0). On the premises of interval arithmetic	Contraction outside the window and as $\hbar \to 0$. The boundary stage (pin $\in (0, \frac{1}{2})$)
Proved here The five steps of the sector framework, and the lower bound on the core gap on the assumption row (the Window Log-Sobolev Threshold Proposition) (23-09). The arithmetic of S_W	The Window Log-Sobolev Threshold Proposition itself. Hence the core Poincaré uniformity assumption, the saddle-tube Poincaré assumption and the saddle-tube condition
Reading The inner chain of the ring $q_{\text{in}} \approx 1.24$–$1.28$, the constants of the route of the window-constant numerics	Upper bounds on these. Interval machinery for the inner chain
That the dependency graph has no circle (the rule of the numbers)	A road by which the computer-assisted rows reach the core rows of the saddle-tube condition (none in the current graph)

Sources and reproduction

Item	Kind	Source / tools
Contraction of the chain outside the window, the outer chain of the ring	Computer-assisted, Lean	Interval arithmetic in Python float64 (outward widths for each operation, our own exponential function). Procedure and cross-checks in the table of sources of 23-08. Lean: `win_qfree_window_le_of_interval_table`, `unc_window_lt_one_of_table`, `vt_window_of_table` (assumption: the table of outputs of interval arithmetic).
Preservation of MTP_2	Cited	S. Karlin, Y. Rinott, J. Multivariate Anal. 10 (1980) 467–498; statement in S. Fallat et al., Ann. Statist. 45 (2017) 1152–1184, arXiv:1510.01290 (checked in Proposition 3.5).
The Muckenhoupt criterion quantity	Cited	B. Muckenhoupt, Studia Math. 44 (1972) 31–38; statement in F. Barthe, C. Roberto, Potential Anal. 29 (2008) 167–193 (checked in Theorems 1 and 2).

Item	Kind	Source / tools
The plateau of the seam of the ring, convexity of lumps, the corner at the wall	Assumption, Proved here, Reading	Continuum variational argument and floating-point minima (Assumption, Reading), smallest eigenvalue of the tridiagonal matrix (Proved here), sign of the corner ratio (Proved here, `corner_eq_one`, `corner_gt_one`).
The inner chain of the ring, the boundary stage	Reading	Floating-point transfer (grid $h = 2^{-10}$, limiting pairs rechecked at 2^{-11}). The regular Gaussian chain reproduces $2/3$ to 3 digits (positive control), and a chain with the wall moved in the direction that pushes outwards gives > 1.2 (negative control).
Reflection	Proved here, Lean	`chain_action_reflect` (symmetry of each term of the action).

Words used in this chapter — route, core, site window, window (of $\hbar$), chain outside the window, Stein-kernel ratio, ring, seam, lump, outer chain, inner chain, boundary stage, corner at the wall, computer-assisted, floor of the product form. Their meanings are in A-02 Glossary.

Next chapter to read: 23-08 What computer-assisted means — the conditions on the chain inside the window — explains the mechanism and premises by which interval arithmetic turns floating-point computations into proofs of inequalities, using the contraction condition of the chain outside the window as an example.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time computer-assisted, interval arithmetic, window · Kiichi, Shiori and Rin

What computer-assisted means — the conditions on the chain inside the window

> This chapter is this book's own investigation. § 01 and § 03 are established knowledge.

What this chapter shows — We make it possible to read what results labelled "computer-assisted" (Computer-assisted) in this part guarantee and what they do not. The example is the contraction condition $q < 1$ for the chain extending outside the core. Instead of writing floating-point values as they are, we hold every number as an interval that contains the true value, make $\hbar$ itself an interval, and check that the inequality holds at every point of the computed window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ of $\hbar$. We describe this mechanism and its premises (the rounding conventions). At the end we look at an example where computer assistance returns "no", and confirm that it says nothing about what lies outside the window.

Prerequisite chapters — 23-07 § 02 (the route of the chain outside the window and the route of the seam of the ring), 23-00 § 01 (the labels of the five grades), 23-04 § 01 (the path measure).

Said first — The level is [Y2]–[Y3]. § 01– § 03 are written to be readable at [Y2], and § 04 onwards lean towards [Y3]. The inequalities treated in this chapter are one input to the construction of the first step with the diffusive-relaxation-time assumption at its axis, and by themselves they do not close the saddle-tube condition (23-07 § 06).

The order of this chapter

1. Floating-point values, and inequalities written with intervals — a small example that can be checked by hand
2. What the chain condition $q < 1$ guarantees — the Stein-kernel ratio, and the geometric series of influence
3. The premises of interval arithmetic — the rounding conventions and our own exponential function
4. Making $\hbar$ an interval — why it breaks when done naively, and three lemmas
5. Example 1: the outer chain of the core of the bridge — 25 subintervals (Table 1)

6. Example 2: the outer chain of the ring — from the conditioned chain to the unconditioned chain
7. Cross-checks, controls, pre-registration — reproduction by another implementation, and positive and negative controls
8. What Lean checked — the construction with the table of outputs of interval arithmetic placed as an assumption
9. Examples where computer assistance returns "no" — the inner chain, and a subinterval with first-order tilting
10. What computer assistance does not say, and the cost — how to read the label (box)

01

Floating-point values, and inequalities written with intervals

This section and § 03 summarise existing knowledge (floating point and interval arithmetic in general). The line between theorem and observation is introduced in 5-07 § 09.

Real numbers on a computer are floating-point numbers, and the result of most operations is the true value rounded to the nearest representable number. For example, $0.1 + 0.2$ becomes 0.30000000000000004 in double precision. Each rounding is small, so in many computations it can be ignored. But when claiming an inequality "$q < 1$" about a value close to 1, such as $q \approx 0.9989$, one must show that accumulated rounding does not flip the direction of the inequality.

Interval arithmetic is computation in which every number is held as "an interval that is sure to contain the true value". Intervals are built from intervals: addition is $[a, b] + [c, d] = [a + c, b + d]$, multiplication takes the minimum and maximum of the four products of the ends, and so on. When the ends are computed, rounding is taken **outwards** (the lower end downwards, the upper end upwards). Then the final interval $[q^-, q^+]$ is sure to contain the true q, and if $q^+ < 1$, then $q < 1$ follows as an inequality. In this part, inequalities obtained in this way carry the label Computer-assisted . Floating-point values written as they are carry Reading .

An example that can be checked by hand. The true range of $g(x) = x - x^2$ on $x \in [0, 1]$ is $[0, \frac{1}{4}]$. A naive evaluation in interval arithmetic gives, from $x \in [0, 1]$ and $x^2 \in [0, 1]$, $x - x^2 \in [0 - 1,\ 1 - 0] = [-1, 1]$. This contains the true value, but the width has grown eightfold. The cause is that the intervals forget the **dependence** that x and x^2 come from the same x. There are two ways to fix this.

- **Split finely.** Splitting into $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ gives $[0 - \frac{1}{4}, \frac{1}{2} - 0] = [-\frac{1}{4}, \frac{1}{2}]$ and $[\frac{1}{2} - 1, 1 - \frac{1}{4}] = [-\frac{1}{2}, \frac{3}{4}]$, together $[-\frac{1}{2}, \frac{3}{4}]$, and the width shrinks to 1.25.
- **Expand about the centre.** With $g(x) = g(c) + g'(\xi)(x - c)$, $c = \frac{1}{2}$ and $g'(\xi) = 1 - 2\xi \in [-1, 1]$, we get $\frac{1}{4} + [-1, 1] \cdot [-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{4}, \frac{3}{4}]$, and the width is 1. This takes the first-order part of the dependence outside the interval.

What we meet in § 04 when making $\hbar$ an interval is a large-scale version of these two fixes. We split the window of $\hbar$ into 25 subintervals (split finely), and within each subinterval expand the expectations in $\hbar$ to take the first- and second-order movement outside the interval (expand about the centre).

02

What the chain condition $q < 1$ guarantees

Seen from a site in the core, consider the chain $x_{i-1}, x_{i-2}, \dots$ extending to one side. The term of the action between neighbours is $(x_{i-1} - x_i)^2/(2\varepsilon)$, and the measure is $e^{-S/\hbar}$. Fixing $x_i = y$ and integrating out the far side, the conditional law of x_{i-1} takes the form

$$\rho_y(x) \;\propto\; \exp\Big(-\frac{1}{\hbar}\Big[\frac{(x-y)^2}{2\varepsilon} + U(x)\Big]\Big) \qquad (x \text{ in the box } [L, R]\ ) \quad (1)$$

(U is the effective potential with the far side integrated out, and does not depend on y). Let $m(y)$ be the mean of ρ_y, and let the Stein kernel be

$$\tau_y(x) = \frac{1}{\rho_y(x)} \int_L^x \big(m(y) - t\big)\rho_y(t)\,dt \quad \geq 0 \quad (2)$$

. Integration by parts gives $\mathrm{Cov}_y(\varphi(X), X) = \mathbb{E}_y[\varphi'(X)\tau_y(X)]$ (the boundary term vanishes since $\int_L^R (m - t)\rho_y\,dt = 0$). On the other hand, the logarithmic derivative of (1) in y is $(x - m(y))/(\varepsilon\hbar)$, so

$$\frac{d}{dy}\,\mathbb{E}_y[\varphi(X)] = \frac{\mathrm{Cov}_y(\varphi(X), X)}{\varepsilon\hbar} = \frac{\mathbb{E}_y[\varphi'(X)\tau_y(X)]}{\varepsilon\hbar}, \qquad \Big|\frac{d}{dy}\,\mathbb{E}_y[\varphi(X)]\Big| \;\leq\; q\,\sup|\varphi'|, \quad q :=$$

Proved here (differentiation under the integral sign and integration by parts; the box is bounded and the density is smooth). (3) reads: "each step back, the slope of a function (its Lipschitz constant) becomes at most q times as large". Going back d steps gives at most q^d times. **If $q < 1$, the influence of fluctuations at distant sites on nearby sites decreases as a geometric series.** This is the content of "the chain condition", and we call q the Stein-kernel ratio.

A check. If ρ_y is a normal distribution with variance σ^2, computing (2) gives $\tau_y \equiv \sigma^2$, and $q = \sigma^2/(\varepsilon\hbar)$. For the harmonic chain (U a quadratic), one can check by hand that this ratio converges stage by stage to a fixed value.

What it is used for. In the construction of the logarithmic Sobolev constant of the site window (the product form of 23-07 § 04), the quantity SW that adds up the influence of all stages outside the window enters as a factor. Adding the contribution of the stages inside the window and the contribution of the stages outside the window $\sum_{d\ge 0} q^d = 1/(1-q)$ on the two sides, left and right, gives

$$SW = 1 + 2\Big(16 + \frac{1}{1-q}\Big) \quad (4)$$

(the form with the ratio of the stages inside the window set to 1. The paper construction is not placed in the book, so the step that uses (4) sits on an Assumption row). If $q < 1$, then SW is finite, and if $q \le 0.998854904$, then $1/(1-q) \le 873.3$ and $SW \le 1779.6$ Proved here (arithmetic).

Why q is close to 1. The right wall of the box of the outer chain of the core is the top 0, and there $V'(0) = 0$. At the corner of this wall the classical ratio is exactly 1 (the corner at the wall in 23-07 § 03), and the departure from 1 is set by the width $\eta \approx 0.5\hbar$ of the thin layer near the wall, of size $1 - q \approx \eta \ln(1/\eta)$ Reading (an expansion of the classical map, cross-checked with transfer-matrix values). The smaller $\hbar$ is, the closer q comes to 1. **This also shows that checking $q < 1$ inside a fixed window of $\hbar$ leaves the behaviour as $\hbar \to 0$ as a separate question.**

03 The premises of interval arithmetic

The computer assistance of this part is interval arithmetic built on Python's double-precision floating point (float64). We collect the premises in one place.

- **Rounding conventions.** We assume that each arithmetic operation returns, as the IEEE 754 standard specifies, the floating-point number nearest the true value (correct rounding). The result of each operation is made into an interval by moving it outwards by one to the neighbouring floating-point number (the lower end one down, the upper end one up). The error of correct rounding is at most half a unit, so moving one unit outwards contains the true value. It is not a scheme that switches the rounding direction for each operation. Fused multiply–add (FMA) is not used.
- **Summation error.** Long sums are widened outwards by a known upper bound on summation error.

- **The exponential function.** The error of the standard mathematical functions is not guaranteed, so we use our own implementation of the exponential function and add an upper bound on its error to the interval. The arithmetic of the constants of the upper bound is checked in Lean.
- **Integrals.** For each grid cell we build upper and lower envelopes of the integrand (with a concavity correction) and squeeze the integral outwards. Truncation of the integral (cells with weight below e^{-40} of the maximum) enters the interval as an upper bound on the truncated mass.

In this part this premise row is Assumption (the row of interval-arithmetic premises in 23-07 § 05). That the computer and the language implementation keep correct IEEE 754 rounding is placed as a premise, not something we checked. When citing a computer-assisted result, we write this premise in the same row.

04

Making $\hbar$ an interval — why it breaks when done naively, and three lemmas

To get $q^{+} < 1$ at a point value of $\hbar$ (for example $\hbar = 3 \cdot 10^{-3}$), the interval arithmetic of § 03 is enough. To say it for **every** $\hbar$ in the window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$, $\hbar$ itself must be put into an interval. Even if a downward trend is seen at three points, that the upper bound is monotone in $\hbar$ cannot be derived on paper (the formula for the upper bound contains two terms that are largest at opposite ends of $\hbar$).

We fix notation. $\beta = 1/(\varepsilon\hbar) = 2/\hbar$; for a subinterval $\hbar \in [\hbar_1, \hbar_2]$, $\beta \in [\beta_2, \beta_1]$; midpoint $\beta_0 = (\beta_1 + \beta_2)/2$, half-width $\Delta = (\beta_1 - \beta_2)/2$, and $\tau = \beta - \beta_0 \in [-\Delta, \Delta]$. $\hbar$ enters only through β.

Naive intervals break. Multiplying the exponent of the density $\beta(sx - A(x))$ by the interval of β as it is allows the weight of each grid cell to move independently by factors $e^{\pm\Delta|sx - A|}$, and the width of the exponent is of order 10^2. It is the same loss of dependence as $x - x^2$ in § 01. Indeed, for the chain of the double well, at the second stage the interval of the mean spread over the whole box (width 0.89), and the map of the stages lost its meaning.

Linear-Fractional Lemma (making the expectation a linear-fractional function of τ). Putting $h = g - g^*$ (g^* a constant) and $\varphi_0 = \beta_0 h$, with $\beta = \beta_0 + \tau$,

$$\mathbb{E}_\beta[f] = \frac{\int f e^{\beta g}}{\int e^{\beta g}} = \frac{S_0 + \tau S_1 + R_S(\tau)}{Z_0 + \tau Z_1 + R_Z(\tau)}, \qquad 0 \le e^{\tau h} - 1 - \tau h \le \frac{(\Delta|h|)^2 e^{\Delta|h|}}{2} \quad (5)$$

($S_0 = \int f e^{\varphi_0}$, $S_1 = \int f h e^{\varphi_0}$, $Z_0 = \int e^{\varphi_0}$, $Z_1 = \int h e^{\varphi_0}$; R are the integrals of the remainders). With the values of the remainders fixed, $t \mapsto (S_0 + R_S + tS_1)/(Z_0 + R_Z$

$+ tZ_1$) is linear-fractional, and if the denominator is positive at $t = \pm\Delta$ it is monotone on $[-\Delta, \Delta]$. Hence for all $|\tau| \le \Delta$ the expectation lies between its two values at the ends $t = \pm\Delta$, and evaluating those two in intervals encloses it Proved here Lean `mobius_between` (assumptions: the denominator is positive at $t = \pm\Delta$, $-\Delta \le t \le \Delta$). The first-order term $\tau S_1/Z_0$ represents the real change with $\hbar$, and only Δ times the quadrature error and the remainder (of order Δ^2) are left in the width of the interval.

Tilted-Enclosure Lemma (first and second forms; holding the enclosure tilted in τ). The fixed points of the map of the stages (the conditional means m_k of each stage of the chain) really move with $\hbar$. When the width of a subinterval is 3%, that movement alone is 1.5×10^{-4}, and putting it into the width of the enclosure ruins the downstream estimates. So we hold the enclosure in the form "$m_k - \tau\mu_k - \tau^2\nu_k$ lies in an interval" (μ_k and ν_k are floating-point sequences). On the output side the identity

$$\frac{A + tB}{C + tD} = \frac{A}{C} + t\,\frac{B - (A/C)D}{C + tD} \tag{6}$$

extracts the first-order movement Proved here Lean `shear_identity` (assumptions: $C \neq 0$, $C + tD \neq 0$). Tilting to second order, the remaining width is of order Δ^3. This is the form "expand about the centre" of § 01.

Moving-Centre Lemma (moving the centre with τ). The upper bound on the variance has the form of a sum of two one-sided parts $\beta\,\mathbb{E}[(X - c)_+^2] + \beta\,\mathbb{E}[(c - X)_+^2]$, and when the mean moves by $\tau\mu$ one increases and the other decreases. Maximising the two separately at the ends of τ adds movements that in truth cancel. Moving the centre as $c_\tau = c + \tau\kappa$ and bounding by $(y - t)_+^2 \le y_+^2 - 2ty_+ + t^2$ brought this loss down from 3.6×10^{-3} to 2.2×10^{-4} Proved here Lean `dvtau_pos` (no assumptions).

Invariant sets. The chain is infinitely long, so we cannot run the map of the stages a finite number of times and stop. The map $\Phi_{[\hbar_1,\hbar_2]}$ on a subinterval is built to contain the results of the maps $\Phi_\hbar$ for all $\hbar \in [\hbar_1, \hbar_2]$, so if interval arithmetic confirms $\Phi_{[\hbar_1,\hbar_2]}(E^*) \subset E^*$ for some enclosure E^*, then $\Phi_\hbar(E^*) \subset E^*$ for every $\hbar$ Proved here (transitivity of inclusion). All later stages lie inside E^*, and it is enough to evaluate the upper bound on q once on E^*.

Table 1 The balance on one subinterval (outer chain of the core of the bridge; the upper bound at the point $\hbar = 3 \cdot 10^{-3}$ is 0.995611069)

Version	$\hbar \in [3/1030, 3/1000]$ **(width 3%)**	$\hbar \in [3/1100, 3/1000]$ **(width 10%)**
Naive intervals	Breaks at the second stage	Breaks at the second stage
Linear-Fractional Lemma only	Computer-assisted $q^+ =$ 0.998757877	—

Version	$\hbar \in [3/1030, 3/1000]$ **(width 3%)**	$\hbar \in [3/1100, 3/1000]$ **(width 10%)**
Linear-Fractional Lemma + first-order tilt + Moving-Centre Lemma	Computer-assisted $q^{+} = 0.996449644$	The upper bound is 1.00114, above 1, and no certificate is produced (§ 09)
Linear-Fractional Lemma + second-order tilt + Moving-Centre Lemma	—	Computer-assisted $q^{+} = 0.997808483$

Second-order tilting made subintervals of width 10% usable, and the window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ was covered by 25 subintervals with ends $\hbar = 3/N$ ($N = 1000, 1100, 1210, \dots, 9100, 10000$, ratio between neighbours ≤ 1.1). The grid width is 2^{-11}, and the number of stages is 20 on the upper half and 25 on the lower half.

05

Example 1: the outer chain of the core of the bridge — $q_{\mathrm{free}} \le 0.998854904$ on the whole window

Write q_{free} for the ratio of the unmarked chain extending outside the site window of the core of the bridge (box $(-x_c, 0)$, $x_c = 0.89$) (the route of the chain outside the window in 23-07 § 02). On each of the 25 subintervals we built the enclosures of the map of the stages and an invariant set in interval arithmetic, and obtained an upper bound q_i^{+} on q for all chain lengths $n \ge 2$ and all pins of the box (the values at the end of the chain). The results are the left column of Table 2.

Result (computer-assisted)

For the path measure of the 0+1-dimensional double well ($V = (x^2 - 1)^2/8$, $\varepsilon = \frac{1}{2}$, $x_c = 0.89$, mark threshold $\theta = \frac{1}{2}$), under the premises of § 03, for every $\hbar$ in the window $\hbar \in [3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$

$$q_{\mathrm{free}}(\hbar) \;\le\; \max_i q_i^{+} \;=\; 0.998854904 \;<\; 1 \qquad (7)$$

On the upper half $\hbar \in [10^{-3}, 3 \cdot 10^{-3}]$, ≤ 0.998098180. The maximum is on the subinterval at the lower end of the window. Regularity and the inputs of the sector framework are Assumption .

The values in the table (rounded up at 9 digits) increase as $\hbar$ decreases, with a bulge only at the two at the upper end. The limiting place is the cell where the value at the end of the chain lies just below the wall 0. It is exactly the place in § 02 where "the corner at the wall is exactly 1".

Table 2 Upper bounds on the 25 subintervals (from the large-$\hbar$ side. Left = outer chain of the core of the bridge q_i^{+}, right = outer chain of the ring, unconditioned form Q_i^{unc}. Both rounded up at 9 digits; computer-assisted)

#	Subinterval of $\hbar$	Bridge q_i^{+}	Ring Q_i^{unc}
01	$[3/1100,\ 3/1000]$	0.997808483	0.993773312
02	$[3/1210,\ 3/1100]$	0.998098180	0.993531447
03	$[3/1330,\ 3/1210]$	0.996740073	0.993402112
04	$[3/1460,\ 3/1330]$	0.995689889	0.993847141
05	$[3/1600,\ 3/1460]$	0.994979345	0.994267993
06	$[3/1760,\ 3/1600]$	0.995310186	0.994690320
07	$[3/1930,\ 3/1760]$	0.995450880	0.995072576
08	$[3/2120,\ 3/1930]$	0.995739245	0.995442678
09	$[3/2330,\ 3/2120]$	0.996022449	0.995791525
10	$[3/2560,\ 3/2330]$	0.996294103	0.996116234
11	$[3/2810,\ 3/2560]$	0.996551317	0.996416208
12	$[3/3000,\ 3/2810]$	0.996695721	0.996606914
13	$[3/3300,\ 3/3000]$	0.996969923	0.996887371
14	$[3/3630,\ 3/3300]$	0.997198625	0.997139630
15	$[3/3990,\ 3/3630]$	0.997410737	0.997371392
16	$[3/4380,\ 3/3990]$	0.997607042	0.997583254
17	$[3/4810,\ 3/4380]$	0.997791698	0.997780790
18	$[3/5290,\ 3/4810]$	0.997966990	0.997966387
19	$[3/5810,\ 3/5290]$	0.998126215	0.998135077
20	$[3/6390,\ 3/5810]$	0.998276990	0.998293614
21	$[3/7020,\ 3/6390]$	0.998414660	0.998437926
22	$[3/7720,\ 3/7020]$	0.998559421	0.998572746
23	$[3/8490,\ 3/7720]$	0.998691543	0.998697045
24	$[3/9100,\ 3/8490]$	0.998741829	0.998780022
25	$[3/10000,\ 3/9100]$	**0.998854904**	**0.998888530**

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06
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Example 2: the outer chain of the ring — from the conditioned chain to the unconditioned chain

The outer chain of the seam of the ring (23-07 § 03) is a chain moving in a region of fixed sign, and mapped to $(-\infty, 0]$ by reflection it becomes the same type as Example 1. But there is no lower wall $-x_c$ as in the box of the bridge, so the certificate of Example 1 cannot be used as it is. We swapped the grid of the same machinery and closed it in two stages.

1. **The chain conditioned on a box.** For a chain conditioned on a finite box for each subinterval ($[-7/5, 0]$ on the upper half, $[-6/5, 0]$ on the lower half), the upper bound on the whole window is 0.998888525 Computer-assisted . As it stands, four assumption rows are needed to pay separately for configurations that leave the box.
2. **The unconditioned chain.** Widen the box to $[-3, 0]$, and put the contribution of the tail $(-\infty, -3]$ into the remainder of the interval arithmetic by a tangent upper bound from the convexity of the tail. The upper bound on the tail variance, in a form that pushes the law onto a single row at the end of the tail, is ≤ 0.357608022 at every $\hbar$ (within < 5.25 needed for convexity and ≤ 1 needed for the correlation inequality). Pins deeper than -3 are covered by a closed-form upper bound, ≤ 0.592898559. The results are the right column of Table 2, with maximum 0.998888530 Computer-assisted (all pins $p \le 0$, evaluation points $s \le 0$, end value $z_0 = -1$; the seam limit assumption, the random-end-value assumption and regularity are Assumption).

Comparing the two columns, the upper bound for the bridge bulges at the two at the upper end, while that for the ring is smallest on the third subinterval and then increases monotonically. At the lower end the ring is slightly closer to 1 (difference 3.4×10^{-5}), and the order of the floating-point point values (the ring closer to 1) appears in the upper bounds as well. Each row of the unconditioned chain is higher than the corresponding row of the conditioned chain by only 2.4×10^{-9}–6.7×10^{-7}, with the same limiting place and shape.

The boundary stage is not covered. The claim of the right column is a claim for pins $p \le 0$. In configurations where the lump touches the end of the window, the pin of the first outer stage falls in $(0, \frac{1}{2})$, and there the floating-point reading is $q \approx 1.004$–1.045 Reading . It is a single finite stage, so it does not break the finiteness of the construction, but when citing the right column of Table 2 the condition "pin ≤ 0" cannot be dropped.

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07
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Cross-checks, controls, pre-registration

The correctness of the interval-arithmetic scripts themselves has not been formally verified. Instead they pass the following checks. All are checks to see "whether what we think we checked has drifted from what we wanted to check".

- **Cross-check with another implementation.** On every subinterval of Example 1, we ran the map of the stages by a floating-point recursion separate from the interval arithmetic, at three values of $\hbar$ (the two ends and the midpoint of the subinterval), and counted whether its values fell inside the interval enclosures at every stage and every grid point (75 runs, 0 misses). For the unconditioned chain of the ring, the same check at one $\hbar$ per subinterval (25 runs, 0 misses).
- **Cross-check with point values.** The floating-point point values obtained from the transfer matrix (26 ends) all lay below the upper bound of the subinterval containing that end (the smallest difference, at the lower end of the window, was 2.85×10^{-4}).
- **Positive control.** Whether the machinery reproduces the answer where it is known in closed form. For the pure Gaussian chain (variance per stage $v_k = (k-1)/k$, mean $m_k(s) = (k-1)s/k$, neither depending on $\hbar$), every point fell inside the enclosures with $\hbar$ kept as an interval. With the width of the interval of $\hbar$ set to 0, it gave the same values as the machinery at a point $\hbar$ to order 10^{-12}. The unconditioned machinery for the ring reproduced the limiting place and shape of the version conditioned on a box on the same subintervals, with values within an upward difference of order 10^{-8} (they are not the same chain, so this is a check of reproduction, not agreement).
- **Negative control.** Whether the machinery rejects claims that should not hold. We confirmed that lowering the target upper bound at the limiting cell by 10^{-7} makes the Lean check fail to close, that replacing one entry of the table by 1 makes < 1 on the whole window fail to close, and that setting the tail variance above the convexity line stops the tail stage.
- **Pre-registration.** Before running the computations, we wrote down the expected range of values, the limiting place, and the pass lines of the positive and negative controls, and recorded after the results whether they were right. In Example 1 we expected 60–120 subintervals and 25 sufficed (wrong), and the limiting place coming at the lower end of the window was right. That a prediction was wrong does not weaken the result, but it is a record that the pass lines were not moved after seeing the results.

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08
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What Lean checked — the construction with the table of outputs of interval arithmetic placed as an assumption

The Lean column attached to computer-assisted results splits into the following three layers.

1. **The algebraic core.** The monotonicity of the linear-fractional function in the Linear-Fractional Lemma (`mobius_between`), the tilting identity (`shear_identity`), the inequality of the Moving-Centre Lemma (`dvtau_pos`), and the algebra of passing from the cover to the whole window (`cover_le`, assumption: $f \le c$ on each subinterval).
2. **Recomputation of cells.** From the input values output by the interval arithmetic (exact values of the floating-point numbers) and the rational ends of the subintervals, for the limiting cells and randomly chosen cells (46 in Example 1, 23 in the conditioned version for the ring), the formula for the upper bound at the last stage was recomputed inside Lean, closing "at most the interval-arithmetic value + 2×10^{-9}". This is not verification of every cell (a few dozen out of more than several million cells in one subinterval).
3. **Assembly of the table.** `win_qfree_window_le_of_interval_table` (Example 1), `unc_window_lt_one_of_table` (Example 2) and `vt_window_of_table` (tail variance) place as an **assumption** "$q(\hbar) \le q_i^+$ for every $\hbar$ in each subinterval" (the table of outputs of interval arithmetic; q is an abstract function), and give as conclusion "$q \le \max_i q_i^+ < 1$ for every $\hbar$ in the window".

Outside Lean lie the map of the stages (the interval evaluation of the Linear-Fractional Lemma, the Tilted-Enclosure Lemma and the Moving-Centre Lemma), quadrature, the invariant sets, the layers of the formula for the upper bound on q, and the choice of evaluation points. So the machine-checking column guarantees "if the outputs of the interval arithmetic are correct, the conclusion on the whole window is correct as algebra", and "the outputs of the interval arithmetic are correct" rests on the premises of § 03 and the checks of § 07.

09

Examples where computer assistance returns "no"

When the upper bound from interval arithmetic exceeds 1, all one can say is "this construction produces no certificate", not "the true value exceeds 1". We distinguish two cases.

(a) An example where it fails on the side of the tool. On the subinterval of width 10% in Table 1, the version with first-order tilting only gave the upper bound 1.00114. On

the same subinterval the version with second-order tilting shows the true value is ≤ 0.997808483, so this "no" comes from the width of the tool (having put the curvature of the true m in $\hbar$ into the interval). It is the kind of "no" that disappears with finer subintervals or a higher order of expansion.

(b) An example where it fails on the side of the object — the inner chain of the ring. For the stages inside the site window of the ring (the inner chain), measuring q_{in} by floating-point transfer gave $1.2707/1.2765/1.2421$ at $\hbar = 3\cdot 10^{-3}/10^{-3}/3\cdot 10^{-4}$ Reading , moving upwards as the grid was refined. The limit is set by the stage with the pin at the edge of the far wall of the lump box, the corner where the wall force points out of the box (classical value 1.2857) (23-07 § 03). Here, even building interval machinery is not expected to give $q^{+} < 1$. What it would give is only a finite upper bound $\bar{q} = \max(q_{\mathrm{in}}, 1) \approx 1.3$. This "no" is of a kind that does not disappear however fine the tool is made, and it can only be absorbed on the side of the construction (making a form that does not need $q_{\mathrm{in}} < 1$, or paying a coefficient $\bar{q}^{16}$ on each of S_W and T_W, and $\bar{q}^{32}$ on A_W).

When reading a computer-assisted result and seeing a report that the upper bound exceeded 1, check whether (a) or (b) is what is written.

10

What computer assistance does not say, and the cost

Box: how to read the "Computer-assisted" label

An inequality carries Computer-assisted only when all of the following are in place. (1) It is the output of a computation that holds every number as an outward interval, not a floating-point value written as it is. (2) The premises (correct IEEE 754 rounding, upper bounds on summation error, upper bounds on the error of our own exponential function, regularity) are written in the same row. (3) The scope of the claim (the window of $\hbar$, the chain lengths, the range of pins) is written in the same row. It says nothing outside the window of $\hbar$, as $\hbar \to 0$, or for pins outside the range.

What it does not say. (i) $\hbar$ outside the window $[3\cdot 10^{-4}, 3\cdot 10^{-3}]$. As seen in § 02, q approaches 1 as $\hbar \to 0$, so the further the window is extended downwards, the finer the subintervals and grid must be, and no finite computation can cover the whole of $\hbar \to 0$. The double-well relaxation-time power statement is a statement as $\hbar \to 0$, so that is work for paper asymptotics. (ii) Anything outside the model on the lattice $\varepsilon = \frac{1}{2}$ (the difference from the continuum path measure). (iii) Regularity (smoothness of eigenfunctions and conditional expectations) and the inputs of the sector framework.

(iv) The boundary stage of Example 2 (pin $\in (0, \frac{1}{2})$). (v) The saddle-tube condition itself (23-07 § 06).

Table 3 Cost (one core; "estimate" means not run)

Computation	Runs	Time	Size of one run
Example 1, outer chain of the core of the bridge (whole window)	27 runs	About 4 hours 10 minutes	6–16 minutes, 107 MB or less
Example 2, outer chain of the ring, form conditioned on a box	25 runs	About 3 hours 19 minutes	5–12 minutes, 114 MB or less
Example 2, outer chain of the ring, unconditioned form	25 runs	About 6 hours 44 minutes	13–22 minutes, 123 MB or less
Putting the inner chain of the ring into intervals (estimate)	About 50 runs	About 42 hours	About 50 minutes
Remaining numbers of the route of the window-constant numerics (estimate; decided not to run)	About 42 runs	About 21 hours	Within 30 minutes

This part is a study with the diffusive-relaxation-time assumption at its axis, and within it computer assistance is used as a tool that raises individual inequalities inside the window from floating-point readings to inequalities. Another view is treated in Part 24.

What this chapter can say

What is said	What is not said
Computer-assisted $q_{\mathrm{free}} \le 0.998854904$ at every $\hbar$ of the window, and ≤ 0.998888530 for the outer chain of the ring (pin ≤ 0). On the premises of § 03	Outside the window, $\hbar \to 0$, the boundary stage
Proved here The derivation of the chain condition (3), the algebra of the Linear-Fractional Lemma, the Tilted-Enclosure Lemma and the Moving-Centre Lemma, the inclusion of invariant sets Lean (the algebraic core and the assembly of the table)	The formal correctness of the interval-arithmetic scripts (only as far as checked by cross-checks and controls)
Reading The inner chain of the ring $q_{\mathrm{in}} \approx 1.24$–$1.28$	An interval upper bound on it

Sources and reproduction

Item	Kind	Source / tools
The ideas of interval arithmetic, loss of dependence and the centred form	—	Standard methods (interval arithmetic, Taylor-form enclosures). This chapter cites no bibliography and places the example of § 01 and the lemmas of § 04 in the text.

Item	Kind	Source / tools
The Stein kernel and the chain condition	Proved here	(2)(3) of § 02 (integration by parts and differentiation under the integral sign).
Implementation of interval arithmetic	Computer-assisted	Python float64, one floating-point step outwards per operation, our own exponential function, grid 2^{-11}, stages 20 / 25, 25 subintervals.
Cross-checks	Reading	A separate floating-point recursion (three values of $\hbar$), transfer-matrix point values (26 points).
Lean	Lean	`mobius_between`, `shear_identity`, `dvtau_pos`, `cover_le`, `win_qfree_window_le_of_interval_table`, `unc_window_lt_one_of_table`, `vt_window_of_table` (the assumptions are the rows of the text). The list of statements is in 23-13.

Words used in this chapter — computer-assisted, interval arithmetic, outward rounding, loss of dependence, expanding about the centre, Stein-kernel ratio, chain condition, subinterval, tilting, invariant set, cross-check, positive and negative controls, pre-registration, boundary stage. Their meanings are in A-02 Glossary.

Next chapter to read: 23-09 The sector framework and the three strands of the window log-Sobolev route — states the five steps of the sector sum and the closing condition, the derivation of the needed value, how far the three strands that attacked the one remaining proposition got, and the common wall, the shortfall of the curvature at one site at the top.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time sector framework, closing condition, the common wall · Kiichi, Shiori and Rin

The sector framework and the three strands of the window log-Sobolev route

> This chapter is this book's own investigation. The established knowledge it builds on is in 2-05 and 2-07.

What this chapter shows — We prove, split into five steps, the construction that controls the gap of the core uniformly in the core length k (the sector framework), and derive the condition under which it closes, $b(A_W c_H + D_W + \ln 2) < \kappa \ell_{\min}$, and the needed value $A_{\max}$ determined by it. We sort the constants of the condition into those that come from the structure and those that come from the choice of tools, and list the directions that raise the needed value and that lower the insufficient side. We put in one table how far the three strands (componentwise, scalar, Feynman–Kac) that attacked the one remaining proposition (that the logarithmic Sobolev constant of the site window lies within the needed value) got, and state in one line of diagonal dominance why all three stop at the same place: the curvature at one site at the top falls $\frac{1}{4}$ short of convexity.

Prerequisite chapters — 23-07 (the positions of the route of the sector framework and the route of the window log-Sobolev inequality), 23-08 § 02 (the chain condition), 23-05 § 03 (the saddle-tube condition and the core Poincaré uniformity assumption), 1-10 § 07– § 09 (Poincaré inequalities).

Said first — The level is [Res], and only § 01 is written to be readable at [Y3]. § 02 onwards are sections at the research level, and you can skip ahead to Table 4 in § 05 and the common wall in § 06 and still read the conclusions of this chapter. This part places the diffusive-relaxation-time assumption at the axis of the study (23-01), and this chapter says how far the core row of the saddle-tube condition of its first step could be built. The sector framework has been shown, but the one remaining proposition is not closed by any of the three strands.

The order of this chapter

1. Summary — a section readable at [Y3]
2. The five steps of the sector framework — sum over sectors, Gibbs in marginal form, componentwise localisation, locality of volume, composition

3. The closing condition and the needed value $A_{\max}$ — where the constants come from, and two directions (Tables 1 and 2)
4. The threshold mismatch — an example where the condition, taken literally, gave nothing
5. The three strands of the window log-Sobolev route — componentwise, scalar (the Scalar-Strand Proposition, Table 3), Feynman–Kac, and how far they got (Table 4)
6. The common wall — the curvature at one site at the top, $15/4$ against 4
7. The table of the five grades (Table 5)
8. The dependency graph

01

Summary

The core $Q_k = (-x_c, 0)^k$ is an interval of k sites where the path stays near the top, and here the Hessian of the action has negative directions. To control the gap of the measure of the core (the core Poincaré uniformity assumption) uniformly in k, one must pay using the fact that places with negative curvature are **rare**. The sector framework splits that payment into five steps.

1. **Sum over sectors.** Even when the space of configurations is split into a direct product of boxes, the Dirichlet form, the variance and the entropy split into sums over sectors, and the number of sectors does not enter the constants.
2. **Gibbs in marginal form.** The weight on a rare set B (measure at most $e^{-\ell}$) can be paid with the entropy and $\ln 2$: $\ell \int_B g^2 \le \mathrm{Ent} + \ln 2 \int g^2$.
3. **Componentwise localisation.** Applying the Gibbs step in marginal form to each component of the gradient with the logarithmic Sobolev constant A_W of the marginal of the marks on the 17 sites around that site (the site window), k does not enter the sum.
4. **Locality of volume.** The rarity exponent $\ell_{\min}$ does not depend on k.
5. **Composition.** Combining the above with the two inequalities of Barta and Bochner gives, for the gap of the core λ, $\lambda(1 + bA_W/\ell_{\min}) \ge \kappa - b(A_W c_H + D_W + \ln 2)/\ell_{\min}$. The condition for the right-hand side to be positive is the **closing condition** $b(A_W c_H + D_W + \ln 2) < \kappa \ell_{\min}$.

The closing condition rewritten for A_W is the needed value $A_{\max} = (\kappa \ell_{\min}/b - D_W - \ln 2)/c_H$. At the three values $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$, the original form on the road that defines defects by three sites is negative at $3 \cdot 10^{-3}$ (it closes for no A_W), 0.20 and 7.13, and correcting two choices of tools raises it to $25.1/90.3/312.3$ Reading (arithmetic on the measured rarity exponents). On the other hand, in the current construction the upper bound on A_W is of order $1857/2794/9106$ Reading (including estimates), 29–74 times short.

The one remaining proposition "$A_W < A_{\max}$" was attacked by three strands. The componentwise strand stopped at 1.4–5.5 times the needed value, the scalar strand became a proposition with a table of assumptions (the Scalar-Strand Proposition) for $\hbar \in [3 \cdot 10^{-4}, 10^{-3}]$, and the Feynman–Kac strand lies within 0.54 times the threshold by a Monte Carlo reading, but its upper bound is 10^3 times the threshold. All three stop at the same place: at the top, the diagonal at one site, $4 - \frac{1}{4} = 15/4$, does not reach the sum of the couplings to the neighbours, 4 (§ 06).

02

The five steps of the sector framework

Setting. Take the core $Q_k = (-x_c, 0)^k$ ($x_c = 0.89$), fix the end values $z_0, z_{k+1} \le -x_c$, and let $\mu \propto e^{-S/\hbar} \mathbf{1}_{Q_k}$ and $\mathcal{E}(f) = \hbar \int |\nabla f|^2 d\mu$ (Neumann on the faces of the box). Let λ be the first non-zero eigenvalue of $-L$, f the eigenfunction, $u = \nabla f$, and put

$$X = \int |u|^2 d\mu, \qquad Z = \hbar \int \|\nabla u\|^2 d\mu, \qquad Y = \sum_{j=1}^{k} \int_{N_j} u_j^2 \, d\mu \quad (1)$$

. N_j is the defect set of site j (configurations where j or one of its neighbours lies on the top side of the threshold $-\theta_B$, $\theta_B = 0.57$). Attach a mark $\sigma_j = \mathbf{1}[z_j > -\theta]$ to each site, and write σ_{W_j} for the set of marks on the window W_j around site j. A sector is a direct product of boxes for each type of marks, $\Omega_\sigma = \prod_j I_{\sigma_j}$.

Sector sum

For any partition $\mathcal{P}$ into direct products of boxes, (a) $\sum_P \mu(P) \mathcal{E}_P(f) = \mathcal{E}(f)$ (faces have measure 0, and $|\nabla f|^2$ is the same function on each sector), (b) $\mathrm{Var}_\mu f = \sum_P \mu(P) \mathrm{Var}_{\mu_P} f + \mathrm{Var}_\nu(\mathbb{E}[f \mid \mathcal{P}])$ (the law of total variance; the entropy has a chain rule of the same form), (c) if $\mathrm{Var}_{\mu_P} f \le C_P \mathcal{E}_P(f)$ on each sector, the first term is at most $(\sup_P C_P)\, \mathcal{E}(f)$ Proved here . The number of sectors appears only in the weights $\mu(P)$, which

sum to 1. Lean `var_split_cells` (no assumptions), `cell_sum_sup` (assumptions: $w_i \geq 0$, $E_i \geq 0$, $C_i \leq C_{\max}$).

Gibbs decomposition — the Gibbs step in marginal form

The Gibbs variational inequality. For a probability measure ν, $G \geq 0$ and a function F,

$$\int FG\,d\nu \;\leq\; \mathrm{Ent}_\nu(G) + \Big(\int G\,d\nu\Big)\ln \int e^F\,d\nu \tag{2}$$

Proved here Proof. Put $\tilde{G} = G/\int G$. Jensen's inequality gives $\ln \int e^F\,d\nu \geq \ln \int_{\tilde{G}>0} e^{F-\ln\tilde{G}}\tilde{G}\,d\nu \geq \int (F-\ln\tilde{G})\tilde{G}\,d\nu$. Multiplying both sides by $\int G$ gives (2). ∎

Taking $F = t\mathbf{1}_B$ and $p = \nu(B) \leq e^{-\ell}$ in (2) gives $\ln \int e^F = \ln(1-p+pe^t)$, and with $t = \ell$ this is $\leq \ln 2$ Lean `gibbs_ln2_cell` (assumptions: $p \geq 0$, $\ell \geq 0$, $pe^\ell \leq 1$). If B is a set determined by a coarse variable $\mathcal{F}$, taking $G = \mathbb{E}[g^2 \mid \mathcal{F}]$ gives

$$\ell \int_B g^2\,d\mu \;\leq\; \mathrm{Ent}_\nu\big(\mathbb{E}[g^2 \mid \mathcal{F}]\big) + \ln 2 \int g^2\,d\mu \tag{3}$$

. This is the form that pays for the weight on a rare set with the entropy of the marginal of the coarse variable.

Componentwise localisation

As input we assume, for all j and g, the logarithmic Sobolev inequality of the marginal on the window

$$(\mathrm{LS}_W) \qquad \mathrm{Ent}_{\nu W_j}\big(\mathbb{E}[g^2 \mid \sigma W_j]\big) \;\leq\; A_W\,\mathcal{E}(g) + D_W \int g^2 \tag{4}$$

Assumption , and let $N_j \subset B_j$ (B_j a set determined by σW_j, $\ell_{\min} = \min_j \ln(1/\mu(B_j))$). Applying (3) to $g = u_j$ and summing over j, from $\sum_j \mathcal{E}(u_j) = Z$ and $\sum_j \int u_j^2 = X$,

$$\ell_{\min}\,Y \;\leq\; A_W\,Z + (D_W + \ln 2)\,X \tag{5}$$

holds Proved here Lean `comp_sum_le` (assumptions: $0 < \ell \leq \ell_j$, the inequality for each component). k **does not enter.** For contrast, if (3) is applied to a single union $\bigcup_j B_j$ with a scalar g, without splitting the gradient into components, just putting in the constant function $g \equiv 1$ makes the defect term at least $\sum_j \mu(B_j)$, and it is not uniform in k Proved here Lean `scalar_not_uniform` (assumption: $p > 0$; the conclusion is "there is no d with $kp \leq d$ for all k"). **Uniformity in k and having A_W enter the condition are a trade-off.**

Volume locality

$\ell_{\min}$ is the rarity exponent of the defect set, and does not depend on k. On the road that defines defects by three sites, the measured values $\hbar \ln \mu(N_j)$ give $\ell_{\min} = 38.3/121.9/406.8$ ($\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$) Reading (floating-point transfer matrix; $3 \cdot 10^{-4}$ is an extrapolation, and the remeasured value is 420.0). On the road that defines defects by lumps of length 8 or more, $\ell_{\min} \geq E_8/\hbar - \ln V_{\mathrm{loc}}$ ($E_8 = 0.402$, $\ln V_{\mathrm{loc}} = 2.3$) Reading . To say the same on the bridge, one needs that the remainder of the transfer operator of a scrap is small uniformly in the end values (the bridge volume-locality input) Assumption .

Sector composition

We combine the following three inequalities.

- **Barta**: $\lambda X \geq \kappa X - bY$. It comes from the row inequality $(\mathcal{M}\psi)_j \geq (\kappa - b\mathbf{1}_{N_j})\psi_j$ obtained by applying a weight ψ to the componentwise operator $\mathcal{M} = -L + \operatorname{Hess} S$ ($u = \nabla f$ satisfies $\mathcal{M}u = \lambda u$). $\kappa = 3/1000$ is the lower bound on the rows outside the defects, and $b = 493/1000$ is the depth of the defects, determined by the worst row over the whole box Assumption (the paper estimate of the rows is not placed in the book).
- **(5)**: componentwise localisation itself.
- **Bochner**: $Z = \lambda X - \int u \cdot \operatorname{Hess} S\, u\, d\mu \leq (\lambda + c_H)X$, $\operatorname{Hess} S \geq -c_H$, $c_H = \frac{1}{4}$. The identity is the integrated form of Bochner's formula Cited (Proposition 1.3 of Ledoux's lectures, checked in the original text), and the vanishing of the boundary terms on the flat faces of the box and the smoothness of the eigenfunction up to the corners are Assumption .

Putting Bochner into (5), and that into Barta, gives

$$\lambda\Big(1 + \frac{bA_W}{\ell_{\min}}\Big) \;\geq\; \kappa - \frac{b\,(A_W c_H + D_W + \ln 2)}{\ell_{\min}} \quad (6)$$

Proved here Lean `comp_close` (assumptions: $X > 0$, $b, A \geq 0$, $\ell > 0$, and the three inequalities Barta, (5), Bochner). If the right-hand side is positive, that is, if

$$b\,(A_W c_H + D_W + \ln 2) \;<\; \kappa \ell_{\min} \quad (7)$$

, the gap of the core λ is bounded below by a positive number that does not depend on k. (7) is the **closing condition**. Of the five steps, the constructions of the sector sum, the Gibbs decomposition, componentwise localisation and sector composition have elementary proofs placed in the text, and what remains as assumptions is the marginal log-Sobolev input for the window, the bridge volume-locality input, the Barta rows, the

regularity for Bochner, and whether A_W is small enough to satisfy (7). **What remains as an assumption is not structure but the size of a constant.**

03

The closing condition and the needed value $A_{\max}$

Solving (7) for A_W gives the needed value

$$A_W \;<\; A_{\max}(\hbar) := \frac{\kappa \ell_{\min}/b - D_W - \ln 2}{c_H} \tag{8}$$

follows Proved here (rearranging (7)). That A_W is smaller than this is the one remaining proposition (the Window Log-Sobolev Threshold Proposition), the route of the window log-Sobolev inequality in 23-07 § 02.

Where the constants come from. We sort the six constants of (7) into those that come from the structure and those that come from the choice of tools.

Table 1 Where the constants of the closing condition (7) come from

Constant	Value	Origin	Side
κ	$3/1000$	Lower bound on the Barta rows outside the defects (small because the threshold $\theta_B = 0.57$ is placed just above the one-site threshold 0.5653)	Structure
b	$493/1000$	The worst value of the same rows over the whole box (-0.49 on the top side)	Structure
$\ell_{\min}$	Table 2	The rarity exponent of the defect set (not a length)	Structure
c_H	$\frac{1}{4}$	In the Bochner step, the minimum $w(0) = -\frac{1}{4}$ of the negative curvature $w = \varepsilon V''$ paid **uniformly over the whole** core	Choice of tools
$\ln 2$	0.693	The value with the tilt t in the Gibbs step fixed at $t = \ell$	Choice of tools
D_W	0	A construction that puts no defect term in the marginal log-Sobolev input for the window	—

The reason A_W enters the condition multiplied by c_H is this: when $A_W Z$ on the right of (5) is replaced by $(\lambda + c_H)X$ using Bochner, the λ part can be moved to the left side, but $c_H X$ remains without a factor λ and can only be paid by κ. The amplification $b/\kappa \approx 164$ multiplies $A_W c_H$ here.

Two directions — raising the needed side, lowering the insufficient side

Raising the needed side (correcting the choice of tools).

- **Localising Bochner.** The negative curvature $-w(z) = (1 - 3z^2)/4$ exists only for $z > -1/\sqrt{3}$, and that region is already inside the defect set N_j. From the one-line interval bound $-w(z) \le \delta + \gamma\,\mathbf{1}[z > -\theta_R]$ ($\delta = \max(0, (1 - 3\theta_R^2)/4)$, $\gamma = c_H - \delta$) Proved here Lean `negw_le_localized` (assumptions: $\theta \ge 0$, $z \le 0$, $(1 - 3\theta^2)/4 \le \delta \le \frac{1}{4}$), and using that the kinetic term $2T_k$ is positive semi-definite, we get

$$Z \;\le\; (\lambda + \delta)X + \gamma Y \quad (9)$$

  Proved here Lean `bochner_localized` (assumptions: the identity $Z = \lambda X - H$, $-H \le \delta X + \gamma Y$). Y is a term already paid by (5), so the coefficient of A_W goes down from $bc_H = 0.123$ to $b\delta + \kappa\gamma = 0.0038$ ($\theta_R = 0.57$; about 1/32).
- **Optimising the Gibbs tilt.** t in (2) may also satisfy $t < \ell$, and at $t^* = \ell - \ln((b - \kappa)/\kappa)$ $= \ell - 5.10$, which maximises $\kappa t - b\ln(1 + e^{t-\ell})$, the cost $b \ln 2 = 0.342$ is replaced by 0.0183 Proved here (maximum of a concave function of one variable). Lean's `tilt_cost_le` (assumptions: $b \ge 0$, $L \le 1/178$) shows only an upper bound of 0.0184 on the cost at t = ℓ − 5.2, not at the optimal point Lean (the optimisation itself is not in Lean).

The closing condition with both combined is $b(A_W\delta + D_W + L(t^*)) < \kappa(t^* - A_W\gamma)$, and the needed value is $A_{\max} = (\kappa\ell_{\min} - 0.0183 - bD_W)/(b\delta + \kappa\gamma)$. Raising the threshold θ further to near $1/\sqrt{3}$ gives $\delta \le 0$, and the condition becomes in effect $A_W \lesssim 4\ell_{\min}$.

Table 2 The needed value $A_{\max}$ ($D_W = 0$; triples are $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$; values are arithmetic on the measured $\ell_{\min}$, Reading)

Form	$A_{\max}$	Remarks
Original form (road defining defects by lumps of length 8 or more)	0.43/6.96/29.8	$\ell_{\min} \ge E_8/\hbar - \ln V_{\text{loc}}$
Original form (road defining defects by three sites)	Negative / 0.20 / 7.13	At $3 \cdot 10^{-3}$ it does not close even with $A_W = 0$
Gibbs tilt only	0.78/2.82/9.75	Three-site road
Localised Bochner only	Negative / 6.23 / 228	As above

Form	$A_{\max}$	Remarks
Both combined	25.1/90.3/312.3	As above. The value is Reading . What Lean's `judge_i_inside` shows goes only as far as this: for the two values of ħ on the small side (the cases where the lower bound on ℓ is 406 and 121), the condition for closing holds if $0 \le A \le 22$ (that is, the needed value, namely the upper limit of the A for which it closes, is 22 or more) Lean . The third ħ, and the values themselves, are not in Lean (assumptions: $\ell \ge 406$, $\ell \ge 121$, $0 \le A \le 22$, $0 \le L \le 1/178$)
Raising the threshold to $1/\sqrt{3}$	128/452/1604	$\kappa(\theta)$ and $\ell(\theta)$ are floating-point lattice values

Lowering the insufficient side (rebuilding the upper bound on A_W). The upper bound on A_W for the three-site window $|W| = 3$ is 1857/2794/9106 in the current construction Reading (including estimates), which is **74 / 31 / 29 times** outside the needed value with both combined, and 14.5 / 6.2 / 5.7 times outside the needed value with the threshold raised (the latter is an estimate that assumes the same upper-bound technique gives the same value when the threshold is raised). On the other hand, the lower bound on A_W measured by test functions is 1.71/1.74/2.59 and does not exclude the needed value. **That the true A_W lies within the needed value is not excluded; what falls short is the way the upper bound is built.** This is the starting point of the three strands of the route of the window log-Sobolev inequality.

04

The threshold mismatch — an example where the condition, taken literally, gave nothing

Componentwise localisation uses $N_j \subset B_j$, with B_j a set determined only by the marks of the window σ_{W_j}. In Table 1, κ, b and $\ell_{\min}$ were values measured with the defect threshold $\theta_B = 0.57$, while A_W was measured with the mark threshold $\theta_\sigma = \frac{1}{2}$. If $\theta_\sigma < \theta_B$, the atom of marks "all marks in the window on the valley side" contains points with $z_i \in (-\theta_B, -\theta_\sigma]$, that is, points of N_j. The only set determined by the marks that contains N_j is the whole space, so $\mu(B_j) = 1$ and $\ell = 0$ Proved here (one line of measurability). **With the two thresholds mismatched, (7), taken literally, gives nothing.** The fix is not to lower θ_B to $\frac{1}{2}$ (at $\theta = \frac{1}{2}$ the lower bound κ on the Barta rows becomes negative), but to measure A_W with marks at $\theta_\sigma = \theta_B$, which is an order to the insufficient side. It is an example showing that, when numerical constructions are stacked, a step is needed that checks literally whether each of the conditions being composed holds on the same definitions.

05

The three strands of the window log-Sobolev inequality

The one remaining proposition was attacked in three forms that differ in the functional form of the condition. None of the strands has shown sector composition, the core Poincaré uniformity assumption or the saddle-tube condition. All numbers are values on the 0+1-dimensional double well on the lattice $\varepsilon = \frac{1}{2}$. The full list of detour plans and the types of breakage are sent to 23-10.

The componentwise strand — the componentwise form

On top of the localised Bochner and the Gibbs tilt, adding a form that pays the negative curvature at each point by the rarity of that point (paying the curvature with a function of height $h = (-w(z_j))_+$) raises the needed value to $537/1754/6052$ Reading . The upper bound is built in the product form of "a coarse step" and "a propagation step".

- **The coarse step is exhausted.** With a mark chain that is multi-valued only in the middle, the logarithmic Sobolev constant of the coarse step is $\leq 3.584/3.544/3.613$, a ratio of $1.016/1.005/1.003$ to the lower bound from the indicator of a single point Reading (a rational decomposition on double-precision data). Nothing more can be gained from this step.
- **The propagation step has no upper bound.** Moving the rest into a weighted conditional Poincaré constant C_w, the needed value is $C_w \leq 39.3/138.2/476.3$, and the lower bound from a family of test functions is 15.4–15.6 (almost independent of $\hbar$) Reading . With the Lyapunov-function criterion (of the Bakry–Barthe–Cattiaux–Guillin type Cited , checked in a restatement in a secondary source), the family of Lyapunov functions that do not depend on the far end was $16.6/14.7/14.7$ times outside the needed value on the top face $x \equiv 0$. Measuring by duality the best of a split family extended to the far end goes beyond this leading term, but on the finest grid it stops at $215/355/654 = 5.5/2.6/1.4$ times the needed value Reading (grid values; the lower $\hbar$ goes, the more sensitive to the grid). The floor that holds for all Lyapunov functions is 31 (uniform in $\hbar$) Reading Lean `tilt_g1` (assumption: $\hat{\theta}\,\mathrm{Var} \leq \hbar b$), and the form of the criterion itself can get below the needed value.

The remaining step: a Lyapunov function with a coupling term between neighbours that lowers the finite-$\hbar$ correction. Grades: Reading and Assumption .

The scalar strand — the scalar form, and the Scalar-Strand Proposition

This form does not split the gradient into components: it applies Bakry–Émery to the convex bottom (the box on the valley side) and pays for the band at the top by the Gibbs step at each site. The condition is only $\ln k < \ell_{\mathrm{band}}$, and A_W, κ, b and c_H are not

needed, but k enters as $\ln k$, so it works only over a finite range of core lengths. The Poincaré constant of the core becomes the sum of $C_P^{(1)} \approx 50$–64 and a remainder (Rem_2), and, restricted to the first eigenfunction, the remainder only needs to be small, $R_2(f)/\int g^2 < \frac{1}{4}$. The Scalar-Strand Proposition turns this into a proposition with a table of assumptions.

Scalar-Strand Proposition (with a table of assumptions)

0+1-dimensional double well, $\varepsilon = \frac{1}{2}$, $x_c = 0.89$, core length $k = 20$, central lump $\{9, 10\}$, f the first non-zero eigenfunction. Under the rows of Table 3, at each node $\{3, 4, \ldots, 10\} \cdot 10^{-4}$ of $\hbar \in [3 \cdot 10^{-4}, 10^{-3}]$, the remainder quantity R satisfies $R \le 2A$, with $A = 2kP_v e^{-Ev} \int f^2$ and $2kP_v e^{-Ev} \le \frac{1}{4} e^{-3}$.

Lean `path5_conditional` (assumptions, named: the remainder inequality, the proxy ratio, the rarity, the arithmetic margin $c_0 + q_g + l_\mu \le -0.7$), `p5_assembly` (assumptions: the proxy ratio, the rarity, the threshold, arithmetic, monotonicity).

Table 3 The rows of the Scalar-Strand Proposition (five grades)

Row	Statement (one line)	Grade	Notes
Spectral regularity	Discrete spectrum, existence and regularity of eigenfunctions, simplicity of λ_1	Assumption	Regularity of the Neumann problem on a box with corners does not follow from standard theorems
Logarithmic Sobolev at one site of the band	Logarithmic Sobolev with a defect at one site of the band	Assumption	Not yet
Convex bottom and propagation in the band	Convex bottom and propagation in the band ($C_P^{(1)} \approx 50$–64)	Assumption	The paper argument is not placed in the book
Upper bound on λ_1	Upper bound on λ_1 uniform in k (Rayleigh quotient)	Reading	Interval-arithmetic scheme (1 run per node), not yet run
Localisation in the direction of exit	Estimate of localisation in the direction of exit	Assumption	A paper estimate. Interval-arithmetic scheme with 1 closed formula per node, not yet run. The IMS localisation formula is Cited (checked in a secondary source)
Rarity of the target	Rarity of the target $\mu(\mathrm{Rem}_v) \le e^{l_\mu(\hbar)}$, $l_\mu \le -4.97$	Reading	Floating-point transfer. Interval-arithmetic scheme with 7 runs per node, not yet run

Row	Statement (one line)	Grade	Notes
Rarity on non-convex fibres	Rarity on a non-convex fibre (the proportion of f^2 is at most e^{-7})	Assumption	Markov's inequality and transfer. Interval-arithmetic scheme with 2 runs per node, not yet run. The lower bound on the action difference ΔS_{nc} is not in interval-arithmetic form
Lifting the proxy ratio	Proxy ratio $c_1 \le c_*(\hbar)$ ($c_* \ge 5.39$, $\hbar \le 10^{-3}$)	Assumption	**The only assumption that lifts beyond the measured range**: a linear lift from the measured values for two-site lumps, 0.002–0.033, by +0.048 per step (tolerance ≥ 0.67)
Comparison of level sets	Comparison of level sets $\ln \Phi \le -4.209$	Reading	Interval-arithmetic scheme with 3 runs per node, not yet run
Arithmetic of the margin	Arithmetic of the margin $c_1 + \ln \Phi \le -3.7$	Proved here	Arithmetic
Enclosure between the nodes	Enclosure of $\hbar$ between the nodes	Assumption	Not yet. The range $[3 \cdot 10^{-4}, 10^{-3}]$ is point values at 8 nodes

The number of runs to move to interval arithmetic is 14 per node — upper bound on λ_1 1, localisation in the direction of exit 1, rarity of the target 7, rarity on non-convex fibres 2, comparison of level sets 3 — making 112 over the 8 nodes, plus the enclosures between the nodes, and not a single one has been run yet. Slightly beyond 10^{-3} the two conditions no longer hold together on the same window (they do not hold together at $1.35 \cdot 10^{-3}$ Reading), and it does not extend towards $\hbar \to 0$ either, because the estimate in the direction of exit is not uniform, by $\hbar^{-1/2}$. **The Scalar-Strand Proposition is a proposition with a table of assumptions on a finite range at the lower end of the window.**

The Feynman–Kac strand — the Feynman–Kac form

In the form that keeps the Feynman–Kac weight on the diagonal of the comparison, what is needed is that the logarithmic Sobolev constant of the whole core is $A_{\mathrm{core}} \lesssim a/\hbar$ with $a < 1.29/1.47/1.55$ (the question extends to the coefficient of $1/\hbar$). The lower bound from a family of test functions is $\hbar A_{\mathrm{core}} \approx 0.78$–$0.83$ Reading , a margin of 1.6–1.9 times. Splitting into radial chains, the chain constant is 0.54–0.56 times the needed value (on the model chain) Reading , and what remains is the quantity $K^2 = \sup |\nabla \psi|^2$ in the Helffer–Sjöstrand representation Cited (bibliographic only; the statement is unchecked, so this book treats it as Assumption), with threshold $1 + K^2 \le W$, that is, $K^2 \le W - 1 = 0.83$–0.86 (W is the ratio of the needed value to the chain constant).

- **Monte Carlo reading**: values from below after searching for starting points, $L = 0.4333 \pm 0.0067$ (10^{-3}) and 0.4163 ± 0.0079 ($3 \cdot 10^{-4}$); adding 3σ gives 0.453 = **0.54 times** the threshold 0.835 Reading (the maximum over starting points, not an upper

bound over the whole fibre = the assumption that the maximum over searched starting points suffices).

- **Paper upper bound**: a pathwise Grönwall-type upper bound counted at the same points along the same paths gives $U^2 = 798$ = about 10^3 times the threshold Reading . Most of the width comes from the step where the scalar Grönwall discards the direction of the gradient and reduces every direction at the softest rate. The version that controls the contraction by the smallest eigenvalue of the time-averaged matrix is false Lean `mbar_bound_false` (a two-dimensional counterexample, no assumptions).

The remaining step: an upper bound that keeps the direction (a matrix Lyapunov function along the path, or an estimate split into soft and hard subspaces).

The three side by side

Table 4 How far the three strands got (triples are $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$)

Strand	How far it got	Grade	Range of $\hbar$	The remaining step
2 Scalar	The Scalar-Strand Proposition (with a table of assumptions, one lifting assumption, 112 interval-arithmetic runs not yet made)	Assumption + Reading	8 nodes of $[3 \cdot 10^{-4}, 10^{-3}]$	Interval arithmetic at each node, enclosures between the nodes, making the lifting assumption explicit
3 Feynman–Kac	0.54 times the threshold by the Monte Carlo reading. The paper upper bound is 10^3 times the threshold	Reading	$10^{-3}, 3 \cdot 10^{-4}$	An upper bound that keeps the direction
1 Componentwise	Went beyond the leading term of the Lyapunov criterion, but 5.5 / 2.6 / 1.4 times the needed value. The floor for all Lyapunov functions is 31	Reading	Three points	A Lyapunov function with coupling terms

Table 4 is not a ranking of which strand is "close". The scalar strand has a form only at the lower end of the window, and only the componentwise strand and the Feynman–Kac strand have forms that cover the whole window. Read progress by the grade of each row.

06

The common wall — the curvature at one site at the top, $15/4$ against 4

The Hessian of the action of the core is, on the lattice $\varepsilon = \frac{1}{2}$, the tridiagonal matrix

$$\operatorname{Hess} S = \begin{pmatrix} 4+w_1 & -2 & & \\ -2 & 4+w_2 & -2 & \\ & \ddots & \ddots & \ddots \\ & & -2 & 4+w_k \end{pmatrix}, \qquad w_j = \varepsilon V''(z_j) = \frac{3z_j^2 - 1}{4} \quad (10)$$

(the neighbour term $(z_{j+1} - z_j)^2/(2\varepsilon)$ gives 4 on the diagonal and -2 next to it). Looking at one row, the diagonal is $4 + w_j$, and the sum of the absolute values of the couplings to the neighbours is $2 + 2 = 4$. At the top $z_j = 0$, $w_j = -\frac{1}{4}$, so

$$\text{diagonal} \;=\; 4 - \tfrac{1}{4} \;=\; \tfrac{15}{4} \;<\; 4 \;=\; \text{sum of couplings} \quad (11)$$

, and **diagonal dominance fails by $\frac{1}{4}$** Proved here . Row-by-row estimates (Gershgorin discs) only go as far as "smallest eigenvalue $\geq \min_j w_j = -\frac{1}{4}$", and in fact on the face $x \equiv 0$ where every site is at the top, the smallest eigenvalue of a lump of k sites with both ends fixed is $8\sin^2\frac{\pi}{2(k+1)} - \frac{1}{4}$, negative for $k \geq 8$ Proved here (the closed formula for the eigenvalues of $\operatorname{tridiag}(-2, 4, -2)$). To check by hand: at $k = 7$, $8\sin^2(\pi/16) \approx 0.304 > \frac{1}{4}$; at $k = 8$, $8\sin^2(\pi/18) \approx 0.241 < \frac{1}{4}$.

The three strands came back to this one-line shortfall in changed forms.

- However the Barta weight is chosen, the floor of the row of the one-site weight at the top is $F(0,0) = w(0) = -\frac{1}{4}$.
- The criterion of the uniform road of the componentwise strand was $A - 2B = -\frac{1}{4}$ with diagonal $A = 4 - \frac{1}{4}$ and coupling $B = 2$.
- The top face where the Lyapunov criterion of the componentwise strand stopped is exactly the face $x \equiv 0$ where the gradient of the action vanishes.
- In the Feynman–Kac strand, the coefficient of $1/\hbar$ lived in the radius from the corner at the top (the non-convex direction of the plateau).

Why the wall is universal (one line). Tools whose entrance is convexity (a positive lower bound on the Hessian) — Bakry–Émery, Brascamp–Lieb, the Barta rows, diagonal dominance — must, since the Hessian really has a negative direction on the plateau at the top, pay this $\frac{1}{4}$ somewhere with the rarity (entropy) of the plateau, and $\hbar$ enters that payment. A detailed study of this view, and other viewpoints that do not rely on convexity, are treated in Part 24 (24-06).

07

The table of the five grades — the rows of the sector framework and the route of the window log-Sobolev inequality

Table 5 The rows of this chapter ("—" in the range of $\hbar$ marks an assumption row not shown at any $\hbar$; the numbers in the row column are the numbers of the dependency graph in § 08)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[01]	Marginal log-Sobolev input for the window (the form of logarithmic Sobolev for the marginal on the window)	Assumption	§ 02 (4)	—	—
[02]	Bridge volume-locality input (volume locality on the bridge)	Assumption	§ 02 Volume locality	—	—
[03]	Barta rows $\kappa = 3/1000$, $b = 493/1000$	Assumption	§ 02 Sector composition (the paper argument is not placed in the book)	Independent of $\hbar$	—
[04]	Vanishing of the Bochner boundary terms and regularity of eigenfunctions	Assumption	§ 02 Sector composition	—	—
[05]	The lifting assumption of the Scalar-Strand Proposition and the other Assumption rows of Table 3	Assumption	§ 05 Table 3	—	—
[07]	Integrated Bochner Formula	Cited	Integrated Bochner Formula (Proposition 1.3 of Ledoux's lectures; checked in the original text)	Independent of $\hbar$	—
[08]	$\ell_{\min} = 38.3/121.9/406.8$	Reading	§ 02 Volume locality (transfer matrix, including extrapolation)	Three points	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[09]	Upper bound $1857/2794/9106$ and lower bound $1.71/1.74/2.59$ on A W	Reading	§ 03 (including estimates / family values)	Three points	—
[10]	Sector sum	Proved here	§ 02	Independent of $\hbar$	`var_split_cells` (no assumptions), `cell_sum_sup` (assumptions: $w_i, E_i \geq 0$, $C_i \leq C_{\max}$)
[11]	Gibbs decomposition: the Gibbs variational inequality (2) and (3)	Proved here	Proof in § 02	Independent of $\hbar$	`gibbs_ln2_cell` (assumptions: $p \geq 0$, $\ell \geq 0$, $pe^{\ell} \leq 1$)
[12]	Componentwise localisation (5) ⟸ [01]	Proved here	§ 02	Independent of $\hbar$	`comp_sum_le` (assumptions: $0 < \ell \leq \ell_j$, the inequality for each component)
[13]	The scalar form is not uniform in k	Proved here	§ 02 Componentwise localisation	Independent of $\hbar$	`scalar_not_uniform` (assumption: $p > 0$)
[17]	Sector composition (6) and the closing condition (7)	Proved here	§ 02	Independent of $\hbar$	`comp_close` (assumptions: $X > 0$, $b, A \geq 0$, $\ell > 0$, Barta, (5), Bochner)
[18]	The needed value (8)	Proved here	§ 03 (the values are arithmetic on the measured ℓ_min, hence Reading)	Three points	—
[15]	Localised Bochner (9)	Proved here	§ 03	Independent of $\hbar$	`negw_le_localized`, `bochner_localized` (the assumptions are in the text of § 03)
[16]	Optimising the Gibbs tilt	Proved here	§ 03	Independent of $\hbar$	`tilt_cost_le` (up to the upper bound on the cost at t = ℓ − 5.2; assumptions: $b \geq 0$, $L \leq 1/178$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[19]	The needed value with both combined $25.1/90.3/312.3$	Reading	§ 03 Table 2	Three points	`judge_i_inside` (for two values of $\hbar$, closes if $A \leq 22$; assumptions: lower bounds on the measured ℓ, $A \leq 22$)
[20]	With mismatched thresholds $\ell = 0$	Proved here	§ 04	Independent of $\hbar$	—
[21]	Scalar-Strand Proposition	Assumption	§ 05 the scalar strand (a construction on the assumption lifting the proxy ratio and the row of spectral regularity)	8 nodes of $[3 \cdot 10^{-4}, 10^{-3}]$	`path5_conditional`, `p5_assembly` (the assumptions are in § 05)
[22]	How far the componentwise strand got (1.4–5.5 times the needed value, floor 31)	Reading	§ 05 the componentwise strand	Three points	`tilt_g1` (assumption: $\hat{\theta}\,\mathrm{Var} \leq \hbar b$)
[23]	How far the Feynman–Kac strand got (reading 0.54 times, paper upper bound 10^3 times)	Reading	§ 05 the Feynman–Kac strand	Two points	`mbar_bound_false` (counterexample, no assumptions)
[24]	Diagonal dominance fails by $\frac{1}{4}$ at one site at the top	Proved here	§ 06 (11)	Independent of $\hbar$	—
[06]	Window Log-Sobolev Threshold Proposition $A_W < A_{\max}$	Assumption	The route of the window log-Sobolev inequality	—	—

The "Proved here" rows of Table 5 are all either algebra on assumption rows or general inequalities whose proofs are placed in the text. They do not mean that a lower bound on the gap of the core has been shown unconditionally.

08

The dependency graph

The format is the same as 23-05 § 07, and **the numbers to the right of "⟸" are always placed smaller than the number on the left**.

```
[01] Assumption        marginal log-Sobolev input for the window
[02] Assumption        bridge volume-locality input
[03] Assumption        Barta rows κ = 3/1000, b = 493/1000
[04] Assumption        Bochner boundary terms and regularity
[05] Assumption        Assumption rows of the Scalar-Strand Propositi
[06] Assumption        Window Log-Sobolev Threshold Proposition: A_W
[07] Cited             Integrated Bochner Formula
[08] Reading           ℓ_min = 38.3 / 121.9 / 406.8
[09] Reading           upper and lower bounds on A_W
[10] Proved here       sector sum
                        Lean var_split_cells, cell_sum_sup
[11] Proved here       Gibbs decomposition (Gibbs variational inequal
                        Lean gibbs_ln2_cell (assumptions: p ≥ 0, ℓ ≥
[12] Proved here       componentwise localisation (5) ⟸ [01][10][11]
                        Lean comp_sum_le
[13] Proved here       the scalar form is not uniform in k
                        Lean scalar_not_uniform (assumption: p > 0)
[14] Reading           volume locality: ℓ_min does not depend on k ⟸
[15] Proved here       localised Bochner (9) ⟸ [04][07]
                        Lean negw_le_localized, bochner_localized
[16] Proved here       optimising the Gibbs tilt
                        Lean tilt_cost_le (up to the upper bound)
[17] Proved here       sector composition (6), closing condition (7)
                        Lean comp_close (assumptions: Barta, (5), Boc
[18] Proved here       needed value (8) ⟸ [17]
[19] Reading           needed value 25.1 / 90.3 / 312.3 ⟸ [08][15][1
                        Lean judge_i_inside (for two values of ħ, up
[20] Proved here       with mismatched thresholds ℓ = 0
[21] Assumption        Scalar-Strand Proposition ⟸ [05]
                        Lean path5_conditional, p5_assembly
[22] Reading           how far the componentwise strand got ⟸ [19]
[23] Reading           how far the Feynman–Kac strand got
[24] Proved here       diagonal dominance fails by ¼ at the top
[25] Proved here       positive lower bound on the core gap uniform i
```

How to check. The rows with ⟸ are [12] ([01][10][11]), [14] ([02][08]), [15] ([04][07]), [17] ([03][04][07][12][14]), [18] ([17]), [19] ([08][15][16][18]), [21] ([05]), [22] ([19]) and [25] ([06] [17]). In each the numbers on the right are smaller than the one on the left, so there is no circle.

What to read. The conclusion [25] depends, through [17], on the structural steps ([10]–[14]) and the input assumptions ([01]–[04]), and further on the assumption row [06] (the one remaining proposition). The candidates to support [06] are the three strands [21] [22][23], and none of them shows [06] ([21] is a proposition with a table of assumptions at the lower end of the window, [22][23] are readings). [20] is a checking row: unless [01] and [03] are measured at the same threshold, [12] is empty. [24] explains where the three strands stopped, and is a premise of no conclusion.

We also check that the statements of the assumption rows do not contain the quantity of the conclusion. The quantity of the conclusion is the gap λ of the measure of the core. [01] and [06] are logarithmic Sobolev constants of the marginal of the marks on the window, [02] the remainder of the transfer operator of a scrap, [03] the lower bound on the weighted rows, [04] the boundary terms and regularity, [05] the remainder quantity and the proxy ratio. None of them contains λ.

09

To the next chapters

The full list of detour plans tried in the route of the window log-Sobolev inequality (twenty-two, plus two born along the way, and their derivatives) and the types of breakage that appeared repeatedly are treated in 23-10. The construction that controls the core constant in three steps from the side of the box is 23-11. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter has said how far the core row of the saddle-tube condition of its first step could be built. A viewpoint that handles the negative curvature at the top without relying on convexity is treated in Part 24 (24-06).

What this chapter can say

What is said	What is not said
Proved here The constructions of the sector sum, the Gibbs decomposition, componentwise localisation and sector composition in the sector framework, the closing condition (7), the needed value (8), the localised Bochner and the Gibbs tilt, the consequence of the threshold mismatch, the failure of diagonal dominance at the top Lean (algebra)	The inputs (the marginal log-Sobolev input for the window, the bridge volume-locality input, the Barta rows, regularity)
Reading The needed value $25.1/90.3/312.3$, the gap of 29–74 times to the upper bound on A_W, how far the three strands got	The one remaining proposition (the Window Log-Sobolev Threshold Proposition). Hence the core row (the core Poincaré uniformity assumption) and the saddle-tube condition

What is said	What is not said
Assumption The Scalar-Strand Proposition (8 nodes of $\hbar \in [3 \cdot 10^{-4}, 10^{-3}]$, with a table of assumptions)	Interval arithmetic for the Scalar-Strand Proposition, between the nodes, the upper half of the window and $\hbar \to 0$

Sources and reproduction

Item	Kind	Source / tools
Integrated Bochner Formula	Cited	M. Ledoux, Sém. Probab. XXXV, LNM 1755 (2001) 167–194 (the integrated form of Bochner's formula; text checked).
Lyapunov functions and criteria for local Poincaré	Cited	D. Bakry, F. Barthe, P. Cattiaux, A. Guillin, Electron. Commun. Probab. 13 (2008); D. Bakry, P. Cattiaux, A. Guillin, J. Funct. Anal. 254 (2008) 727–759 (Theorem 3.6 checked in a restatement in a secondary source).
The IMS localisation formula	Cited	H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators*, Springer 1987 (checked in a restatement of § 3).
The Helffer–Sjöstrand representation	Assumption	B. Helffer, J. Sjöstrand, J. Stat. Phys. 74 (1994) 349–409 (bibliographic only; statement unchecked).
The Gibbs variational inequality, sum over sectors, componentwise localisation, composition	Proved here, Lean	Proofs in § 02 of the text. `var_split_cells`, `cell_sum_sup`, `gibbs_ln2_cell`, `comp_sum_le`, `comp_close`, `scalar_not_uniform`.
The two corrections on the needed side	Proved here, Reading, Lean	§ 03. `negw_le_localized`, `bochner_localized`, `tilt_cost_le`, `judge_i_inside`. The rarity exponents from the transfer matrix (midpoint rule, the smaller of grids 600 and 1200).
Numbers of the three strands	Reading, Lean	The componentwise strand: duality and the primal problem on floating-point grids (grids 40–160), `tilt_g1`. The scalar strand: floating-point transfer, `path5_conditional`, `p5_assembly`. The Feynman–Kac strand: Monte Carlo of the adjoint estimator (32 paths, search for starting points), `mbar_bound_false`.

Words used in this chapter — sector framework, sector, mark, site window, defect set, rarity exponent, Gibbs variational inequality, componentwise localisation, closing condition, needed value, choice of tools and structure, threshold mismatch, the three strands, Scalar-Strand Proposition, lifting assumption, the common wall, diagonal dominance. Their meanings are in A-02 Glossary.

Next chapter to read: 23-10 The list of detour plans — types of breakage — puts the detour plans into one table with "the line that killed it", and reads the three sides (the needed side, the form, the floor) and the types of breakage that appeared repeatedly.

The list of detour plans — types of breakage

This chapter is a map of open questions. § 04 is this book's own investigation; § 07 is guide material.

What this chapter shows — We put into one table the twenty-four detour plans tried (the list of detour plans), arranged by which factor of the closing condition of the route of the window log-Sobolev inequality, $b(A_W c_H + D_W + \ln 2) < \kappa \ell_{\min}$, they move. Each row gives "the line that killed it" (for a dead plan) or "the remaining assumption" (for a live plan), and whether the factor moved comes from the structure or from the choice of tools. From the table we extract seven types of breakage that appeared repeatedly, and for each type give the plans it caught and a question with which readers can test their own plans.

Prerequisite chapters — 23-09 § 03 (the closing condition, the needed value, Table 1 of where the constants come from), 23-09 § 06 (the common wall), 23-07 § 02 (the position of the route of the window log-Sobolev inequality).

Said first — The level is [Y3]–[Res], and § 01, § 02 and the names of the types and the questions in § 04 can be read at [Y3]. This part places the diffusive-relaxation-time assumption at the axis of the study (23-01), and this chapter tidies up after the plans tried towards the core row of the saddle-tube condition of its first step. None of the plans has shown the saddle-tube condition. All numbers are values on the 0+1-dimensional double well on the lattice $\varepsilon = \frac{1}{2}$, and triples are in the order $\hbar = 3 \cdot 10^{-3} / 10^{-3} / 3 \cdot 10^{-4}$.

The order of this chapter

1. Summary — a section readable at [Y3]
2. The starting point — the closing condition, and three sides
3. The list of detour plans (Table 1, the count)
4. Types of breakage — seven types, and questions to test your own plan (Table 2)
5. Where the survivors went — into the three strands (Table 3)
6. The table of the five grades — the main lemmas for each type (Table 4)

01

Summary

The one remaining proposition of the route of the window log-Sobolev inequality was that the logarithmic Sobolev constant A_W of the site window lies within the needed value $A_{\max}$ (23-09 § 03). A detour plan moves one side of this inequality, or the form of the inequality itself.

- **The results of the twenty-four.** The twenty-two listed first split into 13 dead, 2 on hold, 3 conditional and 4 alive. Of the 4 alive, 2 (detour plan "localising negative curvature" and detour plan "optimising the Gibbs tilt") enter every later form as parts, and the other 2 (detour plan "scalar payment" and detour plan "replacing the functional inequality"), together with detour plan "paying with a height function", which was born along the way, became the three strands of 23-09.
- **The side of tools and the side of structure.** What moved the needed value a lot were the plans that corrected the choice of tools (localising negative curvature and optimising the Gibbs tilt: $0.20 \to 90.3$ at 10^{-3}, about 460 times), while the plans that moved factors coming from the structure (the depth of defects b, the rarity of the defect set ℓ) all stopped under a ceiling of 1.15 times.
- **Seven types of breakage.** One source (the curvature at one site at the top); a ceiling for structural factors that enter as ratios; paying for the same place twice; mixing up currencies; the address of $1/\hbar$; a mismatch between the direction of fluctuation and the soft direction; transfer. Of the 13 dead plans, 12 fall under one of these.

02

The starting point — the closing condition, and three sides

Copied from 23-09. The sector framework, which controls the gap λ of the core Q_k uniformly in the core length k, combines the three inequalities of Barta, componentwise localisation and Bochner, and closes under the following condition (23-09 (7)).

$$b\,(AW\,cH + DW + \ln 2) \;<\; \kappa\,\ell_{\min}, \qquad AW \;<\; A_{\max} = \frac{\kappa\ell_{\min}/b - DW - \ln 2}{cH} \quad (1)$$

Of the six constants, $\kappa = 3/1000$, $b = 493/1000$ and $\ell_{\min}$ are quantities of the **structure**, determined by the model and by how the defect set is defined, while $cH = \frac{1}{4}$ (the negative curvature paid uniformly over the whole core) and $\ln 2$ (the Gibbs tilt fixed at $t = \ell$) are quantities of the **choice of tools** (23-09 Table 1). In the original form the needed value was negative / 0.20 / 7.13, while the upper bound on AW was $1857/2794/9106$ Reading (including estimates) and the lower bound $1.71/1.74/2.59$ Reading (values from a family of test functions).

What a detour plan is. To close (1), one raises the needed value, lowers the upper bound, or changes the form of the condition. In this chapter we split the plans into three sides by that direction.

Figure 1 The three sides (three columns of text)

The needed side	Form	The insufficient side (floor)
Make $A_{\max}$ larger. Make κ and $\ell_{\min}$ on the right of (1) larger, and b, cH and $\ln 2$ on the left smaller	Change the functional form of (1). A construction in which AW does not enter the condition; a construction with Poincaré instead of logarithmic Sobolev	Lower the upper bound on AW. Reduce the structural loss (floor) of the construction of the upper bound
Localising negative curvature, optimising the Gibbs tilt, raising the threshold, layering defects by depth, narrowing the defect set, defects as long clusters, non-linear Barta weight, extension to the classical limit, Hölder and hypercontractive steps, paying with a height function	Scalar payment, detour plan "Agmon-type payment", detour plan "re-examining downstream needs", replacing the functional inequality	Paying for the summit band, coarse and propagation steps at once, log-Sobolev with clusters fixed, perturbing only the summit band, known uniform log-Sobolev, finite-volume criteria, Glauber log-Sobolev, two-step cluster expansion, one- or two-site windows, localising at the cluster core

The other axis is the character of the factor moved. Factors of the **structure** are quantities determined by the model, and moving them requires changing the definitions of the defect set or the weights. Factors of the **tools** are choices of steps in the proof, and can be changed with the same sets. The "Structure / tools" column of Table 1 is this axis, and "tools → structural floor" marks plans that changed tools but stopped on hitting a structural floor.

03

The list of detour plans

The words of the verdicts in Table 1 mean the following. **Alive** = in that form it came within the needed value (still not a proof of (1); remaining assumptions exist). **Conditional** = the remaining assumptions can be named. **On hold** = not decided on paper. **Dead** = the killing test put it outside the needed value, or no form came out. The line that killed it shrinks to one line the fact that stopped the plan, and all numbers are readings from floating-point grids or transfer matrices Reading (lower bounds are values within a family of test functions).

Table 1 List of detour plans (triples are $\hbar = 3\cdot 10^{-3}/10^{-3}/3\cdot 10^{-4}$; "Type" is the type of breakage in § 04)

Detour plan	Plan (one line)	What it moves	Structure / tools	Result	The line that killed it (dead) / remaining assumption	Type, chapter
Localising negative curvature	Localise the negative curvature of Bochner to the defect set	cH (needed side)	Tools	Alive (absorbed as a part)	Together with optimising the Gibbs tilt, needed value $25.1/90.3/312.3$. Remaining: regularity of the Bochner step; measuring AW with marks at the same threshold as the defects	Double-payment type, 23-09 § 03
Optimising the Gibbs tilt	Set the Gibbs tilt to $t^* = \ell - 5.10$	$\ln 2$ (needed side)	Tools	Alive (absorbed)	Cost $b\ln 2 = 0.342 \to 0.0183$. No remaining assumption (a one-line inequality)	Double-payment type, 23-09 § 03
Raising the threshold	Raise the threshold to $1/\sqrt{3}$, and the remainder of localisation δ to 0	δ, κ (needed side)	Tools	Conditional	The needed value is in effect $AW \lesssim 4\ell$: $128/452/1604$ against the upper bound $1629/2444/8153$ (12.8 / 5.41 / 5.08 times outside). Remaining: the tail outside the window	Type of the ceiling on structural factors
Layering defects by depth	Split the depth of defects b into layers	b (needed side)	Structure	Dead	Factor $1.05/1.01/1.004$. Plans that move only b and κ without changing the set have $A_{\max} < 4\ell$ (ceiling $1.15/1.04/1.01$ times the current value)	Type of the ceiling on structural factors
Narrowing the defect set	Narrow the defect set to the set where the rows fall below κ	$\ell_{\min}$ (needed side)	Structure	Dead	The set became 1.40 times rarer, but the rarity on the Barta side acts only multiplied by $\kappa/(\beta_0 - \kappa) \approx 1/47$ (factor $1.11/1.05/1.02$)	Type of the ceiling on structural factors

Detour plan	Plan (one line)	What it moves	Structure / tools	Result	The line that killed it (dead) / remaining assumption	Type, chapter
Defects as long clusters	Move to the road that defines defects by lumps of length 8 or more	$\ell_{\min}$ (needed side)	Structure	Dead	For short lumps of length ≥ 2, the best lower bound on the Barta rows is negative (-0.0096 at length 2)	Summit-curvature type
Paying for the summit band	Pay for the band at the top as a defect	Floor	Structure	Dead	The rarity of the band and the tail through the band are both $\propto 1/\hbar$ and cancel, and the tail of paths that touch the band and leave equals the whole tail	Type of the address of $1/\hbar$
Scalar payment	Pay for the defects of the sectors in scalar form	Form	Tools	Alive (conditional)	The condition is only $\ln k < \ell_{\text{band}}$; AW, κ, b and c_H are not needed. Remaining: the rows of the table of the Scalar-Strand Proposition	23-09 the scalar strand
Non-linear Barta weight	Make the Barta weight non-linear	b (needed side)	Tools → structural floor	Dead	The worst row rises $-0.4865 \to -0.25$, but the floor at the top $F(0,0) = w(0) = -\frac{1}{4}$ is there for every one-site weight (factor $1.02/1.006/1.002$)	Summit-curvature type
Agmon-type payment	Pay the weight of the defects with the gradient of the true eigenfunction alone (Agmon type)	Form	Tools → structural floor	Dead	Agmon's gain is only where the curvature has a positive margin, and the defect set is the one place where the curvature is negative = the most permissive side	Summit-curvature type
Re-examining downstream needs	Re-examine what is needed downstream	Form	Tools	On hold → absorbed into scalar payment	Downstream, "uniform in k, polynomial in $\hbar$" suffices, but the only tools that give the input uniformly in k were of logarithmic Sobolev type	23-09 the scalar strand
Coarse and propagation steps at once	Do the coarse step and the propagation step at once rather than as a product	Floor	Tools	Conditional	The coarse step is exhausted, with ratio $1.016/1.005/1.003$ to the lower bound. Remaining: the weighted conditional Poincaré constant C_w of the propagation step	23-09 the componentwise strand
Replacing the functional inequality	Poincaré instead of logarithmic Sobolev / modified LSI / Φ-entropy	Form	Tools	Poincaré alive (conditional), the other two dead	Modified LSI only moves $\ln(1/\pi_*)$ to the energy side, and Φ-entropy cannot beat the Poincaré endpoint	Transfer type, the scalar strand
Log-Sobolev with clusters fixed	Logarithmic Sobolev for the conditional measure with the state of the lumps fixed	Floor	Tools	Dead	After conditioning, the measure of the defect set is 0 or 1, and $\ell = 0$	Currency-mix-up type
Perturbing only the summit band	Bakry–Émery outside the window, Holley–Stroock perturbation only for the band at the top	Floor	Tools	Dead	The one-site oscillation $\mathrm{osc}/\hbar = 3.7/11/37$ enters the exponent and is multiplied by the length along the chain; paid in the defect term it is 2000–4900 times the allowance	Type of the address of $1/\hbar$
Known uniform log-Sobolev	Known theorems on uniform logarithmic Sobolev in one dimension	Floor	Tools	Dead	The uniform constant is a finite-volume constant for a box covering the correlation length (length $\gtrsim 230$–810), and is not below the window constant	Transfer type
Finite-volume criteria	Finite-volume criteria (of the Martinelli–Olivieri and Lu–Yau types)	Floor	Tools	Dead	Same as detour plan "known uniform log-Sobolev": the box of the criterion must cover the correlation length	Transfer type
Glauber log-Sobolev	Known logarithmic Sobolev for Glauber dynamics	Floor	Tools	Dead	The coarse step is not the limit, and the Glauber constant ≥ 39–400 is if anything worse	—
Two-step cluster expansion	Two-level cluster expansion, block spins	Floor	Tools	Dead	The pin of the largest path stays at the top, and the tail gain ≤ 1.8 (10^{-3}), ≤ 3 ($3 \cdot 10^{-3}$) falls short of the needed 5–13 times	Transfer type

Detour plan	Plan (one line)	What it moves	Structure / tools	Result	The line that killed it (dead) / remaining assumption	Type, chapter
Extension to the classical limit	Take $\hbar$ below the window (the classical limit)	Needed side (range)	Structure	On hold	Removing the band, the contraction of the boundary map is $0.976/0.952/0.929 < 1$ (band width $0.1/0.2/0.3$), and the tail is bounded independently of $\hbar$. Remaining: the self-loop lemma for the boundary and the band lemma (detour plan "paying for the summit band" is dead)	—
One- or two-site windows	Make the window one or two sites (half rows)	Floor	Tools	Conditional (small gain)	The effect is about 2/3, a factor gap of $8.45/3.59/3.38$. The propagation tail remains	Transfer type
Localising at the cluster core	Localise at the core of the lump instead of the window	Floor	Tools	Dead	Even in the limit with the window tail set to 1, $4.2/1.8/1.7$ times outside (the origin of the floor does not depend on how the window is chosen)	Transfer type
Hölder and hypercontractive steps	Replace the Gibbs step by Hölder and hypercontractivity	The needed side	Tools	On its own, dominated by detour plan "raising the threshold"	$\sqrt{k}$ does not appear, but A enters $\ln C^2$ linearly and the ceiling is the same, $\approx 4\ell$ (needed value 0.80–0.91 times that of raising the threshold). Its merit is that measurability of the set of marks is not needed	Type of the ceiling on structural factors
Paying with a height function	Pay the negative curvature by a function of height $h = (-w(z_j))_+$ (the curvature at each point by the rarity of that point)	The needed side	Tools	Conditional	Needed value $537/1754/6052$ (3.8–4.2 times that of raising the threshold). Dead if Rothaus's lemma is used as it is. Remaining: the propagation tail	The componentwise strand; together with detour plan "Hölder and hypercontractive steps", the Feynman–Kac strand

Counting. The first twenty-two (from localising negative curvature to localising at the cluster core in Table 1) are 4 alive (localising negative curvature and optimising the Gibbs tilt absorbed as parts; scalar payment and the Poincaré form of replacing the functional inequality conditional), 3 conditional (raising the threshold, detour plan "coarse and propagation steps at once", detour plan "one- or two-site windows"), 2 on hold (re-examining downstream needs, absorbed into scalar payment; detour plan "extension to the classical limit") and 13 dead. The 13 dead split into 8 on the side of the floor (paying for the summit band, log-Sobolev with clusters fixed, perturbing only the summit band, known uniform log-Sobolev, finite-volume criteria, Glauber log-Sobolev, two-step cluster expansion, localising at the cluster core), 4 on the needed side (detour plan "layering defects by depth", detour plan "narrowing the defect set", detour plan "defects as long clusters", detour plan "non-linear Barta weight") and 1 on form (Agmon-type payment). Of the two born along the way, Hölder and hypercontractive steps is, on its own, dominated by raising the threshold, and paying with a height function is conditional and entered the componentwise strand; the form combining it with Hölder and hypercontractive steps became the Feynman–Kac strand.

Contrasting the side of tools with the side of structure. Putting the "Structure / tools" column of Table 1 next to the results column, every row that moved the needed value a lot is "tools". Localising negative curvature and optimising the Gibbs tilt kept the same defect set and only corrected how the negative curvature is paid and how the Gibbs tilt is chosen, and moved the needed value from negative / 0.20 / 7.13 to $25.1/90.3/312.3$ (about 460 times at 10^{-3}, about 44 times at $3 \cdot 10^{-4}$, from negative to positive at $3 \cdot 10^{-3}$) Reading . The "structure" rows (layering defects by depth, narrowing the defect set, defects as long clusters, paying for the summit band) and the rows that hit a structural floor (non-linear Barta weight, Agmon-type payment), on the other hand, did not exceed a factor 1.15, or no form came out. Most of the loss on the needed side lay not in the structure of the model but in the choice of tools. Conversely, the gap that remains after the tools have been fully corrected (5–13 times in the form of raising the threshold) goes back to the structural floor of the summit-curvature type in § 04.

▶ List of derivatives (forms tried within the strands, collapsed)

04

Types of breakage

Lining up the lines that killed the plans in Table 1 and Table 1′, the same reasons appear again and again in changed forms. We group them into seven types. For each type we give the plans it caught, a paragraph on why plans break in that way, and a question to test your own plan. The types are not mutually exclusive, and one plan can fall under two types.

The summit-curvature type — one source (the curvature at one site at the top)

Plans caught: defects as long clusters, non-linear Barta weight, Agmon-type payment, and the uniform road and the Lyapunov criterion without sectors in Table 1′.

In one row of the Hessian of the action of the core, at the top the diagonal $4 - \frac{1}{4} = \frac{15}{4}$ falls $\frac{1}{4}$ short of the sum of the couplings to the neighbours 4 (23-09 (11)). This shortfall appears with the same value in the Barta weight rows, in the contraction criterion for chains and in the Lyapunov-function criterion, on the top face $x \equiv 0$ (every site at the top). For example, the Barta row of a one-site weight $\psi_j = \phi(z_j)$ has the form $F(z, n) = w(z) + (\text{log-derivative of the weight}) \times (\text{drift})$, and at the top $z = 0$ with neighbours $n = 0$ the drift $\varepsilon V'(0) + 8(0 - 0)$ is 0, so however the weight is chosen Proved here

$$F(0, 0) = w(0) = -\tfrac{1}{4} \ (2)$$

(a one-line computation on the same assumption row as the Barta rows of 23-09) Lean `floor_top` (assumption: weight value $/= 0$). The Agmon type in Agmon-type payment is a tool that gains where the curvature has a positive margin, and the defect set is exactly where the curvature is negative, so the coefficient of gain there is ≤ 0 Lean `agmon_no_gain_on_negative_curvature` (assumptions: curvature $w \leq 0$, $\lambda \geq 0$ and others; the framework of Agmon estimates is not placed in the book). In defects as long clusters, defining defects by long clusters made the rows of short clusters of length 2 or more negative, and the same shortfall came out on the side of short clusters.

Question to test: what does your plan pay on the top face $x \equiv 0$? If the quantity paid there is a constant independent of $\hbar$, the shortfall of $\frac{1}{4}$ remains as it is. Only the rarity (entropy) of the plateau at the top can pay for this shortfall, and $\hbar$ enters that payment.

The type of the ceiling on structural factors — structural factors that enter as ratios have a ceiling

Plans caught: layering defects by depth, narrowing the defect set, Hölder and hypercontractive steps, and the ceiling of raising the threshold.

Raising the threshold, with remainder of localisation $\delta \geq 0$, remaining coefficient γ, Gibbs tilt t and $L(t) = \ln(1 + e^{t-\ell})$, the closing condition is $b(A\delta + D_W + L(t)) < \kappa(t - A\gamma)$ (the form of localising negative curvature, optimising the Gibbs tilt and raising the threshold).

Ceiling lemma. If $0 < \kappa < b$ and $\delta, D_W \geq 0$, any $A \geq 0$ satisfying this condition satisfies $A\gamma < \ell$.

Proved here Proof. The left-hand side is at least $bL(t) > 0$, so $t - A\gamma > 0$. From $b > \kappa$, $b(t - A\gamma) > \kappa(t - A\gamma) > bL(t) \geq b(t - \ell)$ ($\ln(1 + e^x) \geq x$). Dividing by b gives $A\gamma < \ell$. ∎ Lean `ceiling_same_set` (assumptions: $0 < \kappa < b$, $t - A\gamma > 0$, $t - \ell \leq L$, the closing condition)

Plans that move only b and κ without changing the defect set (hence ℓ) cannot push the needed value above ℓ/γ (4ℓ in the form of raising the threshold). The needed value in the form of raising the threshold is already at $1/1.15/1/1.04/1/1.01$ of the ceiling Reading , and the $1.05/1.01/1.004$ times of layering defects by depth lies within it. In narrowing the defect set, which changes the set, the rarity on the Barta side (ℓ_B of the set where the rows fall below κ) enters the condition only multiplied by $\kappa/(\beta_0 - \kappa) \approx 1/47$ Assumption (the paper argument is not placed in the book) Lean `ceiling_split`, and making the set 1.40 times rarer stopped at a factor of 1.11. Hölder and hypercontractive steps changed the tool of the Gibbs step, but A enters in the same form, so the ceiling is the same.

Question to test: does your plan change the defect set? If not, the needed value does not exceed ℓ/γ. If it does, first write down which coefficient of the condition the rarity of the new set is multiplied by when it enters.

The double-payment type — the same place was paid for twice

Plans caught (on the side that fixed it): localising negative curvature, optimising the Gibbs tilt, and the construction of the window in Table 1′.

The Bochner step of the original construction paid the negative curvature $-\frac{1}{4}$ over the whole core X. But negative curvature exists only for $z > -1/\sqrt{3}$, and that region is inside the defect set already paid by the Gibbs step as Y. Paying the same place again on the X side made the coefficient of A_W equal to $bc_H = 0.123$, through the amplification $b/\kappa \approx 164$. Localisation (localising negative curvature) brings the coefficient down to 0.0038 (23-09 (9)). Fixing the Gibbs tilt at $t = \ell$ (optimising the Gibbs tilt), and the $(1 + \sqrt{C_w})^2$ in the construction of the propagation step (counting the same quantity twice), are of the same type. This is a type that has a large effect when fixed.

Question to test: is the quantity paid in that step on a set already paid for by an earlier step? Drawing the places where the two steps pay in the same figure makes it visible.

The currency-mix-up type — speed of exit and rarity

Plans caught: detour plan "log-Sobolev with clusters fixed", the form that pays on the band by rarity and the double threshold in Table 1′, and the naive decomposition of the variance in 23-11 § 04.

What a Poincaré inequality measures is the speed of exit (the ratio of the variance to the Dirichlet form), and what can pay for rarity $\mu(B) \le e^{-\ell}$ is the currency of entropy (23-09 (3): $\ell \int_B g^2 \le \mathrm{Ent} + \ln 2 \int g^2$). The ratio of the variance to the Dirichlet form does not change when the function is multiplied by a constant, so putting in a function supported on a rare set makes the rarity cancel between numerator and denominator. "The fluctuations are small because the path sticks to the band" is an intuition about the variance, and what the step that pays for rarity needs is an estimate of entropy. Log-Sobolev with clusters fixed is more direct: if the defect set is determined by the σ-algebra of the conditioning, the measure of that set under the conditional measure is 0 or 1 Proved here (the definition of conditional probability), and no rarity ℓ is left to pay with.

Question to test: does the step that pays for the rarity $\mu(B)$ pass through entropy (or an inequality of the same type)? If it uses only the variance and the Dirichlet form, rarity has no effect. Also check whether rarity is still left after conditioning.

The type of the address of 1/ħ — the address of $1/\hbar$

Plans caught: paying for the summit band, detour plan "perturbing only the summit band", and in Table 1′ the linear collective coordinate, the sector-split chain, the multi-stage cut at the wall, the chain of wall marks, and the chain with the mean height added.

What the Feynman–Kac strand needs is the coefficient of $1/\hbar$ in the logarithmic Sobolev constant of the core. If one mistakes which quantity that $1/\hbar$ lives in, the plan only moves $1/\hbar$ to another place. Three facts came up again and again.

- **$1/\hbar$ lives not in $\sum z$ but in the radius.** On the plateau at the top, the action changes not in the direction of the linear collective coordinate $\sum_j z_j/\sqrt{k}$ but along the radius from the corner at the top (the direction of $\sum_j z_j^2$), and its length is the crossing length $\ell_\times \approx 0.34/\hbar$ Reading . In the two-scale split into the collective coordinate and the transverse directions, the smallest curvature in the directions orthogonal to the collective coordinate on the top face, $4 - 4\cos(2\pi/(k+1)) - \frac{1}{4}$, becomes negative at $k = 17$ Proved here (the closed formula for tridiagonal eigenvalues) Lean `transverse_fails_k17` (no assumptions), so convexity holds only for $k \le 16$.
- **Cutting into sectors only allocates $1/\hbar$.** The chain rule of entropy $\mathrm{Ent} = \mathrm{Ent}(\text{marginal of the marks}) + \mathbb{E}\,\mathrm{Ent}(\text{inside the sector})$ is an identity Proved here Lean `chain_rule_cells` (assumptions: the chain rule, upper bounds on the two terms), and cutting the plateau at the position of the wall allocates $\approx 0.8/\hbar$ between the side of the marks and the side of the sectors Reading . The sum does not decrease.
- **A mark represents only the one step that crosses the face.** In the chain of wall marks the constant of the chain itself becomes small, but a mark does not distinguish positions on the plateau and represents only the step across the face, so $1/\hbar$ moved into the constant $\ge 13/26/66$ that compares the mark chain with the original measure Reading . Whether a variable added to the chain is good or bad is decided by whether its rate function is flat at the extremum (if flat, the cost of adding it is $O(1)$) Lean `S_ext_01_iff` (algebra); the mean height is not flat and pays $\propto 1/\hbar$, while the radius passed.

Paying for the summit band (the rarity of the band and the tail through the band are both $\propto 1/\hbar$ and cancel) and perturbing only the summit band (the cost of the Holley–Stroock perturbation $e^{\mathrm{osc}/\hbar}$) are also examples that only moved the place where $1/\hbar$ is paid.

Question to test: in your plan, which quantity is $1/\hbar$ attached to? If it has only moved to a sector, a mark, a comparison constant or the cost of a perturbation, it has not disappeared. Draw the rate function of the variable you add near its extremum.

The type of the misaligned fluctuation direction — mismatch between the direction of fluctuation and the soft direction

Plans caught: in Table 1′, radius and angles on the sphere, the three steps, the variance version of two-scale, and the interior radius; and the paper upper bound of the Feynman–Kac strand.

In a two-scale composition, taking the gradient outside the fibre with $\sup$ (the direction of fluctuation) and paying for it with the constant of the softest direction of the fibre (of the type $\kappa \times A_{\mathrm{fib}}$) makes the upper bound too large by orders of magnitude wherever the two directions are not aligned. Even in the version replaced by the variance, the lower bound on the loss, 0.66–1.8, was outside the needed 0.18–0.50 Reading . What must be paid is the gradient of the solution of the Poisson equation on the fibre ($|M^{-1}\nabla_\perp G|$ in the Helffer–Sjöstrand representation), and it must be estimated while keeping the direction. The gap in the Feynman–Kac strand, where the Monte Carlo reading is 0.54 times the threshold and the paper upper bound 10^3 times the threshold, also came from the step where the scalar Grönwall discards the direction of the gradient and reduces every direction at the softest rate (23-09 the Feynman–Kac strand).

Question to test: somewhere in your upper bound, have you crushed a vector quantity into a scalar? If there is such a step, it pays at the rate of the softest direction there. Draw the direction of large fluctuation and the soft direction in the same figure.

The transfer type — the weight only moves to another column

Plans caught: the modified LSI of replacing the functional inequality, known uniform log-Sobolev, detour plan "finite-volume criteria", detour plan "two-step cluster expansion", one- or two-site windows, detour plan "localising at the cluster core", and the proxy series in Table 1′.

Known uniformity theorems (known uniform log-Sobolev, finite-volume criteria) do not make the constant; they move the problem to the finite-volume constant of a box covering the correlation length (length $\gtrsim$ **230–810**). The window is finite, so existence was trivial from the start, and the reason they do not help is size. Plans that change how the window is taken (one- or two-site windows, localising at the cluster core) reduced the loss of the window, but the floor of the propagation step remained. In the cluster expansion (two-step cluster expansion) the pin of the largest path stayed at the top, and in the modified LSI (replacing the functional inequality) $\ln(1/\pi_*)$ moved to the energy side. In the proxy series of the Scalar-Strand Proposition, when the assumption column became lighter by changing proxy or direction, the computed column became heavier by the same amount (an identity in which the sum of the two columns does not depend on the direction) Assumption (the definitions are not placed in the book) Lean `threshold_gap_invariant` (assumption: the identity of the two sums).

Question to test: in your plan, has another column (the size of a box, the propagation step, a computed quantity) become heavier by as much as one column became lighter? If the sum is invariant, the margin has not moved.

Table 2 Types of breakage (one line each)

Type	One line	Plans caught	Question to test
Summit-curvature type	At one site at the top, the diagonal $15/4$ falls $\frac{1}{4}$ short of the coupling 4	Defects as long clusters, non-linear Barta weight, Agmon-type payment, the uniform road, Lyapunov without sectors	What is paid on the top face $x \equiv 0$
Type of the ceiling on structural factors	Moving b and κ without changing the set still gives $A\gamma < \ell$	Raising the threshold, layering defects by depth, narrowing the defect set, Hölder and hypercontractive steps	Does it change the defect set
Double-payment type	A later step pays again for a place an earlier step paid for	Localising negative curvature, optimising the Gibbs tilt (on the side that fixed it), the construction of the window	Have you drawn the places paid in the same figure
Currency-mix-up type	Poincaré is the speed of exit; rarity is the currency of entropy	Log-Sobolev with clusters fixed, the band by rarity, the double threshold, the naive decomposition of the variance	Does the step that pays for rarity pass through entropy
Type of the address of 1/ħ (the address of $1/\hbar$)	$1/\hbar$ lives in the radius, and sectors and marks only allocate it	Paying for the summit band, perturbing only the summit band, collective coordinate, sector-split chain, multi-stage, mark chain, mean height	Which quantity is $1/\hbar$ attached to
Type of the misaligned fluctuation direction	Paying a $\sup$ gradient in the softest direction is too large by orders of magnitude	Angles and radius, three steps, variance version, interior radius, Grönwall	Have you crushed a vector into a scalar
Transfer type	Another column becomes heavier by as much as one becomes lighter	Replacing the functional inequality, known uniform log-Sobolev, finite-volume criteria, two-step cluster expansion, one- or two-site windows, localising at the cluster core, the proxy series	Is the sum an invariant

Among the dead and on-hold plans, those whose type column in Table 1 is "—" are detour plan "Glauber log-Sobolev", which fixed a step that was not the limit, and extension to the classical limit, which is on hold (for the alive and conditional plans that column gives the chapter they went to). This table format, which keeps the line that killed each dead plan, can be used as it is to tidy up after sorting out detour plans for other problems too.

05

Where the survivors went — into the three strands

The alive and conditional plans were gathered into the table of needed values and the three strands. How far the three strands got and their remaining steps are in 23-09 Table 4, and here we list only the correspondence from the plans.

Table 3 Where the surviving plans went

Plan	Destination	What remains there
Localising negative curvature, optimising the Gibbs tilt (, raising the threshold)	The table of needed values (23-09 Table 2). Parts of every later form	Regularity of the Bochner step; measuring A_W with marks at the same threshold as the defects (23-09 § 04)
Paying with a height function, coarse and propagation steps at once (and the derivatives of the componentwise strand in Table 1′)	The componentwise strand — the componentwise form	An upper bound on C_w of the propagation step (a Lyapunov function with coupling terms)
Scalar payment, the Poincaré form of replacing the functional inequality, re-examining downstream needs	The scalar strand — the scalar form (the Scalar-Strand Proposition)	The interval-arithmetic rows, between the nodes, the one lifting assumption

Plan	Destination	What remains there
The form combining Hölder and hypercontractive steps with paying with a height function	The Feynman–Kac strand — the Feynman–Kac form	An upper bound that keeps the direction (the type of the misaligned fluctuation direction)

The place where all three strands stopped is the source of the summit-curvature type, the same as the common wall of 23-09 § 06. Another construction that controls the core constant from the side of the box rather than by the sector framework is treated in 23-11.

06

The table of the five grades — the main lemmas for each type

This chapter is a list, so its rows are limited to the lemmas supporting the types and the values of Table 1. None of the rows is a premise of the conclusion of the route of the window log-Sobolev inequality, and the dependence of the conclusion is in the graph of 23-09 § 08. So this chapter has no dependency graph.

Table 4 The rows of this chapter

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
Floor at the top	The floor of the Barta row of a one-site weight $F(0,0) = w(0) = -\frac{1}{4}$	Proved here	§ 04 (2) (the form of the row is on the Assumption row of 23-09 (the assumption of the Barta row))	Independent of $\hbar$	`floor_top` (assumption: weight value $/= 0$)
Agmon's gain	Where the curvature is negative, the coefficient of Agmon's gain is ≤ 0	Assumption	§ 04 summit-curvature type (the framework of Agmon estimates is not placed in the book)	—	`agmon_no_gain_on_negative_curvature` (algebra)
Ceiling lemma	The ceiling for plans that do not change the set, $A\gamma < \ell$	Proved here	Proof of the type of the ceiling on structural factors in § 04	Independent of $\hbar$	`ceiling_same_set` (assumptions: $0 < \kappa < b$, $t - A\gamma > 0$, $t - \ell \leq L$, the closing condition)
Rarity on the Barta side	The rarity on the Barta side enters only multiplied by $\kappa/(\beta_0 - \kappa)$	Assumption	§ 04 type of the ceiling on structural factors (the paper argument is not placed in the book)	—	`ceiling_split` (algebra)
Fixing the double payment	Localised Bochner and the Gibbs tilt (fixing the double payment)	Proved here	23-09 § 03	Independent of $\hbar$	`bochner_localized`, `tilt_cost_le` (the assumptions and scope are in 23-09)
Rarity after conditioning	Under the conditional measure, a set determined by the σ-algebra of the conditioning has measure 0 or 1	Proved here	§ 04 currency-mix-up type	Independent of $\hbar$	—
Transverse convexity	On the top face, the smallest curvature orthogonal to the collective coordinate is negative at $k = 17$	Proved here	§ 04 type of the address of $1/\hbar$ (closed formula)	Independent of $\hbar$	`transverse_fails_k17` (no assumptions)
Chain rule of entropy	The chain rule of entropy (cutting into sectors does not decrease the sum)	Proved here	§ 04 type of the address of $1/\hbar$ (identity)	Independent of $\hbar$	`chain_rule_cells` (assumptions: the chain rule, upper bounds on the two terms)
Values of the address of $1/\hbar$	The address of $1/\hbar$ (crossing length $0.34/\hbar$, allocation $\approx 0.8/\hbar$, comparison constant $\geq 13/26/66$)	Reading	§ 04 type of the address of $1/\hbar$	Three points	—
Loss from the mismatch of directions	The loss from the mismatch of directions (lower bound 0.66–1.8 against needed 0.18–0.50)	Reading	§ 04 type of the misaligned fluctuation direction	Three points	—
Invariance of the proxy series	Invariance of the sum of the two columns of the proxy series	Assumption	§ 04 transfer type (the definitions are not placed in the book)	—	`threshold_gap_invariant` (assumption: the identity of the two sums)
Reading of known uniformity	Known theorems on uniform logarithmic Sobolev in one dimension give the constant of a box	Assumption	§ 04 transfer type (the statements in the original sources have not been checked)	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
Values of Table 1	Factors, ceilings and needed values of Tables 1 and 1′	Reading	§ 03 (floating-point grids and transfer matrices; lower bounds within families)	Three points	—

The "Proved here" rows are general facts that identify the types, and do not mean that any detour plan succeeded. None of the rows of Table 4 shows the closing condition (1).

07

To the next chapter

The construction that controls the core constant in three steps from the side of the box-reduction condition and the gap-inside-the-box condition rather than the sector framework, and the drawing of the boundary of the end sets, are in 23-11. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter tidied up after the plans aimed at the core row of the saddle-tube condition of its first step. Another viewpoint that handles the negative curvature at the top without relying on convexity is treated in Part 24 (24-06).

What this chapter can say

What is said	What is not said
Proved here The floor at the top (2), the ceiling for plans that do not change the set, the disappearance of rarity after conditioning, the threshold of transverse convexity, the chain rule of entropy Lean (algebra)	The closing condition (1). Hence the one remaining proposition of the route of the window log-Sobolev inequality and the saddle-tube condition
Reading The factors and needed values of Table 1; that plans on the side of tools moved the needed value by 44–460 times while plans on the side of structure stopped under a ceiling of 1.15 times	That the types exhaust all plans. There are plans that fit no type (Glauber log-Sobolev, extension to the classical limit)
Assumption Agmon's gain, how the rarity on the Barta side enters, the invariance in the proxy series, the reading of known uniformity theorems	That the propositions of the dead plans are false (death is a property of the tool, not the truth of the proposition)

Sources and reproduction

Item	Kind	Source / tools
The integrated form of Bochner's formula	Cited	M. Ledoux, Sém. Probab. XXXV, LNM 1755 (2001) 167–194 (Proposition 1.3; text checked).
The Holley–Stroock perturbation	Cited	R. Holley, D. Stroock, J. Stat. Phys. 46 (1987) 1159–1194 (type checked in the restatement as the Holley–Stroock Perturbation Lemma (Proposition 6.1 of Ledoux 2001)).
Rothaus's lemma	Cited	O. S. Rothaus, J. Funct. Anal. 64 (1985) 296–313 (statement checked in a restatement).
Uniform logarithmic Sobolev for unbounded spins, finite-volume criteria	Assumption	Th. Bodineau, B. Helffer, J. Funct. Anal. 166 (1999) 168–178; F. Martinelli, E. Olivieri, Comm. Math. Phys. 161 (1994) 447–486, 487–514 (bibliographic only; statements unchecked).
Glauber dynamics of the one-dimensional Ising model	Assumption	R. J. Glauber, J. Math. Phys. 4 (1963) 294–307 (bibliographic only; values are readings from diagonalisation on the lattice).
The Helffer–Sjöstrand representation	Assumption	B. Helffer, J. Sjöstrand, J. Stat. Phys. 74 (1994) 349–409 (bibliographic only; statement unchecked).
Lemmas of the types	Proved here, Lean	§ 04 of the text. `floor_top`, `ceiling_same_set`, `ceiling_split`, `agmon_no_gain_on_negative_curvature`, `transverse_fails_k17`, `chain_rule_cells`, `S_ext_01_iff`, `threshold_gap_invariant`.
Numbers of Tables 1 and 1′	Reading	Floating-point grids, transfer matrices (midpoint rule), lower bounds from families of test functions, rational decompositions. Interval arithmetic is not used.

Words used in this chapter — detour plan, the line that killed it, needed side, form, insufficient side (floor), structure and choice of tools, ceiling, types of breakage, paying twice, currency (speed of exit and entropy), the address of $1/\hbar$, crossing length, direction of fluctuation and soft direction, transfer. Their meanings are in A-02 Glossary.

Next chapter to read: 23-11 Box reduction and the gap inside the box — the constant of the core — turns box reduction and the gap inside the box into propositions, and controls the core constant in three steps.

Box reduction and the gap inside the box — the constant of the core

> This chapter is this book's own investigation. The established knowledge it builds on is in 2-05 and 2-07.

What this chapter shows — We state Step 1 (box reduction) and Step 2 (inside the box) of the Double-Well Relaxation-Time Upper-Bound Proposition of the first step (23-05 § 04) as the Box-Reduction Proposition (box reduction, using only the Poincaré inequality) and the Gap-Inside-the-Box Proposition (the gap inside the box, with a table of assumptions), and we give the proof of the product inequality for boxes, which 23-05 graded "Proved here". Next, we bound the constant of the core that enters the fast side of the Gap-Inside-the-Box Proposition in three stages by the length of the core (convexity, two blocks, and the reduction to the box applied to the core). Finally, we state how far the set of values at the ends of the core is covered, and how the bistability that appears at ends beyond the top is paid for by the margin of the action barrier.

Prerequisite chapters — 23-05 § 03, § 04 (the statements of the box-reduction condition and the gap-inside-the-box condition, and the single line of argument), 23-04 (the path measure, the correlation length ξ, kinks), 23-09 § 06 (the curvature of one site at the top).

Said first — The level is [Res], and only § 01 is written so that it can be read at [Y3]. § 02 onwards are sections at the research level; you can skip them and go to the table of remaining rows in § 07 and still read the conclusion of this chapter. This part places the diffusive-relaxation-time assumption at the axis of the study (23-01), and this chapter states how far the box-reduction condition and the gap-inside-the-box condition of its first step could be put in the form of propositions. The Box-Reduction Proposition and the Gap-Inside-the-Box Proposition are propositions that leave assumption rows in place; they do not show the box-reduction condition or the gap-inside-the-box condition. All numbers are values on the 0+1-dimensional double well $V = (x^2-1)^2/8$ on the lattice $\varepsilon = \frac{1}{2}$, and triples are in the order $\hbar = 3\cdot 10^{-3}/10^{-3}/3\cdot 10^{-4}$.

The order of this chapter

1. Summary — a section readable at [Y3]
2. Box-Reduction Proposition — box reduction (Figure 1, Table 1)
3. Proof of the product inequality for boxes — the Markov property and Efron–Stein
4. Gap-Inside-the-Box Proposition — the gap inside the box (Table 2)
5. Three stages for the constant of the core — the Three-Range Core-Constant Proposition (Tables 3 and 4)
6. The set of ends — what is covered, and how to pay for bistability
7. The list of remaining rows (Table 5)
8. The table of the five grades (Table 6)
9. The dependency graph

01

Summary

The box-reduction condition bounds the Poincaré constant of a long path by the Poincaré constant of a box of length $C\xi$, and the gap-inside-the-box condition bounds the gap inside that box at the order of $D/(C\xi)^2$.

1. **Box-Reduction Proposition.** Cover the line with overlapping boxes and split them into even boxes and odd boxes; then $C_P \le (1+1/n)\,C_{\mathrm{box}}/(1-c_{\mathrm{alt}})$. It uses only the Two-Projection Lemma, the product inequality for boxes and averaging over shifts, and does not use the logarithmic Sobolev inequality. The remaining assumption rows are that the maximal correlation between even and odd, $c_{\mathrm{alt}} < 1$, is uniform, and the sup over the boundary values. The correlation length is read as $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$, and a box does not cut the plateau at the top.
2. **Gap-Inside-the-Box Proposition.** Splitting the inside of the box by the configuration of kinks Y gives $C_{\mathrm{box}} \le C_{\mathrm{loc}} + (1+\chi) C_{\mathrm{kg}}$. If the reduced kink gas is put in the same type as the Glauber dynamics of the one-dimensional Ising model, the gap is the sum of "creation and annihilation of pairs" and "diffusion of kinks", $(C^2+\pi^2)D/(C\xi)^2$. The remaining rows are the cross term (the cross-term uniformity assumption), the type of the gas (the kink-gas gap assumption), the version for segments (the segment local Poincaré assumption), and the constant of the core.
3. **Three stages for the constant of the core.** Core length $k \le 7$ is bounded by convexity, ≤ 9.18 (independent of $\hbar$ and of the end values); $8 \le k \le 14$ by the lemma

of two blocks; and $k \geq 15$ by applying the reduction to the box to the core. Taking the end values as samples in $[0, x_c]$ gives $\sup_k C_P(\text{core}) \leq 22.1/30.5/66.4$ Reading .

4. **The set of ends.** In the core of the sector framework, the ends are only on the side of the well, and there is no hole. In the core on the box side, the ends can lie beyond the top, and there the wall of conditioning creates bistability; but its barrier $\Delta S_* \approx 10^{-2}$ is about 95 times smaller than $2S_0$, and the local term of the Gap-Inside-the-Box Proposition vanishes as $\hbar \to 0$ (on the reading at $\hbar = 0$).

02

Box-Reduction Proposition

Setting. Take the action $S = \sum_i (x_{i+1} - x_i)^2/(2\varepsilon) + \varepsilon \sum_i V(x_i)$ on the lattice line $\mathbb{Z}$ (spacing $\varepsilon = \frac{1}{2}$) at $T = \infty$, the infinite-volume measure μ of $e^{-S/\hbar}$, and the Dirichlet form $\mathcal{E}(f) = (\hbar/\varepsilon) \sum_i \mathbb{E}_\mu |\partial_i f|^2$ (the uniqueness of μ and the nearest-neighbour Markov property are in the row of common premises (Step 0 in 23-05 § 04)). A box B is an interval of L sites, the boundary values (a, b) are the values at the neighbouring sites to the left and right of B, $\mu_B^{a,b}$ is the conditional measure with these fixed, and $\mathcal{E}_B$ is the sum over the sites of B only. We put

$$C_{\text{box}}(L) := \sup_{a,b \in \mathbb{R}} C_P\left(\mu_B^{a,b}; \mathcal{E}_B\right)$$

The maximal correlation of two σ-algebras is $c(\mathcal{G}_1, \mathcal{G}_2) := \sup\{\text{corr}_\mu(F, G) : F \in L^2(\mathcal{G}_1), G \in L^2(\mathcal{G}_2)\}$.

Two-Projection Lemma. If $c = c(\mathcal{G}_1, \mathcal{G}_2) < 1$, then for every $f \in L^2(\mu)$

$$\operatorname{Var} f \;\leq\; \frac{1}{1-c}\Big(\mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_1) + \mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_2)\Big) \quad (1)$$

Proved here (proof in Step 1 in 23-05 § 04) Lean `two_proj_sum_le` (assumptions: Cauchy–Schwarz and the inequality of maximal correlation), `two_proj_var` (assumption: $A + B \leq (1 + c)V$).

The family of boxes. Line up boxes B_m of L sites so that neighbours overlap in l sites ($l \leq L/3$), and let E be the family of even-numbered boxes and O the family of odd-numbered ones. $\mathcal{G}_E$ is the σ-algebra generated by the values at the sites outside E, $\mathcal{G}_O$ likewise, and $c_{\text{alt}}(L, l) := c(\mathcal{G}_E, \mathcal{G}_O)$ (independent of translation).

```
Figure 1  Even and odd boxes, and the shift (example with L = 8, l =
sites       ....|....|....|....|....|....|....|....
even E      [ B0   ]       [ B2   ]       [ B4   ]
odd O            [ B1   ]       [ B3   ]
overlap          ^^     ^^      ^^     ^^          ← l sites each
shift s=1   move the whole row of boxes l sites right (the overlaps m
with n = ⌊(L−l)/l⌋ shifts, the overlap sites are mutually disjoint
```

The product over boxes. Even boxes are $L - 2l \geq 1$ sites apart from each other, and the following inequality holds (proof in § 03).

$$\mathbb{E}\,\mathrm{Var}(f \mid \mathcal{G}_E) \;\leq\; C_{\mathrm{box}}(L) \sum_{B \in E} \mathcal{E}_B(f) \quad (2)$$

Box-Reduction Proposition. $n := \lfloor (L-l)/l \rfloor$. If $c_{\mathrm{alt}}(L,l) < 1$, then

$$C_P(\mu) \;\leq\; \Big(1 + \frac{1}{n}\Big) \frac{C_{\mathrm{box}}(L)}{1 - c_{\mathrm{alt}}(L,l)} \quad (3)$$

Proved here Proof. Fix one shift s and let its even and odd families be E^s, O^s. From (1) and (2) for the two families, $(1 - c_{\mathrm{alt}})\,\mathrm{Var}\, f \leq C_{\mathrm{box}} \big(\sum_{B \in E^s} \mathcal{E}_B + \sum_{B \in O^s} \mathcal{E}_B \big)$. The boxes cover the line, each site lies in the even family at most once and in the odd family at most once, and only the overlap sites lie in both, so the bracket is $\mathcal{E}(f) + \mathcal{E}_{\mathrm{ov}^s}(f)$ (the sum over the overlap sites only). Averaging over the shifts $s = 0, \ldots, n-1$, the overlap sites are mutually disjoint across shifts, so $\sum_s \mathcal{E}_{\mathrm{ov}^s} \leq \mathcal{E}$, hence $(1 - c_{\mathrm{alt}})\,\mathrm{Var}\, f \leq C_{\mathrm{box}}(1 + 1/n)\mathcal{E}(f)$. ∎ Lean `box_two_family` (assumptions: the Two-Projection Lemma, (2) for each family, $\mathcal{E}_E + \mathcal{E}_O \leq \mathcal{E} + \mathcal{E}_{\mathrm{ov}}$, $C_{\mathrm{box}} \geq 0$), `box_shift_average` (assumptions: the two-family inequality for each shift, the sum over overlaps is at most $\mathcal{E}$).

(3) is built only from Poincaré inequalities and does not use the logarithmic Sobolev inequality. The in-phase local Poincaré condition and the saddle-tube condition enter only through C_{box} (§ 04). Putting in $C_{\mathrm{box}} \leq K_4 (C\xi)^2 / D$ of the gap-inside-the-box condition together with $L\varepsilon = C\xi$ and $D = \hbar / S_0$ gives $C_P \leq (1 + 1/n) K_4 C^2 / (1 - c_{\mathrm{alt}}) \cdot \xi^2 S_0 / \hbar$, the form of the First-Step Upper-Bound Proposition Lean `H3_with_H4`, `H3_H4_compose` (assumptions: (3), the gap-inside-the-box condition, $D = \hbar / S_0$, $n > 0$, $c_{\mathrm{alt}} < 1$). It is the same type as the box dynamics of the Glauber dynamics of discrete spins (Propositions 3.4 and 3.5 of Martinelli's lectures) Cited ; here we rewrite it in one dimension, in L^2, and in the form of maximal correlation.

The remaining assumption rows, and the numbers

- **Alternating maximal-correlation assumption for boxes: $c_{\mathrm{alt}} < 1$ uniformly in $\hbar$.** There is no paper proof Assumption . As values, on the reading that the slow part is a two-state chain of the sign (the telegraph chain), boxes with $\theta = l/L = \frac{1}{4}$ give $c_{\mathrm{alt}} = 0.731/0.410/0.138$ for $C = 2/4/8$ Reading (values of a family that copies the values of the closed formula of the two-state chain). These values agree with $\varphi(s,t)$ of the Closed Formula for the Alternating Maximal Correlation in § 05 with $s = e^{-C\theta}$ and $t = e^{-C(1-2\theta)}$ (at $C = 2$, $\varphi = 0.7308$).
- **The sup over boundary values.** C_{box} takes the sup over all real boundary values. We see in § 04 that boundary values near the top are the worst.
- **The size of ξ.** Computing the difference of the even and odd eigenvalues of the transfer operator in the logarithmic domain gives $\ln \xi = 218.08/661.04/2212.72$ and $\ln \xi - S_0^{\mathrm{lat}}/\hbar - \frac{1}{2}\ln\hbar \to -0.778$ Reading (two independent computational routes agree to six digits at intermediate $\hbar$). That is, $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$, and its ratio to the length of the plateau at the top (the crossing length $\ell_\times \approx 0.34/\hbar$, 23-10 type of the address of $1/\hbar$) is $\xi/\ell_\times \geq e^{213}$ Lean `xi_over_ellx` (assumption: the measured $\ln\xi - \ln\ell_\times \geq 213$). **A box of length $C\xi$ does not cut the plateau; it contains it whole.** In exchange, the constant of the core inside the box has to be uniform in the length of the core (§ 05).

Table 1 Positive controls for the inequality of the Box-Reduction Proposition (models in which both sides are known from a closed formula or by diagonalisation)

Model	What was compared	Result
Pure Gaussian ($V = x^2/2$)	The right-hand side $(1 + 1/n)C_{\mathrm{box}}/(1 - c_{\mathrm{alt}})$ against $CP = 1$	Right-hand side 1.10–1.33 ≥ 1 ($L = 4$–128) Reading . Approaches 1 as $l/L \to 0$, $l \to \infty$ Lean `gauss_box_le_one`
Heat-bath Glauber for the one-dimensional Ising model (rings of 12 and 16 sites, couplings $K = 0.3, 0.6, 1.0$)	Both sides of (3)	Holds in all 9 cases, with a loss of 1.17–1.35 times Reading (even at $c_{\mathrm{alt}} = 0.87$) Lean `ising_K1_holds` (assumption: measured values)
Negative control	$\mathcal{G}_1 = \mathcal{G}_2$ ($c = 1$)	The reduction says nothing Lean `no_bound_when_c_one`, `blowup_as_c_to_one`

In the Glauber dynamics examples, even when the gap of the ring (the lower edge of the band) was only about a sixth of the gap of the box, $1 - c_{\mathrm{alt}}$ carried exactly that difference. The band structure of the kink gas does not break the form of the box.

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03
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Proof of the product inequality for boxes

We give the proof of (7) of 23-05 ((2) of this chapter). The box-product inequality row was graded "Proved here" in the table of the five grades in 23-05 because of the proof in this section.

Proposition (product over boxes)

When the boxes of the even family E are $L - 2l \geq 1$ sites apart from each other, (2) holds for smooth functions f that depend on finitely many sites only.

Step (a) Conditional independence. The action S is the sum of terms between neighbours $(x_{i+1} - x_i)^2/(2\varepsilon)$ and one-site terms $\varepsilon V(x_i)$. If all values outside a box B are fixed, the conditional density of x_B is determined only by the terms of $e^{-S/\hbar}$ that contain x_B, which are the terms inside B and the two terms joining the two ends of B to the two neighbours a_B, b_B. So the conditional law is $\mu_B^{a_B, b_B}$, and it depends on the values outside B only through the two neighbours (the nearest-neighbour Markov property). Between even boxes there is at least one site fixed by $\mathcal{G}_E$, so given $\mathcal{G}_E$, the conditional density of the even boxes splits into a product of factors, one per box,

$$\prod_{B \in E} \frac{1}{Z_B} \exp\Big(- \frac{S_B(x_B; a_B, b_B)}{\hbar} \Big)$$

and the values of the even boxes $(x_B)_{B \in E}$ are mutually independent with the product law $\bigotimes_{B \in E} \mu_B^{a_B, b_B}$ Proved here (one line on the density splitting into a product in finite volume; that the measure at $T = \infty$ has the same conditional law is in the row of common premises).

Step (b) The Efron–Stein inequality for product measures. Let $\nu = \nu_1 \otimes \cdots \otimes \nu_m$ be a product measure on $X_1 \times \cdots \times X_m$, $F \in L^2(\nu)$, and $\mathrm{Var}_j F$ the variance obtained by moving only the j-th coordinate (the other coordinates fixed). Then

$$\mathrm{Var}_\nu F \;\leq\; \sum_{j=1}^{m} \mathbb{E}_\nu \big[\mathrm{Var}_j F \big] \qquad (4)$$

Proved here Proof. Put $F_j := \mathbb{E}[F \mid X_1, \ldots, X_j]$ ($F_0 = \mathbb{E}F$, $F_m = F$) and $\Delta_j := F_j - F_{j-1}$. The Δ_j are mutually orthogonal, so $\mathrm{Var}\, F = \sum_j \mathbb{E}\Delta_j^2$. Since the coordinates are independent, F_{j-1} is F_j integrated over the j-th coordinate with ν_j, and $\mathbb{E}\Delta_j^2 = \mathbb{E}[\mathrm{Var}_j F_j]$. Moreover F_j is F averaged over the later coordinates $X_{j+1}, \ldots, X_m$, and $G \mapsto \mathrm{Var}_j G = \| G - \int G \, d\nu_j \|^2_{L^2(\nu_j)}$ is convex (the square of a seminorm), so Jensen's

inequality gives $\mathrm{Var}_j F_j \le \int \mathrm{Var}_j F \, d\nu_{j+1} \cdots d\nu_m$. Taking expectations, $\mathbb{E}\Delta_j^2 \le \mathbb{E}\,\mathrm{Var}_j F$, and summing over j gives (4). ∎

Step (c) Poincaré for the box. Apply (4) to the conditional product of Step (a) (with the coordinate X_j the value x_B of the box B). The variance obtained by moving only the coordinates of a box B is the variance of the measure $\mu_B^{a_B,b_B}$, so

$$\mathrm{Var}_B f \;\le\; C_P(\mu_B^{a_B,b_B})\,\frac{\hbar}{\varepsilon}\sum_{i\in B}\int |\partial_i f|^2\, d\mu_B^{a_B,b_B} \;\le\; C_{\mathrm{box}}(L)\,\frac{\hbar}{\varepsilon}\sum_{i\in B}\int |\partial_i f|^2\, d\mu_B^{a_B,b_B}$$

by the sup in the definition of C_{box}.

Step (d) Putting it together. From (4) and Step (c), $\mathrm{Var}(f \mid \mathcal{G}_E) \le C_{\mathrm{box}} \sum_{B\in E} \mathbb{E}\big[(\hbar/\varepsilon) \sum_{i\in B} |\partial_i f|^2 \mid \mathcal{G}_E\big]$. Taking expectations, by the tower property of conditional expectation each term on the right becomes $\mathcal{E}_B(f)$, which is (2). Since f depends on finitely many sites only, finitely many boxes are involved, and the terms of boxes that do not touch the support of f are 0. ∎

The extension to general f in the domain of $\mathcal{E}$ (approximation by functions depending on finitely many sites) is included in the row of common premises. The odd family is the same. (4) of Step (b) is the Efron–Stein inequality (Efron–Stein 1981); since we give the proof here, the grade is "Proved here".

04

Gap-Inside-the-Box Proposition

Fix a box B ($L\varepsilon = C\xi$) and boundary values $(a,b) \in \mathbb{R}^2$, and let $\mu = \mu_B^{a,b}$ and $D = \hbar/S_0$. The expected number of kinks in the box is $C/2$ from the kink density $1/(2\xi)$, which is $O(1)$ Proved here Lean `pair_fugacity` (assumption: $\xi \;/= 0$).

The naive split goes round in a circle. Splitting by a rare set E (for example, configurations with a long core), with $p = \mu(E)$, gives $\mathrm{Var}\, f = p\,\mathrm{Var}_E f + (1-p)\,\mathrm{Var}_{E^c} f + p(1-p)(m_E - m_{E^c})^2$ Proved here (an identity). The second term can be paid with a constant on E^c, but the first term is as large as $\mathrm{Var}\, f$ if f is concentrated on E, and the smallness of p does not come out. What can be paid is up to $p\cdot \mathrm{osc}(f)^2/4$, which is not a gap but a weak form of the Poincaré inequality Lean `naive_split_weak` (algebra). It is an example of 23-10 currency-mix-up type: rarity cannot be paid with variance. So we take the split not by a rare set but by a slow variable.

Splitting by the slow variable. Let Y be the configuration of kinks in the box (the number of kinks, and the position of each kink defined by projection onto the manifold of kinks), and take the identity for conditional expectation Proved here

$$\operatorname{Var} f = \mathbb{E}\operatorname{Var}(f \mid Y) + \operatorname{Var}\big(\mathbb{E}[f \mid Y]\big)$$

as the starting point.

- **The first term (the fast side).** By the nearest-neighbour Markov property, the conditional measure with Y fixed splits into a product of scraps of the phase without kinks (length $\le C\xi$, with values at both ends), the tubes around the kinks, and the cores inside them. The Poincaré constant of a product measure is at most the maximum of the constants of the components ((4) and Poincaré for each component), so $\mathbb{E}\operatorname{Var}(f \mid Y) \le C_{\mathrm{loc}}\,\mathcal{E}(f)$ with $C_{\mathrm{loc}} = \max(C_{(\mathrm{H1}^{\mathrm{seg}})}, C_{(\mathrm{H2}^{\mathrm{tube}})}, C_{(\mathrm{T0}'_{\mathrm{P}})})$ Proved here (the algebra of products). The three constants are assumption rows Assumption, and the constant $C_{(\mathrm{T0}'_{\mathrm{P}})}$ of the core is bounded in § 05 and § 06.
- **The second term (the slow side).** $\mathbb{E}[f \mid Y]$ is a function of Y alone and relaxes by diffusion of the kink positions (mobility D, the kink diffusion-coefficient row Cited) and by the creation and annihilation of kink pairs. The kink-gas gap assumption: the Poincaré constant of the box of the reduced kink gas, $C_{\mathrm{kg}} \le (C\xi)^2/(c_{\mathrm{kg}}(1-\eta_{\mathrm{KG}})D)$ Assumption. The cross-term uniformity assumption: the cross term $\mathcal{E}_Y(\mathbb{E}[f \mid Y]) \le (1+\chi)\mathcal{E}(f)$ Assumption (of the type of the cross constant of a two-scale inequality; a statement of this form has not been drawn from a known theorem).

Gap-Inside-the-Box Proposition. Under the rows above, for every (a, b)

$$C_{\mathrm{box}} \;\le\; C_{\mathrm{loc}} + (1+\chi)\,C_{\mathrm{kg}} \;\le\; K_4\,\frac{(C\xi)^2}{D}, \qquad K_4 = \frac{1+\chi}{c_{\mathrm{kg}}(1-\eta_{\mathrm{KG}})} + C_{\mathrm{loc}}\,\frac{D}{(C\xi)^2} \quad (5)$$

Proved here (the sum of the two terms above) Lean `H4_two_scale` (assumptions: the decomposition inequality, first term $\le C_{\mathrm{loc}}\mathcal{E}$, the kink-gas gap assumption, the cross-term uniformity assumption, $C_{\mathrm{kg}} \ge 0$), `H4_gap_form` (assumptions: $D > 0$, $C\xi \neq 0$, $c_{\mathrm{kg}} > 0$, $\eta_{\mathrm{KG}} < 1$, $1+\chi \ge 0$). (5) is the same formula as (12) of 23-05.

Comparison with the Glauber dynamics of the one-dimensional Ising model

The constant c_{kg} of the kink-gas gap assumption is fixed by the assumption that puts the reduced kink gas in the same type as the heat-bath Glauber dynamics of the one-dimensional Ising model. The gap of a box of L sites with the spins at both ends fixed has the closed formula

$$\mathrm{gap}_{\mathrm{box}} = 1 - \gamma\cos\frac{\pi}{L+1} = (1-\gamma) + \gamma\Big(1-\cos\frac{\pi}{L+1}\Big), \qquad \gamma = \tanh 2K \quad (6)$$

Assumption (the original source is bibliographic only; diagonalisation on the lattice agrees to the 6 digits shown, and it is the same whether the boundary values are $(+,+)$

or $(+,-)$ Reading) Lean `glauber_split` (the right-hand equality, no assumptions). In the limit of long correlation length, the first term $1-\gamma \approx 1/(2\xi^2)$ is the infinite-volume gap = **creation and annihilation of pairs**, and the second term $\approx \pi^2/(2L^2)$ is the **diffusion of kinks** in an interval of length L. Written in the Glauber unit $D=\frac{1}{2}$, it is $D/\xi^2+\pi^2 D/L^2=(C^2+\pi^2)D/(C\xi)^2$, that is, $c_{\mathrm{kg}}=C^2+\pi^2$ Proved here (algebra) Lean `kg_constant` (assumption: $C\xi \;/=0$). The pair term is not a small perturbation; it enters additively as one of the main terms of the gap. The local term of the part without kinks does not appear at the lower edge of the gap.

When the boundary values are near the top, a half kink can sit at the end of the box. This corresponds to a free end of the Ising model, and there is no closed formula. Read by diagonalisation on the lattice, the gap does not fall below $1-\gamma$ for any type of boundary (ratio ≥ 1.17), but a free end is slower than a fixed end.

Table 2 The box constant of the reduced gas $c_{\mathrm{kg}}=\mathrm{gap}\cdot(C\xi)^2/D$ and the coefficient K of the First-Step Upper-Bound Proposition ($\chi=\eta\mathrm{KG}=0$, $n=3$, $\theta=\frac{1}{4}$; all are values of models independent of $\hbar$ Reading)

C	1	2	3	4	6
Fixed ends $C^2+\pi^2$	10.87	13.87	18.87	25.87	45.87
Free ends c_{ff} (lattice values)	2.86	7.17	12.94	20.32	39.07
c_{alt} (two-state chain)	0.915	0.731	0.550	0.410	0.234
K (fixed ends only)	1.44	1.43	1.41	1.40	1.37
K (the sup including boundary values at the top)	5.46	2.76	2.06	1.78	1.60

$K=(1+1/n)C^2/(c_{\mathrm{kg}}(1-c_{\mathrm{alt}}))$ Lean `K_fixed_C2`, `K_ff_C2`, `K_ff_C4` (assumptions: the measured c_{ff} and c_{alt}; arithmetic only). Since K_4 decreases with C, with fixed ends only K hardly depends on C, and including boundary values at the top, the larger C is taken, the better.

The rows remaining in the gap-inside-the-box condition. The cross-term uniformity assumption, the kink-gas gap assumption (the Glauber type and free ends), the segment local Poincaré assumption (the Six-Node In-Phase Poincaré Proposition of 23-06 is a statement on the ring, and there is no version for segments with end values), the saddle-tube Poincaré assumption, and the row of the core. The second term C_{loc} $D/(C\xi)^2$ of K_4 vanishes if C_{loc} is sufficiently smaller than $e^{2S0/\hbar}$, because D/ξ^2 is of size $e^{-441.6}/e^{-1328.6}/e^{-4433.1}$ Reading . We use this margin for the core in § 06.

05

The three-stage proposition for the constant of the core

Setting. A core is a maximal interval of sites at which the path stays between the top and the well, $(0, x_c)$ ($x_c = 0.89$). In this chapter we reverse the direction by symmetry and write the core as $(0, x_c)^k$ (the mirror image of $Q_k = (-x_c, 0)^k$ in 23-09). The measure of the core $\mu_k^{\alpha,\beta}$ is $e^{-S/\hbar}$ with the outside values at both ends α, β fixed, restricted to $(0, x_c)^k$ with Neumann boundary, and the Dirichlet form is $\mathcal{E} = (\hbar/\varepsilon) \sum_i \mathbb{E}|\partial_i f|^2$ (converted from the convention $\hbar \int |\nabla g|^2$ of 23-09 by $C^{(23\text{-}09)} = C/\varepsilon = 2C$). What the fast side of the gap-inside-the-box condition needs is that $C_P(\mu_k^{\alpha,\beta})$ is bounded uniformly in the length of the core k and in the end values (the row of the core Poincaré uniformity assumption).

The short-core step $k \le 7$: convexity (the Short-Core Convexity Lemma)

The Hessian of the action of the core is

$$\operatorname{Hess} S = \frac{1}{\varepsilon} T_k + \varepsilon \operatorname{diag} V''(x_i), \qquad T_k = \operatorname{tridiag}(-1, 2, -1)$$

, and the end values α, β enter S only in first-order terms, so they do not appear in the Hessian. From $\lambda_{\min}(T_k) = 4 \sin^2 \frac{\pi}{2(k+1)}$ Proved here (the closed formula for the eigenvalues of a tridiagonal matrix) and $\inf V'' = V''(0) = -\frac{1}{2}$, at $\varepsilon = \frac{1}{2}$

$$\operatorname{Hess} S \;\ge\; \kappa(k) := 8 \sin^2 \frac{\pi}{2(k+1)} - \frac{1}{4} \quad (7)$$

. This is the same formula that appeared in 23-09 § 06 as the smallest eigenvalue on the surface of the top; it is positive for $k \le 7$ and negative at $k = 8$ ($8 \sin^2(\pi/16) \approx 0.304 > \frac{1}{4}$, $8 \sin^2(\pi/18) \approx 0.241 < \frac{1}{4}$) Lean `kappa7_pos`, `kappa8_neg` (no assumptions).

Short-Core Convexity Lemma. If $\kappa(k) > 0$, then for all end values $\alpha, \beta \in \mathbb{R}$ and all $\hbar > 0$, $C_P(\mu_k^{\alpha,\beta}) \le \varepsilon/\kappa(k)$.

Proved here Proof. By (7), $S/\hbar$ is uniformly convex with $\kappa/\hbar$. A uniformly log-concave measure restricted with Neumann boundary to a convex domain (the box $(0, x_c)^k$) satisfies $\operatorname{Var} f \le (\hbar/\kappa) \int |\nabla f|^2$ Cited (Bakry–Émery on convex domains, checked in Theorem 1.2 (1) of Kolesnikov–Milman), and rewritten in $\mathcal{E} = (\hbar/\varepsilon) \int |\nabla f|^2$ this is $\le (\varepsilon/\kappa)\mathcal{E}$. ∎ Lean `lemma_BV` (assumptions: the lower bound of the Hessian is $\ge \kappa > 0$ independently of the end values, and the pointwise Bakry–Émery inequality).

The values are $\varepsilon/\kappa(k) = 0.133/0.286/0.543/0.973/1.749/3.42/9.18$ ($k = 1, \dots, 7$), independent of both $\hbar$ and the end values. The sup measured at $k = 1, 2$ (0.128 and 0.26)

lie within these Reading . **If the blocks are cut to 7 sites or fewer, the difficulty of the sup over end values disappears by convexity.** The price moves to the side of the correlation at the seams of the cut blocks.

The medium-core step $8 \le k \le 14$: two blocks (the Two-Block Lemma)

End-Reduction Lemma. If the chain $x_1, \dots, x_k$ is Markov, then for $a < b$, $c\big(\sigma(x_1, \dots, x_a), \sigma(x_b, \dots, x_k)\big) = c\big(\sigma(x_a), \sigma(x_b)\big)$.

Proved here Proof. Right-hand side $\le$ left-hand side follows from the inclusion of σ-algebras. Conversely, let F be a centred function of the front piece and G one of the back piece. By the Markov property $\mathbb{E}[G \mid x_1, \dots, x_a] = \mathbb{E}[G \mid x_a] = \mathbb{E}\big[\mathbb{E}[G \mid x_b] \mid x_a\big]$, so putting $\hat F = \mathbb{E}[F \mid x_a]$ and $\hat G = \mathbb{E}[G \mid x_b]$ gives $\mathbb{E}[FG] = \mathbb{E}[\hat F\, \mathbb{E}[\hat G \mid x_a]] = \mathbb{E}[\hat F \hat G] \le c(\sigma(x_a), \sigma(x_b))\|\hat F\|\|\hat G\| \le c(\sigma(x_a), \sigma(x_b))\|F\|\|G\|$. ∎

Two-Block Lemma. Let $1 \le a < b \le k$, $k - a \le 7$, $b - 1 \le 7$. Then

$$C_P(\mu_k^{\alpha,\beta}) \;\le\; \frac{M(a,b)}{1 - c(x_a, x_b)}, \qquad M(a,b) = \begin{cases} \varepsilon/\kappa(k-a) + \varepsilon/\kappa(b-1) & (b \ge a+2) \\ \max\big(\varepsilon/\kappa(k-a),\, \varepsilon/\kappa(b-1)\big) & (b = a+1) \end{cases} \tag{8}$$

Proved here Proof. Apply the Two-Projection Lemma to $\mathcal{G}_1 = \sigma(x_1, \dots, x_a)$ and $\mathcal{G}_2 = \sigma(x_b, \dots, x_k)$. By the End-Reduction Lemma, $c(\mathcal{G}_1, \mathcal{G}_2) = c(x_a, x_b)$. The conditional law given $\mathcal{G}_1$ is the chain $x_{a+1}, \dots, x_k$ of length $k - a \le 7$ (with ends x_a and β) restricted to the box, and the Short-Core Convexity Lemma gives $\mathrm{Var}(f \mid \mathcal{G}_1) \le (\varepsilon/\kappa(k-a))(\hbar/\varepsilon) \sum_{i>a} \mathbb{E}[|\partial_i f|^2 \mid \mathcal{G}_1]$. $\mathcal{G}_2$ likewise gives $\sum_{i<b}$ and $\varepsilon/\kappa(b-1)$. Taking expectations and putting them into (1), the coefficient of site i is $[i > a]\varepsilon/\kappa(k-a) + [i < b]\varepsilon/\kappa(b-1)$; both are switched on only when $a < i < b$, which happens only when $b \ge a + 2$. ∎ Lean `tb_far`, `tb_adjacent` (assumptions: the conclusion of the Two-Projection Lemma, the Short-Core Convexity Lemma for each block, $\mathcal{E}_i \le \mathcal{E}$, $c < 1$).

From the conditions $k - a \le 7$, $b - 1 \le 7$, $a < b$ we get $k \le 14$, and at $k = 15$ there is no (a, b) Lean `tb_range`, `tb_none_at_15` (arithmetic of natural numbers). The input is a single **two-point** maximal correlation $c(x_a, x_b)$, which is the second singular value of the normalised matrix of the joint distribution of two sites Cited (the singular-value representation of Witsenhausen and Rényi, checked in a restatement). For the measure on the lattice, this value can be computed from products of transfer matrices.

Values. Taking the ends as samples in $[0, x_c]$, the sup over $8 \le k \le 14$ of the upper bound (the Two-Block Lemma) is $22.1/30.5/66.4$; at 10^{-3} and $3 \cdot 10^{-4}$ it is attained at $k = 8$ with both ends 0, and at $3 \cdot 10^{-3}$ at $k = 13$ (at $3 \cdot 10^{-3}$, the value for $k = 8$ with both ends 0 is 15.5). The lower bound by polynomial test functions (Rayleigh–Ritz) is

$8.39/17.3/38.5$ Reading . $k = 8$ lies just outside the threshold of convexity; on the reading at $\hbar = 0$, the smallest eigenvalue of the Hessian at the constrained minimiser is 0.0175, heading towards $C_P \to \varepsilon/\lambda_{\min} = 28.55$. There is a peak at $\hbar \approx 10^{-4}$ (lower bound 57, upper bound 99) Reading , and it is not monotone in $\hbar$.

Why the form of the long-core step is not used here. In a core with $k = 8$ and both ends at the top, the alternating maximal correlation of subboxes of length 5 rises to $0.879/0.938/0.971$, and the coefficient of the long-core step becomes 76 or more Reading Lean `coeff_k8` (assumption: the measured $\bar{c} \geq 0.9713$). This is because the soft direction ($\lambda_{\min} = 0.0175$) pushes the correlation at the seams towards 1; the form of two blocks avoids the alternating correlation and can be paid with the two-point correlation alone.

The long-core step $k \geq 15$: applying box reduction to the core (the Box-Reduction Inequality for Long Cores)

The measure of the core is also one-dimensional and nearest-neighbour Markov (restriction to a product box preserves the Markov property), so the construction of § 02 and § 03 can be used literally inside the core. Taking subboxes of length $L_c = 5$ with overlap $l_c = 1$ ($n_c = 4$), the constant of the subboxes (all of 5 sites or fewer, including short ones cut at the ends) is $\leq \varepsilon/\kappa(5) = 1.749$ uniformly in the end values by the Short-Core Convexity Lemma, so for every k and all end values

$$C_P(\mu_k^{\alpha,\beta}) \;\leq\; \Big(1 + \frac{1}{4}\Big)\,\frac{\varepsilon/\kappa(5)}{1 - \bar{c}_{\text{core}}} = \frac{2.186}{1 - \bar{c}_{\text{core}}} \quad (9)$$

holds Proved here (the proof of the Box-Reduction Proposition transferred to the core) Lean `core_uniform_L5` (assumptions: the conclusion of the Box-Reduction Proposition, $C_{\text{box}} \leq 7/4$, the measured $\bar{c}_{\text{core}}$). $\bar{c}_{\text{core}}$ is the alternating maximal correlation between the even and odd subboxes inside the core; the values measured at $k \geq 15$ are ≤ 0.535 over end values (including negative ends beyond the top), and at $k = 20$–30 they are saturated to 7 digits Reading (a measurement of a **lower bound** of the maximal correlation by additive polynomial functions and two points at the seams). Putting this in gives $C_P \leq 4.70$.

The form of the alternating maximal correlation (the Closed Formula for the Alternating Maximal Correlation). In a stationary Gaussian chain $\ldots, L_j, R_j, L_{j+1}, R_{j+1}, \ldots$ with correlation t inside a piece and correlation s at the seams ($\text{corr}(L_j, R_j) = t$, $\text{corr}(R_j, L_{j+1}) = s$, $|s|, |t| < 1$; the chain is taken to continue infinitely on both sides), the maximal correlation between the even pieces and the odd pieces is $\varphi(|s|, |t|)$, where

$$\varphi(s,t) = \frac{s(1+t)}{1+s^2t}, \qquad 1 - \varphi(s,t) = \frac{(1-s)(1-st)}{1+s^2t} \quad (10)$$

. Derivation: for the uniform type $F = \sum_{j\ \mathrm{even}}(L_j + R_j)$, $G = \sum_{j\ \mathrm{odd}}(L_j + R_j)$, the covariance per period is $S_{AB} = 2s(1+t)^2/(1-s^2t^2)$, because by the Markov property a two-point correlation is the product of the correlations of the links in between; the variance is $S_{AA} = 2(1+t) + 2(1+t)^2 s^2 t/(1-s^2t^2)$, and the ratio is φ Proved here (sum of a geometric series) Lean `gauss_mode0`, `one_sub_phi`. This ratio falls within the framework of the convergence rate of two-block Gibbs sampling for a Gaussian target distribution Cited . The rate of Gibbs sampling that alternately redraws all the even pieces and all the odd pieces is the square of the largest eigenvalue of the block Jacobi matrix (the Two-Block Gibbs-Sampler Rate Theorem (Theorem 5 of Roberts–Sahu) of Roberts–Sahu 1997, and § 2.4), and that largest eigenvalue is the maximal correlation φ of the two pieces. The case of pieces of length 1 ($L_j = R_j$; the value $2s/(1+s^2)$ obtained by putting $t = 1$ in (10)) is in Theorem 1 of Pitt–Shephard 1999 and Corollary 2 of Mider–Jenkins–Pollock–Roberts 2022, its square being the rate in the limit of infinite chain length Cited . The value of (10) for general piece length is the same quantity as the convergence radius of line Jacobi iteration in numerical linear algebra (the bibliography on that side has not been cross-checked). That the maximal correlation of a Gaussian chain equals the linear canonical correlation is Cited (Kolmogorov–Rozanov, checked in a restatement); that the uniform type is the maximum within it has only been confirmed by computation Reading . Since $\varphi \geq s \geq st$, the naive product $c \leq s \cdot t$ is false Lean `naive_product_false` (assumption: $0 < s, t < 1$). The $c_{\mathrm{alt}} = 0.7308$ of the two-state chain in § 02 is the φ at $s = e^{-1/2}$, $t = e^{-1}$.

It does not reduce to two-point quantities. In the core, the Gaussian proxy φG built from the two-point maximal correlation agrees with the measured value at the order of 10^{-4} Reading . Even so, for a general Markov chain there is no inequality that bounds the alternating maximal correlation by two-point quantities alone. A chain of three variables X_1–X_2–X_3: $X_2 \sim \mathrm{Bern}(p)$; if $X_2 = 1$ then $X_1 = X_3 = 1$; if $X_2 = 0$ then X_1, X_3 are independently 1 with probability e; $p = 10^{-4}$, $e = 10^{-3}$. A function of a binary variable is affine, so the maximal correlation between binary variables is $|\,\mathrm{corr}\,|$, and $\mathrm{corr}(X_1, X_2)^2 = p(1-p)(1-e)^2/(m_1(1-m_1)) \leq 1/10$ ($m_1 = p + (1-p)e$). But $Z = X_1 X_3$ has $\mathrm{corr}(Z, X_2)^2 = p(1-p)(1-e^2)^2/(m_Z(1-m_Z)) \geq 98/100$ ($m_Z = p + (1-p)e^2$) Proved here (computation of covariances) Lean `rareR_pair`, `rareR_joint` (arithmetic of rational numbers). It is an example in which a non-linear function that holds a rare state from both sides is effective; the paper for the row of the alternating maximal-correlation assumption for the core needs a comparison theorem that uses the fact that the conditional law of the core is close to log-concave.

The Three-Range Core-Constant Proposition and the table of the three stages

Three-Range Core-Constant Proposition. For a set of end values $\mathfrak{E}$, let $B_2(\hbar;\mathfrak{E}) := \sup_{8\le k\le 14}\sup_{(\alpha,\beta)\in\mathfrak{E}^2}\min_{(a,b)} M(a,b)/(1-c_{k,\alpha,\beta}(x_a,x_b))$ and $B_3(\hbar;\mathfrak{E}) := 2.186/(1-\bar{c}_{\ge 15}(\mathfrak{E}))$. Then for every $k\ge 1$ and $(\alpha,\beta)\in\mathfrak{E}^2$

$$C_P(\mu_k^{\alpha,\beta}) \;\le\; \max\{\varepsilon/\kappa(7),\; B_2(\hbar;\mathfrak{E}),\; B_3(\hbar;\mathfrak{E})\} \quad (11)$$

Proved here (split into cases by k and apply the three steps) Lean `three_stage` (assumption: the upper bound of each stage).

Table 3 Core length and stage (ends sampled in $[0,x_c]$; triples are $\hbar = 3\cdot 10^{-3}/10^{-3}/3\cdot 10^{-4}$)

Stage	k	Tool	Grade	Upper bound	Lower bound
1	≤ 7	Short-Core Convexity Lemma (convexity)	Proved here + Cited (Bakry–Émery on convex domains)	≤ 9.18 (independent of $\hbar$ and the ends)	—
2	8–14	Two-Block Lemma + two-point maximal correlation	Proved here + Reading	$22.1/30.5/66.4$	$8.39/17.3/38.5$
3	≥ 15	Box-Reduction Inequality for Long Cores + alternating maximal correlation	Proved here + Reading + Assumption (the alternating maximal-correlation assumption for the core)	≤ 4.70 ($\bar{c}$ is a measurement of a lower bound)	—

From Table 3, $\sup_k C_P(\text{core}) \le 22.1/30.5/66.4$ Reading Lean `three_stage_3em4` (assumption: the measured values of the medium-core step and the long-core step). The ratios of upper to lower bound are $2.6/1.76/1.72$. The input of the medium-core step (the two-point maximal correlation) is a product of matrices and a second singular value, so it is of a type that can be put on interval arithmetic; but it needs a paper estimate of the difference between the lattice measure and the continuum measure, and the cost of running it is estimated at the order of 10^3 CPU hours (not yet). In the long-core step, what is measured is a lower bound, and there is no paper for the upper bound.

06

The set of ends — what is covered, and how to pay for bistability

Table 3 took the end values as samples in $[0,x_c]$. A core is a **maximal** interval of sites in $(0,x_c)$, so the neighbouring sites outside it actually take values in $(-\infty,0]\cup[x_c,\infty)$. We separate which ends are needed in which construction.

The core of the sector framework — ends only on the side of the well

The core of the sector framework (23-09) is defined as having both end values on the side of the well ($[x_c, \infty)$ in the direction of this chapter). So ends beyond the top do not enter there Proved here (by definition) Lean `p5_end_not_negative` (assumption: $x_c = 0.89$), and the set of ends has no hole. In the stationary chain, a fraction $1 - e^{-0.63/\hbar}$ of the mass of runs of length k is of this type, and the minimiser on the reading at $\hbar = 0$ is a single shape with every site pressed against the wall x_c Reading .

The core on the box side — ends beyond the top, and the wall of conditioning

In the core on the fast side of the Gap-Inside-the-Box Proposition (the core after splitting by the configuration of kinks), the outside neighbour can take negative values beyond the top. There, two local minimisers at $\hbar = 0$ appear in cores with $k = 9, 10$, the barrier is $\Delta S \in [1.1 \cdot 10^{-4}, 2.5 \cdot 10^{-3}]$ (depending on the end values), and the lower bound of CP for $k = 9$ with ends $(-0.035, -0.035)$ is $13.9/44.6/526$, growing in the manner of $e^{\Delta S/\hbar}$ Reading .

- **Both minimisers sit on the wall of conditioning.** One is a bump spanning the core interval as it is; the other has the zeros at both ends moved inwards by one site each. Because we conditioned the core interval to be "exactly this interval", a wall stands in the direction in which the zeros want to move, and the two types compete. The Peierls–Nabarro barrier of a discrete kink on this lattice is 10^{-12} or less, a different order from the bistability barrier $0.5\text{–}2.0 \cdot 10^{-3}$ Reading Lean `pn_vs_bistab` (assumption: measured values). **The bistability is not physical metastability; it is made by the choice of conditioning.**
- **Cutting by the binary of the bump brings the fibres back.** Splitting by the binary Y_3 of whether a site x_m near the top is at or above the threshold θ^* (whether there is a bump), the CP of each fibre returns, by the Two-Block Lemma (the same proof applies to products of intervals), to upper bound $25.8/46.2/101$ and lower bound $8.9/17.4/32.8$, the same order as the medium-core step of Table 3 Reading .
- $e^{\Delta S/\hbar}$ **only moves.** In exchange, the Poincaré constant of the marginal of the binary Y_3 takes on $e^{\Delta S/\hbar}$, with $\mathrm{Var}(\mathbb{E}[f \mid Y_3]) \geq 470\,\mathcal{E}(f)$ $(3 \cdot 10^{-4})$ Reading Lean `y_side_k9_3em4` (assumption: measured values). Cutting does not remove the barrier; it takes the form of transfer of 23-10 transfer type. Extending the constant of the core to the actual set of ends while keeping it $O(1)$ cannot be done by this split.

Paying with the margin of the action barrier — the row of the core action barrier

In the Gap-Inside-the-Box Proposition the constant of the core enters C_{loc}, and appears in K_4 only in the form $C_{\mathrm{loc}} \cdot D/(C\xi)^2$ ((5)). $D/(C\xi)^2$ is of the order of $e^{-2S_0/\hbar}$,

so the constant of the core need not be $O(1)$; it suffices that the action barrier is smaller than $2S_0$. We put this as an assumption row.

The row of the core action-barrier assumption. There are $c_*, p \geq 0$ and $\Delta S_* < 2S_0$ such that, for the conditional measures of the core on the box side and of the windows containing it (the core, its two neighbours and m sites outside, with arbitrary outer ends), $C_P \leq c_* \hbar^{-p} e^{\Delta S_*/\hbar}$ (for small $\hbar$) Assumption .

Reading. On the reading at $\hbar = 0$ (numerical minimisers checked for stationarity and positive definiteness, and the height of the connection bracketed from above and below), the sup of the depth of non-global minimisers over the range $k = 8$–14, $m \leq 12$, outer ends $[-1.3, -0.01]$ is $\Delta S_*^0 \in [9.6 \cdot 10^{-3}, 1.4 \cdot 10^{-2}]$; its ratio to $2S_0 = 4/3$ is $\leq 1.05 \cdot 10^{-2}$, a margin of about 95 times Reading Lean `measured_ratio` (assumption: the measured upper bound $\Delta S^0 \leq 0.01401$). It is 5–8 times deeper than the value with the ends fixed ($1.77 \cdot 10^{-3}$), and the sup is attained where moving the outer ends makes the two types degenerate. The lower bound of $\hbar \ln C_P$ approaches ΔS^0 from below ($0.97 \Delta S^0$ by a fit), but the prefactor $\hbar^{-p}$ of the upper bound has not been measured.

Proposition (the local term vanishes). Under the core action-barrier assumption and the reading of ξ, $D/(C\xi)^2 \leq d\, e^{-2S_0/\hbar}$, for any $\delta > 0$

$$C_{\mathrm{loc}} \cdot \frac{D}{(C\xi)^2} \;\leq\; c_* d\, \frac{p!}{\delta^p}\, e^{-(2S_0 - \Delta S_* - \delta)/\hbar} \qquad (12)$$

Proved here Proof. From $e^x \geq x^p/p!$ ($x = \delta/\hbar > 0$), $\hbar^{-p} \leq (p!/\delta^p) e^{\delta/\hbar}$. Multiplying this by the two assumptions gives $e^{\Delta S_*/\hbar} e^{-2S_0/\hbar} e^{\delta/\hbar}$, which is (12). ∎ Lean `cloc_term_poly`, `cloc_term_vanishes` (assumptions: the core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$); negative control `no_vanish_if_no_gap` (if $\Delta S_* \geq 2S_0$, the estimate does not vanish).

If δ is taken smaller than $2S_0 - \Delta S_*$, the right-hand side of (12) goes to 0 as $\hbar \to 0$, and the measured values leave a margin of $2S_0 - \Delta S_*^0 \geq 1.319$. **The bistability of the core on the box side does not affect the coefficient K of the First-Step Upper-Bound Proposition, whether one conditions or not** (on the core action-barrier assumption and the reading at $\hbar = 0$). Cores with negative ends appear in the sector framework inside the sector of kink positions (the bridge), and their mass in the stationary chain has the rarity $e^{-1.13/\hbar}$ ($k = 9$) Reading .

07

The list of remaining rows

Table 5 The rows remaining in the box-reduction condition, the gap-inside-the-box condition and the core

Row	What remains	Present support	Grade
Row of common premises	Uniqueness of the measure at $T = \infty$, the nearest-neighbour Markov property, extension from local functions, regularity of the conditional measures and the coordinates of kinks	That the kernel of the transfer operator is positive (the original source for the version on $\mathbb{R}$ not yet checked)	Assumption
Alternating maximal-correlation assumption for boxes	The alternating maximal correlation between boxes $c_{\mathrm{alt}} < 1$ is uniform in $\hbar$ and the boundary values (a paper on the mixing of boxes)	The reading 0.731 of the two-state chain ($C = 2$), the form of the Closed Formula for the Alternating Maximal Correlation	Assumption
Uniformity in the boundary values	The $\sup$ over boundary values: near the top (the free-end type) and far away (steep descent)	The reading of c_{ff} for free ends; far away, only the prospect that the gap does not vanish	Assumption
Cross-term uniformity assumption	The cross term $\mathcal{E}_Y(\mathbb{E}[f \mid Y]) \le (1 + \chi)\mathcal{E}(f)$, with χ uniform	None (the type of a two-scale inequality)	Assumption
Kink-gas gap assumption	That the reduced kink gas falls in the Glauber type, and the loss at free ends	The closed formula of the Ising model and diagonalisation	Assumption
Segment local Poincaré assumption	Local Poincaré for segments with end values	The Six-Node In-Phase Poincaré Proposition in the ring version (23-06)	Assumption
Saddle-tube Poincaré assumption	Poincaré for the tube, uniform in the tube length $\le C\xi$	None	Assumption
Mechanisation of the medium-core step	Enclosing the two-point maximal correlation in interval arithmetic, and a paper on the difference between lattice and continuum	Floating-point transfer (cost estimated at the order of 10^3 CPU hours)	Reading
Alternating maximal-correlation assumption for the core	An upper bound for $\bar{c}_{\ge 15} < 1$ of the long-core step	Measurement of a lower bound ≤ 0.535. Does not reduce to two-point quantities (counterexample)	Assumption
Core action-barrier assumption	$C_P \le c_* \hbar^{-p} e^{\Delta S_*/\hbar}$ for the core on the box side and its windows, in particular the prefactor $\hbar^{-p}$ and windows with $k \ge 15$, $m > 12$	The reading at $\hbar = 0$, $\Delta S_*^0 \le 1.4 \cdot 10^{-2}$, and the lower bound of $\hbar \ln C_P$	Assumption

The construction of the steps of the box-reduction condition and the gap-inside-the-box condition itself ((1)–(5), (7)–(12)) is algebra for which elementary proofs are given, and what remains as assumptions are the rows of Table 5. There is still no $\hbar$ at which all the rows of the four conditions are in place together (23-05 § 08).

08

The table of the five grades — the rows of box reduction, the gap inside the box and the core

Table 6 The rows of this chapter ("—" in the range of $\hbar$ marks an assumption row that has not been shown at any $\hbar$; the numbers in the row column are the numbers of the dependency graph in § 09, and "—" marks a row not in the graph)

Row	Statement (one line)	Grade	Basis	Range of $\hbar$	Machine checking
[01]	Row of common premises (uniqueness, Markov property, extension, regularity)	Assumption	§ 03, Step 0 in 23-05 § 04	—	—
[16]	Two-Projection Lemma (1)	Proved here	Proof of Step 1 in 23-05 § 04	Independent of $\hbar$	`two_proj_var` (assumption: $A + B \le (1 + c)V$)
[17]	Efron–Stein for product measures (4)	Proved here	Proof in § 03 Step (b)	Independent of $\hbar$	—
[18]	Product over boxes (2) $\Longleftarrow$ [01], [17]	Proved here	§ 03 Steps (a)–(d)	Independent of $\hbar$	—
[19]	Box-Reduction Proposition (3)	Proved here	Proof in § 02	Independent of $\hbar$	`box_two_family`, `box_shift_average` (assumptions: the two-family inequality, $C_{\mathrm{box}} \ge 0$, the sum over overlaps $\le \mathcal{E}$)
[02]	Alternating maximal-correlation assumption for boxes	Assumption	§ 02	—	—
[12]	$c_{\mathrm{alt}} = 0.731/0.410/0.138$ ($C = 2/4/8$)	Reading	§ 02 (values of a family copying the values of the two-state chain)	Model independent of $\hbar$	—
[11]	$\ln \xi = 218.08/661.04/2212.72$, $\xi/\ell_\times \ge e^{213}$	Reading	§ 02	Three points	`xi_over_ellx` (assumption: the measured difference)

Row	Statement (one line)	Grade	Basis	Range of $\hbar$	Machine checking
—	Positive controls (Gaussian, Ising)	Reading	Table 1	—	`gauss_box_le_one`, `ising_K1_holds` (assumption: measured values)
[20]	Decomposition of the variance and the product on the fast side	Proved here	§ 04	Independent of $\hbar$	—
[03], [04]	Segment local Poincaré assumption, saddle-tube Poincaré assumption	Assumption	§ 04	—	—
[05], [06]	Kink-gas gap assumption, cross-term uniformity assumption	Assumption	§ 04	—	—
[09]	Mobility of a single kink $D = \hbar / S_0$	Cited	23-04 § 03, the step inside the box in 23-05	$\hbar \to 0$	—
—	The gap of the Glauber box with fixed ends (6)	Assumption	§ 04 (original source bibliographic only; a 6-digit reading by diagonalisation on the lattice)	—	`glauber_split` (equality)
[21]	$c_{\mathrm{kg}} = C^2 + \pi^2$	Proved here	§ 04 (algebra of the limit of (6))	Independent of $\hbar$	`kg_constant` (assumption: $C\xi \neq 0$)
—	c_{ff} for free ends and the values of K (Table 2)	Reading	§ 04	Model independent of $\hbar$	`K_ff_C2`, `K_ff_C4` (assumption: measured values)
[31]	Gap-Inside-the-Box Proposition (5)	Proved here	§ 04	Independent of $\hbar$	`H4_two_scale`, `H4_gap_form` (assumptions in § 04)
[10]	Bakry–Émery on convex domains	Cited	Kolesnikov–Milman Theorem 1.2 (1)	Independent of $\hbar$	—
[22]	Short-Core Convexity Lemma ($C_P \leq \varepsilon / \kappa(k)$ for $k \leq 7$)	Proved here	§ 05 (7) and proof	Independent of $\hbar$	`lemma_BV`, `kappa7_pos`, `kappa8_neg`
[23]	End-Reduction Lemma	Proved here	Proof in § 05	Independent of $\hbar$	—

Row	Statement (one line)	Grade	Basis	Range of $\hbar$	Machine checking
[24]	Two-Block Lemma (8), for $k \leq 14$ only	Proved here	Proof in § 05	Independent of $\hbar$	`tb_far`, `tb_adjacent`, `tb_range`, `tb_none_at_15`
—	Singular-value representation of maximal correlation	Cited	Witsenhausen 1975, Rényi 1959 (checked in a restatement)	—	—
[13]	The two-point maximal correlation of the medium-core step and B_2 $(22.1/30.5/66.4)$	Reading	§ 05 Table 3	Three points	—
[27]	Box-Reduction Inequality for Long Cores (9)	Proved here	§ 05 (a copy of the Box-Reduction Proposition)	Independent of $\hbar$	`core_uniform_L5` (assumption: the measured $\bar{c}$)
[07]	Alternating maximal-correlation assumption for the core: an upper bound for $\bar{c}_{\geq 15} < 1$	Assumption	§ 05	—	—
[14]	$\bar{c}_{\geq 15} \leq 0.535$ (measurement of a lower bound)	Reading	§ 05	Three points	—
[25]	The ratio of the Closed Formula for the Alternating Maximal Correlation (10)	Cited (framework) + Proved here (closed formula)	§ 05 (geometric series), the Two-Block Gibbs-Sampler Rate Theorem of Roberts–Sahu 1997	Independent of $\hbar$	`gauss_mode0`, `one_sub_phi`, `naive_product_false`
—	For Gaussians, maximal correlation = linear canonical correlation	Cited	Kolmogorov–Rozanov (checked in a restatement)	—	—
[26]	Two-point quantities cannot bound the alternating maximal correlation (example with three variables)	Proved here	§ 05	—	`rareR_pair`, `rareR_joint` (arithmetic of rational numbers)

Row	Statement (one line)	Grade	Basis	Range of $\hbar$	Machine checking
[28]	Three-Range Core-Constant Proposition (11)	Proved here	§ 05	Independent of $\hbar$	`three_stage` (assumption: the upper bound of each stage)
[29]	The ends of the core of the sector framework are only on the side of the well	Proved here	§ 06 (by definition)	Independent of $\hbar$	`p5_end_not_negative`
[15]	Bistability of the core on the box side, the fibres of the binary of the bump, the marginal of Y_3	Reading	§ 06	Three points	`pn_vs_bistab`, `y_side_k9_3em4` (assumption: measured values)
[08]	Core action-barrier assumption	Assumption	§ 06	—	—
—	$\Delta S_*^0 \le 1.4 \cdot 10^{-2}$, ratio to $2S_0 \le 1.05 \cdot 10^{-2}$	Reading	§ 06 (reading at $\hbar = 0$)	$\hbar = 0$	`measured_ratio` (assumption: the measured upper bound)
[30]	The local term vanishes (12)	Proved here	Proof in § 06	$\hbar \to 0$	`cloc_term_poly`, `cloc_term_vanishes` (assumptions: [08], [11], $\Delta S_* < 2S_0$)

The "Proved here" rows of Table 6 are either algebra on top of assumption rows or general inequalities whose proofs are given in the text. They do not mean that the box-reduction condition and the gap-inside-the-box condition have been shown, nor that the constant of the core has been bounded unconditionally.

09

The dependency graph

The format is the same as 23-05 § 07: **the numbers to the right of "⟸" are always smaller than the number on the left**.

[01]	Assumption	row of common premises: uniqueness at T=∞, Mar
[02]	Assumption	alternating maximal-correlation assumption for
[03]	Assumption	segment local Poincaré assumption
[04]	Assumption	saddle-tube Poincaré assumption: Poincaré for
[05]	Assumption	kink-gas gap assumption: box gap of the reduce
[06]	Assumption	cross-term uniformity assumption: cross term 𝛿
[07]	Assumption	alternating maximal-correlation assumption for
[08]	Assumption	core action-barrier assumption: C_P ≤ c ħ^(−p)
[09]	Cited	kink diffusion-coefficient row: mobility D = ħ
[10]	Cited	Bakry–Émery on convex domains
[11]	Reading	row of the growth of the correlation length: ξ
[12]	Reading	values of c_alt
[13]	Reading	two-point maximal correlation of the medium-co
[14]	Reading	c̄ ≤ 0.535 (measurement of a lower bound)
[15]	Reading	bistability, fibres of the binary of the bump
[16]	Proved here	Two-Projection Lemma (1) Lean two_proj_var
[17]	Proved here	Efron–Stein for products (4)
[18]	Proved here	product over boxes (2) ⟸ [01][17]
[19]	Proved here	Box-Reduction Proposition (3) ⟸ [02][16][18] Lean box_two_family, box_shift_average
[20]	Proved here	decomposition of the variance ⟸ [01]
[21]	Proved here	c_kg = C²+π² Lean kg_constant
[22]	Proved here	Short-Core Convexity Lemma (7) ⟸ [10] Lean lemma_BV (assumptions: lower bound of th
[23]	Proved here	End-Reduction Lemma ⟸ [01]
[24]	Proved here	Two-Block Lemma (8) ⟸ [16][22][23] Lean tb_far, tb_adjacent, tb_range
[25]	Proved here	ratio of the Closed Formula for the Alternatin
[26]	Proved here	example that does not reduce to two-point quan
[27]	Proved here	Box-Reduction Inequality for Long Cores (9) ⟸
[28]	Proved here	Three-Range Core-Constant Proposition (11) ⟸ Lean three_stage (assumption: upper bound of
[29]	Proved here	ends of the core of the sector framework on th
[30]	Proved here	the local term vanishes (12) ⟸ [08][11] Lean cloc_term_poly, cloc_term_vanishes
[31]	Proved here	Gap-Inside-the-Box Proposition (5) ⟸ [03][04] Lean H4_two_scale, H4_gap_form
[32]	Proved here	K_4 independent of ħ ⟸ [28][30][31]

```
[33] Proved here        composition C_P ≤ Kξ²S₀/ħ ⟸ [19][32]
                         Lean H3_H4_compose
```

How to check. The rows with ⟸ are [18] ([01][17]), [19] ([02][16][18]), [20] ([01]), [22] ([10]), [23] ([01]), [24] ([16][22][23]), [27] ([01][16][17][22]), [28] ([07][13][14][22][24][27]), [30] ([08][11]), [31] ([03]–[06][09][17][20][21]), [32] ([28][30][31]) and [33] ([19][32]); in every one the numbers on the right are smaller than the one on the left, and there is no circle.

What to read. The conclusion [33] depends on all the assumption rows [01]–[08] ([07] [08] through the constant of the core [28] and the local term [30]). [12][15][25][26][29] are premises of no conclusion; they are rows that explain the reading of values, the nature of the bistability, the form of the alternating correlation and the drawing of the line at the ends. The range of $\hbar$ of [28] is "three points" because the values of the medium-core step and the long-core step are readings at three values of $\hbar$; the form of (11) itself does not depend on $\hbar$.

We also check that the statements of the assumption rows do not contain the quantity of the conclusion. The quantity of the conclusion is $C_P(\mu)$ of the whole measure at $T = \infty$. [02][07] are quantities of correlation, [03][04][08] are constants of scraps, tubes, and cores and windows, [05] is a constant of the reduced gas, [06] is a comparison of forms, and [01] is a property of the measure; none of them contains $C_P(\mu)$.

Correspondence with the graph of 23-05. In the integrated graph (23-12), the leaves [02] (the segment local Poincaré assumption), [03] (the saddle-tube Poincaré assumption), [05] (the kink-gas gap assumption), [06] (the cross-term uniformity assumption) and [07] (the alternating maximal-correlation assumption for boxes) of 23-05 are the same nodes as [03][04][05][06][02] of this chapter; [04] (the core Poincaré uniformity assumption) of 23-05 is replaced by [28] (the Three-Range Core-Constant Proposition) and [30] (the local term) of this chapter, and the leaves move to [07][08]. [11] (the variance of the product of even boxes), [14] (the Box-Reduction Proposition), [15] (the Gap-Inside-the-Box Proposition) and [16] (the First-Step Upper-Bound Proposition) of 23-05 are [18], [19], [31] and [33] of this chapter.

10

To the next chapter

23-12 joins the dependency graphs of 23-05 to 23-11 into one, puts every row into one table of the five grades, and shows in a figure of bands that there is still no $\hbar$ at which all rows are in place. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter was the step that put the box-reduction condition and the gap-inside-the-box condition of its first step in the form of propositions. The

viewpoint of carrying the form of box reduction over to the four-dimensional lattice or to other quantities is taken up in Part 24 (24-03).

What this chapter can say

Can say	Cannot say
Proved here The product over boxes (2) and its proof, the Box-Reduction Proposition (3), the Gap-Inside-the-Box Proposition (5), the Short-Core Convexity Lemma, the End-Reduction Lemma and the Two-Block Lemma, the Box-Reduction Inequality for Long Cores, the ratio of the Closed Formula for the Alternating Maximal Correlation, the example that does not reduce to two-point quantities, the construction of the Three-Range Core-Constant Proposition, the local term vanishes (12) Lean (algebra)	The box-reduction condition and the gap-inside-the-box condition. The rows of the alternating maximal-correlation assumption for boxes, the cross-term uniformity assumption, the kink-gas gap assumption, the segment local Poincaré assumption, the saddle-tube Poincaré assumption, the alternating maximal-correlation assumption for the core and the core action-barrier assumption
Reading The size of ξ, the values of c_{alt}, c_{ff} and K, the constant of the core $22.1/30.5/66.4$, the bistability barrier and the 95-fold margin	Mechanisation of the upper bound for the constant of the core, uniformity as $\hbar \to 0$, the prefactor of the core action-barrier assumption
Cited Bakry–Émery on convex domains, the singular-value representation of maximal correlation, maximal correlation of Gaussians, the type of the box dynamics	A $\hbar$ at which all rows of the four conditions are in place

Sources and reproduction

Item	Kind	Source, tool
Bakry–Émery on convex domains	Cited	D. Bakry, M. Émery, Sém. Probab. XIX, LNM 1123 (1985) 177–206; A. V. Kolesnikov, E. Milman, J. Geom. Anal. 27 (2017) 1680–1702 (Theorem 1.2 (1), checked in the text).
Box dynamics (discrete spins)	Cited	F. Martinelli, "Lectures on Glauber dynamics for discrete spin models", LNM 1717 (1999) 93–191 (Propositions 3.4 and 3.5, checked in the text of the preprint).
Singular-value representation of maximal correlation	Cited	H. S. Witsenhausen, SIAM J. Appl. Math. 28 (1975) 100–113; A. Rényi, Acta Math. Acad. Sci. Hungar. 10 (1959) 441–451 (statement checked in a restatement).
For Gaussians, maximal correlation = linear canonical correlation; maximal correlation of stationary chains	Cited	A. N. Kolmogorov, Yu. A. Rozanov, Theory Probab. Appl. 5 (1960) 204–208; R. C. Bradley, Probab. Surveys 2 (2005) 107–144 (checked in a restatement).

Item	Kind	Source, tool
The rate of two-block Gibbs sampling (the framework of the Closed Formula for the Alternating Maximal Correlation, and the case of pieces of length 1)	Cited	G. O. Roberts, S. K. Sahu, J. R. Stat. Soc. B 59 (1997) 291–317, doi:10.1111/1467-9868.00070 (the Two-Block Gibbs-Sampler Rate Theorem (Theorem 5), Corollary 3, § 2.4, checked in the text); M. K. Pitt, N. Shephard, J. Time Series Anal. 20 (1999) 63–85 (Theorem 1; checked in the text of the 1996 working-paper version, the published version not cross-checked); M. Mider, P. A. Jenkins, M. Pollock, G. O. Roberts, arXiv:2009.10440v2 (2022) (Theorem 2, Corollary 2, checked in the text). The bibliography on the line Jacobi side has not been cross-checked.
The Efron–Stein inequality	Proved here	B. Efron, C. Stein, Ann. Statist. 9 (1981) (bibliographic; the proof is given in § 03).
Glauber dynamics of the one-dimensional Ising model	Assumption	R. J. Glauber, J. Math. Phys. 4 (1963) 294–307 (bibliographic only; the values are readings by diagonalisation on the lattice).
Eyring–Kramers type (the barrier and $\hbar \ln CP$)	Cited	G. Menz, A. Schlichting, Ann. Probab. 42 (2014) (the statement in the case without boundary checked in the text; the version for minimisers on a wall has not been obtained, and the core action-barrier assumption is an assumption row).
Algebra of the construction	Proved here, Lean	Proofs in the text. `two_proj_var`, `box_two_family`, `box_shift_average`, `H4_two_scale`, `H4_gap_form`, `H3_H4_compose`, `lemma_BV`, `tb_far`, `tb_adjacent`, `three_stage`, `gauss_mode0`, `cloc_term_poly`.
Numbers	Reading	Floating point for the transfer operator (two routes: the power method in the logarithmic domain and dense eigenvalues), diagonalisation of the Ising model on the lattice, polynomial Rayleigh–Ritz, numerical minimisers and heights of connection at $\hbar = 0$. Interval arithmetic is not used.

Terms used in this chapter — box dynamics (reduction to the box), even and odd boxes, averaging over shifts, maximal correlation, alternating maximal correlation, the Two-Projection Lemma, the Efron–Stein inequality, the reduced kink gas, free end, cross term, core, the threshold of convexity, the Two-Block Lemma, reduction to the ends, the Three-Range Core-Constant Proposition, the wall of conditioning, the binary of the bump, the action barrier. Their meanings are in the A-02 glossary.

Next chapter: 23-12 The integrated table and the dependency graph — puts every row of the four conditions into one table of the five grades, and confirms that the dependency graph of the steps has no circle.

The integrated table and the dependency graph

> This chapter is this book's own investigation. § 01 and § 08 are guide material; § 06 and § 07 are a map of open questions.

What this chapter shows — For the four double-well conditions of the first step, we put every row of the tables of the five grades of 23-05, 23-06, 23-07, 23-09 and 23-11 into one table, and renumber the five dependency graphs and join them into one. We put it in a form in which readers can check by the rule of numbering (the number of a premise is smaller than the number of a conclusion) that there is no circle, and we see in figures where the computer-assisted rows lie on the road to the conclusion, and that there is still no $\hbar$ at which all rows are in place together.

Prerequisite chapters — 23-05 § 07 (the format of the dependency graph), 23-06 § 05, 23-07 § 05, 23-09 § 07, 23-11 § 08 (the table of each condition), 23-00 § 01 (the labels of the five grades).

Said first — The level is [Y3]. The integrated table is a map of "how far things are shown, and where assumption begins"; it is not a ruler for measuring progress. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter lays out on one sheet the whole construction of its first step.

The order of this chapter

1. How the integrated table is made — the rules for putting five tables on one sheet, how identical rows are merged, the meaning of the columns
2. The integrated table — every row of the first step (Table 1), and the number of rows per grade (Table 2)
3. The dependency graph — the trunk (Figure 1) and the whole, how to check that there is no circle, and that the assumption rows do not contain the quantity of the conclusion
4. The family of bridges — the one-dimensional and multi-dimensional one-sided bridges
5. The position of the computer-assisted rows — what is on the road to the conclusion and what is not
6. There is no $\hbar$ at which all rows are in place — bands of the range (Figure 2, Table 4), and what is needed to bring them into line
7. The rows needed to reach the power statement — two ways of supporting the row of the core (Table 5)
8. A note on how to read it — a map, not a fraction

01

How the integrated table is made — putting five tables on one sheet

The claim of the first step is the Double-Well Relaxation-Time Upper-Bound Proposition (23-05 § 04): on the four double-well conditions and the premise rows, $C_P \le K\xi^2 S_0/\hbar$, and hence the double-well relaxation-time power statement, $\ln C_P / \ln \xi \to 2$. Each of the four conditions is written in its own chapter as a table of rows in the five grades (Proved here, Cited, Computer-assisted, Reading, Assumption) and as a dependency graph of numbered nodes. In this chapter we put them on one sheet by the following rules.

1. **Count by rows.** The unit is each row of the tables, not the name of a condition (whether the in-phase local Poincaré condition is "closed"). This is because it is usual for rows with proofs and assumption rows to be mixed within one condition.
2. **Make identical rows one.** When the same statement appears in two chapters, we make it one row and write the nodes of both chapters in the source column. For example, [14] of 23-05 and [19] of 23-11 are both the Box-Reduction Proposition, and the window log-Sobolev threshold row of 23-07 and [06] of 23-09 are both the remaining single proposition (the Window Log-Sobolev Threshold Proposition). The core Poincaré uniformity assumption of 23-05 ([04]) is supported in 23-11 by the pair of the Three-Range Core-Constant Proposition and the local term, and in 23-07 and 23-09 by the lower bound of the core on the sector side, so we made it one node with an "or" of the two ways of support.
3. **Rows left out.** The rows of the list of detour plans (23-10) are not included, because none of them is a premise of any conclusion of the present construction. Of the rows of 23-04, the uniqueness of the measure is included in the row of common premises, and the remaining spectrum of the fluctuation operator (of Pöschl–Teller type) is included as a single explanatory row.
4. **Line up the family of bridges.** The One-Dimensional One-Sided Bridge Proposition (23-04 § 04) and the one-sided bridge (23-01 § 06) are the one-dimensional and multi-dimensional versions of the same bridge, "the relaxation time of the path measure is at least the square of the correlation length", so we included them as a family of adjacent nodes (§ 04). The other rows of Table 8 of 23-01 (the relaxation-time upper-bound assumption, the relaxation-time lower-bound assumption and the rereading of mass, the chain, the version with discrete time, the infinite-volume step) do not enter the road to the conclusion of the first step, so they stay in the graph of 23-01.

The columns of Table 1 are as follows. **No.** is the number of the node in the dependency graph of § 03; **Grade** is the label of the five grades; **Range of $\hbar$** is what the row speaks about (nodes, window, three points, independent of $\hbar$, $\hbar \to 0$, or "—" for not shown at any $\hbar$); **Condition** is which condition the row belongs to; **Premise of which conclusion** gives the numbers of the nodes that take the row as a direct premise, and whether following the edges of implication reaches the final conclusion (the double-well power statement); **Source** gives the tables and graph nodes of each chapter. Machine checking (Lean theorem names and assumptions) is placed under each row of the graph in § 03.

02

The integrated table — every row of the first step

Table 1 is divided into five blocks by grade (within a block, in the order of the graph numbers). The smaller the number, the more it is a leaf (Assumption, Cited, Computer-assisted, Reading); the larger, the more it is a step of the construction; the last, [111], is the double-well power statement.

Table 1 The integrated table (every row of the first step. Numbers are the node numbers of the dependency graph in § 03, and the order of the table is the same as the graph. "To the double-well power statement" is whether following the edges of implication from the row reaches the conclusion [111])

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
Assumption	**Assumption rows (put without proof) (30 rows)**					
[001]	Uniqueness of the measure at T=∞, Markov property, extension from local functions, the limit step, regularity	Assumption	—	Common	[071][073][077][081][106]. To the double-well power statement: reaches	23-05 [01], 23-11 [01], 23-04 § 01
[002]	Saddle-tube Poincaré assumption: Poincaré for the tube, uniform in tube length ≤ Cξ	Assumption	—	Saddle-tube condition	[075]. To the double-well power statement: reaches	23-05 [03], 23-11 [04]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[003]	Kink-gas gap assumption: the box gap of the reduced kink gas is of the Glauber type	Assumption	—	Gap-inside-the-box condition	[075]. To the double-well power statement: reaches	23-05 [05], 23-11 [05]
[004]	Closed formula for the gap of the Glauber box with fixed ends (original source bibliographic only)	Assumption	—	Gap-inside-the-box condition	Premise of no node. Part of the content of the kink-gas gap assumption (a reading not made an independent node in the graph)	23-11 Table 6 (the row of the Glauber box with fixed ends)
[005]	Cross-term uniformity assumption: cross term $\mathcal{E}_Y(E[f\|Y]) \le (1+\chi)\mathcal{E}(f)$, χ uniform	Assumption	—	Gap-inside-the-box condition	[075]. To the double-well power statement: reaches	23-05 [06], 23-11 [06]
[006]	Alternating maximal-correlation assumption for boxes: the alternating maximal correlation between boxes c_alt < 1 is uniform	Assumption	—	Box-reduction condition	[072]. To the double-well power statement: reaches	23-05 [07], 23-11 [02]
[007]	Alternating maximal-correlation assumption for the core: upper bound for the alternating maximal correlation in the core $\bar{c} < 1$ ($k \ge 15$)	Assumption	—	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [07]
[008]	Core action-barrier assumption: C_P ≤ c ħ^(−p) e^(ΔS/ħ) for core and window	Assumption	—	Saddle-tube condition	[084]. To the double-well power statement: reaches	23-11 [08]
[009]	Continuous version of Holley's comparison (the tail lemma)	Assumption	Six nodes	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches	23-06 [05]
[010]	The Poincaré–Miranda theorem (covering by boxes)	Assumption	(0,0.2]	In-phase local Poincaré condition	[063]. To the double-well power statement: reaches	23-06 [08]
[011]	Variance-decomposition row: regularity and density of the functions used	Assumption	—	In-phase local Poincaré condition	[060]. To the double-well power statement: reaches	23-06 [10]
[012]	Margin row (large-end side): neighbour and end values > 2.5	Assumption	—	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2
[013]	Margin row (between the nodes): ħ between the nodes (in uniform form)	Assumption	Outside the nodes	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2
[014]	Margin row (small ħ): ħ < 0.005 (in uniform form)	Assumption	—	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[015]	Two-site gap row (large-end side): end values of the two sites > 4	Assumption	—	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2
[016]	Transfer from the ring to segments (with end values)	Assumption	—	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2
[017]	Premises of interval arithmetic (IEEE correct rounding, errors of sums, our own exp)	Assumption	—	Saddle-tube condition	[042][043]. To the double-well power statement: **does not reach**	23-07 [01], 23-08 § 03
[018]	Seam limit assumption, random-end-value assumption, regularity at the seam of the ring	Assumption	—	Saddle-tube condition	[043][099][103]. To the double-well power statement: reaches	23-07 [02]
[019]	The small assumptions of the route of the sector input (inputs to the sectors)	Assumption	—	Saddle-tube condition	[099]. To the double-well power statement: reaches	23-07 [04]
[020]	Product Bound for the Window Log-Sobolev Constant (construction of the upper bound for A_W of the product form)	Assumption	—	Saddle-tube condition	[102]. To the double-well power statement: **does not reach**	23-07 [05]
[021]	Marginal log-Sobolev input for the window (the form of LSI for the marginals of site windows)	Assumption	—	Saddle-tube condition	[087]. To the double-well power statement: reaches	23-09 [01]
[022]	Bridge volume-locality input (locality of volume on the bridge)	Assumption	—	Saddle-tube condition	[089]. To the double-well power statement: reaches	23-09 [02]
[023]	Barta's row $\kappa = 3/1000$, $b = 493/1000$	Assumption	Independent of ħ	Saddle-tube condition	[092]. To the double-well power statement: reaches	23-09 [03]
[024]	The boundary term of Bochner and regularity of eigenfunctions	Assumption	—	Saddle-tube condition	[090][092]. To the double-well power statement: reaches	23-09 [04]
[025]	The provisional rows of the Scalar-Strand Proposition (lifting assumption, regularity of the spectrum and others)	Assumption	—	Saddle-tube condition	[096]. To the double-well power statement: **does not reach**	23-09 [05]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[026]	Window Log-Sobolev Threshold Proposition: A_W < A_max(ħ); the remaining single proposition of the route of the window log-Sobolev inequality	Assumption	—	Saddle-tube condition	[099]. To the double-well power statement: reaches	23-09 [06], 23-07 [03]
[027]	The next eigenvalue 3/4 of the fluctuation operator (of Pöschl–Teller type)	Assumption	Independent of ħ	Explanation	Premise of no node. An explanatory row that does not enter the premises of the conclusion of the first step	23-04 Table 3
[028]	Ground-state assumption: discreteness of the spectrum, simplicity and positivity of the ground state (finite spatial lattice)	Assumption	—	Bridge	[107][108]. To the double-well power statement: **does not reach**	23-01 [06]
[029]	κ of the metric comparison assumption is uniform in β (the graduation of the multi-dimensional version)	Assumption	—	Bridge	[109]. To the double-well power statement: **does not reach**	23-01 [08]
[096]	Scalar-Strand Proposition (the lower end of the window, with a table of assumptions)	Assumption	8 nodes	Saddle-tube condition	Premise of no node. A candidate to support the remaining single proposition (the Window Log-Sobolev Threshold Proposition), with no edge of implication	23-09 [21]
Cited	**Rows citing an original source (10 rows)**					
[030]	Mobility of a single kink D = ħ/S_0 (at the level of the process)	Cited	ħ→0	Gap-inside-the-box condition	[075][105]. To the double-well power statement: reaches	23-05 [08], 23-11 [09]
[031]	Bakry–Émery on convex domains (Kolesnikov–Milman)	Cited	Independent of ħ	Saddle-tube condition	[076]. To the double-well power statement: reaches	23-11 [10]
[032]	Muckenhoupt's bracket B ≤ C_P ≤ 4B	Cited	Independent of ħ	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches	23-06 [03]
[033]	Correlation inequality for MTP_2 (monotonicity of e, ħκ ≥ 0)	Cited	Independent of ħ	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches	23-06 [04]
[034]	Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux)	Cited	Independent of ħ	In-phase local Poincaré condition	[062]. To the double-well power statement: reaches	23-06 [06]
[035]	Brascamp–Lieb (the cross κ_f, the crossing row)	Cited	Independent of ħ	In-phase local Poincaré condition	[063][064]. To the double-well power statement: reaches	23-06 [09]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[036]	Integrated Bochner Formula (Proposition 1.3 of Ledoux)	Cited	Independent of ħ	Saddle-tube condition	[090][092]. To the double-well power statement: reaches	23-09 [07]
[037]	Singular-value representation of maximal correlation (Witsenhausen)	Cited	—	Saddle-tube condition	Premise of no node. A representation used for how the reading values of the medium-core step are obtained	23-11 Table 6
[038]	For Gaussians, maximal correlation = linear canonical correlation (Kolmogorov–Rozanov)	Cited	—	Saddle-tube condition	Premise of no node. Used for reading the Closed Formula for the Alternating Maximal Correlation	23-11 Table 6
[039]	Stationary path-measure assumption: formula for correlations of the stationary path measure	Cited	T=∞	Bridge	[108]. To the double-well power statement: **does not reach**	23-01 [07]
Computer-assisted	**Computer-assisted rows (4 rows)**					
[040]	Margin row: margin $\hbar\rho - 2\hbar\kappa \geq 0.067$ (neighbour and end in $[0,2.5]^2$)	Computer-assisted	Six nodes	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches (only by the road on which it joins the Assumptions [012]–[016] at [068])	23-06 [01]
[041]	Two-site gap row: $\lambda_c \geq 1.3\times10^{-4}$ (ends in $[0,4]^2$)	Computer-assisted	(0,0.2]	In-phase local Poincaré condition	[063][064]. To the double-well power statement: reaches (only by the road on which it joins the Assumptions [012]–[016] at [068])	23-06 [07]
[042]	$q_{free} \leq 0.998854904$ (the chain outside the window)	Computer-assisted	Window	Saddle-tube condition	[100]. To the double-well power statement: **does not reach**	23-07 [09]
[043]	$q_{out} \leq 0.998888530$ (the outer chain of the ring, pin ≤ 0)	Computer-assisted	Window	Saddle-tube condition	[101]. To the double-well power statement: **does not reach**	23-07 [10]
Reading	**Reading rows (19 rows)**					
[044]	$\ln \xi \geq s/\hbar$ ($\xi \propto \hbar^{1/2} e^{(S_0/\hbar)}$), the size of $D/(C\xi)^2$	Reading	Three points, leading term	Conclusion	[084][111]. To the double-well power statement: reaches	23-05 [09], 23-11 [11]
[045]	c_{alt} = 0.731 / 0.410 / 0.138 (C = 2 / 4 / 8)	Reading	Model	Box-reduction condition	Premise of no node	23-11 [12]
[046]	Two-point maximal correlation of the medium-core step and B_2 (22.1 / 30.5 / 66.4)	Reading	Three points	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [13]
[047]	$\bar{c}_{(\geq 15)} \leq 0.535$ (measurement of a lower bound)	Reading	Three points	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [14]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[048]	Bistability of the core on the box side, the fibres of the binary of the bump	Reading	Three points	Saddle-tube condition	Premise of no node	23-11 [15]
[049]	Positive controls for the Box-Reduction Proposition (pure Gaussian, Ising)	Reading	—	Box-reduction condition	Premise of no node	23-11 Table 6
[050]	K ≈ 2.76 / 1.43 (C = 2), 1.78 (C = 4), and c_ff for free ends	Reading	Model	Gap-inside-the-box condition	Premise of no node	23-05 Table 2, 23-11 Table 6
[051]	$\Delta S^{0}_{*} \le 1.4\cdot10^{-2}$ (ratio to $2S_0 \le 1.05\cdot10^{-2}$)	Reading	ħ = 0	Saddle-tube condition	Premise of no node	23-11 Table 6
[052]	The constants of the route of the window-constant numerics π_*, Λ, C_1D, gap_W, the inner chain of the bridge	Reading	Three points	Saddle-tube condition	[102]. To the double-well power statement: **does not reach**	23-07 [06]
[053]	The inner chain of the ring q_in ≈ 1.24–1.28	Reading	Four points	Saddle-tube condition	[101]. To the double-well power statement: **does not reach**	23-07 [07]
[054]	The boundary stage q ≈ 1.004–1.045	Reading	Four points	Saddle-tube condition	[101]. To the double-well power statement: **does not reach**	23-07 [08]
[055]	ℓ_min = 38.3 / 121.9 / 406.8	Reading	Three points	Saddle-tube condition	[089][094]. To the double-well power statement: reaches	23-09 [08]
[056]	Upper bound of A_W 1857 / 2794 / 9106, lower bound 1.71 / 1.74 / 2.59	Reading	Three points	Saddle-tube condition	Premise of no node	23-09 [09]
[057]	How far the Feynman–Kac strand got (reading 0.54 times, paper upper bound 10^3 times)	Reading	Two points	Saddle-tube condition	Premise of no node	23-09 [23]
[067]	C ≤ 658 (numerical input)	Reading	Six nodes	In-phase local Poincaré condition	Premise of no node	23-06 Table 2
[089]	Volume locality: ℓ_min does not depend on k	Reading	Three points	Saddle-tube condition	[092]. To the double-well power statement: reaches	23-09 [14]
[094]	Needed value 25.1 / 90.3 / 312.3	Reading	Three points	Saddle-tube condition	[097]. To the double-well power statement: **does not reach**	23-09 [19]
[097]	How far the componentwise strand got (1.4–5.5 times the needed value)	Reading	Three points	Saddle-tube condition	Premise of no node	23-09 [22]
[102]	A_W of the product form ≈ $2.5\cdot10^{6}$–$4.7\cdot10^{7}$	Reading	Three points	Saddle-tube condition	Premise of no node	23-07 [14]
Proved here	**Rows with proofs (elementary proofs, algebra on named premises) (48 rows)**					

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[058]	Tilted Second-Order Taylor Lemma (second-order Taylor of the tilt and the Hölder remainder)	Proved here	Independent of ħ	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches	23-06 [02]
[059]	Nearest-neighbour chain row: the marginal of the kept sites is a nearest-neighbour chain	Proved here	Independent of ħ	In-phase local Poincaré condition	[062]. To the double-well power statement: reaches	23-06 [11]
[060]	Variance-decomposition row: decomposition of the variance	Proved here	Independent of ħ	In-phase local Poincaré condition	[065]. To the double-well power statement: reaches	23-06 [12]
[061]	The continuum statement of the margin λ	Proved here	Six nodes	In-phase local Poincaré condition	[062]. To the double-well power statement: reaches	23-06 [13]
[062]	Gap on the coarse side $\geq \lambda$ (Gershgorin on the ring)	Proved here	Six nodes	In-phase local Poincaré condition	[065]. To the double-well power statement: reaches	23-06 [14]
[063]	Nesting $\lambda_S \geq 2.3\times10^{-5}$	Proved here	(0,0.2]	In-phase local Poincaré condition	[065]. To the double-well power statement: reaches	23-06 [15]
[064]	Crossing row: $v/\hbar \leq 5888$	Proved here	(0,0.2]	In-phase local Poincaré condition	[065]. To the double-well power statement: reaches	23-06 [16]
[065]	Algebra of rates $C_P \leq \varepsilon/t$	Proved here	Six nodes	In-phase local Poincaré condition	[066][067]. To the double-well power statement: reaches	23-06 [17]
[066]	Six-Node In-Phase Poincaré Proposition: $C \leq 1.6\times10^{10}$ (ring version)	Proved here	Six nodes	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 [18]
[068]	In-phase local Poincaré condition, segment local Poincaré assumption: assembly when the rows [012]–[016] hold in uniform form	Proved here	(0,0.2]	In-phase local Poincaré condition	[075]. To the double-well power statement: reaches	23-05 [02], 23-11 [03], 23-06 § 07
[069]	Two-Projection Lemma	Proved here	Independent of ħ	Box-reduction condition	[072][078][081]. To the double-well power statement: reaches	23-05 [10], 23-11 [16]
[070]	Efron–Stein for product measures (martingale differences and Jensen)	Proved here	Independent of ħ	Box-reduction condition	[071][075][081]. To the double-well power statement: reaches	23-11 [17]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[071]	Product inequality for boxes (conditional independence of even boxes)	Proved here	Independent of ħ	Box-reduction condition	[072]. To the double-well power statement: reaches	23-05 [11], 23-11 [18]
[072]	Box-Reduction Proposition: C_P ≤ (1+1/n)C_box/(1−c_alt)	Proved here	Independent of ħ	Box-reduction condition	[105]. To the double-well power statement: reaches	23-05 [14], 23-11 [19]
[073]	Decomposition of the variance and splitting into a product on the fast side	Proved here	Independent of ħ	Gap-inside-the-box condition	[075]. To the double-well power statement: reaches	23-05 [12], 23-11 [20]
[074]	c_kg = $C^2+\pi^2$ (algebra of the limit of the closed formula)	Proved here	Independent of ħ	Gap-inside-the-box condition	[075]. To the double-well power statement: reaches	23-05 [13], 23-11 [21]
[075]	Gap-Inside-the-Box Proposition: C_box ≤ C_loc + (1+χ)C_kg	Proved here	Independent of ħ	Gap-inside-the-box condition	[104]. To the double-well power statement: reaches	23-05 [15], 23-11 [31]
[076]	Short-Core Convexity Lemma (convexity for k ≤ 7)	Proved here	Independent of ħ	Saddle-tube condition	[078][081][082]. To the double-well power statement: reaches	23-11 [22]
[077]	End-Reduction Lemma	Proved here	Independent of ħ	Saddle-tube condition	[078]. To the double-well power statement: reaches	23-11 [23]
[078]	Two-Block Lemma (two blocks for 8 ≤ k ≤ 14)	Proved here	Independent of ħ	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [24]
[079]	The ratio in the Closed Formula for the Alternating Maximal Correlation (geometric series)	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-11 [25]
[080]	Example in which two-point quantities cannot bound the alternating maximal correlation	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-11 [26]
[081]	Box-Reduction Inequality for Long Cores (k ≥ 15)	Proved here	Independent of ħ	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [27]
[082]	Three-Range Core-Constant Proposition	Proved here	Three points	Saddle-tube condition	[103]. To the double-well power statement: reaches	23-11 [28]
[083]	The ends of the core of the sector framework are only on the side of the well	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-11 [29]
[084]	The local term vanishes as ħ→0	Proved here	ħ→0	Saddle-tube condition	[103]. To the double-well power statement: reaches	23-11 [30]
[085]	Sector sum	Proved here	Independent of ħ	Saddle-tube condition	[087]. To the double-well power statement: reaches	23-09 [10]
[086]	Gibbs decomposition: the Gibbs variational inequality	Proved here	Independent of ħ	Saddle-tube condition	[087]. To the double-well power statement: reaches	23-09 [11]
[087]	Componentwise localisation	Proved here	Independent of ħ	Saddle-tube condition	[092]. To the double-well power statement: reaches	23-09 [12]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[088]	The scalar form is not uniform in k	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-09 [13]
[090]	Localised Bochner	Proved here	Independent of ħ	Saddle-tube condition	[094]. To the double-well power statement: **does not reach**	23-09 [15]
[091]	Optimising the Gibbs tilt	Proved here	Independent of ħ	Saddle-tube condition	[094]. To the double-well power statement: **does not reach**	23-09 [16]
[092]	Sector composition and the closing condition	Proved here	Independent of ħ	Saddle-tube condition	[093][099]. To the double-well power statement: reaches	23-09 [17], 23-07 [11]
[093]	The formula for the needed value A_max	Proved here	Independent of ħ	Saddle-tube condition	[094]. To the double-well power statement: **does not reach**	23-09 [18]
[095]	If the thresholds disagree, $\ell = 0$	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-09 [20]
[098]	At the single site at the top, diagonal dominance fails by ¼	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-09 [24], 23-07 Table 4
[099]	A positive lower bound for the gap of the core, uniform in k (sector side)	Proved here	—	Saddle-tube condition	[103]. To the double-well power statement: reaches	23-09 [25], 23-07 [15]
[100]	S_W of the bridge $\leq$ 1779.6	Proved here	Window	Saddle-tube condition	[102]. To the double-well power statement: **does not reach**	23-07 [12]
[101]	S_W of the ring $\leq$ 122745	Proved here	Window	Saddle-tube condition	Premise of no node	23-07 [13]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[103]	Core Poincaré uniformity assumption: Poincaré for the core, uniform in core length and end values	Proved here	—	Saddle-tube condition	[104]. To the double-well power statement: reaches. For the core on the sector side (both ends in the well), it is supported for every $\hbar \in (0,1]$ by the Dobrushin criterion of the coarse chain: conditional on the paper's proof (Coarse-Chain Dobrushin Theorem (Theorem CC of the paper)) + Computer-assisted ($\hbar \in [10^{-5},1]$, 239 intervals, interval arithmetic, $\rho_{col} \leq 0.400$) + a paper proof and Computer-assisted (Small-ħ Coarse-Chain Proposition (Proposition PIN of the paper): $\hbar \leq 10^{-5}$, $\rho_{col} \leq 0.874$) + Cited (Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux) in Bobkov–Ledoux 2000) and the Coarse-Chain Brascamp–Lieb Proposition (Proposition BLc of the paper). No Reading rows. The core on the box side is not covered (the paper "The coarse-chain Poincaré inequality")	23-05 [04], 23-07 [16], 23-11 [28][30]
[104]	K_4 does not depend on ħ	Proved here	$\hbar \to 0$	Gap-inside-the-box condition	[105]. To the double-well power statement: reaches	23-11 [32]
[105]	First-Step Upper-Bound Proposition: $C_P \leq K\xi^2 S_0/\hbar$	Proved here	$\hbar \to 0$	Conclusion	[110]. To the double-well power statement: reaches	23-05 [16], 23-11 [33]
[106]	Family of bridges, one-dimensional version: One-Dimensional One-Sided Bridge Proposition ($R \geq 1$)	Proved here	Every ħ	Bridge	[110]. To the double-well power statement: reaches	23-05 [17], 23-04 § 04
[107]	Identity of the ground-state transformation	Proved here	Finite spatial lattice	Bridge	[108]. To the double-well power statement: **does not reach**	23-01 [11]
[108]	Family of bridges, multi-dimensional version: the one-sided bridge ($C_P \geq \kappa\xi^2$)	Proved here	Continuous time, $T=\infty$	Bridge	[109]. To the double-well power statement: **does not reach**	23-01 [12]
[109]	The lower half of the power for the multi-dimensional version	Proved here	$\xi > 1$	Bridge	Premise of no node	23-01 [17]
[110]	$1 \leq R \leq K'/\hbar$	Proved here	$\hbar \to 0$	Conclusion	[111]. To the double-well power statement: reaches	23-05 [18]
[111]	Double-well power statement: $\ln C_P / \ln \xi \to 2$	Proved here	$\hbar \to 0$	Conclusion	The final conclusion	23-05 [19]

Counting the rows by grade gives Table 2.

Table 2 The number of rows of Table 1 (by grade; the breakdown in the condition column is counted by rows, not by the names of conditions)

Grade	Rows	Main content
Assumption	30	The row of common premises, the saddle-tube Poincaré assumption, the kink-gas gap assumption, the cross-term uniformity assumption, the alternating maximal-correlation assumption for boxes, the alternating maximal-correlation assumption for the core, the core action-barrier assumption, [012]–[016] extending the in-phase local Poincaré condition beyond the nodes, the premises of interval arithmetic, the inputs to the sectors, the remaining single proposition (the Window Log-Sobolev Threshold Proposition), the Scalar-Strand Proposition (with a table of assumptions) and others
Cited	10	Mobility of kinks, Bakry–Émery on convex domains, Muckenhoupt, MTP_2, Helffer's criterion, Brascamp–Lieb, Bochner, two representations of maximal correlation, the formula for stationary correlations
Computer-assisted	4	The margin of the in-phase local Poincaré condition (six nodes), the gap of two sites, the chain outside the window, the outer chain of the ring
Reading	19	How the correlation length grows, maximal correlations and the values of the steps of the core, the constants of the route of the window-constant numerics, the inner chain, the rarity exponent, the needed value, the coefficient K of the reduced model and others
Proved here	48	The assembly of the Six-Node In-Phase Poincaré Proposition, the Two-Projection Lemma, the product over boxes, the Box-Reduction Proposition, the Gap-Inside-the-Box Proposition, the Three-Range Core-Constant Proposition, the five stages of the sectors, the First-Step Upper-Bound Proposition, the One-Dimensional One-Sided Bridge Proposition, the algebra to the double-well power statement and others
Total	111	Breakdown by condition: in-phase local Poincaré condition 25, saddle-tube condition 57, box-reduction condition 7, gap-inside-the-box condition 9, common 1, family of bridges 7, conclusion 4, explanation 1

There are many rows of the saddle-tube condition in Table 2 because the rows of the core of the saddle-tube condition are built from two sides (the three steps on the box side and the five stages on the sector side); it does not mean that the saddle-tube condition is further along than the others. Conversely, there are few rows of the box-reduction condition because the construction of the Box-Reduction Proposition is short and the remaining assumption rows are gathered into one, the alternating maximal-correlation assumption for boxes. The number of rows counts where there are proofs and where assumptions remain; it does not measure how many of the four conditions are done (§ 08).

03

The dependency graph — joining five graphs into one

We turn the 111 rows of Table 1 into a list of nodes in the same format as 23-05 § 07. **The numbers of the premises to the right of "⟸" are always smaller than the number of the conclusion on the left.** There is just one node, [103] (the core Poincaré uniformity assumption), for which either of two ways of support suffices, and there we write "(...) or (...)". The numbers in both brackets are smaller than [103], so the rule works in the same way.

The whole is long, so we first give Figure 1, which extracts only the trunk. The numbers in the trunk are the same as in the whole graph.

Figure 1 The trunk of the integrated graph (of all 111 nodes, only the nodes on the main roads to the conclusion [111]; numbers as in the whole)

```
[001] ASM  row of common premises (uniqueness, Markov property, extension, limit, regula
[002][003][005] ASM saddle-tube Poincaré assumption, kink-gas gap assumption, cross-term
[006] ASM alternating maximal-correlation assumption for boxes      [007][008] ASM alter
[012]–[016] ASM rows extending the in-phase local Poincaré condition beyond the nodes, b
[026] ASM Window Log-Sobolev Threshold Proposition (the remaining single proposition)
[040][041] CMP margin of the in-phase local Poincaré condition, gap of two sites (six nc
[044] RDG ln ξ ≥ s/ħ
[066] PRV Six-Node In-Phase Poincaré Proposition (ring version, six nodes) ⟸ [065] (wi
[068] PRV in-phase local Poincaré condition, segment local Poincaré assumption ⟸ [066]
[072] PRV Box-Reduction Proposition ⟸ [006][069][071]
[075] PRV Gap-Inside-the-Box Proposition ⟸ [068][002][003][005][030][070][073][074]
[082] PRV Three-Range Core-Constant Proposition (core on the box side) ⟸ [007][046][04
[084] PRV the local term vanishes ⟸ [008][044]
[099] PRV lower bound of the core (sector side) ⟸ [026][092][019][018]
[103] PRV core Poincaré uniformity assumption ⟸ ([082][084]) or ([099][018])
[104] PRV K₄ independent of ħ ⟸ [103][075]
[105] PRV First-Step Upper-Bound Proposition ⟸ [030][072][104]
[106] PRV family of bridges, One-Dimensional One-Sided Bridge Proposition (one-dim. vers
[110] PRV 1 ≤ R ≤ K'/ħ ⟸ [105][106]
[111] PRV double-well power statement ⟸ [044][110]

(side road) [042][043] CMP chain outside the window, outer chain of the ring → [100][101
```

▶ Open the whole graph (111 nodes)

How to check. There are three steps by which readers can check for themselves. (1) Look one by one at the rows of the whole graph that have "⟸", and check that the numbers on the right are all smaller than the number on the left. At [103], look at both brackets. (2) If this rule is kept in every row, the number strictly decreases each time a premise is followed, so you eventually reach a leaf (a row without "⟸") and never return to the original row. That is all there is to "there is no circle". (3) If even one edge is found that does not keep the rule, it is a candidate for a circle of dependence, and the text has to name which step takes the conclusion of which step as a premise. In the graph of this chapter there was no such edge (all edges of the 111 nodes were checked by comparing numbers).

The assumption rows do not contain the quantity of the conclusion. The rule of numbering does not prevent the form in which the conclusion has slipped into the *content* of an assumption row (the form that puts the conclusion as a premise). So we read the statements of the assumption rows one by one. The quantities of the conclusion are three: the Poincaré constant $C_P(\mu)$ of the whole measure at $T=\infty$, the ratio $R = C_P/\xi^2$, and the power $\ln C_P / \ln \xi$. The 30 assumption rows fall into (i) properties of the measure ([001], [011], [018], [024]), (ii) constants of partial measures (scraps, tubes, cores, site windows: [002], [008], [012]–[016], [021], [022], [026]), (iii) quantities of correlation ([006], [007]), (iv) reduced models and comparisons of forms ([003], [004], [005]), (v) premises of computation and citation ([009], [010], [017], [019], [020], [023], [025], [096]), and (vi) the multi-dimensional version and explanatory rows ([027], [028], [029]); none of them contains $C_P(\mu)$, R or the power. The Poincaré constants contained in the rows of (ii) belong to the *partial* measures of scraps, tubes, cores and windows, not to the whole measure. This check is done by eye; machine checking does not replace it (23-13 § 05).

04

The family of bridges — the one-dimensional and multi-dimensional one-sided bridges

[106] the One-Dimensional One-Sided Bridge Proposition and [108] the one-sided bridge are the one-dimensional and multi-dimensional versions of a sentence of the same form. Both say "for the path measure,

the ceiling of the Rayleigh quotient of test functions local in time is the square of the correlation length", and from this come lower bounds for the relaxation time, $C_P \geq \xi^2$ (the One-Dimensional One-Sided Bridge Proposition) and $C_P \geq \kappa\xi^2$ (the one-sided bridge).

Table 3 The two versions of the family of bridges

Version	Statement	Premises	Role in this part
One-dimensional version [106]	One-Dimensional One-Sided Bridge Proposition: $R = C_P/\xi^2 \geq 1$ for the 0+1-dimensional double well	[001] (continuous time, $T \to \infty$; proof in 23-04 § 04)	The left inequality of (14). Gives the lower half of the double-well power statement without conditions
Multi-dimensional version [108]	One-sided bridge: $C_P \geq \kappa\xi^2$ on a finite spatial lattice	[028] assumption on the spectrum, [039] formula for correlations, [107] ground-state transformation (23-01 § 06)	[109] the lower half of the power for the multi-dimensional version (with the uniformity of the graduation [029]). Does not enter the road to the conclusion [111] of the first step

Putting the two side by side, one reads in the same form that the content of the diffusive-relaxation-time assumption is only "the upper half". The lower half is given automatically by the bridge (the one-dimensional version needs only the premise [001]; the multi-dimensional version needs [028][039] and the uniformity of the graduation [029]), and what the four double-well conditions take on is the upper half $C_P \leq K\xi^2 S_0/\hbar$. For the multi-dimensional version to apply to the four-dimensional Wilson measure, the rows of discrete time, infinite volume and graduation remain ([08][09][10] of the graph of 23-01).

05

The position of the computer-assisted rows — what is on the road to the conclusion and what is not

For the four "Computer-assisted" rows of Table 1, we look at the road to the conclusion [111].

- **[040] the margin, [041] the gap of two sites (in-phase local Poincaré condition).** There is a road [061] the continuum statement of the margin → [062], [063], [064] → [065] the algebra of rates → [066] the Six-Node In-Phase Poincaré Proposition → [068] the in-phase local Poincaré condition → [075] the Gap-Inside-the-Box Proposition → ... → [111]. But it is only the road on which they join the assumption rows [012]–[016] at [068]; there is no road that reaches the conclusion without passing through [068]. The computer assistance at the six nodes becomes a premise of the conclusion only when combined with the assumption rows extending it beyond the nodes, beyond the ends and to segments.
- **[042] the chain outside the window, [043] the outer chain of the ring (saddle-tube condition; the route of the chain outside the window and the route of the seam of the ring).** [042] → [100] S_W of the bridge → [102] A_W of the product form (Reading) stops there, and [043] → [101] S_W of the ring stops there. **Neither is on the road to the conclusion.** This is because the value of A_W of the product form lies outside the needed value that the remaining single proposition [026] asks for (23-09 § 03).

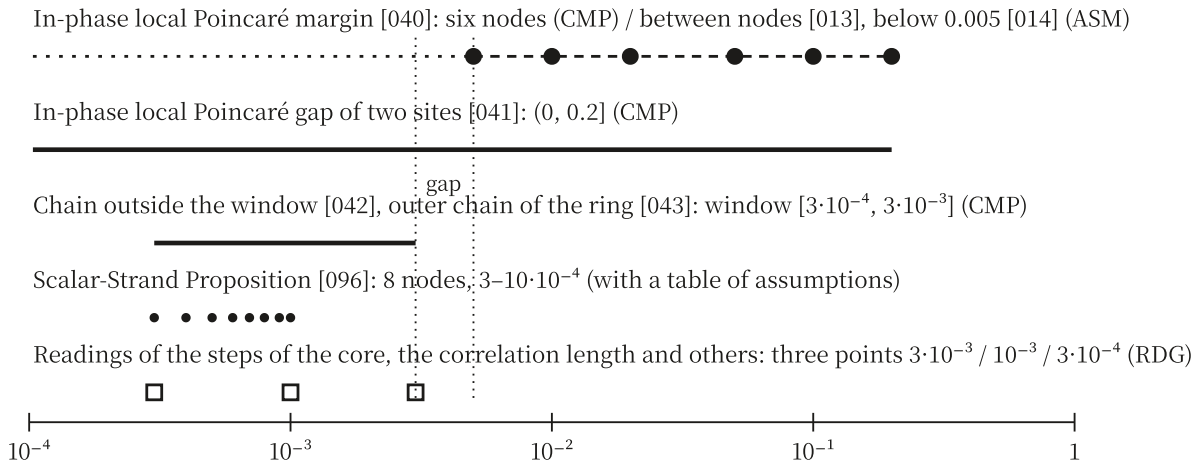
These two rows turn the contraction of the chains into inequalities over the whole window $\hbar \in [3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$, and they stand as results of computer assistance within the window in their own right (23-08). But in the present construction they are not used as premises of the conclusion. They enter the road when an upper bound for A_W not of the product form satisfies [026] and the ratio of the chain q enters that upper bound (23-07 § 06). Put the other way round, the number of computer-assisted rows does not show how close one is to the conclusion.

06

There is no $\hbar$ at which all rows are in place

We overlay the range of $\hbar$ of each row on one number line (Figure 2). Assumption rows whose range is "—" ([002], [003], [005], [006] and others, most of the 30 assumption rows) are rows not shown at any $\hbar$, so they are not drawn; only the rows that have a range are drawn.

Figure 2 Bands of the range of $\hbar$ (the horizontal axis is $\log_{10}\hbar$, from 10^{-4} to 1). Solid lines and circles are computer-assisted, dashed lines are assumptions, squares are readings, and small dots are the node values of a proposition with a table of assumptions



There are three things to read from Figure 2. (a) The computer assistance for the margin of the in-phase local Poincaré condition is at six nodes with $\hbar \geq 0.005$, and the smallest node 0.005 is larger than the upper end of the window $3 \cdot 10^{-3}$ (the "gap" between the vertical dotted lines in the figure). (b) The Scalar-Strand Proposition is point values at 8 nodes at the lower end of the window, $[3 \cdot 10^{-4}, 10^{-3}]$, and a little beyond 10^{-3} the two conditions can no longer hold together in the same window (the reading that they do not hold together at $1.35 \cdot 10^{-3}$, 23-09 § 05). (c) The six nodes of the in-phase local Poincaré condition and the 8 nodes of the Scalar-Strand Proposition do not meet, and there is no node of the in-phase local Poincaré condition inside the window. Moreover, the assumption rows whose range is "—" (the saddle-tube Poincaré assumption, the kink-gas gap assumption, the cross-term uniformity assumption, the alternating maximal-correlation assumption for boxes and others) remain assumptions at every $\hbar$. Therefore **there is still not a single $\hbar$ at which all the rows of the four conditions are in place together.**

Table 4 Three kinds of range of $\hbar$ (do not mix them; four kinds together with the three points of 23-04 § 07)

Kind	Range	Which rows	Grade
The nodes of the in-phase local Poincaré condition	$\{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$	[040], [061]–[066]	Computer-assisted, Proved here (between the nodes and below 0.005, Assumption of [013] [014])
The computer-assisted window	$[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ (25 subintervals)	[042], [043], [100], [101]	Computer-assisted, Proved here
The range of the Scalar-Strand Proposition	8 nodes in $[3 \cdot 10^{-4}, 10^{-3}]$ (no enclosure between the nodes)	[096]	Assumption (a proposition with a table of assumptions)
The three points of reading	$\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$	[044], [046], [047], [052]–[056] and others	Reading

What is needed to bring them into line. To bring them into line over the same range of $\hbar$, one of the following is needed.

1. **Extend the in-phase local Poincaré condition to $\hbar \leq 10^{-3}$.** That is, show row [014] ($\hbar < 0.005$). The type of computer assistance by subintervals cannot be used, because with a fixed ratio the number of subintervals, $\ln(0.005/\hbar)/\ln r$, grows without bound as $\hbar \to 0$; a paper argument uniform as $\hbar \to 0$ is needed (the small-ħ remaining item of 23-06 § 07).
2. **Extend the Scalar-Strand Proposition to $\hbar \geq 0.005$.** This corresponds to the route of the range of ħ (outside the window), a route decided to be outside the scope of this part (23-07 § 02). The form of the Scalar-Strand Proposition itself also stops at $1.35 \cdot 10^{-3}$.

Whichever is done, they only come into line over a fixed range. The double-well power statement is a statement about $\hbar \to 0$, so in the end every row has to be stated in a form uniform as $\hbar \to 0$, and computer

assistance in the window alone cannot reach it. Outside the window, paper asymptotics are needed. The assumption rows whose range is "—" remain at the stage before ranges are brought into line: first being shown at a single $\hbar$.

07

The rows needed to reach the power statement — two ways of supporting the row of the core

Following the graph from the conclusion [111] in the direction of the premises picks up all the leaves the conclusion needs. The only fork is the "or" at [103] (the core Poincaré uniformity assumption), so there are two required sets of leaves (Table 5). This applies to the graph of the first step the way of thinking that counts the sets that reach the conclusion when the assumption rows are put as "provisionally true" (for the list of walls on the side of the four-dimensional lattice, see 23-02 § 06).

Table 5 The leaves needed for the conclusion [111] (for each way of supporting [103]. In either set, every row listed is needed)

Grade of leaf	Support A: the core on the box side ([082] and [084])	Support B: the core on the sector side ([099])
Assumption	15 rows: [001], [002], [003], [005], [006], [007], [008], [009], [010], [011], [012]–[016]	20 rows: [001], [002], [003], [005], [006], [009], [010], [011], [012]–[016], [018], [019], [021], [022], [023], [024], [026]
Cited	[030], [031], [032], [033], [034], [035]	[030], [032], [033], [034], [035], [036]
Computer-assisted	[040], [041]	[040], [041]
Reading	[044], [046], [047]	[044], [055] (and [089], which depends on it)

What to read from Table 5. (1) The assumption rows common to both ways of support are the common premise [001], the rows inside the box [002][003][005], [006] between boxes, and [009]–[016] extending the in-phase local Poincaré condition. These remain whichever road is taken. (2) Support A puts the alternating maximal correlation of the core [007] and the action barrier [008] as assumptions, and needs the readings of the medium-core step and the long-core step, [046][047], to be replaced by upper bounds. Support B puts the remaining single proposition [026] and the inputs to the sectors [019], [021], [022], [024] and Barta's row [023] as assumptions. (3) Both sets still contain "Reading" rows. For the conclusion to become unconditional, every "Assumption" row has to be replaced by a proof, and every "Reading" row by an upper bound (Proved here, Cited or Computer-assisted).

08

A note on how to read it — a map, not a fraction

The integrated table is a map that shows where in the claim of the first step there are proofs and where assumptions remain. A proportion such as "48 of 111 rows are Proved here" does not show progress. There are two reasons. One is that the rows do not have equal weight (the single row [001] holds everything from the uniqueness of the measure to regularity, while the single row [074] is the algebra of a sum of two terms). The other is that many of the "Proved here" rows are algebra that takes assumption rows as input, and as long as one assumption row remains, the conclusion stays conditional (the same as the note after Table 2 of 23-05).

When reading it as a map, the following order helps. (a) Follow the trunk of Figure 1 from the conclusion [111], (b) look at the required set of leaves in Table 5, and (c) trace the "Assumption" rows in it from the source column of Table 1 to the text of each chapter. What remains and its cost are laid out row by row of this table in 23-15.

This part places the relaxation-time assumption at the axis of the study, and has built the upper half of the first step by dividing it into four conditions. This map, which puts the whole on one sheet, is also a map within that way of dividing. Other viewpoints are taken up in Part 24.

What this chapter can say

Can say	Cannot say
The integrated table putting the rows of the five chapters of the first step on one sheet (111 rows: Assumption 30, Cited 10, Computer-assisted 4, Reading 19, Proved here 48), and the dependency graph that keeps the rule of numbering on every edge (no circle)	Any of the 30 assumption rows. Hence the conclusion of the First-Step Upper-Bound Proposition and the double-well power statement of the first step
That the statements of the assumption rows do not contain the quantities of the conclusion $CP(\mu)$, R or the power (a check by reading the statements)	That the assumption rows are compatible with each other, or that none of them is false
The position of the four computer-assisted rows: the two rows of the in-phase local Poincaré condition become premises of the conclusion only on the road where they join [012]–[016], and the two rows of the saddle-tube condition are not on the road to the conclusion	Carrying the computer-assisted results outside the window or to $\hbar \to 0$
That there is no $\hbar$ at which all rows are in place together (by comparing ranges), and the two things needed to bring them into line	Either of the two things to bring them into line. Rows in a form uniform as $\hbar \to 0$
That there are two required sets of leaves for the conclusion (Table 5)	Which of the two ways of support is easier

Sources and reproduction

Item	Kind	Source, tool
The statement, grade and range of each row of Table 1	This book	23-05 Table 2, 23-06 Table 2, 23-07 Table 4, 23-09 Table 5, 23-11 Table 6, 23-04 Table 3, 23-01 Table 8. The original sources of the cited theorems are in the source table of each chapter
Edges of the dependency graph	This book	The "⟸" of each chapter's dependency graph copied as it is, with identical rows merged (the rules of § 01). The "or" at [103] comes from the notes on correspondence in 23-05, 23-07 and 23-11
Checking the rule of numbering, counting the rows, the required set of leaves	Text processing	A small script that, from data copying out the rows and edges of the table, compares the numbers on every edge, counts rows by grade, and collects leaves from the conclusion in the direction of the premises (not a mathematical computation)
Lean theorem names and assumptions	Lean	Under each row of the graph. The list of statements and the limits of checking are in 23-13

Terms used in this chapter — integrated table, dependency graph, node, leaf, trunk, the rule of numbering, circle of dependence, the form that puts the conclusion as a premise, family of bridges, way of support, required set of leaves, bands of the range of $\hbar$ (nodes, window, the range of the Scalar-Strand Proposition, the three points of reading). Their meanings are in the A-02 glossary.

Next chapter: 23-13 Lean statements and the limits of machine checking — looks, through the list of bundles and examples, at what machine checking guarantees, "the conclusion follows from the propositions as written", and what it does not guarantee (the truth of the assumptions, the encoding of the objects, closing by putting the conclusion into a definition, circles of dependence).

Lean statements and the limits of machine checking

> This chapter is a guide. § 01, § 02 and § 05 are established knowledge.

What this chapter shows — We confirm, with the list of bundles of the first step, the convention that what machine checking (Lean) guarantees is only that "the conclusion follows from the propositions as written (the list of assumptions)", and that it guarantees neither that the assumptions are true nor that the real numbers inside Lean represent physical quantities. We look, through examples with their sources withheld, at two types that cannot be detected even when `#print axioms` stays within the three standard axioms (putting the conclusion into a definition, and a circle of dependence), and we give a procedure by which readers can inspect statements themselves.

Prerequisite chapters — A-04 Lean and machine checking (the meaning of the label "Lean"), 23-12 § 03 (the dependency graph and the rule of numbering), 23-08 § 08 (assemblies that put the output of interval arithmetic as an assumption). For how to read Lean itself, see the Lean guide.

Said first — The level is [Y2]–[Y3]. § 01– § 02 and § 06 can be read at [Y2]. This part places the diffusive-relaxation-time assumption at the axis of the study, and Lean is used as a tool to check the algebraic steps of its construction. This chapter shows neither the mass gap nor the continuum limit.

The order of this chapter

1. What machine checking guarantees — the arrow from a list of assumptions to a conclusion (example: the composition of the Double-Well Relaxation-Time Upper-Bound Proposition)
2. The three axioms and `#print axioms` — the criterion of cleanness, why `native_decide` is not used
3. The bundles of the first step — the number of theorems in each bundle, what algebra it is, how the assumptions are put (Table 1), the side of the four-dimensional lattice (Table 2)
4. Assumptions next to theorem names — the discipline of reading, and the fact that the layer of interval arithmetic is outside Lean
5. Four limits — the truth of the assumptions, the encoding of the objects, putting the conclusion into a definition, circles of dependence (Table 3)
6. A procedure for readers to inspect for themselves — six questions
7. Translating conventions — places where the same letter or the same name refers to different quantities (Table 4)
8. How the bundles are divided — by article, paper and subject

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01

What machine checking guarantees — the arrow from a list of assumptions to a conclusion

Lean is a system in which a mathematical statement is written as a type and its proof as a term of that type, and a small core (the kernel) mechanically checks that the proof matches the statement (Lean guide 01). The rows in this book labelled "`Lean`" refer to theorems that have passed that check. The label guarantees only one thing.

The convention of this book

What is guaranteed in a row labelled "Lean" is that **if all the assumptions written in the statement of the theorem hold, then the conclusion written there holds**. It is not guaranteed that the assumptions hold, that the symbols in the statement represent the quantities of the text, or that the statement means the same as the claim in the text.

As an example, we copy the statement of the theorem for the composition of the First-Step Upper-Bound Proposition (23-05 § 04 Step 3 (composition)).

```
theorem H3_H4_compose (CP Cbox n ca K4 C ξ D hbar S0 : ℝ) (hn : 0 < n) (hca : ca < 1)
    (hh : 0 < hbar) (hS : 0 < S0) (hD : D = hbar / S0)
    (hH3 : CP ≤ (1 + 1 / n) * Cbox / (1 - ca)) (hH4 : Cbox ≤ K4 * (C * ξ) ^ 2 / D) :
    CP ≤ (1 + 1 / n) * K4 * C ^ 2 / (1 - ca) * (ξ ^ 2 * S0 / hbar)
```

How to read it. Everything in the brackets of the first line is a real number (`ℝ`); `CP` and `ξ` are just real numbers with names. `hH3` takes the conclusion of the Box-Reduction Proposition, $C_P \le (1+1/n)C_{\mathrm{box}}/(1-c_{\mathrm{alt}})$, and `hH4` the conclusion of the Gap-Inside-the-Box Proposition, $C_{\mathrm{box}} \le K_4(C\xi)^2/D$, as **assumptions**. What the theorem checked is the algebra by which (13) $C_P \le K\xi^2 S_0/\hbar$ follows from these two and $D = \hbar/S_0$. That `CP` is the Poincaré constant of the path measure at $T=\infty$, and that the conclusions of the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition actually hold (that is, the four double-well conditions and the premise rows), are outside Lean.

This character of "only the arrow" is not a weakness but a premise of how it is used. Many of the "Proved here" nodes in the dependency graph (23-12 § 03) are algebra that takes assumption rows as input, and Lean checks that this algebra has no copying errors or sign errors. Because what enters as an assumption is named in the statement, it can be cross-checked against the list of assumption rows (the integrated table).

02

The three axioms and `#print axioms`

This section summarises existing knowledge (the three axioms of Lean and #print axioms). The introduction is in A-04 § 04.

On top of the type theory of Lean's kernel, the standard toolkit adds three axioms: propositional extensionality `propext` (equivalent propositions are equal), soundness of quotients `Quot.sound` (equivalent elements are equal in the quotient), and choice `Classical.choice` (extracting an element from a proof of existence). Which axioms a theorem depends on is listed by the command `#print axioms theorem_name`.

In the bundles of this book, we put `#print axioms` for the main theorems at the end of each file, and our criterion of cleanness is that **every line of its output is a subset of the three axioms**. Using `sorry`, which skips a proof, adds `sorryAx` to the list, and declaring an `axiom` oneself adds its name, so this criterion is a mechanical confirmation that "the proof has no hole and no arbitrary axiom has been added". In every bundle of Table 1 (the "copy of the axiom output" column of Table 1; including the copies retaken on a second machine on 2026-10-09), we reread that there is not a single axiom name other than the three. Some theorems do not use `Classical.choice` among the three (for example, `tb_range` uses only `propext` and `Quot.sound`).

`native_decide`: why it is not used
`decide` checks a proposition decided by finite case analysis by computation inside the kernel. `native_decide` translates the same computation into a Lean program, runs it as machine code, and trusts the answer. It is fast, but in exchange the range of trust widens from the kernel to the translator and the execution environment, and that trust appears in the list as an additional axiom (depending on the version, an axiom with a different name appears for each declaration). In the bundles of this book we do not use `native_decide`, and we put the criterion of cleanness as "every line of the list is a subset of the three axioms". Looking for the presence or absence of a particular axiom name would miss things when the naming changes between versions.

03

The bundles of the first step — what algebra was checked

The Lean theorems of the first step are divided into files (bundles) by subject. Table 1 lists the main bundles in the order of the chapters that cite them in the text. "Number of theorems" is the number of `theorem` / `lemma` declarations in the file, and the column "Representative theorem" gives the theorem name with that theorem's assumptions in the same column.

Table 1 The main bundles of the first step (the number of theorems is the number of declarations. Copy of the axiom output: ○ = the output of `#print axioms` at the end remains separately, and we reread that every line is a subset of the three axioms. Includes the copies retaken on a second machine on 2026-10-09)

Bundle	Number of theorems	What algebra	Representative theorem (assumptions)	Chapter	Copy of the axiom output

Bundle	Number of theorems	What algebra	Representative theorem (assumptions)	Chapter	Copy of the axiom output
`LatticeGaugeOneScale`	12	The relaxation-time upper-bound assumption, the relaxation-time lower-bound assumption and the rereading of mass (the equality version)	`hup_is_gap_positivity` ($0 < m$, $CP = 1/m^2$, $CP \le K\xi^2$)	23-01	○
`LatticeGaugeHupBridge`	18	The one-sided bridge, the lower half of the power, negative controls	`hup_is_gap_positivity_of_lower` ($0 < m_{\mathrm{lat}}$, $\kappa/m_{\mathrm{lat}}^2 \le CP$, $CP \le K\xi^2$), `first_power_does_not_bridge` (negative control)	23-01	○
`LatticeGaugeDoubleWellTestbed`	7	The skeleton of the finite sum of the One-Dimensional One-Sided Bridge Proposition	`local_rayleigh_ratio_le_one` ($0 < m$, $a_k \ge m$, $c_k \ge 0$, $\sum c_k a_k > 0$)	23-04	○
`LatticeGaugeTwoScalePoincare`	8	Splitting of the variance, curvature of the convex core, values of the reduced two-state model, the band ratio of Glauber	`split_poincare_le` (the splitting formula, two local Poincaré inequalities, the mean difference, additivity of forms, non-negative constants)	23-04, 23-05	○
`LatticeGaugeDoubleWellKramers`	10	From the bracketing of the ratio to the double-well relaxation-time power statement	`power_H_of_kink_ratio` ($0 < K, s, \delta$, $1 \le R \le K/\hbar$, $s/\hbar \le \ln \xi$)	23-04, 23-05	○
`LatticeGaugeH1Assembly`	17	The algebra of rates of the Six-Node In-Phase Poincaré Proposition, the arithmetic of the margin at six nodes, Gershgorin on the ring	`H1_assembly` (seven named ones: decomposition, the fine side, Helffer's criterion, the margin, $0 < \lambda$, the crossing row, the condition on rates)	23-06	○
`LatticeGaugeCoarseSideUniform`	20	Arithmetic of the crossing row, the two-sided form of Muckenhoupt	`T4_paper_numeric` ($\lambda_c \ge 1.3 \times 10^{-4}$)	23-06	○

Bundle	Number of theorems	What algebra	Representative theorem (assumptions)	Chapter	Copy of the axiom output
`LatticeGaugeIntervalQfreeWindow`	150	The linear-fractional map of the Linear-Fractional Lemma, the identity of the tilt, from covering to window, recomputation of the rate-limiting cells	`win_qfree_window_le_of_interval_table` (the table of interval-arithmetic output)	23-08	○
`LatticeGaugeRingUnconditionedWindow`	20	Assembly of the window for the outer chain of the ring	`unc_window_lt_one_of_table` (the table of interval-arithmetic output)	23-07, 23-08	○
`LatticeGaugeRingInnerChain`	22	The ratio at the corner of the wall, S_W of the ring	`sw_inner_le` ($q_{\mathrm{in}} \le 13/10$, $q_{\mathrm{out}} \le 0.998888530$)	23-07	○
`LatticeGaugePartitionFramework`	14	The sector sum, the Gibbs step, composition and the closing condition	`comp_close` ($X > 0$, b, $A \ge 0$, $\ell > 0$, and the three inequalities of Barta, componentwise localisation and Bochner)	23-09	○
`LatticeGaugeRemVProxy`	23	Assembly of the Scalar-Strand Proposition	`path5_conditional` (with the remainder inequality, the proxy ratio, rarity and the arithmetic margin named)	23-09	○
`LatticeGaugeDetourKills`, `…2`	17, 16	The arithmetic of "the line that killed it" for the detour plans	Lemmas for each type (23-10 Table 4)	23-10	○
`LatticeGaugeBoxReduction`	23	The Two-Projection Lemma, even and odd boxes, averaging over shifts, positive and negative controls	`box_shift_average` (the two-family inequality for each shift, $C_{\mathrm{box}} \ge 0$, the sum over overlaps $\le \mathcal{E}$)	23-05, 23-11	○
`LatticeGaugeBoxGap`	18	The Gap-Inside-the-Box Proposition, the algebra of c_{kg}, the composition of the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition	`H3_H4_compose` (the statement of § 01)	23-05, 23-11	○

Bundle	Number of theorems	What algebra	Representative theorem (assumptions)	Chapter	Copy of the axiom output
`LatticeGaugeCoreMixing`, `CoreMaxCorr`, `CoreThreeStage`	15, 20, 18	The three stages of the core (the Short-Core Convexity Lemma, the Closed Formula for the Alternating Maximal Correlation, the Two-Block Lemma, combining the three stages)	`three_stage` (the upper bound of each stage)	23-11	○
`LatticeGaugeCoreActionGap`	15	The vanishing of the local term, drawing the line at the ends	`cloc_term_vanishes` (the core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$)	23-11	○
`LatticeGaugeGlobalLyapunov`	21	The algebra of the Lyapunov criterion and the positive control of the one-dimensional Gaussian	`gauss1d_bbcg`, `gauss1d_equality` (no assumptions)	23-14	○

The theorems of Table 1 are all inequalities and equalities on real numbers, finite sums and sequences. There is no theorem that has measures, operators or spaces of paths as types. The bundle of interval arithmetic (most of the 150 declarations of `IntervalQfreeWindow`) recomputes the formula for the upper bound of the last step cell by cell in rational numbers; there are many declarations, but the content is arithmetic. The index of all bundles is shared with Appendix A-04.

The side of the four-dimensional lattice. The theorems of the roads tried on the four-dimensional lattice (23-02 § 07 Table 4), divided by the type of the object their statements speak about directly, are as in Table 2.

Table 2 The type of object and the character of the theorems on the side of the four-dimensional lattice (rows are theorems; the character symbols correspond to the classification of Table 4 of 23-02: R = reproof, A = arrangement, C = conditional, N = negative control, M = lattice theorem)

Type of object	Number of theorems	Examples
Real numbers (and natural numbers) only	23	`chain_band`, `drain_exponent`
Sequences, real functions	11	`S4_two_sided`, `conditional_scale_bound`
Sums over finite sets and finite groups	8	`intact_frac_le`, `meff_ge_of_corr`
Operators on an inner-product space	1	`tau_int_le`
Lattice gauge fields (quaternions, Haar measure)	3	`one_link`, `hess_gauge`
Total	46	Counted by character (41 rows, counting two paired theorems as one row): R 11, A 15, C 7, N 5, M 3

There are three theorems that have lattice gauge fields as a type, and there is no theorem about expectations, correlations or gaps of the Wilson measure. As for the correspondence of characters, "reproof, arrangement, conditional, negative control, lattice theorem" of Table 4 of 23-02 correspond one to one to R, A, C, N, M of this table.

04

Assumptions next to theorem names

In the text and tables of this part, when we write a theorem name, **we list the assumptions of that theorem in the same column** (example: `H1_assembly` (assumptions: seven named ones)). If only the theorem name is written, it is read at the stage of summary as "settled in Lean", and conditional algebra swells into an unconditional result. In the integrated table (23-12), too, we wrote the theorem names and assumptions under each node of the graph.

In the computer-assisted rows, the Lean column splits into three layers (23-08 § 08): (1) the algebraic core (monotonicity of the linear-fractional map, the identity of the tilt, the algebra from covering to window); (2) recomputation of the rate-limiting cells and of randomly chosen cells (not all cells); (3) the assembly that puts the table of interval-arithmetic output as an **assumption**

and derives the conclusion for the whole window. Interval arithmetic itself (interval evaluation of the map of a step, quadrature, invariant sets, the conventions of rounding) is outside Lean, and its correctness rests on cross-checks and control tests (23-08 § 07). The label of a theorem that puts the table as an assumption is read as "if the output of interval arithmetic is correct, the conclusion for the whole window is correct as algebra".

05

Four limits — what cannot be detected even within the three axioms

Even for a theorem that meets the criterion of cleanness (§ 02), the following four are not guaranteed. The first two restate the convention of § 01; the last two are types in which the way the statement is written itself empties the conclusion.

(1) The truth of the assumptions. A theorem does not prove a single step of its assumptions. If `hH4` of `H3_H4_compose` is false, the conclusion says nothing.

(2) The encoding of the objects. `CP` inside Lean is a real number with a name, and that it is the Poincaré constant of the path measure is a reading of the text. The check passes even if the same name refers to a different quantity in another bundle. For example, a theorem named `measured_ratio` exists in two bundles of the first step (`CoreActionGap` and `HierarchicalLink`), and they are statements about different quantities in different namespaces. A name is no guarantee of content.

(3) Putting the conclusion into a definition. If the desired property is written as it is inside a definition and that definition is taken as an assumption, the conclusion becomes a copy of the assumption, and the proof closes in one line. The following is pseudo-Lean showing only the type.

Putting the conclusion into a definition closes anything (pseudo-Lean, an example of the type)

```
def InSpectrum (E : ℝ) : Prop := E = 0 ∨ c ≤ E          -- write the desired property into the definition of "spectrum"
theorem gap (E : ℝ) (h : InSpectrum E) : E = 0 ∨ c ≤ E := h   -- just returns the assumption as it is
def massGap : ℝ := 1 / 2                                  -- put the value literally, and positivity closes too
```

All three lines pass the check, and the list from `#print axioms` is either empty or within the three axioms. Even so, "spectrum" is not tied to the spectrum of any operator, so there is no mathematical content. One document wrote one of the conjuncts of its main theorem in this type, and another as a number put literally. It is an example in which the theorem name contains "mass gap" and every check passes, yet the statement does not encode the object.

(4) A circle of dependence. Lean checks theorems one at a time. Even if theorem A takes the conclusion of B as an assumption and theorem B takes the conclusion of A as an assumption, both pass the check. Neither conclusion has been derived unconditionally, yet putting the two side by side makes it look "shown". The same holds when proofs are written in prose: one document put the final conclusion, that the gap is positive, as a premise of intermediate steps (clustering, uniqueness of the vacuum), "to be shown later". Drawn as a dependency graph, an edge goes back from a conclusion with a large number to a step with a small number, and the rule of numbering (23-12 § 03) is broken.

Table 3 The four limits, and how this book guards against them

Limit	Visible to the check?	How this book guards against it
(1) The truth of the assumptions	Not visible (assumptions are arguments)	Write the assumptions next to the theorem name (§ 04); count the assumption rows one by one in the integrated table (23-12 Table 1)
(2) The encoding of the objects	Not visible (it only gives names to real numbers)	State the types of the Lean objects in the text ("real numbers, finite sums" of § 03); translate conventions (§ 07); write, chapter by chapter, the correspondence between the quantities of the text and the symbols of the theorem statements
(3) Putting the conclusion into a definition	Not visible (stays within the three axioms)	Copy the theorem statement into the text and read the definitions back (questions 3 and 4 of § 06); give negative-control theorems (examples that collapse when an assumption is removed)
(4) A circle of dependence	Not visible (theorems are checked one at a time)	The rule of numbering of the dependency graph; checking that the assumption rows do not contain the quantity of the conclusion (23-12 § 03)

Negative-control theorems work as a guard against (3). For example, `no_bound_when_c_one` (if the two σ-algebras are the same, $c = 1$, the reduction to the box says nothing) and `first_power_does_not_bridge` (with the first power, the bridge cannot be built) show that the conclusion collapses when an assumption is removed, confirming that the assumption is not decoration but acts on the conclusion.

06

A procedure for readers to inspect for themselves — six questions

When reading a claim labelled Lean, ask the following six in order. They apply equally to the theorems of this book and to theorems of other documents.

1. **Cleanness.** Is every line of the output of `#print axioms` a subset of `propext`, `Classical.choice`, `Quot.sound`? If you do not have the output at hand, it is "unconfirmed".
2. **The list of assumptions.** Write out all the brackets before the colon in the theorem statement (the arguments whose types are propositions). Does a claim that the text calls "unconditional" still have propositional arguments?
3. **Trace back the definition of the conclusion.** Open the definitions of the names that appear in the conclusion (such as `InSpectrum`), and see whether they contain the same sentence as the conclusion. Are the assumption and the conclusion unfoldings of the same proposition?
4. **The type of the object.** Do the objects the claim should speak about (measures, operators, fields) appear as types in the theorem statement? If what appears is only real numbers, natural numbers and finite sums, the tie to the objects lies in the reading of the text.
5. **The direction of dependence.** If the claim joins several theorems, arrange as numbered nodes which theorem takes the conclusion of which theorem as an assumption, and see whether the numbers of premises can be placed smaller than the numbers of conclusions. If not, it is a candidate for a circle.
6. **Negative control.** Is an example given in which the conclusion collapses when one assumption is removed? If not, check for yourself whether that assumption acts on the conclusion.

Example. Applied to `H3_H4_compose` of § 01: (1) a subset of the three axioms; (2) the propositional arguments are the seven `hn`, `hca`, `hh`, `hS`, `hD`, `hH3`, `hH4`; (3) the conclusion is an inequality of real numbers and no tracing back of definitions is needed; (4) the objects are all real numbers; (5) `hH3` and `hH4` are the conclusions of the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition, and in the integrated graph [072], [075] and [104] are smaller than [105]; (6) `no_bound_when_c_one` shows that at $c_{\mathrm{alt}} = 1$ the reduction to the box says nothing (`hca` is not decoration). The answer is "if the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition hold, the inequality of the First-Step Upper-Bound Proposition follows as algebra", and no more.

07

Translating conventions — places where the same letter or the same name refers to different quantities

Lean theorems do not know the meaning of symbols, so the check passes even if the same letter refers to different quantities in different files. Table 4 translates the conventions inside the theorems into the notation table of this book (Appendix A-01) and writes discrepancies as discrepancies. The upper six rows are on the side of the four-dimensional lattice, the lower five on the side of the first step.

Table 4 Conventions inside Lean and the notation of this book

Quantity	Inside Lean	This book	Discrepancy
β	One-link integration $e^{\beta\langle u,M\rangle}$, $\langle u,M\rangle = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}(UM\dagger)$	$\beta = 2N/g^2$	None ($\beta = 4/g^2$ at $N=2$). Against the convention of Shen–Zhu–Zhu, $\beta = 4\beta_{\mathrm{SZZ}}$
tr and angle	Link angle $U = e^{i\vartheta\cdot\sigma}$	$T^a = \sigma^a/2$, $U = e^{i\theta^a T^a}$	$\theta = 2\vartheta$, $C_P^{(\theta)} = 4C_P^{(\vartheta)}$. A constant factor, so it does not affect the power
a	`a` of `drain_exponent`	Lattice spacing	Lean's a is the reciprocal of the bookkeeping length, $1/\xi_{\mathrm{RG}}$
η	The static exponent, the exponent of susceptibility, the exponent of overlap	η_s and η_{dyn} separated by subscripts	Present (the same letter for three quantities)
κ	The one-loop slope, the coefficient of a model, the coefficient of the Casimir term, the contribution of faces sharing no edge	Only the one-loop slope $11/(3\pi^2)$	Present (the same letter for four quantities)
C_P (four dimensions)	A real number with a name	The Poincaré constant of Langevin. The normalisation is named	With the normalisations of the field A and the angle ϑ, $C_P^{(A)} = \beta C_P^{(\vartheta)}$
$\hbar$ and ε	`hbar` (a real number). The lattice spacing enters as a number	$\mathcal{E}(f) = (\hbar/\varepsilon)\sum_i \mathbb{E}\lvert\partial_i f\rvert^2$, $\varepsilon = \frac{1}{2}$	None. The conversion between the readings at $\varepsilon = 0.25$ and at $\varepsilon = \frac{1}{2}$ is not written (23-14 § 05)
The graduation of C_P for the core	C_P of the theorems of the stages of the core	23-11 uses the convention of $\mathcal{E}$, 23-09 the convention of $\hbar\int\lvert\nabla g\rvert^2$	$C^{(23\text{-}09)} = C/\varepsilon = 2C$ (23-11 § 05)

Quantity	Inside Lean	This book	Discrepancy
K	`K` of `power_H_of_kink_ratio`	K of the First-Step Upper-Bound Proposition and $K' = KS_0$	Lean's K is K' of the text (23-05 § 04 Step 5)
Theorem names	`measured_ratio`	—	Different theorems with the same name in two bundles ((2) of § 05)
Reading values	Numerical theorems such as `K_ff_C2`, `three_stage_3em4`	Reading	Arithmetic with measured values put as assumptions. The grade of the values stays "Reading" (it does not become "Proved here" through the Lean label)

08

How the bundles are divided — by article, paper and subject

The bundles (files) are divided by subject (one proposition and its lemmas). It is usual for the theorems cited by one chapter or article to span several bundles, and the "Chapter" column of Table 1 gives the correspondence. There are four disciplines for dividing them.

- **One subject per bundle.** Even mutually dependent subjects such as the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition are put in separate bundles, and the composition goes into a bundle of its own. The boundaries between bundles become the boundaries between nodes of the dependency graph, and one can follow which bundle takes the conclusion of which bundle as an assumption.
- **`#print axioms` at the end.** At the end of each bundle we put `#print axioms` for the main theorems, and keep the output under the same name as the bundle (the "copy of the axiom output" of Table 1).
- **No internal names in names.** Bundle names and theorem names are given by mathematical content (`LatticeGauge…` followed by the name of the subject), and names visible to readers do not contain working names or names of people.
- **Decide the bundles handed to articles and papers.** When an article or paper cites theorems of some bundle, the list of the bundles containing the cited theorems is also placed on the article's side, together with the version of the bundles (the output of which check). If a bundle is later corrected, the list on the side of the citing article is corrected too.

This part places the diffusive-relaxation-time assumption at the axis of the study, and Lean was used for the algebraic steps of the construction of the first step. The viewpoint of doing proof checking in another form (for example, a formalisation that makes the objects themselves types) is taken up in Part 24.

What this chapter can say

Can say	Cannot say
`Lean` The theorems of the bundles of Table 1 pass machine checking in a form that makes the assumptions explicit as arguments. In every bundle of Table 1, every line of the remaining `#print axioms` output is a subset of the three axioms	The truth of the assumption rows. A check of reading Lean's real numbers as quantities of the path measure itself
The Lean theorems of the first step are all statements on real numbers, finite sums and sequences; there is no theorem that has measures or operators as types	A formalisation that makes measures and operators types
The four limits and how to guard against them (Table 3), the six questions for inspection	That a claim that has passed the six questions is true (the questions are necessary checks, not sufficient ones)
That the layer of interval arithmetic is outside Lean, and only the assembly that puts the table as an assumption is inside Lean	Formal verification of the implementation of interval arithmetic

Sources and reproduction

Item	Kind	Source, tool
The declarations and numbers of the theorems of the bundles	Lean	We read the declarations of the source of each bundle (Lean 4, Mathlib) and counted the `theorem` / `lemma` lines. Theorem names and assumptions are in each row of the text
Copy of the axiom output	Lean	We read the copy of the `#print axioms` output of each bundle and counted the absence of axiom names other than the three (`sorryAx` and others), after confirming with a control copy that contains such a name that it would be detected
The three axioms and `#print axioms`	Cited	J. Avigad, L. de Moura, S. Kong, S. Ullrich et al., "Theorem Proving in Lean 4", chapter Axioms and Computation

Item	Kind	Source, tool
The layers of type and the characters on the side of the four-dimensional lattice (Table 2)	This book	The count of theorems in 23-02 § 07 (the same 46 theorems)
Examples of the type that puts the conclusion into a definition and the type of a circle of dependence	This book	A reading of two external documents that claim a construction or a proof (the sources are withheld and only the types are written; no judgement is made on whether the documents are right or wrong)

Terms used in this chapter — machine checking, bundle, the criterion of cleanness, the three axioms, `#print axioms`, `native_decide`, encoding of the objects, putting the conclusion into a definition, circle of dependence, negative-control theorem, translating conventions. Their meanings are in the A-02 glossary.

Next chapter: 23-14 Reconciliation — the numbers of the record and the known numbers — lays out, row by row against the numbers of the record, the distance between each condition of the first step and the nearest known theorem, a summary of the bibliographic checks, and the known numbers used for positive controls.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time reconciliation, distance to known theorems, bibliography · Kiichi, Shiori and Rin

Reconciliation — the numbers of the record and the known numbers

> This chapter is this book's own investigation. § 01 and § 07 are guide material; § 02 is established knowledge; § 05 is a map of open questions.

What this chapter shows — For each of the four conditions of the first step, how far the nearest known theorem reaches and where it stops. A summary of the results of checking the statements of the known theorems cited in the first step against original sources and secondary sources, and the types of discrepancy that had occurred before the check. We lay out, row by row, the known numbers (closed formulae) used as positive controls against the numbers of the record, and divide them into agreement, conditional agreement, disagreement and not comparable. The nineteen-row cross-check on the side of the four-dimensional lattice is kept as it is.

Prerequisite chapters — 23-05 § 05 (where known theorems reach), 23-12 (the integrated table), 23-13 § 07 (translating conventions).

Said first — The level is [Y3]. Numbers agreeing shows neither that the assumptions are true nor that the mass gap exists. Many of the numbers that agreed are checks that recompute known values by another road. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter holds the numbers and tools used in its construction up against outside numbers and tools.

The order of this chapter

1. The conventions of reconciliation — four verdicts, and three things to align
2. Distance to known theorems — for each condition, the nearest known result and what differs (Table 1)
3. Summary of the bibliographic checks — the support of the main citations, and the types of discrepancy (Table 2)
4. The numbers of the record and the known numbers — cross-check against the closed formulae used as positive controls (Table 3)
5. Rows that cannot be compared — numbers with no counterpart to check against
6. The nineteen rows on the side of the four-dimensional lattice — one table of cross-checks (Table 4)

7. A way of reading dependency graphs learned from outside documents — two types

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01

The conventions of reconciliation — four verdicts, and three things to align

There are four verdicts.

- **Agreement**: the same quantity compared under the same conditions agrees within the error or the digits shown.
- **Conditional agreement**: it agrees, but how it agrees depends on one choice (the lattice spacing, the box, the convention, the form of extrapolation).
- **Disagreement**: compared as the same quantity, it differs beyond the error. Even when the reason is known, we write disagreement.
- **Not comparable**: the quantities or conditions differ, and no counterpart to check against is found.

Before comparing, we align three things: the **convention** (the same CP changes value with the normalisation of the Dirichlet form, Table 4 of 23-13), the **conditions** (the lattice spacing ε, the boundary, whether inside or outside a model), and the **kind of number** (closed formula, computer-assisted upper bound, floating-point reading, lower bound of a family). Especially in the first step, the counterpart is often "a model whose answer is known by a closed formula"; agreement there is a check that the tool works correctly (a positive control), not a check of the claim on the double well.

02

Distance to known theorems — condition by condition

This section summarises existing knowledge (known criteria and theorems for Poincaré inequalities). The introduction is in 1-10 § 08.

Table 1 of 23-05 listed where known theorems reach for the Double-Well Relaxation-Time Upper-Bound Proposition as a whole. Table 1 divides it by condition, and writes in one line the nearest known result for each condition and what differs.

Table 1 The nearest known result for each condition (the grade is the grade of the row citing the known theorem; statements checked in the text)

Condition	Nearest known result	Where it reaches	What differs	Grade

Condition	Nearest known result	Where it reaches	What differs	Grade
The in-phase local Poincaré condition	Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux), Bakry–Émery on convex domains (Kolesnikov–Milman Theorem 1.2)	The two-scale construction of the chain on the coarse side and the bridge on the fine side, the convex core at the bottom of the valley	The inputs to the criterion (the margin, the gap of two sites) are values at each node of $\hbar$, and it gives no input uniform in $\hbar$	Cited
The saddle-tube condition (transport along the family of saddles)	Poincaré version of Eyring–Kramers (Menz–Schlichting Lemma 2.4, Theorems 2.9 and 2.12)	Asymptotics and prefactor of the gap between valleys of a Morse function	**Fixes the dimension**: the constants are not uniform in dimension, and a degenerate family of saddles such as the translations of a kink is outside the assumptions	Cited
The saddle-tube condition (same as above)	Eyring–Kramers for the stochastic Allen–Cahn equation on a fixed interval (Di Gesù–Le Peutrec, Brooks–Di Gesù)	Asymptotics of the gap with the length of the interval fixed short of the bifurcation threshold	**Excludes the family**: the regime in which the interval becomes infinitely long, and the family of translations, are stated by the authors themselves to be outside the scope	Cited
The box-reduction condition	The box dynamics of Glauber dynamics (Propositions 3.4 and 3.5 of Martinelli's lectures)	The type that bounds the whole gap by the box dynamics and the gap inside the box	Discrete spins, in the form of total-variation coupling. The rewriting into one dimension, L^2 and the form of maximal correlation is on this book's side (23-11 § 02)	Cited (type)
The gap-inside-the-box condition	The closed formula for the Glauber dynamics of the one-dimensional Ising model, diffusion of a single interface (Bertini–Brassesco–Buttà Theorem 2.1)	The formula for the gap of a box with fixed ends, the diffusion coefficient D of the position of a single kink	The former is for discrete spins, and that the reduced kink gas falls in that type is an assumption (the kink-gas gap assumption). The latter is a statement at the level of the process, not a statement on the box gap with many kinks	Assumption (the original source of the closed formula has not been checked), Cited

Condition	Nearest known result	Where it reaches	What differs	Grade
The whole (uniform in volume)	A criterion uniform in volume (Bauerschmidt–Dagallier Theorem 2.1)	Applies directly to the lattice of paths, an upper bound uniform in length	**Uniform in volume but doubly exponential**: the value is $CP \le e^{c\xi}$ ($c \approx 1.4 \to 1.8$, read at $\varepsilon = 0.25$, $\hbar = 0.4 \to 0.1$), and since ξ grows exponentially in $\hbar$, it is doubly exponential in $\hbar$. It does not reach the double-well relaxation-time power statement	Cited, Reading (values)
The whole (an analogous case of a theorem of the same type)	The gap of the dynamics of hierarchical $\lvert\varphi\rvert^4_4$ (Bauerschmidt–Bodineau)	A theorem of the same type as the diffusive-relaxation-time assumption: the gap vanishes with the same power as the free field (with a logarithmic correction)	**Hierarchical**: a model of nested boxes, neither the lattice model itself nor the path measure of the double well. Cited as evidence that a question of this type stands as mathematics (23-01 § 06)	Cited

From Table 1, the known tools lie in four places, "asymptotics between valleys with the dimension fixed", "a single kink", "an upper bound uniform in volume but coarse" and "a theorem of the same type in another model", and within the range searched we found no theorem that gives a sharp upper bound on a long path up to the diffusion time of the kink gas, ξ^2/D. No row can replace any one of the four conditions wholesale by a known theorem; they take the form of citing a step within a condition (a criterion, a type, a coefficient).

03

Summary of the bibliographic checks — support, and the types of discrepancy

We picked up every row in the assumption tables of the four conditions of the first step marked "cites a known theorem", and checked it against the original source or a secondary source restating the statement. 55 items were picked up, of which 26 are main items entering the columns of the condition tables directly. Table 2 gives where the 26 main items went.

Table 2 Where the 26 main citations went (level of checking. "Changed into assumptions" means renamed as an assumption row rather than a citation row)

Destination	Items	Content

Destination	Items	Content
Statement checked in the text of the original source	4	Poincaré version of Helffer's criterion (Ledoux), Bakry–Émery on convex domains and Brascamp–Lieb (by a single theorem of Kolesnikov–Milman), Sturm's oscillation theorem
Statement checked in a restatement in a secondary source	8	Muckenhoupt's bracket (in the form of Barthe–Roberto), correlation inequality for MTP_2 (Karlin–Rinott), the type of Lyapunov and local Poincaré, the Poincaré constant of the exponential distribution, IMS localisation, the bathtub principle, the maximal correlation of n steps of a reversible chain, maximal correlation of Gaussians
Proof given in this book (elementary)	7	Decomposition of the variance and the steps around it, the direction of min-max, the ground-state transformation (integration by parts), the lemma of two projections, the Efron–Stein type
Original source needs to be obtained	4	The Poincaré–Miranda theorem, regularity of the weighted Neumann operator on a box, stochastic monotonicity (Herbst–Pitt), the principal eigenvalue of a positive integral kernel on $\mathbb{R}$
Changed into assumptions	3	The cross term (cross-term uniformity assumption), the type of the kink gas (kink-gas gap assumption), Eyring–Kramers for minimisers on a wall (core action-barrier assumption)
Total	26	19 with support (the top three rows)

The 4 items that need to be obtained are counted in the integrated table as "Assumption" rows ([010], [024], part of [001] and others in 23-12 Table 1). The rule of this part is that citations that cannot be checked in the text of the original source do not go into "Cited" (23-00 § 01).

The type of discrepancy — one word of the setting. Between the record before checking and the original sources there were 15 discrepancies. 12 were reattributions that do not change the mathematics (citing the version for convex domains through another theorem, correcting attributions, correcting pages), and 3 needed rewriting of assumption rows. The 3 rewrites were (a) a row that had put regularity of eigenfunctions on a box up to the corners as "standard" (for a box with corners and an asymmetric weight it does not follow from the standard results, so it was split off as an assumption row), (b) a row in which the theorem number of the continuous version of monotonicity in the end values was unconfirmed (left in the row as unconfirmed), and (c) a row that used a logarithmic Sobolev lemma in uncentred form (the restatement of the original source is in centred form).

The discrepancies all arose from **one word of the setting** — whether there is a boundary, whether it is finite, whether it is centred. And they arose not in rows where the record wrote the source as "memory" but in rows where it wrote "standard". Rows written as "standard" have a weak motive to be checked. This is why the five grades of this part have no grade "standard". Every cited theorem is always assigned to one of "statement checked in an original or secondary source (Cited)", "proof given in this book (Proved here)" and "not checked (Assumption)".

04

The numbers of the record and the known numbers — closed formulae used as positive controls

The tools of the first step (criteria, the machinery of interval arithmetic, reduction to the box) were first applied to models whose answers are known by closed formulae, and applied to the double well only after confirming that they work correctly. Table 3 is the list of those cross-checks; the ratio is "record side ÷ known side".

Table 3 Positive controls and calibration of the first step (grades on the record side by label; row numbers continue after Table 4 of § 06)

#	Quantity	Record side	Known side	Ratio	Verdict	Conditions, notes
20	The best constant from applying the criterion of Lyapunov and local Poincaré (of BBCG type) to the one-dimensional Gaussian	Reading 5.828427 by scanning weights ($u = 0.41422$)	Closed formula $3 + 2\sqrt{2}$ ($u = \sqrt{2} - 1$)	1.000000	Agreement	6 digits. 8 digits in the dual form too. Lean `gauss1d_bbcg`, `gauss1d_equality` (no assumptions)
21	The true Poincaré supremum of the same model	Reading 4.4979 (400 finite elements), 4.4994 (800)	The value of the criterion $3 + 2\sqrt{2}$	0.77	Conditional agreement	The loss of the criterion is 1.30 times, and the direction (criterion $\geq$ true value) is right
22	CP of the pure Gaussian $V = x^2/2$ and the right-hand side of the Box-Reduction Proposition	Reading right-hand side 1.10–1.33 ($L = 4$–128)	$CP = 1$ (equality in Bakry–Émery)	1.10–1.33	Agreement	A check of the direction of the inequality. Approaches 1 as the proportion of overlap $\to 0$ Lean `gauss_box_le_one`
23	Both sides of the Box-Reduction Proposition for heat-bath Glauber of the one-dimensional Ising model	Reading holds in all 9 cases, loss 1.17–1.35 times	The gap by diagonalisation over all states	–	Agreement	Rings of 12 and 16 sites, couplings 0.3, 0.6, 1.0. A check of the direction of the inequality

#	Quantity	Record side	Known side	Ratio	Verdict	Conditions, notes
24	The gap of the Glauber box with fixed ends	Reading diagonalisation over all states of small chains	Closed formula $1-\gamma\cos(\pi/(L+1))$	1.00000	Agreement	The 6 digits shown. The same for boundary values $(+,+)$ and $(+,-)$. The original source is bibliographic only
25	Alternating maximal correlation ($C=2$, $\theta=\frac{1}{4}$)	Proved here $\varphi(e^{-1/2},e^{-1})$ of the Closed Formula for the Alternating Maximal Correlation	The closed formula of the two-state telegraph chain, 0.7308	1	Agreement	Two expressions of the same quantity (23-11 § 02)
26	The action of the kink on the lattice	Reading $S_0^{\mathrm{lat}}=0.665266$ ($\varepsilon=\frac{1}{2}$)	The continuum $S_0=2/3$	0.998	Conditional agreement	The difference of 0.2% is due to the lattice spacing. There is no estimate of the difference between continuum and lattice
27	The logarithm of the correlation length $\ln\xi$	Reading two routes: the difference of eigenvalues, and the transfer-operator version of Herring's formula	The semiclassical leading term $\xi\propto\hbar^{1/2}e^{S_0/\hbar}$	–	Conditional agreement	The two routes agree to 6 digits at intermediate $\hbar$. $\ln\xi-S_0^{\mathrm{lat}}/\hbar-\frac{1}{2}\ln\hbar$ settles towards a constant, $-0.774/-0.777/-0.778$, which agrees with the form of the leading term. There is no counterpart for comparing the constant term
28	The diffusion coefficient of a single kink	Reading $D/(\hbar/S_0)=1.00$–1.14	$D=\hbar/S_0$ (the coefficient of the theorem at the level of the process)	1.00–1.14	Conditional agreement	Point values at $\varepsilon=0.25$, $\hbar=0.02$, 0.07, 0.1. Corrections for finite $\hbar$ and the spacing

#	Quantity	Record side	Known side	Ratio	Verdict	Conditions, notes
29	The machinery for fitting the criterion uniform in volume	Reading 1.0049 for the harmonic model	Closed formula 1	1.005	Agreement	A positive control for the transfer-matrix computation applying the criterion to the lattice of paths ($\varepsilon = 0.25$)
30	The machinery of interval arithmetic: the pure Gaussian chain	Computer-assisted every point is inside the enclosure with $\hbar$ kept as an interval	Variance per step $(k-1)/k$, mean $(k-1)s/k$	1	Agreement	With the interval width set to 0, the same as the point machinery to the order of 10^{-12} (23-08 § 07)
31	The machinery of interval arithmetic: another implementation and point values	Computer-assisted upper bound	Reading 75 floating-point recursions, 26 point values of the transfer matrix	—	Agreement	0 outliers. All point values below the upper bound (smallest difference 2.85×10^{-4})
32	The gap of the hierarchical Gaussian free field	—	L^{-2N} at depth N of the nested boxes (known)		Not comparable	A known number put as a counterpart for comparison. The number on the side of the free field of the Langevin dynamics of links has not yet been computed

The rows of agreement in Table 3 are all checks that "the tool worked correctly on a model whose answer is known"; they do not show that the rows on the double well are correct. The conditions of the rows of conditional agreement are the lattice spacing (rows 26 and 28), finite $\hbar$ (row 28), and the missing constant term of the counterpart (row 27). Row 21 measures, on a model whose answer is known, the loss by which the criterion comes out 1.30 times larger than the true value; it serves as a rough guide when comparing the upper bound of the criterion with the needed value on the double well (23-09 § 05).

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05
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Rows that cannot be compared — numbers with no counterpart to check against

We list the numbers of the first step for which no counterpart to check against is found. All of them are numbers at the places where the construction comes closest to the upper half of the diffusive-relaxation-time assumption, and only there is there no counterpart.

- **Conversion between the two lattice spacings.** The readings of the diffusion coefficient of kinks and of the lower bound of the ratio R are at spacing $\varepsilon = 0.25$, while the table of correlation lengths and the steps of the First-Step Upper-Bound Proposition are at $\varepsilon = \frac{1}{2}$, and the conversion between the two spacings is not written (the table of ranges in 23-04).
- **The coefficient of $1/\hbar$ of the logarithmic Sobolev constant of the core.** A quantity needed in the Feynman–Kac strand; the Monte Carlo reading and the pathwise upper bound are three orders of magnitude apart (23-09 § 05).
- **The reading of $\sup |\nabla \psi|^2$.** A reading that it is within 0.54 times the threshold; it is not an upper bound. No known value of the same quantity was found in the range searched.
- **The lifting assumption of the Scalar-Strand Proposition.** An assumption that lifts the measured values of the two-site lump linearly beyond the measured range; the counterpart for comparison is the lifted value itself (23-09 § 05).
- **Comparison with the hierarchical free field.** Row 32 of Table 3. The known number L^{-2N} exists, and the number on the record side does not yet.

06

The nineteen rows on the side of the four-dimensional lattice

We keep one table that holds the numbers computed by the record on the side of the four-dimensional lattice (the one-loop slope κ, the shortfall $\delta(b)$, the ratio $m/\sqrt{\sigma}$, upper bounds of mass, C_P and autocorrelation times, strong-coupling thresholds, the exponent of stochastic time) up against existing computations and the numbers of primary lattice sources. The ratio is "record side ÷ existing side", and it is left blank for rows that cannot be compared. The details of each row and the conversion of conventions are in the chapter on the side of the four-dimensional lattice (23-02) and the source tables of Part 5.

Table 4 Cross-checks on the side of the four-dimensional lattice (rows 1–19)

#	Quantity	Record side	Existing side	Ratio	Verdict	Conditions, notes
1	κ (window model)	0.371521	0.371511	1.00003	Agreement	Within the one-loop model

#	Quantity	Record side	Existing side	Ratio	Verdict	Conditions, notes
2	Two-loop exponent	51/121	$b_1/(2b_0^2)$	1	Agreement	Arithmetic of rational numbers
3	Lattice slope / one loop	0.304(2)	0.3715	0.819	Disagreement	$\beta \leq 2.60$, before asymptotic scaling
4	Lattice slope / two loops	0.304(2)	0.3972	0.766	Disagreement	Two loops widen the gap
5	c_2 of $\delta(b)$	0.0197, 0.4200	—		Not comparable	Different variables from Symanzik and MCRG
6	$m/\sqrt{\sigma}$ (p=2) / LT	3.844(43)	3.844(61)	1.000	Conditional agreement	The same data table; choosing p=2
7	$m/\sqrt{\sigma}$ (p=1) / LT	4.053(73)	3.844(61)	1.054	Conditional agreement	χ^2 cannot choose p
8	$m/\sqrt{\sigma}$ (p=2) / AT	3.844(43)	3.781(23)	1.017	Conditional agreement	1.3 times the combined error
9	Upper bound β=2.40 (APE)	1.025(140)	0.953(19)	1.076	Agreement	The same spatial side 12 ($L\sqrt{\sigma} \approx 3.2$)
10	Upper bound β=2.55 (APE)	0.531(50)	0.586(10)	0.906	Conditional agreement	-1.1σ. $12^3 \times 24$ ($L\sqrt{\sigma} \approx 1.9$) against 20^4
11	Upper bound β=2.60 (APE)	0.558(51)	0.514(8)	1.086	Conditional agreement	$12^3 \times 24$ ($L\sqrt{\sigma} \approx 1.6$) against 24^4
12	Upper bound β=2.60 (flow)	0.501(71)	0.514(8)	0.975	Conditional agreement	12^4 ($L\sqrt{\sigma} \approx 1.6$) against 24^4
13	Ratio m(2.40)/m(2.60)	1.837(302)	1.854	0.991	Conditional agreement	β=2.60 is a small box with $L\sqrt{\sigma} \approx 1.6$
14	Straight-line threshold / SZZ	1/12	1/48 (converted by $\beta = 4\beta_{SZZ}$)	1.000	Agreement	Rederivation of a known result
15	Threshold after one-link integration	$\sqrt{(2/27)} = 0.2722$	—		Not comparable	Threshold of the marginal measure. The constant moves by 3.27 times within the same criterion. Not a claim of a widened range

#	Quantity	Record side	Existing side	Ratio	Verdict	Conditions, notes
16	C_P of the free field	C_P^(A)=ξ², 4C_P(time 0)²	Gaussian measure, ground-state transformation	1	Agreement	Recomputation
17	τ_int/ξ² of Langevin	0.5–1.4	—		Not comparable	No primary source for the same quantity at zero temperature towards the continuum limit was found in the range searched
18	The pole in stochastic time	13/6, −11/3	Baulieu–Zwanziger	1	Agreement	Only one independent primary source
19	Exponent s	−9/22	g₀^{9/11} (Lüscher–Schaefer equation (1.1))	1	Conditional agreement	Conditional on the one-loop leading logarithm and the relaxation-time assumption. Not an independent check

On the side of the four-dimensional lattice there are 6 agreements, 8 conditional agreements, 2 disagreements and 3 not comparable (rows 3 and 4 are two readings of the same disagreement). On the side of the first step (rows 20–32 of Table 3) there are 8 agreements, 4 conditional agreements and 1 not comparable. Read together, the rows of agreement are recomputations of known values or checks of tools on models whose answers are known, and the rows with no counterpart gather, on both sides, at the numbers that come closest to the relaxation-time assumption ($\tau_{\mathrm{int}}/\xi^2$ on the side of the four-dimensional lattice; the coefficient of the core and the lifting assumption on the side of the first step).

07

A way of reading dependency graphs learned from outside documents

We read two outside documents that claim to solve the existence and mass gap of Yang–Mills, and drew from them, not their correctness, but two types useful as **ways of reading the construction of a claim**. We do not give the sources; we write only the types. This book does not judge whether those documents are right or wrong.

1. **The type that puts the conclusion as a premise of intermediate steps.** One document piled up steps in the form of theorems and proofs; intermediate steps (clustering, uniqueness of the vacuum, exponential decay) took as a premise the

positivity of the mass gap, "to be shown later", and the positivity of the mass gap itself was supported by physical relations and dimensional analysis. Read step by step, each step derives its conclusion from its premises, but when the steps are arranged as numbered nodes, an edge goes back from the final conclusion to the intermediate steps, and the rule of numbering is broken. It is a type that is **invisible to step-by-step checking, and visible once arranged in a dependency graph**.

2. **The type with an object for its name and numbers for its content.** The other document closed its main theorem in Lean. Tracing the conjuncts of the main theorem back to their definitions, "lies in the spectrum" was defined as the same proposition as the conclusion, and the value of the gap was a rational number put literally; operators, fields and measures appeared only as names. It is a type that is **invisible even when machine checking passes and the list of axioms is clean, and visible once the definitions are traced back** (23-13 § 05).

These two types should be applied in the same way to the construction of this book. In the integrated graph of the first step, we checked the rule of numbering on every edge, and checked by reading that the statements of the assumption rows do not contain the quantities of the conclusion (23-12 § 03). For the Lean theorems, that there is no name on the side of the conclusion whose definition needs tracing back (only inequalities of real numbers) can be checked by the six questions of 23-13.

This part places the relaxation-time assumption at the axis of the study, and has held the numbers and tools of its first step up against outside numbers and tools. Other viewpoints are taken up in Part 24.

What this chapter can say

Can say	Cannot say
Cited Where the nearest known result for each condition stops (Table 1). Within the range searched, no known theorem gives a sharp upper bound on a long path up to the diffusion time of the kink gas	That no such theorem exists (outside the range searched)
Of the 26 main citations, 19 are supported by an original source, a secondary source or a proof in this book, 4 need to be obtained, and 3 were changed into assumption rows (Table 2). The type of discrepancy is one word of the setting	The statements of the 4 items that need to be obtained
The cross-check of positive controls (rows 20–32 of Table 3): that the tools worked correctly on models whose answers are known	That this shows the rows on the double well are correct

Can say	Cannot say
The nineteen-row cross-check on the side of the four-dimensional lattice (Table 4)	Explaining the disagreement between the slope of the lattice data tables and the one-loop value beyond the reason that it is before asymptotic scaling
The two types drawn from outside documents, and the check of applying them to the construction of this book	Whether the outside documents are right or wrong

Sources and reproduction

Item	Kind	Source, tool
Statements of known theorems (Table 1)	Cited	G. Menz, A. Schlichting, Ann. Probab. 42 (2014), arXiv:1202.1510. G. Di Gesù, D. Le Peutrec, arXiv:1506.04434; M. Brooks, G. Di Gesù, arXiv:2606.16083. F. Martinelli, LNM 1717 (1999) 93–191. L. Bertini, S. Brassesco, P. Buttà, Arch. Ration. Mech. Anal. 190 (2008) 307–345. R. Bauerschmidt, B. Dagallier, Comm. Pure Appl. Math. 77, arXiv:2202.02295. R. Bauerschmidt, T. Bodineau, arXiv:1809.02075. M. Ledoux, LNM 1755 (2001) 167–194. A. V. Kolesnikov, E. Milman, J. Geom. Anal. 27 (2017), arXiv:1310.2526. The same as the source table of each chapter
Closed formula for Glauber dynamics	Assumption	R. J. Glauber, J. Math. Phys. 4 (1963) 294 (bibliographic only). The formula was checked independently by diagonalisation over all states of small chains
Summary of the bibliographic checks (Table 2)	This book	A summary of the table checking the 55 items cited in the assumption tables of the first step against original sources, secondary sources and bibliographic queries. The bibliography of each item is in the source table of each chapter
Numbers on the record side of Table 3	Reading, Computer-assisted, Proved here	Floating-point computation (transfer matrices, finite elements, diagonalisation over all states), interval arithmetic, closed formulae in the text. In the source table of each chapter
The nineteen rows of Table 4	This book	One table of cross-checks on the side of the four-dimensional lattice (the bibliography of the original materials is in the source tables of Part 5 and 23-02)
The two types	This book	A reading of two outside documents that claim a construction or a proof (sources withheld)

Terms used in this chapter — reconciliation, agreement, conditional agreement, disagreement, not comparable, positive control, calibration, distance to known theorems, one word of the setting, the type that puts the conclusion as a premise of intermediate steps, the type with an object for its name and numbers for its content. Their meanings are in the A-02 glossary.

Next chapter: 23-15 What remains and its cost — the distance to a theorem — lays out what remains and its cost for each assumption row of the integrated table, and

separates the distance within the first step from the distance from the second step on.

What remains and its cost — the distance to a theorem

> This chapter is a map of open questions. It marks the boundary between the parts that close and the parts that do not.

What this chapter shows — That the rows needed to reach the conclusion of the first step (the upper side of the double-well relaxation-time power statement) split two ways at the single fork of the integrated table. For each of the two roads (the core on the box side, the core on the sector side), the assumption and reading rows needed, and the kind of work needed to close each row together with cost estimates. What is needed to produce a $\hbar$ at which all rows are in place. Outside the first step, what the distance from the second step on looks like when measured by three levels of the goal on the four-dimensional lattice, and the meaning of the one line "what is missing is uniformity".

Prerequisite chapters — 23-12 Table 1 (the integrated table) and 23-12 § 07 (the required sets of leaves). The content of each row is in the chapters 23-05 to 23-11. For the second step on, 23-01 and 23-03 § 06.

Said first — The level is [Y3]. Every cost is an estimate, and nothing has been run. We do not add up numbers of rows or hours and read them as "so many tenths to go". The rows do not have equal weight, and only rows expected to close by computation have time estimates; rows whose method is unknown have no estimate at all. This part places the diffusive-relaxation-time assumption at the axis of the study, and what this chapter measures is what remains of the first step on that axis.

The order of this chapter

1. The conventions of this chapter — distance is a list, cost is an estimate
2. The fork and the two roads — the core on the box side, the core on the sector side (Table 1)
3. Comparing the two roads — counting by kind of work (Table 2, Figure 1)
4. Remaining work not on the road to the conclusion — work worth having (Table 3)
5. What is needed to produce a ħ that fits all — why computer assistance in the window cannot reach it / The third branch of the fork (the road of blocks and mixing) (Tables C1 and C2)
6. Conditions common to the four conditions — five premises (Table 4)
7. The distance from the second step on — three levels of the goal, and the number of steps in β (Tables 5 and 6) / Detour: the lesson of ϕ^4_4
8. How to read the distance — a one-line conclusion

01

The conventions of this chapter — distance is a list, cost is an estimate

The "distance" to a theorem is a metaphor. In this chapter we call distance the **list** of those premises of the target proposition that have no proof.

$$\text{distance} \;=\; \{\text{the rows among the premises of the target proposition that have no proof}\} \qquad \text{(a set; we do not}$$

The goal of the first step is the upper side of the double-well power statement for the path measure of the double well, that is, the conclusion of the Double-Well Relaxation-Time Upper-Bound Proposition, $C_P \le K\xi^2 S_0/\hbar$, and $\ln C_P / \ln \xi \to 2$, which follows from it (23-05 § 04). The premises as a whole are the 111 rows of the integrated table (23-12 Table 1), and the rows without proof are the "Assumption" and "Reading" rows. The "Reading" rows are at hand as numbers, but they are not propositions of upper or lower bounds, so as premises of the conclusion they count as rows without proof.

To each row we attach the kind of work needed to close it. There are six: **interval arithmetic** (replace it by a computer-assisted inequality; 23-08), **paper** (write a proof), **bibliography** (check a known theorem in the original source and raise it to "Cited"), **measurement** (measure a quantity not yet measured), **another method** (it is known that the present tools cannot reach it, and another tool is needed), and **method undecided** (it is not yet decided with which tool to attack it). Time estimates are written only for rows with an estimate in the record, and rows with "no estimate" are left blank.

Example. Row [013] of the in-phase local Poincaré condition, between the nodes, has the estimate "24 subintervals, 23–39 hours in total". Row [002] for the tube, on the other hand, has no estimate. It is wrong to read the former as "closer" than the latter. The former is work expected to close if computed; the latter is work for which it is not yet known what would close it. A sum of computing hours does not show the size of the distance.

02

The fork and the two roads — the core on the box side, the core on the sector side

Following the dependency graph of the integrated table from the conclusion [111] in the direction of the premises, there is only one fork with "or": the core Poincaré uniformity assumption (the Poincaré constant of the core is uniform in core length and end values, [103]). This row is supported in two ways (23-12 § 07). For the road of the core on the sector side, the paper "The coarse-chain Poincaré inequality", which came out of this book, supports the core Poincaré uniformity assumption for every $\hbar \in (0,1]$ on the core whose two ends lie in the well: conditional on the paper's proof (Coarse-Chain Dobrushin Theorem (Theorem CC of the paper)) + Computer-assisted ($\hbar \in [10^{-5},1]$ by interval arithmetic over 239 intervals, $\rho_{\text{col}} \le 0.400$) + a paper proof and Computer-assisted (Small-$\hbar$ Coarse-Chain Proposition (Proposition PIN of the paper): $\hbar \le 10^{-5}$, $\rho_{\text{col}} \le 0.874$) + Cited (Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux) in Bobkov–Ledoux 2000) and the Coarse-Chain Brascamp–Lieb Proposition (Proposition BLc of the paper), with no Reading rows. The core on the box side (the core that enters the Gap-Inside-the-Box Proposition on the road of the core on the box side) is not covered.

- **The road of the core on the box side**: supported by the Three-Range Core-Constant Proposition (three stages for the constant of the core, [082]) and the vanishing of the local term as $\hbar \to 0$ ([084]) (23-11 § 05).
- **The road of the core on the sector side**: supported by a uniform lower bound for the gap of the core of the sector framework ([099]) and regularity at the seam (the seam limit assumption) (23-09 § 02).

Whichever road is taken, the rows needed for the conclusion are fixed as one set. Table 1 lists all the "Assumption", "Reading" and "Computer-assisted" rows in the union of the two roads, and writes the work and cost needed to close each row. The table, the graphs and the counts were made from the same single set of data (rows, edges) as the integrated table, with a column of costs added.

Table 1 The assumption, reading and computer-assisted rows needed for the conclusion [111], and the work needed to close them (numbers and contents as in the integrated table. A = the road of the core on the box side, B = the road of the core on the sector side. Every cost is an estimate; nothing has been run)

No.	Content (integrated table)	A	B	Work needed to close it	Kind	Cost estimate	Text
Assumption	**Assumption rows (22 rows)**						
[001]	Uniqueness of the measure at T=∞, Markov property, extension from local functions, the limit step, regularity	○	○	Cite from the original source or show on paper the uniqueness of the measure (the principal eigenvalue of a positive integral kernel on ℝ), the Markov property, extension, the limit step and regularity	Bibliography, paper	Obtaining 1 original source (positive kernel) and paper. No time estimate	23-05
[002]	Saddle-tube Poincaré assumption: Poincaré for the tube, uniform in tube length ≤ Cξ	○	○	The Poincaré constant of the tube is uniform in tube length ≤ Cξ	Method undecided	No estimate (not started)	23-07
[003]	Kink-gas gap assumption: the box gap of the reduced kink gas is of the Glauber type	○	○	Show the box gap of the reduced kink gas without copying the values of the one-dimensional Ising model	Another method	No estimate (a comparison theorem is needed)	23-11
[005]	Cross-term uniformity assumption: cross term $\mathcal{E}_Y(E[f\|Y]) \le (1+\chi)\mathcal{E}(f)$, χ uniform	○	○	χ of the cross term is uniform	Method undecided	No estimate (not started)	23-11
[006]	Alternating maximal-correlation assumption for boxes: the alternating maximal correlation between boxes c_alt < 1 is uniform	○	○	The alternating maximal correlation between boxes c_alt < 1, uniformly	Another method	No estimate (a comparison theorem avoiding the example that does not reduce to two-point quantities is needed)	23-11
[007]	Alternating maximal-correlation assumption for the core: upper bound for the alternating maximal correlation in the core $\bar{c} < 1$ ($k \ge 15$)	○	—	Upper bound for the alternating maximal correlation in the core $\bar{c} < 1$ ($k \ge 15$)	Another method	No estimate (the saturation is only a floating-point reading)	23-11
[008]	Core action-barrier assumption: C_P ≤ c ħ^(−p) e^(ΔS/ħ) for core and window	○	—	C_P ≤ c ħ^(−p) e^(ΔS/ħ) for core and window	Paper, measurement	The prefactor ħ^(−p) has not been measured. The next move is the capacity at the degenerate point (no estimate)	23-11
[009]	Continuous version of Holley's comparison (the tail lemma)	○	○	Continuous version of Holley's comparison	Bibliography	Confirming the version and the theorem number	23-06
[010]	The Poincaré–Miranda theorem (covering by boxes)	○	○	The Poincaré–Miranda theorem	Bibliography	Obtaining 1 original source	23-06

No.	Content (integrated table)	A	B	Work needed to close it	Kind	Cost estimate	Text
[011]	Variance-decomposition row: regularity and density of the functions used	○	○	Regularity and density of the functions used	Bibliography, paper	No estimate	23-06
[012]	Margin row (large-end side): neighbour and end values > 2.5	○	○	Neighbour and end values > 2.5	Paper	No estimate (the road through monotonicity does not close, and the lemma of tangents to a convex tail is far from the margin)	23-06
[013]	Margin row (between the nodes): ħ between the nodes (in uniform form)	○	○	ħ between the six nodes (0.005–0.2)	Interval arithmetic	After one rerun of the frame (30–50 minutes), 24 subintervals, 23–39 hours in total (one core; cut into jobs of at most 25 minutes each, 60–170 jobs)	23-06
[014]	Margin row (small ħ): ħ < 0.005 (in uniform form)	○	○	ħ < 0.005	Paper + interval arithmetic	Interval arithmetic alone would need infinitely many subintervals. On paper it is reduced to the ħ→0 limit of the margin ħρ − 2ħκ ≥ 0.067 of the margin row alone, and the limiting margin is ≥ 0.758 (interval arithmetic + paper). The quantitative uniform Laplace method and a one-dimensional lemma on the ħρ side are needed (not done). With an explicit $\hbar_0$ the infinitely many become finitely many (§ 05). On the upper-end side, ħ∈[0.001949, 0.005] was filled with six subintervals by interval arithmetic (margin ≥ 0.067; the arithmetic on the subintervals in Lean: LatticeGaugeH1HbarInterval). Below that, the estimate of the corner tail does not close with this computing machinery, and the machinery has to be replaced (§ 05)	23-06
[015]	Two-site gap row (large-end side): end values of the two sites > 4	○	○	End values of the two sites > 4	Paper	No estimate	23-06
[016]	Transfer from the ring to scraps (with end values)	○	○	From the statement on the ring to the statement on scraps (with end values)	Paper	No estimate	23-06
[018]	Seam limit assumption, random-end-value assumption, regularity at the seam of the ring	—	○	Seam limit assumption, random-end-value assumption, regularity at the seam	Paper	No estimate	23-07

No.	Content (integrated table)	A	B	Work needed to close it	Kind	Cost estimate	Text
[019]	The small assumptions of the route of the sector input (inputs to the sectors)	—	○	The small assumptions of the route of the sector input (the vertical term, the small assumptions of the sector input, regularity)	Paper	A list only, not started (no estimate)	23-07
[021]	Marginal log-Sobolev input for the window (the form of LSI for the marginals of site windows)	—	○	Marginal log-Sobolev input for the window	Paper	No estimate	23-09
[022]	Bridge volume-locality input (locality of volume on the bridge)	—	○	Bridge volume-locality input (locality of volume on the bridge)	Paper	No estimate	23-09
[023]	Barta's row $\kappa = 3/1000$, $b = 493/1000$	—	○	Barta's row $\kappa = 3/1000$, $b = 493/1000$ (at threshold 0.57)	Paper, interval arithmetic	No estimate (the paper evaluating the row is not given in this book)	23-09
[024]	The boundary term of Bochner and regularity of eigenfunctions	—	○	The boundary term of Bochner and regularity of eigenfunctions (C^3 up to the corners does not follow from the standard results)	Another method	No estimate	23-09
[026]	Window Log-Sobolev Threshold Proposition $A_W < A_{max}(\hbar)$: the remaining single proposition of the route of the window log-Sobolev inequality	—	○	The remaining single proposition of the route of the window log-Sobolev inequality, $A_W < A_{max}(\hbar)$	Method undecided	No estimate. All three strands stop at the shortfall of ¼ in the curvature of the single site at the top	23-09
Reading	**Reading rows (need to be replaced by propositions of upper or lower bounds) (5 rows)**						
[044]	$\ln \xi \geq s/\hbar$ ($\xi \propto \hbar^{1/2} e^{(S_0/\hbar)}$), the size of $D/(C\xi)^2$	○	○	$\ln \xi \geq s/\hbar$ as a lower-bound proposition	Bibliography, paper	Expected to rise to "Cited" if the original source of the semiclassical asymptotics of the tunnelling splitting is checked (no estimate)	23-04
[046]	Two-point maximal correlation of the medium-core step and B_2 (22.1 / 30.5 / 66.4)	○	—	The two-point maximal correlation of the medium-core step ($8 \leq k \leq 14$) as an upper bound	Interval arithmetic, paper	Interval arithmetic for each of the seven values of k and each point of ħ, and a paper on the difference between lattice and continuum (no time estimate)	23-11
[047]	$\bar{c}_{(\geq 15)} \leq 0.535$ (measurement of a lower bound)	○	—	$\bar{c}_{(\geq 15)} \leq 0.535$ as an upper bound	Another method	No estimate (at present only a measurement of a lower bound)	23-11
[055]	$\ell_{min} = 38.3$ / 121.9 / 406.8	—	○	$\ell_{min} = 38.3$ / 121.9 / 406.8 as a lower bound	Interval arithmetic	No estimate	23-09
[089]	Volume locality: ℓ_{min} does not depend on k	—	○	Volume locality: ℓ_{min} does not depend on k	Paper	No estimate	23-09
Computer-assisted	**Computer-assisted rows (done within the nodes and ranges) (2 rows)**						

No.	Content (integrated table)	A	B	Work needed to close it	Kind	Cost estimate	Text
[040]	Margin row: margin $\hbar\rho - 2\hbar\kappa \geq 0.067$ (neighbour and end in $[0,2.5]^2$)	○	○	(Done) computer-assisted at six nodes	—	Closed within the nodes. Outside the nodes, [013][014]	23-06
[041]	Two-site gap row: $\lambda_c \geq 1.3\times10^{-4}$ (ends in $[0,4]^2$)	○	○	(Done) computer-assisted row	—	The range of $\hbar$ as in the column of the integrated table	23-06

What to read from Table 1. (1) The rows common to the two roads are 13 assumption rows and 1 reading row: the common premise [001], the rows inside the box [002][003][005], [006] between boxes, [009]–[016] extending the in-phase local Poincaré condition, and the reading of how the correlation length grows [044]. These remain whichever road is taken. (2) Needed only for the road of the core on the box side are four rows: the alternating maximal correlation of the core [007], the action barrier [008], and the readings of the medium-core step and the long-core step [046][047]. (3) Needed only for the road of the core on the sector side are the remaining single proposition of the route of the window log-Sobolev inequality [026], 7 assumption rows including the inputs to the sectors [019][021][022][024], Barta's row [023] and regularity at the seam [018], and the two reading rows [055][089]. (4) The two computer-assisted rows [040][041] are done within the nodes and ranges, and common to both roads.

03

Comparing the two roads — counting by kind of work

We compare the costs of the two roads not by the number of rows but by the kind of work (Table 2).

Table 2 The assumption and reading rows of Table 1 counted by the kind of work needed to close them (a row with two kinds is counted in both; the two computer-assisted rows are done and excluded)

Kind of work	Common to both roads	A only	B only
Interval arithmetic	1 [013]	1 [046]	2 [023], [055]
Paper	6 [001], [011], [012], [015], [016], [044]	2 [008], [046]	6 [018], [019], [021], [022], [023], [089]
Bibliography	5 [001], [009], [010], [011], [044]	0	0
Measurement	0	1 [008]	0
Another method	3 [003], [006], [014]	2 [007], [047]	1 [024]
Method undecided	2 [002], [005]	0	1 [026]

There are three ways to read it.

First, **the only row with a time estimate is [013], common to both roads** (24 subintervals, 23–39 hours, not run). There are other rows expected to close by interval arithmetic ([046], [023], [055]), but there are as yet no estimates of the number of jobs or the time. And among the common rows is [014] ($\hbar < 0.005$, infinitely many subintervals needed), which is known to be out of reach of interval arithmetic. So neither of the two is a road that "closes given enough computer time".

Second, **the additions of the road of the core on the box side lean towards "another method", and the additions of the road of the core on the sector side lean towards "paper"**. [007][047] of the road of the core on the box side need a comparison theorem bounding the alternating maximal correlation of the core uniformly in k, and it must take a form that avoids the example that does not reduce to two-point quantities (23-11 § 05). Many of the additions of the road of the core on the sector side are papers on the inputs to the sectors, but behind them lies one "method undecided" row, [026]: the remaining single proposition of the route of the window log-Sobolev inequality, the place where all three strands stopped at the shortfall $\frac{1}{4}$ in the curvature of the single site at the top (23-09 § 06).

Third, **the road of the core on the sector side, with more rows, is not necessarily heavier**. The additional papers of the road of the core on the sector side are all rows for which what has to be shown is fixed as a statement. The two additional rows of the road of the core on the box side, [007][047], and the one row of the road of the core on the sector side, [026], are all rows for which "what to show" is fixed but "how to show it" is not. The difference between the two roads is whether these rows of unknown method lie on the side of the maximal correlation of the core or on the side of the curvature at the top.

Figure 1 is the dependency graph for each of the two roads, keeping only the ancestors of the conclusion. From the graph of the integrated table (111 nodes), only the nodes reached by choosing one side of the "or" at [103] and following the direction of the premises are listed, with the same numbers.

> Figure 1 Dependency graph for each road (number, grade, symbol, content; the numbers to the right of $\Longleftarrow$ are premises. The rule of numbering is the same as in the integrated table: the numbers of premises are smaller than that of the conclusion)
> ▶ The road of the core on the box side: the core on the box side (55 nodes)
> ▶ The road of the core on the sector side: the core on the sector side (59 nodes)

Example. In the graph of the road of the core on the box side the premises of [103] are [082][084]; in the graph of the road of the core on the sector side they are [099][018]. The rest of the trunk (the gap of the box and the composition from [104] on) consists of the same nodes in both graphs, and the work before the fork acts in common on both roads.

04

Remaining work not on the road to the conclusion — work worth having

The integrated table contains work that is not on the road to the conclusion yet has been named as remaining. Closing any of it would not move the conclusion [111], but it is worth having for the certainty of the numbers or as material for other roads.

Table 3 Remaining work not on the road to the conclusion (costs copied from the estimates in the record; none has been run)

Remaining work	Row of the integrated table	Work and cost estimate	What could be said if it closed	Text
Putting the inner chain of the ring on interval arithmetic	[053], [054]	With 1–2 rounds of paper study and 1–2 rounds of scripting, build a machine separating the lattice of pins from the lattice of positions and a proposition for the layer of the outward-pushing wall; about 50 jobs, about 42 hours over the 25 subintervals of the window (about 85 hours without the paper on bands)	The coefficient of the inner chain $\bar{q} \approx 1.3$ becomes an upper bound. Not a road that meets the closing condition of the sectors ((7) of 23-09)	23-07 § 03
Extending the pin of the boundary stage to $(0, \frac{1}{2})$	[054]	Part of the machine above (extending the domain of the outer machine)	The condition "pin ≤ 0" on the outer chain is removed	23-07 § 03
The remaining numbers of the route of the window-constant numerics	[052]	About 42 jobs, about 21 hours, for the constants of the window, the minimal weight and the congestion. Since the floor of the product form is 4–5 orders of magnitude above the needed value, the decision is not to run them	The upper bound of the product form becomes Computer-assisted (it does not reach the conclusion)	23-07 § 04

Remaining work	Row of the integrated table	Work and cost estimate	What could be said if it closed	Text
Aligning the thresholds and remeasuring A_W	[056]	Measure A_W with the mark of the defect threshold $\theta_B = 0.57$ (no estimate)	The closing condition (7) of 23-09 has literal meaning	23-09 § 04
Making the rows of the Scalar-Strand Proposition computer-assisted	[025], [096]	12 jobs per node (3 for the bathtub column, 7 for rarity, 2 for non-convex fibres), estimated 60–250 for the nodes. The lower bound of the action difference of non-convex fibres is not yet in the form of interval arithmetic. The lifting assumption remains	The rows of the Scalar-Strand Proposition at the lower end of the window become Computer-assisted. There is no edge of implication to the remaining single proposition [026]	23-09 § 05
Sharpening the constant of the in-phase local Poincaré condition	[066]	Reworking the two-site gap row (no estimate)	$C \le 1.6 \times 10^{10}$ becomes smaller (the form of the conclusion does not change)	23-06 § 07
Conversion between the two spacings	—	Connect the readings at lattice spacing $\varepsilon = 0.25$ with the tables at $\varepsilon = \frac{1}{2}$ (no estimate)	One fewer "not comparable" row in the reconciliation	23-14 § 05
Obtaining bibliography	[010] and others	4 items to obtain (Poincaré–Miranda, elliptic regularity of the Neumann problem on a box, stochastic monotonicity, the principal eigenvalue of a positive integral kernel)	Rows that can be obtained rise from "Assumption" to "Cited"	23-14 § 03

The first three rows of Table 3 are, in amount of computation, larger work than [013] of Table 1. Even so, they are not on the road to the conclusion because closing them does not remove a single "Assumption" row of the integrated table (the same reason as "the two Computer-assisted rows of the saddle-tube condition are not on the road to the conclusion" in 23-12 § 05). The size of the computation and the closeness to the conclusion are different rulers.

05

What is needed to produce a ħ that fits all — why computer assistance in the window cannot reach it

Even if all the rows of Table 1 closed, if the ranges of $\hbar$ in which each row holds do not intersect, the conclusion cannot be stated at even a single $\hbar$. The present ranges are of four kinds (23-12 § 06): the nodes of the in-phase local Poincaré condition $\{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$, the computer-assisted window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$, the nodes in the range of the Scalar-Strand Proposition $[3 \cdot 10^{-4}, 10^{-3}]$, and the three points of reading $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$. The side of the in-phase local Poincaré condition has nothing below 0.005 and the side of the saddle-tube condition has nothing above $3 \cdot 10^{-3}$, so the intersection is empty.

There are only two roads to bring them into line: (a) extend the in-phase local Poincaré condition to $\hbar \le 10^{-3}$ (row [014]), or (b) extend the side of the saddle-tube condition to $\hbar \ge 0.005$. (b) was decided to be outside the scope of this part (the route of the range of ħ, 23-07 Table 1). (a) cannot be reached by interval arithmetic: the width of subintervals shrinks with $\hbar$, and covering down to $\hbar \to 0$ would need infinitely many subintervals. What is needed is another method, writing the asymptotics as $\hbar \to 0$ on paper. That line has come as far as the following form. Of the inputs of the Six-Node In-Phase Poincaré Proposition, the two-site gap row, the crossing row, λ_S and the nesting close for $\hbar \in (0, 0.2]$, and the

condition on the rate is monotone in the margin, so what remains is only to show the margin of the margin row, $\hbar\rho - 2\hbar\kappa \geq 0.067$, as $\hbar \to 0$ [Paper] . The limiting formula is $\kappa_0(a,b) = (H^{-1})_{1,11}/\varepsilon^2$ (the Hessian at the minimiser of the 11-site bridge), and for all ends $(a,b) \in [0,2.5]^2$ there is a unique global minimiser in the interior with $\lambda_{\min} H \geq 0.2$, $\kappa_0 \leq 0.1282$, and in the coarse landscape $U_0''(t^*) \geq 1.943$ [Computer-assisted] (interval arithmetic; the prerequisite paper lemmas — KKT, the truncation $x \leq \max(a,b,1)$, the leading principal minors of Jacobi — are not placed in the book; the action gap g is floating point [Computed]). From this the limit of the margin is $\geq 1.014 - 2 \times 0.1282 = 0.758$ (after paying for Muckenhoupt), larger than the 0.067 of the six nodes, and the true margin at finite $\hbar$ (1.67–1.79) is also nearly flat [Computed] . To bring the $\hbar \to 0$ limit down to finite $\hbar$, the quantitative uniform Laplace method and a one-dimensional lemma on the Muckenhoupt asymptotics on the $\hbar\rho$ side are needed [Assumption] (the constants of the uniform Laplace method are not made explicit in Theorem 2.12 of Katsevich arXiv:2406.12706 [Cited]). If an explicit $\hbar_0$ is obtained, the infinitely many subintervals of [014] become finitely many on $[\hbar_0, 0.005]$ [Reading] (the estimate of $\hbar_0$ is 4×10^{-4}–8×10^{-4}).

Subintervals from the upper end. On the upper-end side of the window $[\hbar_0, 0.005]$, we filled it by extending the same interval-arithmetic machinery as for the six nodes to intervals of $\hbar$. On the $\hbar\kappa$ side, the covariance of the bridge of 11 sites is bounded from above over an interval of $\hbar$ with interpolation in β at three points and a cubic remainder. On the $\hbar\rho$ side, the band of the expectation of the boundary values is extended to an interval of $\hbar$ with the same type, and the forward sums appearing in the Muckenhoupt exponent with band are compared cell by cell at the two ends of $\hbar$, giving a lower bound valid for every $\hbar$ in the interval [Paper] . As a result, on all six subintervals $[0.001949, 0.002281]$, $[0.002281, 0.002668]$, $[0.002668, 0.003122]$, $[0.003122, 0.003653]$, $[0.003653, 0.004274]$, $[0.004274, 0.005]$, the margin $\hbar\rho - 2\hbar\kappa$ exceeds 0.067 for every box of the neighbours and the ends $[0,2.5]^2$ (from the lowest interval up: 0.367, 0.353, 0.321, 0.240, 0.154, 0.077) [Computer-assisted] (float64 with outward widths. The comparison of exponents on the cells over the subintervals, the covering by small pieces and the arithmetic of the margin are in Lean: LatticeGaugeH1HbarInterval, 15 theorems; the numbers were put in for the upper two). Hence the margin of the margin row holds by interval arithmetic for $\hbar \in [0.001949, 0.005]$. It does not overlap with $\hbar_0$ of the quantitative uniform Laplace method (about 5.5×10^{-4} in floating point), and the gap between them remains open. The cost of this machinery grows roughly as the -3.5 power of $\hbar$ [Computed] , and moreover from the next subinterval on, because of the truncation of the weights, the upper bound on the tail at the corner $(a,b) = (2.5, 2.5)$ exceeds the integral, and it does not close on a single weight scale in float64 [Reading] . To join the window, the machinery has to be replaced — keeping the weights as logarithms or revisiting the estimate of the corner tail — or the constants of the quantitative uniform Laplace method have to be sharpened.

Moreover, even if a $\hbar$ that fits all could be produced at a single point, the double-well power statement is a statement about the limit $\hbar \to 0$.

$$\lim_{\hbar \to 0} \frac{\ln CP(\hbar)}{\ln \xi(\hbar)} = 2 \quad (2)$$

(2) does not follow from an inequality on any finite window. Computer assistance in the window only checks the inequality at each $\hbar$ in the window and says nothing about the outside of the window (23-08 § 10). So to close the first step, every row of Table 1 has to be stated "in a form uniform as $\hbar \to 0$", and the role of computer assistance is limited to taking on a finite part within such a uniform proposition (for example, the side $\hbar \geq \hbar_1$).

Example. Even if [013] is closed with 23–39 hours, what is closed is the continuum $0.005 \leq \hbar \leq 0.2$. [014] remains untouched, and the limit of (2) is not a single step closer. What computing time can buy is

"between the nodes", not "$\hbar \to 0$".

The third branch of the fork — the road of block log-Sobolev inequalities

Besides the road of the core on the box side and the road of the core on the sector side, a third candidate has been proposed as a road supporting the fork [103] (the core Poincaré uniformity assumption). It passes neither through the LSI constant A_W of the site windows of the sector framework ([026]) nor through the maximal correlation of the core of the Three-Range Core-Constant Proposition ([007], [046], [047], [082]); in this section we call it the road of blocks and mixing. Many of its steps have no written proof, so this section is not a statement of claims: it sets out the proposed road as a table of rows (Table C1) and a dependency graph. The counts of Tables 1 and 2 and Figure 1 are not changed.

Setting. The object is the conditional measure $\mu_k^{a,b} \propto e^{-S/\hbar}$, with the values a, b outside both ends fixed, on the same core $Q_k = (-x_c, 0)^k$ as in 23-09 ($x_c = 0.89$, Neumann faces, $\varepsilon = \frac{1}{2}$, $V(z) = (z^2-1)^2/8$). The Dirichlet form follows the convention $\mathcal{E}(f) = \hbar \int |\nabla f|^2 \, d\mu$ of 23-09 (the form of the overdamped Langevin dynamics $dz = -\nabla S \, dt + \sqrt{2\hbar}\, dW$), and differs from $(\hbar/\varepsilon)\sum_i \mathbb{E}|\partial_i f|^2$ of 23-11 only by a constant factor ε. So "uniform in $\hbar$" means the same under either convention. The goal is a logarithmic Sobolev inequality (LSI) uniform in the core length k, in $\hbar$ and in all end values; since the Poincaré inequality follows from the LSI Cited , if the goal is shown, the core Poincaré uniformity assumption follows.

Three stages.

1. Block LSI. For blocks of length $\ell_B \geq 12$ (avoiding length 8, just outside the threshold of convexity), an LSI uniform in the end values and in $\hbar$. The inputs are the Hidden-Convexity Proposition below (exactly one minimum on the box), the curvature $\lambda_*(12) \geq 0.255$ at the minimiser for the worst end values, Witten's Morse theory that there are no exponentially small eigenvalues under a single non-degenerate minimum (on a Neumann box), and the Menz–Schlichting decomposition for the case of one minimum.
2. Strong mixing. The optimal path of $\hbar \to 0$ reaches the corner $-x_c$ of the box within 7 steps from every starting point, the chain concentrates around that path with a width of order $\sqrt{\hbar}$, and in the boundary layer at the corner the Doeblin condition holds uniformly in $\hbar$. The conclusion has the form $\mathrm{TV}(n) \leq (1-\delta)^{\lfloor n/10 \rfloor}$, with δ independent of $\hbar$.
3. Combine 1 and 2 by the standard theorem for one-dimensional chains (of the Zegarlinski and Martinelli–Olivieri type: block LSI and strong mixing give an LSI uniform in length).

This road does not take the worst configuration as its entrance. It lies outside the fact that tools evaluating at the worst configuration all stop at the shortfall $\frac{1}{4}$ in the curvature of the single site at the top (23-09 § 06), and it goes after uniformity in $\hbar$ through mixing on average.

Hidden-Convexity Proposition (exactly one minimum on the box)

Let $\varepsilon > 0$, $c > 0$, $l \geq 1$, with end values $a, b \in [-c, 0]$, and with $z_0 = a$, $z_{l+1} = b$,

$$S(z) = \sum_{i=0}^{l} \frac{(z_{i+1} - z_i)^2}{2\varepsilon} + \varepsilon \sum_{i=1}^{l} V(z_i), \qquad V(z) = \frac{(z^2-1)^2}{8} \tag{C1}$$

be considered on the box $Q = [-c, 0]^l$. Then S has exactly one local minimum on Q, and it is the global minimum.

Proved here Proof. Put $z_i = -\sqrt{\rho_i}$ ($\rho_i \in [0, c^2]$, $i = 1, \dots, l$), likewise $a = -\sqrt{\rho_0}$ at the ends, and $b = -\sqrt{\rho_{l+1}}$ (with ρ_0, ρ_{l+1} fixed). With $(z_{i+1} - z_i)^2 = \rho_{i+1} + \rho_i - 2\sqrt{\rho_i \rho_{i+1}}$, the geometric mean $(x, y) \mapsto \sqrt{xy}$ is concave on $[0, \infty)^2$, so $-\sqrt{\rho_i \rho_{i+1}}$ is convex. The end term $-\sqrt{\rho_0}\sqrt{\rho_1}$ is also convex, since $\sqrt{\cdot}$ is concave and the coefficient is ≥ 0. In ρ the linear term is convex, and $V(z_i) = (\rho_i - 1)^2/8$ is strictly convex in ρ_i with second derivative $\frac{1}{4}$. Hence $\tilde{S}(\rho) := S(-\sqrt{\rho})$ is the sum of a convex function and $\varepsilon \sum_i (\rho_i - 1)^2/8$, and is strictly convex on $[0, c^2]^l$. The componentwise $\rho \mapsto -\sqrt{\rho}$ is a homeomorphism from $[0, c^2]^l$ to Q, so local minima correspond to each other. A local minimum of a strictly convex function on a convex set is the global minimum, and there is at most one. S is continuous on the compact Q, so the minimum exists. ∎

What the Hidden-Convexity Proposition says is that an action which is not convex in the coordinates z (the Hessian at the top is not positive definite for lengths 8 and above; next paragraph) is convex in the coordinates $\rho = z^2$, independently of length. But the Poincaré constant and the LSI constant are quantities of the flat metric of z, so the convexity in ρ does not by itself give the Bakry–Émery criterion. The density written in ρ carries the Jacobian $\prod_i (2\sqrt{\rho_i})^{-1}$, and the $\frac{1}{2} \ln \rho_i$ added to the potential is a concave term with second derivative $-1/(2\rho_i^2)$. The lower bound on convexity $\varepsilon/(4\hbar)$ in $\tilde{S}/\hbar$ can cancel this only where $\rho_i \geq \sqrt{2\hbar/\varepsilon}$; in thinner layers convexity is not guaranteed Proved here (a one-line computation). The view is that the non-convexity moves from "the lump at the top of length 8 and above" to "rare thin layers at each site". The proposition in the record (the Closed-Core KKT-Point Proposition), which shows that when the end values are below the box ($a, b \leq -x_c$) the only KKT points on the closed core are the bottom corners, uses a different method based on the convexity of sequences and the KKT conditions, and is not among the ancestors of the conclusion of the road of blocks and mixing.

The threshold of convexity is the first conjugate point of the top. The Hessian of S at the top $z \equiv 0$ is tridiagonal, $\mathrm{tridiag}(-1/\varepsilon,\ 2/\varepsilon + \varepsilon V''(0),\ -1/\varepsilon)$, and with $V''(0) = -\frac{1}{2}$, $\varepsilon = \frac{1}{2}$ it is $\mathrm{tridiag}(-2, \frac{15}{4}, -2)$. For length l the eigenvalues are $\frac{15}{4} - 4\cos\frac{\pi j}{l+1}$ ($j = 1, \dots, l$), and the smallest, $j = 1$, is positive only when $\cos\frac{\pi}{l+1} < \frac{15}{16} = 1 - \frac{\varepsilon^2}{4}$, that is, $l + 1 < \pi/\theta$ ($\cos\theta = 1 - \varepsilon^2/4$) Proved here . With $\theta = 0.3554$, $\pi/\theta - 1 = 7.839$, the smallest eigenvalue is 0.0545 at length 7 and -0.0088 at length 8. The solution of the Euler–Lagrange equation linearised at the top (the Jacobi equation) $-2u_{n-1} + \frac{15}{4}u_n - 2u_{n+1} = 0$ is $\sin(\theta n)$, and π/θ is its first zero, in the language of the calculus of variations the first conjugate point. The statement of the Short-Core Convexity Lemma (23-11 § 05), "convex for length ≤ 7, broken from 8", is exactly the test of whether a conjugate point enters the interval. This length is fixed classically and does not depend on $\hbar$.

Table C1 The rows of the road of blocks and mixing (numbers carry a C to distinguish them from the numbers of the integrated table. Labels are the five grades of Part 23. No row has a cost estimate)

Row	Statement (one line)	Label	Grounds, what remains	Work needed to close it
C01	The threshold of convexity 7.84 is the first conjugate point of the top $\pi/\theta - 1$ ($\cos\theta = 1 - \varepsilon^2/4$)	Proved here	The paragraph above	—

Row	Statement (one line)	Label	Grounds, what remains	Work needed to close it
C02	A core of length $k \le 7$ has $\mathrm{Hess}\, S \ge \kappa(k)$ ($\kappa(7) = 0.0545$) at every configuration; Poincaré uniform in $\hbar$ and the end values (the Short-Core Convexity Lemma, integrated table [076])	Proved here	23-11 § 05. Bakry–Émery on convex domains (Kolesnikov–Milman) is Cited. The LSI form is expected to follow from the same convexity; checking against the literature remains. Not among the ancestors of the conclusion of the road of blocks and mixing (blocks have length 12 or more)	Bibliography (the LSI form)
C03	Hidden-Convexity Proposition: if the end values are $a, b \in [-c, 0]$, there is exactly one local minimum on the box	Proved here	The proof in the box above	—
C04	The curvature at the minimiser for the worst end values $\lambda_*(12) \ge 0.255$, $\lambda_*(16) \ge 0.305$ ($\varepsilon = \frac{1}{2}$, all end values)	Reading	Monotonicity of the minimiser in the end values (Topkis's theorem) and a lower bound per cell by Weyl's inequality. The minimisers at the four corners of each cell come from floating-point optimisation	Interval arithmetic (Computer-assisted if the KKT residuals at the four corners are checked with intervals)
C05	A single non-degenerate minimum $\Rightarrow$ no exponentially small eigenvalues, and at $\hbar \to 0$ the LSI constant is fixed by λ_*	Assumption	Candidates to cite (Helffer–Nier, the Neumann version of Le Peutrec, Menz–Schlichting). The form at the corners of the box has not been checked	Bibliography
C06	The LSI of a block ($\ell_B \ge 12$) is uniform in the end values and in $\hbar$	Assumption	C03, C04, C05, and the line of assembly over the range of finite $\hbar$ and the compactness of the end values	Paper
C07	The optimal path of $\hbar \to 0$ reaches the corner $-x_c$ within 7 steps from every starting point	Reading	A record of solving the path from the top in 50-digit floating point (gradient residual 3×10^{-51}) and checking the signs of the KKT conditions. Uniqueness is C03; monotonicity in the starting point is Topkis's theorem. It is not interval arithmetic, so it is not graded Computer-assisted	Computer-assisted by interval arithmetic (enclosing the existence of the solution in intervals)
C08	Convexity of the one-step action $\kappa_0 \ge 0.055$ ($\hbar \to 0$, all starting points)	Assumption	The record has a line of proof (if there are at most 6 free sites, the second derivative of the one-step action is at least the smallest eigenvalue 0.0545 of the worst tridiagonal matrix of length 7). The book gives no proof. Depends on C01 and C07. $\kappa\hbar \ge 1.88$ measured at finite $\hbar$ ($\hbar = 10^{-2}$–10^{-5}) is Reading	Paper
C09	$\lvert W\hbar - W_0 \rvert = O(\hbar \ln(1/\hbar))$ ($W\hbar = -\hbar \ln \varphi_0$, uniform in $[-x_c, 0]$)	Assumption	A candidate to cite (a limit of Varadhan type). Not yet checked against the original	Bibliography
C10	Concentration to a width $\sqrt{\hbar}$, and the Doeblin condition in the boundary layer at the corner (uniform in the starting point)	Assumption	A line of argument by the constrained Laplace method. What remains is a form uniform in the starting point, and the treatment of the finitely many starting points at which the multiplier of the wall is 0	Paper

Row	Statement (one line)	Label	Grounds, what remains	Work needed to close it
C11	Strong mixing $\mathrm{TV}(n) \le (1-\delta)^{\lfloor n/10 \rfloor}$, with δ independent of $\hbar$	Assumption	The line of assembly on top of C07–C10	Paper
C12	Mixing length $\approx$ 9–10 (total variation $< 10^{-2}$ at the worst end values, $\hbar = 10^{-2}$–10^{-5}); rough Langevin relaxation ≈ 0.46 ($k \ge 16$, $\hbar = 10^{-2}$–10^{-4})	Reading	Numbers for the transfer operator (maximum over 41 end values) and numbers for overdamped Langevin (256 chains, ends at the top). Supporting rows that are not among the ancestors of the conclusion	—
C13	Block LSI + strong mixing ⇒ LSI uniform in length	Assumption	Candidates to cite (Zegarlinski, Martinelli–Olivieri). The version for compact continuous spins has not been checked	Bibliography
C14	The LSI of the core is uniform in k, $\hbar$ and the end values, hence the core Poincaré uniformity assumption	Assumption	Assembly on top of C06, C11, C13	Paper

Dependency graph (the rule of numbering is the same as in Figure 1: the numbers of premises are smaller than that of the conclusion. C02 and C12 are nodes that are not among the ancestors of the conclusion):

```
[C01] PRV  threshold of convexity π/θ − 1 = 7.84 is the first conjugate point of th
[C02] PRV  Short-Core Convexity Lemma: convex for k ≤ 7 (integrated table [076])
[C03] PRV  Hidden-Convexity Proposition: exactly one minimum on the box (strictly c
[C04] RDG  λ*(12) ≥ 0.255, λ*(16) ≥ 0.305
[C05] ASM  one non-degenerate minimum ⇒ no small eigenvalues (corners of a Neumann
[C06] ASM  block LSI (ℓ_B ≥ 12) uniform in end values and ħ ⟸ [C03][C04][C05]
[C07] RDG  optimal path as ħ→0 reaches the corner in ≤ 7 steps (50-digit floating p
[C08] ASM  one-step convexity κ₀ ≥ 0.055 ⟸ [C01][C07]
[C09] ASM  |W_ħ − W₀| = O(ħ ln(1/ħ))
[C10] ASM  concentration to width √ħ, Doeblin in the boundary layer at the corner ⇐
[C11] ASM  strong mixing TV(n) ≤ (1−δ)^⌊n/10⌋ ⟸ [C10]
[C12] RDG  mixing length ≈ 9–10, Langevin ≈ 0.46                        outside the
[C13] ASM  block LSI + strong mixing ⇒ LSI uniform in length
[C14] ASM  LSI of the core uniform in k, ħ and end values ⟸ [C06][C11][C13]
[103] PRV  core Poincaré uniformity assumption ⟸ [C14] (LSI ⇒ Poincaré is cited)
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Table C2 The quantities of Part 23, and how the road of blocks and mixing treats them

Quantity of Part 23	Meaning	Treatment in the road of blocks and mixing
Core Poincaré uniformity assumption ([103])	The Poincaré constant of the core is uniform in core length and end values (23-09, 23-11)	Follows from the conclusion C14 of the road of blocks and mixing (LSI ⇒ Poincaré). The convention is aligned with $\hbar \int \lvert\nabla f\rvert^2 d\mu$ of 23-09
Window Log-Sobolev Threshold Proposition ([026])	$A_W < A_{\max}(\hbar)$: the one remaining proposition of the route of the window log-Sobolev inequality	Not passed through. If the road of blocks and mixing is chosen, [026], the 6 sector input rows of the road of the core on the sector side ([018], [019], [021]–[024]) and the readings [055], [089] drop out of the ancestors of the conclusion

Quantity of Part 23	Meaning	Treatment in the road of blocks and mixing
A_W	The LSI constant of the marginals of the marks of a site window	Not used
[007], [008], [046], [047], [082]	The core rows of the road of the core on the box side (the Three-Range Core-Constant Proposition and its inputs)	Not used
The common rows ([001], [002], [003], [005], [006], [009]–[016], [044])	The tube, the reduced kink gas, the cross term, the maximal correlation between boxes, the extension of the in-phase local Poincaré condition, and others	Remain

Counts, and what does not close. If the road of blocks and mixing is chosen, the rows without proof among the ancestors of the conclusion [111] are the 13 common assumption rows and 1 reading row ([044]), plus the 8 assumption rows of the road of blocks and mixing (C05, C06, C08, C09, C10, C11, C13, C14) and its 2 reading rows (C04, C07). The number of rows is not smaller than for the road of the core on the box side (15 assumptions, 3 readings) or the road of the core on the sector side (20 assumptions, 3 readings). The difference is in the kind of work: the rows of the road of blocks and mixing divide into bibliography (C05, C09, C13), interval arithmetic (C04, C07) and paper (C06, C08, C10, C11, C14), and none of them falls under "another method" or "method undecided" of Table 2. But this classification comes from reading the line of argument of each row, and there is no guarantee that new obstacles will not appear while writing it out.

The road of blocks and mixing is a road supporting the single row of the fork [103], and the trunk from [104] on is the same as for the road of the core on the box side and the road of the core on the sector side. **The 13 common rows remain even if the road of blocks and mixing is chosen.** Of these, [002] (Poincaré for the tube) and [005] (the χ of the cross term is uniform) are method undecided, and [003], [006] and [014] (the form uniform in $\hbar < 0.005$, which interval arithmetic cannot reach) are rows that need another method (Table 2). So even if the road of blocks and mixing closes, the first step does not close. The problem of "a $\hbar$ that fits all" of § 05 also remains, through [014].

When $\ln(1/\hbar)$ remains in the mixing length. Numerically, the length at which the influence of the worst end values falls below 10^{-2} stays at 9–10 from $\hbar = 10^{-2}$ to 10^{-5}, but the position at which the total variation drops below 0.5 moves slowly, logarithmically, from 5 to 8 Reading . The possibility that a dependence on $\ln(1/\hbar)$ remains has not gone away. If it remains, the constant in the conclusion of the road of blocks and mixing grows polynomially in $\ln(1/\hbar)$ as a function of $\hbar$, and the two statements are treated differently. The power statement (2) is not affected: even if a factor $(\ln(1/\hbar))^p$ multiplies C_P, under $\ln \xi \geq s/\hbar$ of [044] we have $p \ln\ln(1/\hbar)/\ln \xi \to 0$ Proved here (the algebra on top of [044]). On the other hand, the form in $C_P \leq K\xi^2 S_0/\hbar$ of the First-Step Upper-Bound Proposition, in which K does not depend on $\hbar$, does not allow it. The uniformity of K includes the uniformity of the χ of the cross term (the cross-term uniformity assumption, [005]), and if the cross-term uniformity assumption is shown by the standard two-scale argument, χ is expected to come out proportional to the square of the local constant C_{loc}; so if a logarithmic factor enters C_{loc}, the uniformity of χ breaks Assumption . The second term $C_{\mathrm{loc}} D/(C\xi)^2$ of (12) in 23-05 allows a logarithmic factor, since $D/(C\xi)^2$ is small, of order $e^{-2S_0/\hbar}$. Therefore the right goal for the road of blocks and mixing is the core Poincaré uniformity assumption in a form without logarithmic factors, and if a logarithm remains, what can be reached is only the power statement.

06

Conditions common to the four conditions — five premises

Apart from the work row by row in Table 1, there are five premises that bear on all four conditions. None of them disappears when a single row is closed; they are conditions to be written into the statement when putting the first step in a form that can be called "closed".

Table 4 Premises common to the four conditions

Symbol	Premise	Present state	What is needed to close it
C1	Kind of number	The reading rows are floating-point values (the 19 Reading rows of the integrated table)	Replace the reading rows among the premises of the conclusion by upper and lower bounds from interval arithmetic or paper (the reading rows of Table 1)
C2	Bibliography of the cited theorems	Of the 26 main citations, 19 have their statements checked in an original or secondary source. 4 to obtain, 3 rows changed into assumptions (23-14 Table 2)	Obtain the 4 items and check their statements. If they cannot be obtained, they stay "Assumption"
C3	The range of machine checking	Lean covers only algebra and arithmetic. Analytic theorems are input as assumptions (23-13 § 05)	Widening the range of machine checking does not reduce the conditions of the conclusion. What reduces the assumption rows is paper and bibliography
C4	A $\hbar$ that fits all	There is no $\hbar$ at which all rows hold (§ 05)	Another method to extend the in-phase local Poincaré condition to $\hbar \to 0$
C5	Limit and regularity	Existence of the limit $T \to \infty$, regularity, and simplicity of the first eigenvalue are assumptions ([001] of the integrated table)	The same as [001] of Table 1

07

The distance from the second step on — three levels of the goal, and the number of steps in β

The distance of the first step was the list of rows of Table 1. From the second step (the four-dimensional lattice, non-abelian) on, the statements are not divided finely enough to be written as a list of rows. This section lays out what the goal on the four-dimensional lattice looks like when written at three levels, and the number of steps in β, as a rough guide by physical extrapolation. What can be expected to carry over from the first step to the second step and beyond is only the form of the statements and the form of the tools; why the constants and the range of $\hbar$ do not carry over is in 23-03 § 06.

"The mass of the lightest particle is positive" is the statement on the spectrum of the quantum Hamiltonian H, $\mathrm{spec}(H) \subset \{0\} \cup [\Delta, \infty)$, $\Delta > 0$ (4-11). On the road entering from the lattice, the same goal splits into three levels.

$$(\mathrm{T1})\ \ \xi(\beta) < \infty\ (\forall \beta),\ \ \xi(\beta) \to \infty\ (\beta \to \infty); \qquad (\mathrm{T2})\ \ \frac{m_{\mathrm{lat}}(\beta)}{a(\beta)} \to \Delta_{\infty} \in (0, \infty); \qquad (\mathrm{T3})\ \ \text{gap positive in the}$$

Table 5 Three levels of the goal, and their relation to the relaxation-time assumption

Level	Content	What is added to the level above	Status in four dimensions	Relation to the relaxation-time assumption
Lattice target level ((T1) of the formulas)	Correlation length finite at each β, diverging as $\beta \to \infty$	—	Unsolved (lattice correlation-length problem (Chatterjee's Problem 5.1)) Cited	The relaxation-time assumption is a statement that presupposes the latter half of the lattice target level

Level	Content	What is added to the level above	Status in four dimensions	Relation to the relaxation-time assumption
Physical-unit target level ((T2) of the formulas)	m_{lat}/a tends to a positive finite value	How the lattice spacing $a(\beta)$ is determined	Unsolved	If the relaxation-time upper-bound assumption and the one-sided bridge (23-01 § 06) are both in place, the lower half follows
Continuum target level ((T3) of the formulas)	A non-trivial theory on $\mathbb{R}^4$ with positive gap	Existence of the limit (Wilson-loop limit statement), rotation-recovery statement, OS reconstruction statement, non-triviality statement	Unsolved (the problem statement). On the existence of the limit and the axioms side, the committee's assessment is that the construction published on 2026-11-10 shows them as the limit of Wilson's lattice gauge theory (Part 6). The gap is not shown (6-04)	Does not follow even with the relaxation-time assumption and the bridge in place (23-01 § 08)

Next, we measure the gap between the β where there are proofs and the β on the side of the continuum limit by the number of steps that coarsen the lattice by a factor of 2. It is a physical extrapolation that extends two-loop asymptotic scaling, matched to the value $\xi = 1/0.514$ of Lucini–Teper at $\beta = 2.60$; it is not a distance to a theorem.

$$\xi(\beta) \;\propto\; \Big(\frac{6\pi^2\beta}{11}\Big)^{-51/121} e^{3\pi^2\beta/11}, \qquad k = \log_2 \frac{\xi_{\mathrm{target}}}{\xi_{\mathrm{start}}} \tag{4}$$

Table 6 Target correlation lengths, and the β and number of steps needed Reading (values of the two-loop extrapolation. The window end is the end of the numerical window, $\xi = 1.95$; the curvature end is $\xi \approx 0.093$, the end of $\beta < 0.2722$ that comes from the lower bound of the curvature after one-link integration)

ξ_{target}	β (two loops)	From the window end	From the curvature end
10	3.243	2.36	6.75
100	4.136	5.68	10.07
1000	5.022	9.01	13.39
10^4	5.903	12.33	16.72
10^6	7.654	18.97	23.36

The numbers in Table 6 are all finite. Even so, the distance does not shrink, because what a theorem needs is not "up to some target" but "at every target, with the same constants". In metaphor, what is missing is not the number of steps on the staircase but the guarantee of not missing one's footing on any step. Improving the constants of the theorem on the strong-coupling side ($\beta < 1/12$ of Shen–Zhu–Zhu Cited) only widens the range of β in hand and does not touch this uniformity. The curvature end 0.2722 is a threshold that comes from the lower bound of the curvature (Lean), and the step towards the gap is a sketch Assumption .

Put back into the language of the first step, the same thing happens with $\hbar$. As in § 05, however wide the window of $\hbar$ is made, the double-well power statement (2) does not come out; what is needed is a proposition in a form uniform as $\hbar \to 0$. On the four-dimensional lattice, a form uniform as $\beta \to \infty$; in the first step, a form uniform as $\hbar \to 0$ — on every step of the staircase, the shape of what is missing is the same.

Detour: the lesson of φ^4_4 — even if the limit is reached, the place one arrives at may be trivial

A subsection you may skip. For the four-dimensional scalar field ϕ^4, cutting it off on a lattice and taking the limit naturally lands on a Gaussian (the free field) (Aizenman–Duminil-Copin 2021, 7-08) Cited . Even if the limit exists and the mass stays finite, if the interaction has vanished it is not a solution in the sense of the problem statement. This is why the non-triviality statement is written separately in the continuum target level of Table 5, and it is the same caution as the relaxation-time lower-bound assumption being a different statement from non-triviality (23-01 § 06). In metaphor, one may think one has reached the summit and find it is flat ground.

08

How to read the distance — a one-line conclusion

We put the distance of this chapter into a single diagram of arrows.

$$\underbrace{\text{each row of the first step}}_{\text{Table 1}} \xrightarrow{\hbar\to 0 \text{ in uniform form}} \text{power form of Assumption } (\xi^2) \text{ (first step)} \cdots \underbrace{\text{theorems f}}_{\text{in}}$$

The dotted line is not an implication; it shows that a statement of the same form is placed at two steps. None of the arrows of (5) has been proved. In the first step we have gone as far as listing the rows before the arrow, dividing them into two roads and attaching a kind of work to each row; the dividing is an arrangement, not a shortening of the arrow.

In one line, **what is missing is not the number of steps but uniformity: holding with the same constants at every scale**. In the first step it appears as the in-phase local Poincaré condition and the rows of the core uniform as $\hbar \to 0$; on the four-dimensional lattice, as the upper-bound assumption and the bridge uniform as $\beta \to \infty$ — the same thing in different forms.

This part placed the relaxation-time assumption at the axis and divided that uniformity into the rows of the first step. Roads that ask about the same uniformity from viewpoints other than the relaxation-time assumption are taken up in Part 24. A single table that rearranges, for each of the four steps of the staircase, the hurdle, the known reach, whether the tools of this book reach it, and the kind of cost is in 24-07 § 04.

What this chapter can say

Can say	Cannot say
That the rows needed for the conclusion [111] split into two roads at the fork [103]. The road of the core on the box side has 15 assumption rows and 3 reading rows, the road of the core on the sector side 20 assumption rows and 3 reading rows (counted from the data of the integrated table)	Which road is easier. The number of rows does not show weight
The kind of work needed to close each row (interval arithmetic, paper, bibliography, measurement, another method, method undecided), and the cost estimates in the record	Whether the estimates are right. None of them has been run. The cost of rows without estimates
That there is no $\hbar$ at which all rows are in place, that bringing them into line needs another method to extend the in-phase local Poincaré condition to $\hbar \to 0$, and that computer assistance in the window does not reach (2)	The asymptotics of the in-phase local Poincaré condition as $\hbar \to 0$. (2) itself

Can say	Cannot say
Three levels of the goal from the second step on, and the number of steps by the two-loop extrapolation (reading) Reading	Table 6 is a rough guide by extrapolation, not a theorem on the correlation length. This part shows none of the three levels
Setting out the rows of the road of blocks and mixing (block LSI and strong mixing) in Table C1, and showing the Hidden-Convexity Proposition and threshold of convexity = first conjugate point with proofs Proved here . That even if the road of blocks and mixing is chosen the 13 common rows remain, and the first step does not close	The 8 assumption rows and 2 reading rows of the road of blocks and mixing. Whether $\ln(1/\hbar)$ remains in the mixing length

Sources and reproduction

Item	Kind	Source, tool
Numbers, contents and grades of the rows of Table 1	This book	23-12 Table 1. The source of each row is in the source column of the integrated table
Work and cost estimates of Tables 1 and 3	This book	The sections on what remains in each chapter (23-06 § 07, 23-07 § 04, 23-09 § 05, 23-11 § 07). All are estimates; nothing has been run
Counts of Tables 1 and 2 and Figure 1	Text processing	A small script that adds a column of costs to the same row and edge data as the integrated table, chooses one side of [103], and collects nodes from the conclusion in the direction of the premises (not a mathematical computation)
(3), Table 5, lattice correlation-length problem	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000). S. Chatterjee, arXiv:1803.01950 § 5
(4), Table 6	Reading	Two-loop coefficients (source table of 4-06) and the value at $\beta = 2.60$ in Table 1 of B. Lucini, M. Teper, JHEP 0106 (2001) 050, hep-lat/0103027. The extrapolated numbers are copied from the table in the record of the study
The strong-coupling range $\beta < 1/12$	Cited	H. Shen, R. Zhu, X. Zhu, Commun. Math. Phys. 400 (2023) 805–851, arXiv:2204.12737 (source table of 23-01)
Triviality of ϕ^4_4	Cited	M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163 (source table of 7-08)
The rows of the road of blocks and mixing (Table C1), the numerical rows C04, C07, C12	Proved here, Reading, Assumption	The Hidden-Convexity Proposition and C01 are proved in the text. $\kappa_\hbar$ of C04, C07 and C08, and C12, are floating-point numbers from the record (C07 in 50 digits, not interval arithmetic). The candidates to cite for the assumption rows (Helffer–Nier, Le Peutrec, Menz–Schlichting, a limit of Varadhan type, Zegarlinski, Martinelli–Olivieri, Topkis's monotone comparison theorem) have not been checked against the literature

Terms used in this chapter — distance (a list), fork, road of the core on the box side, road of the core on the sector side, kinds of work (interval arithmetic, paper, bibliography, measurement, another method, method undecided), a $\hbar$ that fits all, three levels of the goal, uniformity. Their meanings are in the A-02 glossary.

Next chapter: 23-16 What is not claimed — what this part does not show, what should not be read strongly, and what is not called false either, with a table of correspondence to sentences of the text.

What is not claimed — not shown, not read strongly, not called false

> This chapter is a map of open questions. § 01 and § 06 are guide material.

What this chapter shows — The list of things Part 23 does not claim (fourteen items), and a table matching each item to the sentence in the body, and the chapter, that supports it. The difference between not claiming something and claiming its opposite. For the first-step proposition, the Lean theorems, the computer-assisted results and the readings, examples of how each is misread once its conditions are removed.

Prerequisite chapters — None. This chapter can be read from anywhere. The terms are defined in 23-01 § 05 (the diffusive-relaxation-time assumption, the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption), 23-00 § 01 (the five labels) and 23-05 (the Double-Well Relaxation-Time Upper-Bound Proposition and the four double-well conditions).

Said first — The level is [Y2]. This book does not solve the problem of Yang–Mills existence and the mass gap. This part contains no construction of the continuum limit and no proof of the mass gap (the existence of the continuum limit is proved in Lean by the construction published on 2026-11-10, along another road — this is the committee's assessment; Part 6. The mass gap is not shown there either; 6-04). This chapter only splits that fact into items and writes them out; it adds no new result. This part places the diffusive-relaxation-time assumption at the axis of the study, and it does not claim that the relaxation-time assumption is the only road or the right road.

The order of this chapter

1. Three meanings — not shown / not read strongly / not called false (Table 1) / Detour: negating a quantifier
2. The fourteen items and the sentences in the body that support them — a table of correspondence (Table 2)
3. On the main problem — the claim of the problem statement, the local-field footnote · weak-existence footnote (footnotes of Jaffe–Witten)

4. On the double-well step — the First-Step Upper-Bound Proposition is algebra on top of four conditions (Table 3)
5. Not reaching is not evidence of falsity — the death of detour plans, the reading for the periodic box, the lower-bound assumption and the free field (Table 4)
6. Conditions fall away in summaries — sentences of the body and one-line summaries (Table 5)
7. Analogies and other documents — what is not judged, and the convention for putting things out

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01

Three meanings — not shown, not read strongly, not called false

When a mathematical text says "not claimed", there are three meanings. The first is **not shown**. On the main problem, this part shows nothing. The second is **not read strongly**. Whatever has been shown always carries conditions or assumptions, and reading it with the conditions removed gives a sentence that has not been shown. The third is **not called false**. That a method did not reach something is a property of the method, not a claim that the proposition it did not reach is false.

The third becomes clear when written in logical form. Writing "P is claimed" as $\vdash P$, we have

$$\neg(\vdash P) \quad \text{and} \quad \vdash \neg P \quad \text{are different sentences} \ (1)$$

On the left: "we do not have a proof of P". On the right: "we have a proof of the negation of P". The "not claimed" of this chapter always has the meaning on the left. As a metaphor: a road not drawn on a map is not evidence that there is no road there.

Table 1 How the items of this chapter are divided (numbers are the items of Table 2)

Kind	What is not claimed	Item
Not shown	Existence and the mass gap on $\mathbb{R}^4$, the construction of the continuum limit, the double-well relaxation-time power statement for the first step	1, 2, 3, 10
Not read strongly	Computer-assisted results, readings, Lean theorems, lifting, the relation between the relaxation-time assumption and the mass gap	4, 5, 6, 7, 11
Not called false	Propositions a tool did not reach, the upper-bound assumption as read to fail in the periodic box	8, 12, 13
Not judged, not counted	Rederivations of known results, the correctness of other documents, putting unclosed propositions out	9, 14

Detour: negating a quantifier

A subsection you may skip. The negation of "for every f, $Q(f) \le C$" is "for some f, $Q(f) > C$". In symbols, $\neg\forall f\, A(f) \iff \exists f\, \neg A(f)$. To show the upper-bound assumption false, a single sequence of counterexamples f suffices; to show the lower-bound assumption false, one must show "for every f the ratio is $o(\xi^2)$". The latter is a sentence of the same "for every" type as the upper-bound assumption, and it does not come out of listing test functions. That some families did not reach it is a "for every" restricted to a family ($\forall f \in \mathcal{F}$), not a "for every" with the family removed.

02

The fourteen items and the sentences in the body that support them

Table 2 writes, one row at a time, what this part does not claim, and lists the places in the body that support each row. "Support" means that the sentence in that chapter already states the conditions, assumptions and scope, and the row merely copies it onto one line.

Table 2 What is not claimed, and the sentences in the body that support it

Item	What is not claimed	Kind	Supporting sentence in the body (gist)
1	The solution of the problem of Yang–Mills existence and the mass gap. Neither a construction of the continuum limit nor a proof of the mass gap in the limit is here	Not shown	23-01 § 08: in four dimensions, what we have (along the line of this part) are only the theorems of the reflection-positivity preservation statement and the OS reconstruction statement, and the infinite-volume uniqueness statement for small β. Even with the relaxation-time assumption and the bridge, the infinite-volume uniqueness statement, the Wilson-loop limit statement, the rotation-recovery statement, the OS reconstruction statement and the non-triviality statement do not follow
2	Even for the first step (the double well in 0+1 dimensions), the relaxation-time assumption (the double-well power statement) has not been shown. The First-Step Upper-Bound Proposition is algebra on top of four conditions, and each of the four conditions leaves assumption rows	Not shown	23-05 § 08: where the four conditions stand. 23-12 § 07: the assumption rows needed for the conclusion number 15 or 20
3	There is not yet a single $\hbar$ at which all rows of the four conditions hold at once	Not shown	23-12 § 06: the nodes of the in-phase local Poincaré condition and the range of the saddle-tube condition do not intersect. 23-15 § 05

Item	What is not claimed	Kind	Supporting sentence in the body (gist)
4	Computer-assisted results are inequalities inside the windows and nodes that were computed, and say nothing about outside the window or about $\hbar \to 0$	Not read strongly	23-08 § 10: what computer assistance does not say (outside the window, $\hbar \to 0$, the difference between continuum and lattice)
5	Readings (floating point, Monte Carlo, fits, lower bounds over families) are used neither as upper bounds nor as proofs of lower bounds	Not read strongly	23-00 Table 1: what the label "Reading" does not guarantee. 23-12 Table 1: all 19 Reading rows keep the grade "Reading"
6	A Lean theorem checks the algebra from assumptions to conclusion; it does not check that the assumptions are true, nor the reading of real numbers as physical quantities	Not read strongly	23-13 § 05: four limits (truth of the assumptions, the encoding of the objects, putting the conclusion into a definition, circles of dependence)
7	From the first step to the second step on, we only state the prospect that the form of the sentences and the form of the tools lift; there is no theorem of implication	Not read strongly	23-03 § 06: what lifts and what does not (constants and the range of $\hbar$ do not lift)
8	That a tool did not reach something is a property of that tool, not a claim that the proposition it did not reach is false	Not called false	the table of scope in 23-10: the "death" of a detour plan is a property of the tool. 23-02 § 08: where the roads on the four-dimensional lattice stopped
9	Rederivations of known theorems are not counted as results. The correctness of other documents that claim a construction is not judged	Not judged	23-14 § 03: the support of the cited theorems. 23-14 § 07: from outside documents we take only patterns, give the source, and do not judge
10	The record on the side of the four-dimensional lattice (the relaxation-time assumption, the three chains, the sum of shortfalls) also remains assumption	Not shown	23-01 § 05: the three chains are all assumptions. 23-02 § 07: what remained is algebra and readings

Item	What is not claimed	Kind	Supporting sentence in the body (gist)
11	Under the assumption of the one-sided bridge, the upper-bound assumption implies a lower bound on the gap of the lattice in physical units, but the converse is false in general and undecided for the class of Wilson measures. The relaxation-time assumption is not the mass gap itself	Not read strongly	(4) of 23-01 § 06 (the one-sided bridge) and (5) (the Free-Particle-on-a-Circle Counterexample: on the circle, $\Delta > 0$ and $CP = \infty$)
12	The "death" of a detour plan means that the ceiling of that construction does not exceed the needed value, not that the one remaining proposition is false	Not called false	23-10 § 03: the line that killed it is a line of a tool's inequality
13	The reading that the upper-bound assumption fails on the four-dimensional lattice in the periodic box is a conditional reading, resting on numbers from the literature and on the conjecture that "HMC and Langevin are in the same universality class"; it is not a theorem that the upper-bound assumption is false	Not called false	the note in 23-01 § 05 (topological sectors and the order of volume), 23-02 § 05
14	A proposition with assumption rows left is not put out as a result. Only what has had all its conditions closed is put out	Not counted	23-12 § 08: the integrated table is a map, not a fraction of progress

Items 1–10 are the skeleton of the part, and 11–14 were added from the text of the individual chapters. Every row compresses the conditions of the body onto one line, and is written so that reading the body gives the conditions in full.

03

On the main problem — nothing is said about the claim of the problem statement

What the problem statement (0-03) asks is that, for every compact simple gauge group, a quantum Yang–Mills theory on $\mathbb{R}^4$ exists, satisfies the axioms (4-09, 4-10), and has a mass gap.

$$\text{existence (satisfying the axioms)} \quad \wedge \quad \operatorname{spec}(H) \subset \{0\} \cup [\Delta, \infty), \ \ \Delta > 0 \ \text{(2)}$$

This part shows nothing about either part of (2) (item 1). The local-field footnote · weak-existence footnote of the problem statement excludes from solutions limits obtained only by weak existence (compactness) Cited . That subsequential limits come for free from compactness is a matter at that level, and neither uniqueness nor the mass gap follows from it.

The relaxation-time assumption, which this part treats, is a sentence that restates Δ of (2) (the gap in Euclidean time) in terms of the relaxation time C_P of Langevin dynamics. The two are different quantities, and only when the one-sided bridge holds can one cross, in one direction, from the upper-bound assumption to a lower bound on Δ (item 11, 23-01 § 06). We placed the relaxation-time assumption at the axis of the study because it is language that can be computed on the lattice, has the tools of convexity and Poincaré inequalities, and is within reach of machine checking; we do not claim that this restatement is the only road to solving the problem. Other views are placed in Part 24.

04

On the double-well step — the First-Step Upper-Bound Proposition is algebra on top of four conditions

The central sentence of the first step is the First-Step Upper-Bound Proposition: "on the four double-well conditions, $C_P \le K\xi^2 S_0/\hbar$, and hence the double-well power statement" (23-05 § 04). Reading this sentence as "the double-well power statement has been shown for the double well" is too strong (item 2). Beneath the four conditions there are 15 or 20 assumption rows of the integrated table (depending on which of the two roads) and 3 reading rows (23-15 Table 1).

The algebra of the construction is checked in Lean, but when only the theorem name remains in a summary, the conditions fall away. Table 3 lists the theorem names of this part that are easily read strongly once their assumptions are dropped (item 6).

Table 3 Theorem names, assumptions, and what is not to be read strongly

Theorem name	Assumptions (gist)	Not to be read strongly
`H3_H4_compose`	The box product inequality, the box gap inequality, $D = \hbar/S_0$, $n, \hbar$, $S_0 > 0$, $c_{\text{alt}} < 1$	Not "the First-Step Upper-Bound Proposition has been shown". The in-phase local Poincaré condition and the saddle-tube condition are inside the assumption of the box gap

Theorem name	Assumptions (gist)	Not to be read strongly
`power_H_of_kink_ratio`	$0 < K, s, \delta$, $1 \le R \le K/\hbar$, $s/\hbar \le \ln \xi$	Not "the double-well power statement has been shown". $R \le K/\hbar$ is the conclusion of the First-Step Upper-Bound Proposition itself
`H1_assembly`	Seven named ones (decomposition, fine side, Helffer's criterion, margin, $0 < \lambda$, the crossing row, the rate condition)	A proposition at six nodes, not the in-phase local Poincaré condition uniform in $\hbar$
`cloc_term_vanishes`	The core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$	The local term vanishes when the core action-barrier assumption is assumed. The value of the action barrier is a reading at $\hbar = 0$
`path5_conditional`	The remainder inequality, the proxy ratio, rarity, the arithmetic margin (named)	The Scalar-Strand Proposition at the lower end of the window; there is no edge of implication to the one remaining proposition
`win_qfree_window_le_of_interval_table`	Table of interval-arithmetic output	Inequalities inside the window (item 4). Not on the road to the conclusion in the integrated table
`hup_is_gap_positivity_of_lower`	$0 < m_{\mathrm{lat}}$, $\kappa/m_{\mathrm{lat}}^2 \le C_P$, $C_P \le K\xi^2$	The one-sided bridge enters as an assumption. That the bridge holds for the Wilson measure has not been shown (item 11)
`drainExp_eq_zero_iff`	$0 \le \eta \le 2$, $\theta \ge 0$	Algebra of exponents on the side of the four-dimensional lattice. Applying it to physical quantities rests on assumptions (item 10)

Example. `power_H_of_kink_ratio` is the algebra "if the ratio R lies between 1 and $K/\hbar$, and $\ln \xi$ is at least $s/\hbar$, then $\ln C_P / \ln \xi \to 2$". Removing the assumption $R \le K/\hbar$ and writing "the double-well power statement in Lean" gives a sentence that has not been shown. The correct summary is "the double-well power statement follows by algebra from the conclusion of the First-Step Upper-Bound Proposition (Lean)".

05

Not reaching is not evidence of falsity

The death of detour plans (items 8, 12). Of the detour plans aimed at the one remaining proposition of the route of the window log-Sobolev inequality, 13 were

judged "dead" (23-10 § 03). Death means that the ceiling of the upper bound obtained by that construction does not exceed the needed value. It does not mean that the one remaining proposition [026] is false, and it says nothing about the possibility of closing it by another construction.

The reading for the periodic box (item 13). On the four-dimensional lattice, it is reported that in a box with periodic boundary the autocorrelation time of the topological charge grows faster than ξ^2, and together with $\tau_{\text{int}} \le C_P$ this reads as the upper-bound assumption failing in the periodic box (the note in 23-01 § 05). This reading stands on numbers from the literature Reading and on the conjecture that HMC and Langevin are in the same universality class Assumption. So this part does not claim "the upper-bound assumption is false on the four-dimensional lattice" either. What it claims is only that how to correct the statement of the relaxation-time assumption remains as a question.

The lower-bound assumption is not non-triviality. What the lower-bound assumption avoids is only that the physical mass collapses to infinity; it does not ask whether there is interaction. This can be confirmed by computing the relaxation-time assumption in a theory without interaction. For the free field with lattice mass m, the Poincaré constant of the path measure is the largest eigenvalue of the covariance,

$$C_P = \frac{1}{m^2}, \qquad m^2 = 4\sinh^2\frac{1}{2\xi}, \qquad \frac{C_P}{\xi^2} = \frac{1}{4\xi^2\sinh^2(1/2\xi)} = 1 - \frac{1}{12\xi^2} + O(\xi^{-4}) \tag{3}$$

Table 4 C_P/ξ^2 for the free field on the lattice Proved here (values of the closed formula (3))

ξ	C_P/ξ^2	$1 - 1/(12\xi^2)$
1	0.920674	0.916667
2	0.979425	0.979167
8	0.998699	0.998698
64	0.999980	0.999980

The ratio tends to 1 as $\xi \to \infty$, so both the upper and the lower side of the relaxation-time assumption hold. The relaxation-time assumption and the lower-bound assumption are sentences that hold in a theory without interaction, and the non-triviality statement does not follow from the relaxation-time assumption (23-01 § 06).

06

Conditions fall away in summaries — sentences of the body and one-line summaries

The error of reading a claim strongly occurs, in most cases, at the stage of compressing a sentence of the body onto one line. Table 5 sets the conditional sentences of the body

beside one-line versions with the conditions dropped completely. Every sentence in the right column is one this part has not shown.

Table 5 Sentences of the body (conditional) and one-liners with the conditions dropped (every one on the right is wrong, or not shown)

Topic	Sentence of the body	One-liner with the conditions dropped
Chain outside the window	$q_{\text{free}} \le 0.998854904$ over the whole $\hbar$ window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ Computer-assisted (assuming IEEE correct rounding, outward widths)	The chain condition of the saddle-tube condition is closed
Chain outside the ring	$q_{\text{out}} < 1$ over the whole window Computer-assisted (conditional on the pin ≤ 0; the boundary stage is a reading of 1.004–1.045)	The seam of the ring is closed
Mixing between boxes	$c_{\text{alt}} = 0.731/0.410/0.138$ ($C = 2/4/8$, values of the reduced model) Reading	The alternating maximal-correlation assumption for boxes holds
In-phase local Poincaré condition	Margin ≥ 0.067 at six nodes (ends $[0, 2.5]^2$) Computer-assisted , constant $C \le 1.6 \times 10^{10}$ (a proposition at the nodes)	The in-phase local Poincaré condition has been shown
First-Step Upper-Bound Proposition	Algebra on top of the four double-well conditions (Lean has the algebra of composing boxes)	The double-well power statement has been shown for the double well
Feynman–Kac strand	A reading of $\sup \lvert\nabla\psi\rvert^2$ from below, 0.433 ± 0.007 (0.54 times the threshold, $\hbar = 10^{-3}$) Reading ; the paper upper bound is about 1000 times the threshold	The assumption of the Feynman–Kac strand falls within the threshold
Periodic box	On numbers from the literature and a conjecture, the upper-bound assumption reads as false in the periodic box	The upper-bound assumption is false in four dimensions

The errors in Table 5 are of two types. One drops the scope (window, nodes, model values); the other drops the grade (Reading, Computer-assisted, Assumption). Both are prevented by keeping the words of the condition in the one-line summary.

07

Analogies and other documents — what is not judged, and the convention for putting things out

Where words are borrowed from the side of physics, they inherit the level of where they were borrowed from. Holography is a conjecture, and a table of correspondences that uses it is a dictionary, not a proved equivalence. The "4" in a complete family existing for $d \le 4$ is a combinatorial number of the lattice, a coincidence unrelated to the dimension of physical spacetime (23-02 § 02). The correspondence between the

first step and the four-dimensional lattice ($\hbar \to 0$ and $\beta \to \infty$) is also an analogy, not a theorem of correspondence (23-03 § 06).

On other documents that claim a construction, this part does not judge correctness (item 9). What we took from outside documents is only patterns for reading dependency graphs (closing by putting the conclusion into a definition, circles of dependence), and even those were written as patterns with the source withheld (23-13 § 05, 23-14 § 07).

Finally, the convention for putting things out (item 14). As long as even one assumption row remains, the propositions of this part are not put out as results. What is put out is only what has had all its conditions closed and has passed machine checking and bibliographic checking. Until then, the text stands as a map of which rows are closed and which are not.

What this chapter can say

Said	Not said
That this part shows nothing about (2) of the problem statement. The fourteen items that are not claimed, and the places in the body that support them (Table 2)	Whether (2) is true. The existence of the continuum limit. A gap uniform in the volume
The assumptions of the theorem names in Table 3 (copies of the statements in each chapter)	The truth of the assumption rows
Proved here That $C_P/\xi^2 = 1 - 1/(12\xi^2) + \cdots$ and the relaxation-time assumption hold for the free field on the lattice (closed formula)	A theorem comparing C_P and ξ^2 for an interacting four-dimensional measure (in neither direction)
The conditions of the reading that the upper-bound assumption fails in the periodic box (numbers from the literature and a conjecture)	That the upper-bound assumption is false, or true, on the four-dimensional lattice

Sources and reproduction

Item	Kind	Source / tools
Problem statement, local-field footnote · weak-existence footnote	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000) (the source table of 0-03)
Each row of Table 2	This book	The chapters and sections in the right column of the table
Theorem names and assumptions of Table 3	Lean	Statements in 23-13 § 03 and the grade tables of each chapter

Item	Kind	Source / tools
(3), Table 4	Proved here	Closed formula. The expansion $1/(4\sinh^2(u/2)) = 1/u^2 - 1/12 + O(u^2)$
Sentences of the body in Table 5	This book	23-07 § 03, 23-08 § 05, 23-11 § 02, 23-06 § 04, 23-09 § 05, 23-01 § 05

Terms used in this chapter — not shown, not read strongly, not called false, negating a quantifier, the one-sided bridge, the death of detour plans, conditions that fall away in summaries. Their meanings are in the A-02 Glossary.

Next chapter to read: 23-17 Insights and small questions — the views that repeatedly proved useful in studying the first step, and questions that can be settled at the size of an undergraduate thesis.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time patterns of insight, discarded hypotheses, small questions · Kiichi, Shiori and Rin

Insights and small questions — views that remain as procedures, and questions that can be settled

> In this chapter, § 01 is guide material, § 02– § 04 are a map of open questions and § 05 is this book's own investigation.

What this chapter shows — The views that repeatedly proved useful while studying the first step, gathered into twelve insights in the form "any problem needs this procedure". For each insight, one small question the reader can check by hand. A table of discarded hypotheses and the reasons. A list of questions that can be settled at the size of an undergraduate thesis, and a worked example that takes one question to the form of a settlement.

Prerequisite chapters — The material for the insights is in the chapters 23-06 to 23-11. At the end of each insight we write the chapter where that view worked. The prerequisites of each question are written in its entry.

Said first — The level is [Y2]–[Y3]. What is listed here are insights and questions before verification, neither theorems nor results. Whichever question is settled, it does not reach a proof of the mass gap. For each question we wrote what does not move even when it is settled.

The order of this chapter

1. How to read the insights — three states, and the criterion for keeping one (Table 1)
2. Twelve insights and small questions — ways of paying, addresses, thresholds, directions, order
3. Discarded hypotheses — the original form, the corrected form, the grounds (Table 2)
4. Questions at the size of an undergraduate thesis — a list of seventeen questions (Table 3)
5. Worked example: one question to the form of a settlement — $1-\lambda_k$ for the Hermite family (Table 4)

01

How to read the insights — three states, and the criterion for keeping one

In the course of a study, "shifts in how things look" get written down apart from results. They fall into three states: **still an insight** (not yet tested), **checked form** (checked once on a small model or a closed formula), and **overlaps the known** (the core was a restatement of a known theorem; not counted as an insight).

There is one criterion for keeping something in this chapter. Not who thought of it or when, but only what can be written in a form that lets us say **another reader who meets a problem of the same type would also need to take this procedure**. For example, "before trimming an upper bound, sort the factors of the needed value into structure and tools" is a procedure that keeps its meaning away from the double well. Conversely, the observation that some number happened to lie near a threshold is not kept.

We read the insights on the side of the four-dimensional lattice in seven patterns (cannot be measured from below / drop to one dimension / drop the units / counterexamples and controls first / look at the power of the difference / the diffusive-relaxation-time assumption is a pair of measure and form / connectedness of the state space). Table 1 shows those seven and how they relate to the twelve insights of this chapter.

Table 1 The seven patterns, and the insights of this chapter (the right column gives the insights of § 02)

Pattern	One line	Insight continuing it in this chapter
Cannot be measured from below	Averages and test functions bound C_P only from below	Before trimming an upper bound, measure one lower bound
Drop to one dimension	Tools for upper bounds on C_P are available in one dimension	The threshold of convexity is decided by whether the parameter enters the Hessian (cut shorter than the threshold)
Drop the units	Make dimensionless ratios the protagonists	What does $1/\hbar$ live in? (the length where $1/\hbar$ lives)
Counterexamples and controls first	First name the models where the proof fails	The wall of conditioning creates minima (count the minima first)
Look at the power of the difference	What fixes the power is not the quantity but what is left after cancellation	Split the ratio of upper bound to reading into a stage of cancellation and a stage of direction
The relaxation-time assumption is a pair of measure and form	Name the normalisation of the Dirichlet form explicitly	Whether the diffusive-relaxation-time assumption is true changes with the coordinates (the Dirichlet form)
Connectedness of the state space	Suspect topology before uniformity of constants	The insight that truth changes with the coordinates, the note in 23-01 § 05

02

Twelve insights and small questions

Are we paying for the same place twice?

In a long chain of inequalities, one bad place (near the top, for the double well) may be paid for by two stages under different names. In the sector framework, the Bochner stage and the Gibbs stage both paid for the same neighbourhood of the top, and the amplification factor of one of them multiplied the whole. Moving the negative curvature to the other side avoids passing through the amplification. In the same pattern, when using gradients in which two terms complement each other, sharing the denominator can sometimes fold $(1+\sqrt{C})^2$ into $(1+C)$ instead. As a procedure: **before tightening a construction, write out place by place what each stage is paying for, and look for double payment** (23-09 § 02, the double-payment type in 23-10). State: checked form.

Small question 1. Compare the best right-hand side, when $\|u\|^2=\|v\|^2=\|w\|^2=1$, of two constructions for the sum of three vectors $u+v+w$: applying $\|u+v\|^2 \le (1+t)\|u\|^2 + (1+1/t)\|v\|^2$ $(t>0)$ twice in a row, and distributing all three terms at once as $\|u+v+w\|^2 \le (1+s_1)\|u\|^2+\cdots$. Where does the twice-in-a-row construction pay for the same quantity twice?

The currency of Poincaré and the currency of entropy

There are situations where one wants to pay for rare events with "rarity". But on the side of the Poincaré inequality, rarity is not a currency. Poincaré looks at the speed of exit, and putting mass on a rare state cancels the mass in both the variance and the Dirichlet form. Rarity can be a currency only on the side of entropy (the logarithmic Sobolev inequality), but there one pays instead $\ln(1/\nu)$ for the rarest state on which test functions can concentrate. As a procedure: **decide between the entropy type and the variance type by setting two rarities side by side (the rarity of the defect one wants to pay for, and the rarity of the states on which test functions can concentrate)** (the currency-mix-up type in 23-10). State: checked form (the two-point model).

Small question 2. Consider the measure $\pi=(1-p,p)$ on two points $\{0,1\}$ and the Dirichlet form with flip rate 1, $\mathcal{E}(f)=\pi_0\pi_1(f_1-f_0)^2$. Show that the Poincaré constant does not depend on p. Check, by citing the known formula for the two-point space (Diaconis–Saloff-Coste), that the logarithmic Sobolev constant grows at the rate $\ln(1/p)$ as $p\to 0$.

What does $1/\hbar$ live in? — read the loss on the same length as the quantity it multiplies

On the plateau at the top, the test function that produces a constant of size $1/\hbar$ was not a function of the collective linear coordinate $\sum z_j$ but of the radius from the corner at the top, $\sum z_j^2$. Choosing the linear collective coordinate gets the direction wrong. Also, $1/\hbar$

lives on the length of the plateau $\ell \approx 0.34/\hbar$, and when measuring a loss it must be measured on the same length as the quantity it multiplies. As a procedure: **before splitting into two scales, check whether the function that produces the lower bound lives on the coarse coordinate or on the fibre** (the type of the address of 1/ħ in 23-10, 23-09 § 06). State: still an insight (in a form supported by numerical readings).

Small question 3. For the measure $\mu \propto e^{-V(|z|)/\hbar}$ on $\mathbb{R}^k$ with $V(r) = (r^2-1)^2/8$, compare the ratios $\mathrm{Var}(\cdot)/\mathcal{E}(\cdot)$ ($\mathcal{E}(f) = \hbar \int |\nabla f|^2 d\mu$) of the linear function $f = z_1 + \cdots + z_k$ and the radial function $g = |z|^2$, for $k = 1$ and $k = 2$, up to the leading term as $\hbar \to 0$. For $k = 2$, when μ concentrates on an annulus, which function captures the slow direction?

The wall of conditioning creates minima

When conditioning by fixing the ends of an interval exactly, all the minima at $\hbar = 0$ may lie on the wall. The "slowness" or "bistability" one then sees is not a property of the model but a shadow created by the conditioning (the bistability at the negative end of the core was the shadow of confining, inside the interval of the core, a configuration in which the kink wants to shift by one site). As a procedure: **each time the way of cutting sectors or conditioning changes, make the first check a count of the $\hbar = 0$ minima lying on the wall** (23-11 § 06). State: checked form.

Small question 4. Where does the measure $\mu_\hbar \propto e^{-V/\hbar}\mathbf{1}_{[0,b]}$, $V(x) = (x^2-1)^2/8$ restricted to the interval $[0, b]$ ($0 < b < 1$), concentrate as $\hbar \to 0$? Near there, approximate $\mu_\hbar$ by an exponential distribution, and write the behaviour of the Poincaré constant of $\mu_\hbar$ (Dirichlet form $\hbar \int f'^2 d\mu$) as $\hbar \to 0$ in terms of $|V'(b)|$. What changes when the wall is moved to $b = 1$?

Before closing the inside uniformly, first write what multiplies it on the outside

In nested estimates, one can get stuck trying to bound an inner constant by $O(1)$. But if, in the outer formula, that constant is multiplied by a small factor, the accuracy needed inside is much looser. In the proposition for the box gap, the local term is multiplied by $e^{-2S_0/\hbar}$, so the constant of the core did not need to be $O(1)$; size $e^{\Delta S/\hbar}$ ($\Delta S < 2S_0$) was enough. As a procedure: **before closing the inside uniformly, first write what multiplies that quantity in the outer formula, and fix the accuracy needed inside by its ratio to the outer scale** (23-11 § 06). State: checked form (the reading of the action barrier is the value at $\hbar = 0$).

Small question 5. Write the condition on B needed for $K(\hbar) = A + e^{-2S_0/\hbar}B(\hbar)$ to be bounded as $\hbar \to 0$. If it has the form $B(\hbar) \le c\,\hbar^{-p} e^{\Delta S/\hbar}$, what must the relation among ΔS, $2S_0$ and p be? What is needed in the exact case $\Delta S = 2S_0$?

The sup over a parameter is attained just outside the threshold

When bounding the constant of a core of length k uniformly in k, the hard case was not a long core but just outside the threshold where convexity is lost ($k = 8$, both ends at the top). Inside the threshold convexity makes it free; far outside it saturates by reduction to the box; and one point in between attains the sup. As a procedure: **when stuck on a sup over a parameter, first find where the threshold is, then name the value just outside it and bound it separately** (23-11 § 05). State: checked form (the threshold is a closed formula).

Small question 6. Find the smallest k for which $\kappa(k) = 8\sin^2\big(\pi/(2(k+1))\big) - \frac{1}{4}$ is negative (the same formula as in 23-09 § 06). How does $\kappa(k)$ behave as $k \to \infty$? Are "the threshold is at $k = 8$" and "the sup is attained at $k = 8$" the same thing or different things?

The threshold of an excluded set is fixed by the mass the downstream inequality allows

When excluding a bad set and estimating on the rest, where to put the threshold is decided not by the convenience of the bad set but by what the downstream inequality, which receives the mass of that set, allows. In the sector framework, the threshold of the window mark and the threshold of the defect disagreed, so the only set determined by the mark that contained the defects was the whole space, and the closing condition, taken literally, gave nothing. As a procedure: **check literally that each of the conditions being composed holds on the same definition of the threshold, and work the threshold back from the mass the downstream can pay** (23-09 § 04). State: checked form (one line of measurability).

Small question 7. At one site $z \in \mathbb{R}$, consider the mark $\sigma = \mathbf{1}\{z \le -\theta_\sigma\}$ and the defect set $N = \{-\theta_B < z \le -\theta_\sigma\}$ ($\theta_\sigma < \theta_B$). List all sets determined by the value of σ (sets measurable with respect to σ) that contain N. What changes if $\theta_\sigma = \theta_B$?

The threshold of convexity is decided by whether the parameter enters the Hessian

When stuck on a sup over end values, the road divides according to whether the end value enters the Hessian. If the end value enters a nearest-neighbour quadratic coupling only linearly, the Hessian does not depend on it, and convexity is uniform in the end value. So cutting shorter than the threshold of convexity makes the sup over end values free, and the whole price moves to the side of the correlation across the cuts (the maximal correlation). As a procedure: **when stuck at the end of a parameter, first check whether that parameter enters the Hessian; if it does not, cut shorter than the threshold of convexity, and account separately only for the correlation at the seams** (23-11 § 05). State: checked form.

Small question 8. For $S(x_1, \dots, x_k) = \sum_{j=0}^{k} \frac{(x_{j+1} - x_j)^2}{2\varepsilon} + \varepsilon \sum_{j=1}^{k} V(x_j)$ and the measure $e^{-S/\hbar}$ with $x_0 = a$ and $x_{k+1} = b$ fixed, check that the Hessian does not depend

on a, b. When $V'' \geq -\frac{1}{2}$ (the value at the top of $V = (x^2-1)^2/8$), write the range of k for which the smallest eigenvalue of the Hessian is positive, in terms of ε (compare with $\kappa(k)$ of small question 6 at $\varepsilon = \frac{1}{2}$).

Split the ratio of upper bound to reading into a stage of cancellation and a stage of direction

When an upper bound U and a reading from below L are far apart, looking at the ratio U/L as a single number does not tell where one is losing. Inserting between them "the quantity along the same path before the cancellation is discarded", R, and splitting into R/L (the stage of cancellation) and U/R (the stage of direction), for the Feynman–Kac strand the former was around 1.2 and the latter over 1000, so almost all the loss was at the stage where the scalar Grönwall discards direction (23-09 § 05). As a procedure: **split the gap between an upper bound and a reading into ratios per stage before choosing the target of improvement**. State: checked form (ratios of readings).

Small question 9. For $\dot{X} = -CX$, $C = \mathrm{diag}(\lambda, \Lambda)$ $(0 < \lambda < \Lambda)$, with initial value $g = (0, 1)$ (the stiff direction), compare the true value of $\int_0^T |X_t|\, dt$ with the upper bound coming from the scalar Grönwall $|X_t| \leq e^{-\lambda t}|g|$. How does the ratio grow with Λ/λ? What is recovered when estimating with the matrix kept as it is?

Before trimming an upper bound, measure one lower bound

When tightening the construction of an upper bound through many stages still does not reach, first measuring one lower bound of the same quantity with a test function tells whether the problem lies in the technique or in the target. If the lower bound exceeds the target, improving the upper bound is pointless. In the sector framework, it turned out that the lower bound on the propagation constant lay at several times the needed value, which changed the meaning of the work of trimming the floor of the upper bound. As a procedure: **before trimming the floor of an upper bound, put a stage that sets the needed value beside the lower bound on the best constant** (23-09 § 03, 23-10 § 02). State: checked form.

Small question 10. On a single road of L cells (uniform distribution, $\mathcal{E}(f) = \frac{1}{L}\sum (f_{i+1} - f_i)^2$), show with the single test function $f_i = i$ that the target "$CP \leq L^2/12$" is false $(\mathrm{Var}(f)/\mathcal{E}(f) = L(L+1)/12)$. Compare also with the true value $1/(2(1-\cos(\pi/L)))$ (23-00 § 04).

Obstacles do not vanish but move — record, line by line, where they moved

Changing the formulation can make an obstacle seem to vanish while it appears elsewhere in the same form. In the first step, a single number — the curvature at one site at the top falls short of the coupling by $\frac{1}{4}$ — came back three times in different forms: the Poincaré constant of the lump of the core, the contraction ratio on the slope approaching

1, and the ratio of the tail pinned at the top exceeding 1 (the common wall of 23-09 § 06). As a procedure: **each time the formulation changes, record line by line "where the obstacle moved", and check whether the formula where it moved can be written with the same number**. If it can, that is not a defect of the tool but the structure of the object. State: checked form (the formula of the common wall).

Small question 11. Find in closed form the smallest eigenvalue of the tridiagonal matrix $M_k = \mathrm{tridiag}(-2,\ 4 - \frac{1}{4},\ -2)$ ($k \times k$, Dirichlet at both ends), and compare it with $\kappa(k)$ (small question 6). How is the failure of diagonal dominance, $\frac{15}{4} < 4$, related to the smallest eigenvalue becoming negative for $k \geq 8$?

Whether the diffusive-relaxation-time assumption is true changes not only with the measure but with the coordinates (the Dirichlet form)

The diffusive-relaxation-time assumption is not a sentence about a measure but about a pair of a measure and a Dirichlet form. Even for the same equilibrium measure, depending on which variables the dynamics moves independently, the size of the gap can be $O(1)$ or inversely proportional to the square of the side of the volume. In a dynamics with a conserved quantity (a quantity only handed between neighbours), its unevenness is smoothed out only by diffusion. As a procedure: **before asking about the relaxation-time assumption, name the coordinates of the dynamics, and first check whether that dynamics has a conserved quantity and at what length the conservation breaks** (23-01 § 05; for how to compare with the length at which conservation breaks, see Part 24). State: still an insight (checked form in the free-field approximation).

Small question 12. For the product measure of L independent standard normals, compare the gaps of (a) the dynamics that moves each coordinate independently by Ornstein–Uhlenbeck, and (b) the dynamics that hands quantity between neighbours and keeps the sum $\sum x_i$ (on the measure with the sum fixed). Check that the gap of (b) has the form $4\sin^2(\pi/(2L))$, and that it is unrelated to the static correlation length (here 0).

03 Discarded hypotheses — the original form, the corrected form, the grounds

Discarded hypotheses and the reasons for discarding them are information for avoiding the same errors. The upper six rows of Table 2 were discarded in the study of the first step, and the lower four on the side of the four-dimensional lattice.

Table 2 Discarded hypotheses and corrected insights

Original form	Corrected form	Grounds
Local Poincaré comes from a convex core	The action of an excursion out of the core is much smaller than S_0, and exponentially many excursions fit into an interval of length ξ	The reading of the action of excursions (towards the two scales of 23-06 § 03)

Original form	Corrected form	Grounds
Convexity improves away from the wall	The opposite. Pressing the end against the wall lowers the variance and raises the Hessian; moving it away removes only this help	Comparison of the Hessian for each end value (23-06 § 04)
Making the lumps larger lets the chain be built from the convexity of each lump	The worst convexity of a lump decreases with the length of the lump, and the coupling between lumps does not decrease	The formula of the common wall (23-09 § 06)
The contraction can be bounded by the smallest eigenvalue of the time-averaged operator	False. What can be used is only the time average of the smallest eigenvalue, and this discards direction	The algebra of the counterexample (Lean `mbar_bound_false`, 23-09 § 05)
If the pointwise reading is singular at rare configurations, the true gradient is singular too	What fixes the upper limit of the true gradient is not the pointwise minimum but the decay rate of the flow of derivatives	The representation $\nabla \psi = \int \mathbb{E}[J_t^{\top} \nabla G]\, dt$ (23-09 § 05)
The road of sharpening constants can lower $\hbar$ indefinitely	It stops at the convex radius (about 0.45); beyond that the error terms become genuinely large	The reading of the margin at the nodes (towards the $\hbar < 0.005$ row of 23-06 § 07)
$C_P \asymp \xi^2$ usually holds	The relaxation-time assumption is a specific claim, the dynamical exponent $z = 2$	$z \approx 2.17$ for the heat bath of the two-dimensional Ising model
Abelian counterexamples show that "non-abelianness is essential"	"An input that forbids a second small parameter" is essential	Three-dimensional $\mathbb{Z}_2$ keeps a single scale even though it is abelian (17-12)
The holographic density is not log-concave	The sign is the opposite ($W'' = -2B''$)	The judgement of the one-dimensional model
Gauge-fixed numbers are a property of confinement	Numbers for compact $U(1)$ in four dimensions; not measured for $SU(2)$	23-02 § 04

There are two types of error, and both occur at the stage of raising something to a one-line summary. One drops the conditions (group, model, range); the other confuses "a pointwise quantity" with "a quantity over the measure". Of the upper six rows of Table 2, four are of the latter type.

04

Questions at the size of an undergraduate thesis — a list of seventeen questions

A "small question" is one that satisfies three things. The form of the answer (exists / does not exist, the value of a constant, whether an inequality holds) is fixed in advance. The tools are at hand (python3 with NumPy and SciPy, a SAT solver, interval arithmetic, Lean and Mathlib, a literature search). There is a prospect that a third- or fourth-year student

of mathematics or physics can settle it within the period of an undergraduate thesis. The first column of Table 3 gives the short name of each question.

Table 3 Seventeen questions ("What does not move when settled" are the rows of this part that do not change even if the question is settled)

No.	Question	One line	Tools	Form of settlement	What does not move when settled
1	Complete families of k=3 cells	Does a complete family of $k = 3$ cells exist in $d = 9$?	SAT, Lean	A construction in the periodic box, or a record refuting satisfiability of the box	That one-link integration is a single step only (23-02 § 02)
2	Largest dimension with a complete family	The largest dimension in which a complete family exists, $d_{\max}(k)$	Exhaustive search, paper	A table for $k = 3, 4, 5$, and the reason the upper bound $3k + 1$ is not sharp	Same as above. Non-existence with period 2 (Lean `no_perfect_k4_d9`, `no_perfect_k5_d8`, assumption: period 2) already exists
3	Constant of the Hermite family	The constants of $1 - \lambda_k$ for the Hermite family (the worked example of § 05)	Paper, python3	The two terms and the remainder of (5)	That test functions bound C_P only from below
4	SU(N) version of the ratio on forests	The $SU(N)$ version, and the Lean statement, of the ratio of functions on the forest saturating at $1/\lambda_1(G)$	Paper, Lean	The statement of Poincaré for products of Haar	The relaxation-time upper-bound assumption (it only closes one family of test functions for the relaxation-time lower-bound assumption on the four-dimensional lattice)
5	C_P(β) on the two-dimensional lattice	$C_P(\beta)$ in the link variables of two-dimensional lattice YM	Paper, python3	The power as $\beta \to \infty$	The relaxation-time assumption in four dimensions (two dimensions is a control)
6	Curvature after one-link integration	The best constant of the curvature after one-link integration, $c^* \in [21, 27]$	Global optimisation, interval arithmetic	c^* and the configuration, a computer-assisted lower bound	That one cannot get out from the strong-coupling side

No.	Question	One line	Tools	Form of settlement	What does not move when settled
7	The N^{−3} coefficient of the column model	The coefficient $C \approx 320$ of N^{-3} in the column model, analytically (finite-size expansion of the XX chain)	Paper, python3	A closed formula agreeing to 4 digits or more	The logic of the arrows of the tube estimate (the tube inequality)
8	Sign of 1/P in the column model	In the column model, the sign of $1/P$ at finite u	python3	A table of signs for $u = 0.1 \sim 0.3$	Same as above
9	Gauge dependence of the shortfall	How much does the shortfall c_2 change with the choice of gauge in the block transformation?	python3	A table of c_2 for each gauge, and whether $c_2 > 0$ is kept for positive kernels	The arrow of the sum of shortfalls (it does not move if $c_2 \neq 0$)
10	C_P and ξ² of the O(3) σ model	C_P and ξ^2 of the two-dimensional $O(3)$ sigma model	python3, literature	The slope, and whether a logarithm is present when the scale is changed	The relaxation-time assumption in four dimensions (an asymptotically free control)
11	C_P/ξ² of three-dimensional Z_2 gauge theory	C_P/ξ^2 of the three-dimensional $\mathbb{Z}_2$ gauge theory	python3	A table, and the correspondence with the dynamical exponent on the dual Ising side	The relaxation-time assumption in the non-abelian case
12	Volume monotonicity of C_P(V)	Monotonicity in volume of $C_P(V)$, and boundary conditions (find all eigenvalues of the generator on small lattices)	python3	A counterexample to monotonicity, or a proof	The question itself of the order of volume in which the relaxation-time assumption is stated (the note in 23-01 § 05)

No.	Question	One line	Tools	Form of settlement	What does not move when settled
13	Dynamical exponent of four-dimensional SU(2)	Collect the primary sources on the dynamical exponent of local algorithms for four-dimensional $SU(2)$	Literature	A table of about ten primary sources (exponent, observable, box, boundary)	The conditions of the reading for the periodic box (the part that is a conjecture)
14	Interval arithmetic for the maximal correlation of the core	Bound from above, by interval arithmetic, the two-point maximal correlation of the medium-core step ($8 \le k \le 14$)	Interval arithmetic, paper	Computer-assisted upper bounds at each point of k and $\hbar$, and a paper on the difference between lattice and continuum	[007] of the road of the core on the box side (the side $k \ge 15$) and [014] (23-15 Table 1)
15	Power of the prefactor for the core and window	Read the power of the prefactor $\hbar^{-p}$ of C_P for the core and the window by the capacity of the degenerate point	python3, paper	The value of p (a reading), and the form of the capacity formula	The side of the action barrier $\Delta S < 2S_0$ (only one of the conditions of small question 5)
16	Comparison for the alternating maximal correlation of boxes	A comparison theorem showing the alternating maximal correlation between boxes $c_{\mathrm{alt}} < 1$, avoiding examples that do not reduce to two-point quantities	Paper	An upper bound on c_{alt} uniform for the reduced model	The rows of the in-phase local Poincaré condition and the saddle-tube condition (only [006] moves)

No.	Question	One line	Tools	Form of settlement	What does not move when settled
17	Length set by a conservation-breaking term	In the dynamics with conservation of small question 12, when a small term that breaks the conservation is added, the length at which the gap departs from the square of the side	Paper, python3	The relation between the strength of the breaking term and the length	The relaxation-time assumption in four dimensions (the smallest model for the questions of Part 24)

For every question, before starting, we put a stage of searching for whether an answer to the same question exists. Write the question in one sentence in the form of a theorem, search with combinations of words from both mathematics and physics, follow the references of reviews to the text of the primary sources, and judge it as one of: the same answer, a close but different result, or not found. If nothing is found, write "not found in the range searched", and leave the words and materials searched in the source table.

05

Worked example: one question to the form of a settlement — $1-\lambda_k$ for the Hermite family

To see the size of a question, we carry the question "constant of the Hermite family" to the end. The Poincaré constant of the standard normal distribution is $C_P=1$, with equality at $f=x$ (1-10). Let λ_k be the maximum of the ratio $\operatorname{Var}(f)/\mathcal{E}(f)$ over the family $F_k=\operatorname{span}\{x^3,\dots,x^{2k+1}\}$, which does not contain x.

Normalising the Hermite polynomials to $\mathbb{E}[h_m h_n]=\delta_{mn}$ gives $h_n'=\sqrt{n}\,h_{n-1}$, so the variance and the Dirichlet form of an odd function $f=\sum_{j=0}^{k}u_j h_{2j+1}$ are diagonal.

$$\operatorname{Var}(f)=\sum_{j=0}^{k}u_j^2,\qquad \mathcal{E}(f)=\sum_{j=0}^{k}(2j+1)\,u_j^2 \tag{1}$$

The condition for being in the family is the single linear constraint "the coefficient of the monomial x is 0", $\sum_j w_j u_j=0$, with $w_j=(-1)^j\sqrt{(2j+1)!!/(2j)!!}$. Finding the minimum μ_k of $\mathcal{E}/\operatorname{Var}$ under the constraint by Lagrange multipliers leaves an equation in one variable [Cited] (the secular equation for stationary values of a quadratic form under a linear constraint, Golub 1973).

$$g_k(\mu) := \sum_{j=0}^{k} \frac{w_j^2}{2j+1-\mu} = 0, \qquad 1 < \mu_k < 3, \qquad \lambda_k = \frac{1}{\mu_k} \tag{2}$$

Setting $\mu_k = 1 + \varepsilon$, the $j = 0$ term is $-1/\varepsilon$ and balances the rest.

$$\frac{1}{\varepsilon} = S_k + O(\varepsilon), \qquad S_k := \sum_{j=1}^{k} \frac{(2j+1)!!}{(2j)!!\, 2j} \tag{3}$$

S_k closes by a generating function. Splitting each term in two by $(2j+1)!!/(2j)!! = (2j+1)\binom{2j}{j}4^{-j}$, and using the partial-sum identity $\sum_{j=0}^{k}\binom{2j}{j}4^{-j} = (2k+1)\binom{2k}{k}4^{-k}$ (by induction on k) and the value $\ln 2$ at $x = 1$ of $\ln \frac{2}{1+\sqrt{1-x}} = \sum_{j\ge 1}\binom{2j}{j}\frac{x^j}{2j\,4^j}$, gives

Proved here

$$S_k = 2\sqrt{\frac{k}{\pi}} + (\ln 2 - 1) + O(k^{-1/2}) \tag{4}$$

Substituting into (3) and expanding $1 - \lambda_k = \varepsilon - \varepsilon^2 + \cdots$ fixes the two terms Proved here .

$$1 - \lambda_k = \frac{\sqrt{\pi}}{2\sqrt{k}} - \frac{\pi \ln 2}{4k} + O(k^{-3/2}), \qquad \frac{\sqrt{\pi}}{2} = 0.886227, \quad \frac{\pi \ln 2}{4} = 0.544397 \tag{5}$$

Table 4 λ_k and a check of (5) Reading (values from solving for the roots of (2) numerically)

k	λ_k	$\sqrt{k}(1-\lambda_k)$	$k\left(1-\lambda_k-\frac{\sqrt{\pi}}{2\sqrt{k}}\right)$
1	0.555556	0.4444	−0.4418
10	0.774921	0.7118	−0.5517
1000	0.972524	0.8689	−0.5489
10^4	0.991192	0.8808	−0.5459
10^6	0.999114	0.8857	−0.5446
(5)	$\to 1$	0.8862	−0.5444

What does not move when it is settled is the fact that test functions bound C_P only from below. Enlarging the family keeps $\lambda_k < 1$ and only approaches 1 at the rate $k^{-1/2}$; that the limit is 1 can be said only from an input outside the family (Gaussian convexity, $V'' = 1$). The settlement of a small question often takes this form.

This part placed the relaxation-time assumption at the axis of the study, and this chapter gathers the views that repeatedly proved useful on that basis. Questions that stand on views other than the relaxation-time assumption are placed in Part 24.

What this chapter can say

Said	Not said
In which stage of which chapter of the body each of the twelve insights worked (the chapter at the end of each insight), and the state of each insight (still an insight, checked form)	That the insights work outside the first step (the four-dimensional lattice, other problems). Every insight is a view before verification
Proved here The formula for the threshold in small questions 6 and 11 ($\kappa(k) < 0 \iff k \geq 8$), and the skeleton of the derivation of (4)(5)	The value of the coefficient of the remainder $O(k^{-3/2})$ in (5)
Reading The numbers of Table 4	Table 4 consists of floating-point values of the roots of (2), not propositions of upper or lower bounds
The list of seventeen questions, and what does not move when they are settled	No question, even when settled, shows anything about the mass gap, the continuum limit or the relaxation-time assumption. "The size of an undergraduate thesis" is a prospect

Sources and reproduction

Item	Kind	Source / tools
Material for the twelve insights	This book	The sections of the chapters given at the end of each insight (23-06 to 23-11)
The two-point logarithmic Sobolev constant of small question 2	Cited	P. Diaconis, L. Saloff-Coste, Logarithmic Sobolev inequalities for finite Markov chains (bibliographic data not checked; the statement has not been checked in the original. It is an object to be checked on the side of the question)
The lower two rows of Table 2 (the heat bath of the two-dimensional Ising model, the holographic density)	This book	The record of hypotheses discarded in this book's study. The z of the two-dimensional Ising model is from M. P. Nightingale, H. W. J. Blöte, Phys. Rev. Lett. 76 (1996) 4548 (abstract only)
Questions 1–13 of Table 3	This book	The fourteen small questions of this book's study, minus the one that became the test bed. Questions 14–17 come from the remaining rows of 23-09 to 23-11
The secular equation (2)	Cited	G. H. Golub, Some modified matrix eigenvalue problems, SIAM Review 15 (1973) 318–334 (bibliographic data only)
Table 4	Reading	Bisection for the roots of (2), cross-checked against the generalised eigenvalues of the Gram matrix of monomials for $k \leq 6$

Terms used in this chapter — the three states of an insight, paying twice, the currency of Poincaré and the currency of entropy, the length where $1/\hbar$ lives, the wall of conditioning, the threshold of convexity, the stage of cancellation and the stage of direction, the moving of obstacles, dynamics with conservation, small questions. Their meanings are in the A-02 Glossary.

Next chapter to read: Part 24 The first road, second half — views other than the relaxation-time assumption, and a list of material for new views that take the relaxation-time assumption as given. To go on to the appendices, A-01 Table of notation.

Entrance to Part 24 — what we take over from Part 23

> This chapter is a guide (the entrance to this part).

This part is a record of the study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — Where Part 24, "Further considerations", stands; the sentences it takes from Part 23; the map of its nine chapters (title, one line, level, and what each chapter can say); the words fixed in this part; and what this part does not treat.

Prerequisite chapters — 23-00 (the entrance to Part 23), 23-01 § 05 (the diffusive-relaxation-time assumption), 23-01 § 06 (what the diffusive-relaxation-time assumption gives and does not give).

Said first — The level is [Y2]. This page is a map and proves nothing. No chapter of Part 24 shows the mass gap. We write neither that the relaxation-time assumption was wrong nor that it was the right road.

The order of this chapter

1. Where this part stands — what to think about after the study of Part 23
2. Sentences taken from Part 23 — five sentences and their labels (Table 1)
3. Map of the part — the nine chapters, and what each can say (Figure 1, Table 2, Table 3)
4. Words fixed in this part — the local diffusive-relaxation-time assumption and the ratio form, the threshold, the length at which conservation breaks
5. What is not treated, the table of notation, returning to Part 23

01

Where this part stands — what to think about after the study of Part 23

Part 23 restated the problem of the mass gap in terms of quantities that can be computed on the lattice and are within reach of machine checking — a power relation between the relaxation time C_P of Langevin dynamics and the correlation length ξ (the relaxation-time assumption) — and used that restatement as a foothold for its study. Setting up the relaxation-time assumption as a sharp sentence made it possible to name where the relaxation-time assumption fails. Which test functions obstruct the upper half of the relaxation-time assumption (the relaxation-time upper-bound assumption) came together as a single pattern. This is not "the relaxation-time assumption was wrong"; the order of events is "because the relaxation-time assumption was set up, the nature of how it breaks became visible".

Part 24 is the "further considerations" that start from there. First we see that what obstructs the relaxation-time assumption comes together in one form — a global quantity not reflected in the gap slows relaxation on a length other than the correlation length (24-01, 24-02). Next, keeping the relaxation-time assumption as a premise, we define two candidates that restate the relaxation-time assumption so as to avoid that form (the relaxation-time assumption for each topological sector, and the local relaxation-time assumption with windows or boxes limited to a few times the ruler), and examine which part the one-sided bridge needs for the mass gap (24-03, 24-04). From there we go on to two views that do not pass through the relaxation-time assumption: that on a lattice with continuous time the gap is exactly the Poincaré constant of the vacuum measure (24-05), and that on groups one needs tools that pay with rarity instead of convexity (24-06). Finally we state where being four-dimensional matters, limited to the two points "the dimensionless coupling" and "topology" (24-07), and close with questions and what is not claimed (24-08).

Many sentences of this part are estimates, and we distinguish them by adding one estimate label, Estimate, to the five labels of Part 23 (Proved here, Cited, Computer-assisted, Reading, Assumption). An estimate is an opinion offered as material for judgement; it carries neither a proof nor numbers.

02

Sentences taken from Part 23

Table 1 lists the five sentences that Part 24 takes from Part 23 and uses. Labels and statements remain as in Part 23; this part only cites equation numbers and sections, and does not copy the proofs.

Table 1 Sentences taken from Part 23

Sentence	Label	Place in Part 23	Chapters that use it

Sentence	Label	Place in Part 23	Chapters that use it
The trade of the restatement: the quantity of the problem statement is the gap in Euclidean time; the quantity of the relaxation-time assumption is the relaxation C_P in Langevin time (a quantity defined by a supremum, bounded from below by a single test function, and whose upper bound can be built from parts). The two are different quantities	Convention	23-01 § 04 (the three C_P)	24-01, 24-04, 24-05
The relaxation-time assumption is the power statement $\lim \ln C_P / \ln \xi = 2$. The halves of the band statement are written the upper-bound assumption and the relaxation-time lower-bound assumption	Assumption	23-01 § 05	All chapters
The one-sided bridge: on a lattice with continuous time, $C_P \geq \kappa \hbar^2 / \Delta^2$. Under the assumption of the bridge, the upper-bound assumption is a stronger sentence than the gap of the lattice in physical units (there is also a version in the form of a paper, before external checking)	Proved here (continuous time), Assumption (on Wilson's lattice with discretised time, the time-step comparison assumption ((D) in the paper "one-sided bridge"))	23-01 § 06 (Lean `hup_is_gap_positivity_of_lower`, assumptions: $0 < m$, $\kappa / m^2 \leq C_P$, $C_P \leq K \xi^2$)	24-04, 24-05
The converse is false in general: for a free particle on a circle, $\Delta > 0$ holds while $C_P = \infty$ (the Free-Particle-on-a-Circle Counterexample)	Proved here	23-01 § 06	24-01, 24-04

Sentence	Label	Place in Part 23	Chapters that use it
In the four-dimensional periodic box, the upper-bound assumption reads as broken by the freezing of the topological charge (on original sources for $SU(3)$ and above, and a reading that extends a conjecture). The note on topological sectors and the order of volume	Assumption , Reading	the note in 23-01 § 05, 23-02 § 05	24-01, 24-03, 24-07

03

Map of the part — the nine chapters, and what each can say

The nine chapters of Part 24 and the table of notation

0. **24-00 Entrance — what is taken from Part 23** [Y2] (this page)
1. 24-01 One form that obstructs the diffusive-relaxation-time assumption — a nearly conserved global quantity [Y3]
2. 24-02 Choosing the ruler and the coordinates again — the length at which conservation breaks [Y3]
3. 24-03 Two ways to repair the relaxation-time assumption — the sector assumption and the local assumption [Res] (summary at [Y3])
4. 24-04 Which form of the relaxation-time assumption the one-sided bridge needs [Res] (summary at [Y3])
5. 24-05 Making time continuous — the Hamiltonian lattice and the vacuum measure [Y3]
6. 24-06 Paying with rarity instead of convexity — renormalisation on groups and log-Sobolev [Res] (summary at [Y3])
7. 24-07 Because it is four dimensions — the dimensionless coupling and topology [Y3]
8. 24-08 Questions, and what is not claimed [Y2]–[Y3]

24-09 Table of notation for the first road, second half (a reference page)

Together with Part 23, this part forms the second half of "The first road — from the direction of time" (the second road is Part 25, the meeting of the two roads is Part 26, and the conditional statements are Part 27). The order of the chapters is: the form of

the breakage (24-01, 24-02), restatements that keep the relaxation-time assumption as a premise (24-03, 24-04), views that do not pass through the relaxation-time assumption (24-05, 24-06), what is specific to four dimensions (24-07), and questions (24-08). From 24-01 to 24-05 is a single chain in which the conclusion of each chapter becomes the question of the next. 24-06 is not a consequence of the chain but a chapter of tools needed to show any of the statements of 24-03 to 24-05, and on the line of the argument it is placed as a branch.

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Figure 1 The routes of Part 24 (an arrow means "the conclusion
of the previous chapter becomes the question of the next";
it is not a theorem of implication)

  24-01 One form ──→ 24-02 Ruler, coordinates ──→ 24-03 Sector assumption,
                        |                            local assumption
                        |                              |
                        |                 (ratio form, threshold)
                        |                              ↓
                        |             24-04 Which form of the relaxation-time
                        |                    assumption the bridge needs
                        |                              |
                        |                              ↓
                        |                  24-05 The vacuum measure
                        |
     Two-dimensional test beds (the U(1) strip, CP^(N−1))
        ── a branch reaching 24-02 and 24-07

  Tool branch: 24-06 Paying with rarity instead of convexity
               (needed by every one of 24-03 to 24-05)
  Specific to four dimensions:
               24-07 The dimensionless coupling and topology
  Closing: 24-08 Questions, and what is not claimed
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Table 2 One line for each chapter, and what that chapter can say

Chapter	One line	What this chapter can say
24-01	Puts in one table, together with their strength, that the number of kinks, the winding number, the flux, the topological charge and the flux through the box are all of the same pattern: "a global quantity not reflected in the gap slows relaxation on another scale"	That the five examples are of the same pattern is a classification (an estimate). The numbers carry the label of each row
24-02	Splits the length to compare with C_P into three rulers, and sets out that the type of relaxation changes with the choice of coordinates (the Dirichlet form), together with numbers for the two-dimensional strip and the test beds	The length at which conservation breaks is a reading on the two-dimensional $U(1)$ strip. Restating what to compare with is a candidate. CP^{N-1} is a conditional reading at one volume and one correlation length

Chapter	One line	What this chapter can say
24-03	Defines the relaxation-time assumption conditioned on topological sectors, and the local relaxation-time assumption with windows or boxes limited to a few times the ruler, and states them in the ratio form against a massless reference	The two relaxation-time assumptions have definitions, and the local relaxation-time assumption has content only in the ratio form. There is no inequality shown at any β
24-04	States that the one-sided bridge uses only the part of the upper-bound assumption that is local in time, and sets out the bridge on a window, the threshold of the ratio form and three references	The threshold of the bridge depends on the reference and the family, and the band between the threshold and 1 is undecided. Everything rests on the local-excitation moment assumption and the assumptions of the One-Sided Bridge Proposition (Theorem 3.1 of the paper)
24-05	Sets out that on a lattice with continuous time the gap is tied, as the exact reciprocal, to the Poincaré constant of the vacuum measure at a single time, and why the tools needed there do not apply	The gap is the Poincaré constant of the vacuum measure itself, and the identity does not make the problem easier
24-06	States that on groups the tools of convexity cannot, in principle, stand at weak coupling, that the non-convex places are pushed into places of small probability, and the tools that pay with rarity and candidate rules	A chapter of tools. The ceiling of convexity has a proof; the class of measures that pay with rarity is a candidate. Results on hierarchical models imply nothing for four dimensions
24-07	Limits what matters in four dimensions to two points, the dimensionless coupling and topology, and sets out the four hurdles and why the third hurdle stayed open for so long, from what primary sources written before the published construction say	The two points are an estimate of a property of the critical dimension. Freezing for $SU(2)$ is undecided. Results in two dimensions imply nothing for four dimensions
24-08	The table of questions, the list of candidate rules, the sixteen items that are not claimed	Questions, and what is not claimed. Candidate rules are candidates

Table 3 lists the material that came up in Part 23 and that the chapters of this part take up as body text.

Table 3 Material taken over from Part 23, and the chapters that take it up

Material	Related in Part 23	Chapter that takes it up
The table of other roads (the renormalisation group and log-Sobolev, stochastic quantisation, the Hamiltonian lattice, hierarchical models, and so on)	23-02 § 08	24-05, 24-06
The upper-bound assumption and the one-sided bridge to the mass gap	23-01 § 06	24-04

Material	Related in Part 23	Chapter that takes it up
Hierarchical gauge models and the problem of coordinates	23-17 Insight 12	24-02 § 02, 24-06 § 05
Conservation of flux and "the length at which conservation breaks"	23-17, the question "length set by a conservation-breaking term"	24-01, 24-02
The study of the relaxation-time assumption on the four-dimensional lattice (freezing of the topological charge, and how to correct it into a sentence per sector or a local sentence)	the note in 23-01 § 05, 23-02 § 05	24-01, 24-03

04

Words fixed in this part

The local relaxation-time assumption and the ratio form. The sentence that states the Poincaré constant $C_P^{\mathrm{loc}}(c)$, with test functions limited to those supported in a window (or box) of c times the ruler ℓ, by its ratio $K_{\mathrm{rel}}(c)$ to the massless reference on the same window with the same graduation (24-03 § 03). The form "bounded by some constant" also holds for the free field and is empty, so we use the ratio form.

The threshold K_*. The value of the threshold such that, if the ratio is below it, the one-sided bridge gives a lower bound on the mass under the assumptions. Depending on the choice of reference it changes: 0.342κ (scalar), 0.502κ (gauge quotient), about 0.62κ (colour singlet of $SU(2)$) (24-04 § 06; the numbers are readings for the free-field reference).

The length at which conservation breaks ℓ_*. The length at which a global quantity not reflected in the gap begins to relax through the breaking of conservation rather than through local transport. It is a ruler different from the static correlation length, and the values measured on the two-dimensional $U(1)$ strip are not proportional to the correlation length (24-02 § 01, § 03).

05

What is not treated, the table of notation, returning to Part 23

What is not treated. Part 24 does not touch the ultraviolet side — the existence of the continuum limit, the recovery of rotations, the OS axioms, non-triviality (the Wilson-loop limit statement, the rotation-recovery statement, the OS reconstruction statement and the non-triviality statement of Part 23, 23-01 § 02). Only 24-07 § 05 sets out, from primary sources, how far the third hurdle has been reached as a difficulty specific to four dimensions, and even there construction is not treated. The existence of the

continuum limit is read in Volume 2 (from Part 6) through the construction published on 2026-11-10 (a Lean formalisation that builds a theory satisfying the Wightman axioms as the limit of Wilson's lattice in four dimensions; the committee's assessment is that it is proved in Lean).

Notation. The notation of Part 24, and its correspondence with symbols of the same letter in Part 23, are in 24-09 Table of notation for the first road, second half. In particular Δ is reserved for the gap of the Hamiltonian $E_1 - E_0$, and the mass in physical units is written m_{phys}.

Returning to Part 23. The map of Part 23 is in 23-00 § 01, what is not claimed in 23-16, and the small questions in 23-17.

What this page can say

Said	Not said
The order of the chapters of Part 24, and one line on what each chapter can say (Table 2). The sentences taken over from Part 23 and their labels (Table 1)	No chapter shows the mass gap. Nor does any show whether the relaxation-time assumption is true
The meaning of the words fixed in this part (§ 04)	Theorems about the local relaxation-time assumption, the threshold, or the length at which conservation breaks (we go as far as definitions, the ratio form, and numbers for the free field and in two dimensions)

Sources and reproduction

Item	Kind	Source / tools
The sentences of Table 1	This book	The sections and source tables of 23-01 and 23-02
The rows of Table 2 and § 04	This book	"What this chapter can say" in each chapter of Part 24

Terms used on this page — further considerations, the trade of the restatement, the one-sided bridge, the estimate label Estimate, the local relaxation-time assumption, the ratio form, the threshold, the length at which conservation breaks, the tool branch. Their meanings are in the A-02 Glossary and 24-09.

Next chapter to read: 24-01 One form that obstructs the diffusive-relaxation-time assumption — a nearly conserved global quantity — sets out, in one table with five

examples, the pattern in which a global quantity not reflected in the gap slows relaxation on another scale.

One form that obstructs the diffusive-relaxation-time assumption — a nearly conserved global quantity

> This chapter is this book's own investigation. The established knowledge it builds on is in 2-10 and 5-09.

What this chapter shows — We set out five examples that obstruct the upper half of the diffusive-relaxation-time assumption (the relaxation-time upper-bound assumption), and see that all of them fall into a single pattern: "a global quantity not reflected in the gap slows relaxation on a scale other than the correlation length". The five examples are the number of kinks in the double well, the winding number of a particle on a circle, the flux in the two-dimensional $U(1)$ strip, the topological charge in the four-dimensional periodic box, and the flux through a box. What differs is the strength (logarithmic, power or exponential, infinite) and the event that does the breaking. How the upper-bound assumption fails in this pattern is decided by the single line $\tau_{\mathrm{int}} \le C_P$. Using the charge of a subregion, we separate two accounts, "transport" and "breaking of conservation", in preparation for the local diffusive-relaxation-time assumption of 24-03.

Prerequisite chapters — 23-01 §05 ((13) for the relaxation-time assumption, (14)(15) for the upper-bound assumption, the note on topological sectors), 23-01 §06 (the Free-Particle-on-a-Circle Counterexample), 23-02 §05 ($\tau_{\mathrm{int}} \le C_P$), the detour in 2-10 (conserving dynamics). The notation is collected in 24-09.

Said first — The level is [Y3]. Calling the five examples one pattern is a classification, not a theorem Estimate . The numbers in each example are numbers under conditions, with a label on each row. This chapter does not claim that the upper-bound assumption is false. The row for the four-dimensional periodic box rests on numbers from original sources for $SU(3)$ and above, and on a reading that extends a conjecture attached to observations with open boundaries to the slowness of topology in the periodic box; for $SU(2)$ it is not settled even at the level of a reading. None of the examples breaks the mass gap.

The order of this chapter

1. The type of a nearly conserved global quantity — three conditions, and the one line by which the upper-bound assumption fails ((1)–(3))
2. Five examples in one table — Table 1, and three rows moved below (Table 2)
3. Three strengths — logarithmic, power or exponential, infinite (Table 3)
4. Two accounts for the charge of a subregion — the transport coefficient d and the coefficient of breaking of conservation h ((4)–(7))
5. The wall of the first step is a scaled-down version of this pattern — the $\frac{1}{4}$ at the top, and the correspondence with the types of breakage in Part 23 (Table 4)
6. What cannot be said — extending the conjecture, $SU(2)$, two dimensions, power and exponential
7. Table of the five grades and the dependency graph — Table 5

01

The type of a nearly conserved global quantity — three conditions, and the one line by which the upper-bound assumption fails

The relaxation-time assumption is the power statement $\lim \ln C_P / \ln \xi = 2$ ((13) of 23-01), and its upper half (the upper-bound assumption) is "for every test function f, $\mathrm{Var}(f) \le C\, \xi_{\mathrm{RG}}^2\, \mathcal{E}(f)$" ((15) there). Since it is a statement about every f, it collapses if there is even one function whose ratio $\mathrm{Var}/\mathcal{E}$ grows faster than ξ^2. Because Part 23 set up the relaxation-time assumption as a sharp sentence, we can name what such a function looks like. The answer of this chapter is the following pattern.

Pattern (an estimate)

For a pair of a measure μ and a Dirichlet form $\mathcal{E}$, there is a global quantity Q such that

(i) $\mathrm{Var}_\mu(Q)$ is large (it grows with the volume or the length of the window).

(ii) $\mathcal{E}(Q)$ is small. Q is handed to neighbours through the flux of a conservation law, or changes only through rare configurations.

(iii) Q is not reflected in the gap Δ. It does not belong to the family of time-integrated single-time functions ($\mathcal{F}_1$ in 24-04), or if it does, it does not overlap the excitations of the gap.

Then the time for Q to relax is fixed not by the correlation length ξ but by the scale of the events that break Q.

(iii) is the condition that separates this pattern from the mass gap. Since the quantity is not reflected in the gap, the gap may stay positive however slow the relaxation is. The smallest example of this separation was the Free-Particle-on-a-Circle Counterexample of Part 23 (a free particle on a circle; the gap is positive and $C_P = \infty$).

The one line by which the upper-bound assumption fails. If the dynamics is reversible (the generator L is self-adjoint on $L^2(\mu)$, $-L \geq 0$), then the integrated autocorrelation time of any observable f does not exceed C_P. Writing ν_f for the spectral measure of $-L$ for $\tilde{f} = f - \mu(f)$ (supported on $[\lambda_1, \infty)$, $\lambda_1 = 1/C_P$), we have

$$\tau_{\mathrm{int}}(f) := \int_0^\infty \frac{\mathrm{Cov}(f(0), f(t))}{\mathrm{Var}\, f}\, dt = \frac{\int \lambda^{-1}\, d\nu_f(\lambda)}{\int d\nu_f(\lambda)} \;\leq\; \frac{1}{\lambda_1} = C_P \quad (1)$$

Proved here (one only integrates $\lambda^{-1} \leq \lambda_1^{-1}$ over $\lambda \geq \lambda_1$; equality holds when $\tilde{f}$ lies in the slowest eigenspace). In Lean the version with a finite spectral sum is checked (LatticeGauge.FourDTrial.tau_int_le_CP, assumptions: weights ≥ 0, each eigenvalue $\geq \lambda_1 > 0$, sum of weights > 0). The version that does not use the dynamics is the definition of the supremum itself:

$$\frac{\mathrm{Var}_\mu(Q)}{\mathcal{E}(Q)} \;\leq\; C_P \quad (2)$$

Proved here. Hence, if the relaxation time of Q ($\tau_{\mathrm{int}}(Q)$ for (1), the static ratio for (2)) grows at least as $c\,\xi^p$ ($p > 2$) along a sequence with $\beta \to \infty$, then $C_P \leq K\xi^2$ fails for every constant K. In the range where the correlation length ξ and the bookkeeping length ξ_{RG} are comparable up to constant factors, this is the negation of the upper-bound assumption.

$$\tau_{\mathrm{int}}(Q) \leq C_P, \quad \tau_{\mathrm{int}}(Q) \geq c\,\xi^p \ (p > 2), \quad \xi \to \infty \qquad \Longrightarrow \qquad \neg\,\exists K:\ C_P \leq K\xi^2 \quad (3)$$

Proved here Lean LatticeGauge.FourDTrial.not_hup_of_fast_tau (assumptions: $0 < c$, $2 < p$, $\xi_n \to \infty$, $\xi_n > 0$, and for each n, $\tau_n \leq C_n$ and $c\,\xi_n^p \leq \tau_n$). Exponential growth $\tau \geq c\,e^{\alpha\xi}$ includes $p = 3$ ($e^x \geq x^3/6$). The caution in the converse direction is also one line. That τ is small for a fast quantity does not support the upper-bound assumption. For an observable whose weight lies on fast modes, τ_{int} stays small however large C_P is (negative control tau_small_but_CP_large).

Let us check the level of this one line. Of the two assumptions on the left of (3), Lean takes only inequalities between sequences of real numbers; that τ_n is the autocorrelation time of a reversible dynamics and that C_n is the Poincaré constant of the Wilson measure lie outside Lean. What supports the second assumption of (3) in four dimensions is the Monte Carlo numbers of the original sources Cited, reinterpreted across the difference of algorithms as Langevin autocorrelations, and that reinterpretation is an assumption (§ 06 (a)).

02

Five examples in one table

Table 1 sets out, in the same columns, five examples that satisfy the three conditions of the pattern. The columns are: the quantity outside the family (the Q of the pattern), the event that moves it (the breaking of conservation), the relation between the scale of slowness and the correlation length, the strength, the numbers measured and their labels, and the chapter where the body text is. The label on a row is the label of that row's numbers; the classification "falls into the same pattern" itself is Estimate on every row.

Table 1 Five examples that obstruct the upper-bound assumption (the strength is how C_P/ξ^2 grows; the conditions on the numbers are in each row's para

Example	Quantity outside the family	What moves it (breaking of conservation)	Scale of slowness and correlation length	Strength	Numbers measured and la
(a) Double well in 0+1	The number of kinks (which well the path is in), the position of a kink in the window	Creation and annihilation of kink pairs (tunnelling), motion across the window	Same origin as ξ. The overall ratio is $S_0/\hbar \approx \ln \xi$. The local ratio is ω_b/m even for the shortest window	Logarithmic	One-site ratio $= \omega_b/m$ Proved here, 4.61, 21.6, 638 Reading, overall rati Assumption
(b) Free particle on a circle	The winding number (integral of the increments of the lifted angle)	Never breaks (continuous paths)	Infinite	Infinite	$C_P = \infty$ with $\Delta = \hbar^2/2 >$ Proved here
(c) The two-dimensional $U(1)$ strip	Flux dipoles, the topological charge Q of the strip	Wrapping of a single plaquette (a barrier of height 2β)	$\ell_c \propto e^{0.916\beta}$, $\xi_{XY} \approx 2\beta$	Exponential in ξ_{XY}	$\ell_c = 3.29$ to 139, $\tau(Q)/\xi_{XY}^2$ $= 0.4 \to 24$ Reading

Example	Quantity outside the family	What moves it (breaking of conservation)	Scale of slowness and correlation length	Strength	Numbers measured and la
(d) Four-dimensional periodic box	The smoothed topological charge Q	Small instantons slipping through the lattice (dislocations)	Faster than ξ^2 in original sources for $SU(3)$ and above. For $SU(2)$, one primary source, close to a^{-2}	Power or exponential ($SU(3)$ and above)	Numbers of the original sources Cited , reading the upper-bound assumption Assumption
(e) Flux through a box	Functionals moved by gauge transformations at the sites on the boundary of the box	They enter the test space if the boundary gauge transformations are not divided out	2.22 to 2.53 times the value of the reference box	Constant factor	Difference between the t choices Proved here , factor Reading

(a) The number of kinks in the double well. In the path measure in 0+1 dimensions (23-04, $V = (x^2-1)^2/8$, $S_0 = 2/3$), which well the path is in changes only when a kink (a path moving from valley to valley) enters. The weight of a kink $e^{-S_0/\hbar}$ fixes the correlation length $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$, and the diffusion of kinks fixes the scale $\xi^2 S_0/\hbar$ on the upper side of C_P, so the slow quantity and the correlation length are born from the same event. The ratio C_P/ξ^2 is of order $S_0/\hbar \approx \ln \xi$, so the band form of the upper-bound assumption fails only by a logarithm while the double-well relaxation-time power statement survives ((14) of 23-04, the accounting for a dilute gas of kinks) Assumption . The same pattern appears in local ratios with limited windows. In a window of a single time, the Poincaré constant of the marginal ψ_0^2 moves by the ground-state transformation to the gap of $(H - E_0)$, and its ratio to the harmonic oscillator with the curvature of the bottom is **exactly** ω_b/m (when ψ_0 is the Schrödinger ground state) Proved here . With the transfer on the lattice $\varepsilon = \frac{1}{2}$, at $\hbar = 0.3,\ 0.15,\ 0.08$ it is $4.61,\ 21.6,\ 638$, 1.04 to 1.05 times ω_b/m Reading . The detailed behaviour of this ratio (lengthening the window moves it to $(\omega_b/m)^2$) is treated in 24-04.

(b) The winding number of a particle on a circle. This is the Free-Particle-on-a-Circle Counterexample of Part 23. For the path lifted to the real line, $\tilde{x}(\tau)$, the integral of its increments with weight w, $f = \int w\, d\tilde{x}$, has $\mathrm{Var}(f) = \hbar \int w^2$ and $\mathcal{E}(f) = \hbar \int w'^2$, and lengthening the window makes the ratio as large as one likes. The winding number never breaks for continuous paths, so the strength is infinite. This f is a functional supported in a window of length L; with the window fixed the ratio is finite, but lengthening the window makes it unbounded Proved here . The winding number is not a time-integrated single-time function, so it satisfies condition (iii) of the pattern.

(c) The flux and the topological charge of the two-dimensional $U(1)$ strip. In a strip of width 2 with the links on the outer boundary fixed, if the links are moved by Langevin, the angles of the plaquettes (the flux) are only handed between neighbours, and unevenness of the flux is smoothed out by diffusion. Because the group is compact, a "wrapping", in which the angle of a single plaquette crosses π, changes the total flux (the topological charge Q of the strip) by one, and conservation is broken. The length at which conservation breaks, ℓ_C, is $3.29,\ 9.88,\ 24.7,\ 53.6,\ 139$ at $\beta = 2, \dots, 6$; the slope of $\ln \ell_C$ is $0.916(21)$, and the ratio to the static length $\xi_{XY} = -1/\ln(I_1(\beta)/I_0(\beta))$ (2.78 to 10.9) opens from 1.2 to 12.8 Reading . When the strip is longer than ℓ_C, relaxation is fixed by the time for Q to move by one, and $\tau(Q)/\xi_{XY}^2$ is about 0.4 at $\beta = 4$ and about 24 at $\beta = 8$ (strip length 32) Reading . Conditions: two-dimensional $U(1)$, width 2, outer boundary frozen, gauge quotient of link Langevin, strip length ≤ 256, τ_{int} as a reading of an upper bound on the gap, fit of the form $A/M^2 + B$. The length at which conservation breaks is discussed in detail in 24-02.

(d) The topological charge in the four-dimensional periodic box. The smoothed topological charge Q cannot move between sectors in the continuum limit; on the lattice it changes only when a small instanton, of the size of the lattice spacing, slips through the lattice (a dislocation). In the periodic box, the autocorrelation time of Q is reported to grow clearly faster than ξ^2 (Del Debbio–Panagopoulos–Vicari 2002 with local heat bath and overrelaxation, Schaefer–Sommer–Virotta 2011 with HMC, Bonanno–Bonati–D'Elia 2021 with local algorithms; if a power, the exponent is about 5, or else exponential in $1/a$, and all three write that power and exponential cannot be distinguished by the data) Cited . The groups in all three are $SU(3)$ and above. For $SU(2)$, the primary sources measuring the dependence of $\tau_{\mathrm{int}}(Q)$ on the lattice spacing number one in the range searched (Abe–Morikawa–Suzuki 2025; HMC, periodic, a small box with side about $2.2/\sqrt{\sigma}$, three lattice spacings), which reports growth close to a^{-2} Cited . The original sources treating $SU(2)$ with heat-bath algorithms were all observations on the side of "freezing not strong enough to measure". The conditions for using this, through (3), in a reading of the upper-bound assumption are collected in § 06.

(e) Flux through a box. An example in which the same pattern appears not through slowness of the dynamics but through the choice of the family of test functions. When measuring the reference for the local Poincaré constant in a four-dimensional box (the value for the massless free gauge field), dividing by gauge transformations at all sites of the box makes the test space functions of the plaquette angles; leaving the gauge transformations at the boundary sites of the box undivided lets a field passing uniformly through the box (a flux whose divergence lies only on two faces) into the test space. The latter value is 2.22 (side 4) to 2.53 (side 12) times the former, which returns roughly to the gauge-fixed value Reading (one-colour Gaussian approximation, $\beta = 1$, canonical graduation, floating point). In a contractible box, the space divided at all sites coincides with the space of functions of the plaquettes Proved here (24-03 § 03). The numbers in this row are numbers for the free field of the reference, not for the target measure. Regarding it as a member of the pattern is Estimate : the flux through the boundary of the box is, seen from inside the box, "a global quantity not reflected in the gap that moves the ratio by one choice of family".

There are three rows that satisfy the conditions of the pattern but were moved below the table of five examples (Table 2). Each has its body text in another chapter, and here we copy only the labels.

Table 2 Three rows moved below Table 1

Example	Quantity outside the family	Slowness	Label	Body text
Conserving dynamics (Gaussian chain)	The total of the field	$\xi^4/2$ for the component at wavenumber $1/\xi$, $C_P = \infty$ for the infinite chain	Proved here	the detour in 2-10
Flux through the plaquettes of the layers of a hierarchical lattice (Gauss approximation, link coordinates)	Flux of each layer	Smallest eigenvalue $\beta \cdot 4\sin^2(\pi/2^{N+1})$. The same slowness even when the layers are decoupled and the static correlation set to 0	Proved here (sector decomposition and the Neumann Laplacian of the two-dimensional dual lattice), Reading (cross-check for $N \le 4$)	24-02
Centre sectors of a small periodic box	Sign of the Polyakov loop	Long autocorrelation under heat bath. A finite-volume artefact that vanishes as the box grows; not the Langevin C_P	Reading	24-08 (questions)

The upper side of the hierarchical-lattice row is checked in Lean (HierLink.lam_le, bundle LatticeGaugeHierarchicalLink; no assumptions: $4\sin^2(\pi/2^{N+1}) \le \pi^2/4^N$. That the smallest eigenvalue equals this expression is outside Lean).

03

Three strengths — logarithmic, power or exponential, infinite

Rearranging the five examples by how the ratio of slowness C_P/ξ^2 (or $\tau(Q)/\xi^2$) grows with the correlation length, the pattern is one but the strengths divide into three.

Table 3 One form, three strengths

Strength	Example	The breaking event and the origin of the correlation length	Effect on the relaxation-time assumption and the upper-bound assumption
Logarithmic	(a) Double well	The same event (kinks) produces the slow quantity and ξ	The band form of the upper-bound assumption fails by a logarithm; the double-well power statement survives Assumption
Power or exponential	(c) the two-dimensional strip, (d) the four-dimensional periodic box	The event that breaks the slow quantity (wrapping, dislocation) and the origin of ξ are different	In (c), $\tau(Q)/\xi_{XY}^2$ grows with β inside the two-dimensional toy Reading . In (d), the upper-bound assumption reads as false under the conditions of § 06 ($SU(3)$ and above)
Infinite	(b) the rotor, and the conserving dynamics and layer flux of Table 2	There is no breaking event	$C_P = \infty$ while $\Delta > 0$ Proved here
(constant factor)	(e) Flux through a box	Not the dynamics but the choice of family	Moves the denominator of the local ratio by a factor of 2.2 to 2.5 Reading

The middle row of Table 3 is the row that matters in four dimensions. In the double well the slow quantity and the correlation length are born from the same kink, so the exponents match and only a logarithmic difference remains. In the two-dimensional strip, the correlation length (the inverse string tension) grows only linearly in β, whereas the wrapping that breaks conservation is an event crossing a barrier of height 2β, and its length grows exponentially in β. Whether the same split occurs in four dimensions, and if so why, is the question of 24-07.

04

Two accounts for the charge of a subregion — the transport coefficient and the coefficient of breaking of conservation

In infinite volume the total topological charge Q is not in L^2, so what goes into (2) of the pattern is the charge of a subregion R. Writing its ratio $\mathrm{Var}/\mathcal{E}$ split into two coefficients makes clear what the local relaxation-time assumption of 24-03 asks.

Setting. Take a translation-invariant lattice measure μ in four-dimensional infinite volume, and the Dirichlet form of link Langevin $\mathcal{E}(f) = \sum_\ell \mathbb{E}|\nabla_\ell f|^2$. The smoothed charge density $q(x)$ is taken to depend only on links within distance r_s of x (made by finitely many smoothings; the Gaussian tail of the Wilson flow is treated separately as a truncation error). The gradient field $J_\ell(x) := \nabla_\ell q(x)$ is 0 for $|x - x_\ell| > r_s$. Put its sum $\bar{J}_\ell := \sum_x J_\ell(x)$ (the gradient of the total charge) and its first moment $D_\ell := \sum_x (x - x_\ell) J_\ell(x)$ (the dipole), and define three coefficients by

$$\chi_t := \sum_y \mathrm{Cov}\big(q(0), q(y)\big), \qquad h := \sum_{\ell \in \text{unit cell}} \mathbb{E}|\bar{J}_\ell|^2, \qquad d := \sum_{\ell \in \text{unit cell}} \mathbb{E}|D_\ell \cdot e_1|^2 \tag{4}$$

Proved here (definitions; by cubic symmetry $\mathbb{E}[D \otimes D]$ is taken to be of the form d times the identity matrix). χ_t is the topological susceptibility in lattice units, h is the density of **breaking of conservation** (in the continuum $\bar{J} = 0$ by topological invariance; only

on the lattice is it non-zero, on dislocation configurations), and d is the coefficient of **transport**. The coefficient of the boundary layer g is the quantity per unit area obtained by subtracting the volume term from $\mathcal{E}$ of the charge cut off by a half-space.

A sharp box. For a box of side R, assuming cluster decay $\sum_z |z|\,|\operatorname{Cov}(q(0),q(z))| < \infty$ Assumption (an assumption equivalent to a gap, so beware of circularity when bringing this back to arguments about the gap),

$$\operatorname{Var}(Q_R) = \chi_t |R| - b|\partial R| + o(|\partial R|), \qquad \mathcal{E}(Q_R) = \sum_\ell \mathbb{E}\Big|\sum_{x\in R} J_\ell(x)\Big|^2 = h|R| + g|\partial R| + O(R^2 r_s^2) \quad (5)$$

(b is the one-sided first moment of the correlation). Note the form of the denominator. The sum over densities of the squared gradients, $\sum_{x\in R}\mathbb{E}|J_\ell(x)|^2$, is not $\mathcal{E}(Q_R)$. It is a quantity that drops the cancellation in the sum $\sum_x J_\ell(x)$ (that in the continuum the total charge does not move locally), and it leaves a volume term even in a perfectly conserving theory. The correct volume term is the density h of $|\sum_x J_\ell(x)|^2$. Inserting $|R| = R^4$ and $|\partial R| = 8R^3$ gives

$$\frac{\chi_t}{h} - \frac{\chi_t R - 8b}{hR + 8g} = \frac{8(g\chi_t + bh)}{h(hR+8g)} \quad (6)$$

Proved here (algebra) Lean LatticeGauge.LocalH.subregion_gap_identity (assumptions: $0 < h$, $0 < hR + 8g$). As $R \to \infty$ the ratio returns to the overall ratio of the periodic box, $\operatorname{Var}(Q)/\mathcal{E}(Q) = \chi_t/h$, and the leading ratio $\chi_t R^4/(hR^4 + gS)$ does not exceed χ_t/h (subregion_ratio_le, assumptions: $0 \le \chi_t$, $0 < h$, $0 \le g$, $0 < R^4$, $0 \le S$).

A smooth window. With $Q_w := \sum_x w(x)q(x)$ and $w(x) = W(x/R)$, we have $\nabla_\ell Q_w = w(x\ell)\bar{J}_\ell + \nabla w(x\ell)\cdot D_\ell + O(R^{-2})$; the cross term vanishes by $\sum \nabla(w^2) = 0$, and

$$\frac{\operatorname{Var}(Q_w)}{\mathcal{E}(Q_w)} = \frac{\chi_t R^2}{hR^2 + d\lambda_W}\big(1 + O(\xi^2/R^2)\big), \qquad \lambda_W := \frac{\|\nabla W\|^2}{\|W\|^2}, \qquad R_*^2 := \frac{d\lambda_W}{h} \quad (7)$$

Assumption (an estimate on top of cluster decay and the definitions of h, d; λ_W is $4\pi^2$ for the Dirichlet window on the unit four-dimensional cube). The two regimes divide at the crossing length R_*. For $R \ll R_*$ the ratio is $\chi_t R^2/(d\lambda_W)$, proportional to R^2; this is the same **transport** type as conserving dynamics (the detour in 2-10). For $R \gg R_*$ the ratio is χ_t/h, the **barrier** type (breaking of conservation). Reading $D_{\text{eff}} := d/\chi_t$ and $r \propto h/\chi_t$ gives $R_*^2 \approx D_{\text{eff}}/r$, a four-dimensional candidate for the "length at which conservation breaks" of 24-02 Estimate .

The two coefficients bear on different questions. The necessary condition for the global upper-bound assumption is $\chi_t/h \lesssim \xi^2$, which asks about the coefficient of breaking of conservation h. The local relaxation-time assumption of 24-03 is the condition obtained by putting (7) into a window of side $c\xi$, and in the regime where h is small it asks about the transport coefficient d (24-03 § 05). The values of h, g and d on the four-dimensional lattice are not in this book.

05

The wall of the first step is a scaled-down version of this pattern

The first step of Part 23 stopped at the common wall: the curvature at one site at the top falls short of convexity by $\frac{1}{4}$ ((11) of 23-09: the diagonal $4 - \frac{1}{4} = \frac{15}{4}$ does not reach the sum of couplings 4). This wall can be read as a 0+1-dimensional scaled-down version of the pattern of this chapter Estimate . The reason is the following correspondence.

Condition (ii) of the pattern was "Q changes only through rare configurations". In the double well, the configurations in which Q (which well) changes are those where the path crosses the plateau at the top, which is exactly where the Hessian of the action has a negative direction. Tools that rely on convexity (Bakry–Émery, Brascamp–Lieb, diagonal dominance) produce a shortfall on these configurations, and the shortfall can be paid only by the rarity (entropy) of the plateau (23-10 currency-mix-up type). That is, $\frac{1}{4}$ is the local face of "on the configurations of the event that breaks a nearly conserved quantity, convexity falls short". What corresponds to this in four dimensions are the configurations where the topological charge changes (dislocations of the size of the lattice spacing, which break the smoothness condition). The prospect of meeting a wall of the same pattern in four dimensions comes from here Estimate .

Table 4 The pattern of this chapter, and its correspondence with the notes and types of breakage of Part 23

Place in Part 23	What is said there	Reading in terms of this chapter's pattern
the note in 23-01 § 05	The topological charge cannot move between sectors, and in the periodic box the upper-bound assumption reads as false under a conjecture. Correcting the relaxation-time assumption into a statement per sector or a local statement is a candidate direction	Row (d) of Table 1. The two corrected forms are in 24-03
the paragraph on the pattern in 23-01 § 06	Relaxation is slower than the gap when a quantity not local in time, not reflected in the gap, is carried by diffusion	Exactly (i)–(iii) of the pattern. This chapter increased the examples to five and divided them by strength

Place in Part 23	What is said there	Reading in terms of this chapter's pattern
23-10 summit-curvature type (one source)	The only source of the shortfall is the curvature at one site at the top	The configurations of the breaking event gather in one place (§ 05)
23-10 currency-mix-up type	Poincaré is the speed of exit; rarity is the currency of entropy. After conditioning, no rarity is left to pay with	The "rare configurations" of (ii) of the pattern cannot be paid for in the Poincaré ratio. Conditioning on sectors moves the rarity into a quantity between sectors (C_{trans} of 24-03)
23-10 transfer type	Another column gets heavier by as much as one got lighter	The sector diffusive-relaxation-time assumption does not remove the obstruction but pushes it into a single quantity between sectors (24-03 § 04)

06

What cannot be said

(a) **Extending the conjecture.** The conjecture "HMC and Langevin are in the same dynamical universality class" is, in the original source (Lüscher–Schaefer 2011), a phrase attached to the observation that with open boundaries for $SU(3)$ the autocorrelation grows as a^{-2}. The same source writes that with periodic boundaries algorithms that integrate Langevin also have difficulty crossing topological sectors, but as grounds it cites original sources on heat bath and HMC, and has no periodic-boundary numbers of its own Cited . Putting row (d) of Table 1 into (3) to say "the upper-bound assumption reads as false in the periodic box" is **a reading that extends the conjecture to the slowness of topology with periodic boundaries** Assumption . Between the Dirichlet form of heat bath and that of Langevin there is only the one-directional one-link comparison $\mathcal{E}_{\text{HB}} \le c_1 \mathcal{E}_L$, and a lower bound on the Langevin C_P does not follow directly from the slowness of heat bath. No primary source stating the relation between HMC and Langevin as a theorem was found in the range searched.

(b) $SU(2)$ **is undecided.** The groups in the original sources are $SU(3)$ and above. There is one primary source for $SU(2)$, and its numbers lie on the side close to a^{-2}. In a rough semiclassical accounting (taking the action of a dislocation as a constant multiple of the one-loop instanton action), with the constant fitted to the numbers for $SU(3)$, $SU(2)$ falls right near the boundary $z = 2$ (24-07). **For $SU(2)$, whether freezing of the topological charge breaks the relaxation-time assumption is not settled.**

(c) **Two dimensions and four.** Row (c) is numbers for a two-dimensional $U(1)$ toy, and two-dimensional $U(1)$ is a trivial theory in the continuum limit. It says nothing about exponents in four dimensions, non-abelian, $\beta \to \infty$.

(d) **Power or exponential.** The three original sources for row (d) themselves write that "power and exponential cannot be distinguished by the data". The "power or exponential" of Table 3 copies that phrasing.

(e) **The pattern is a classification.** That the five examples fall into one pattern is not a theorem. Row (e) is numbers for the free field of the reference, not for the target measure. The coefficients h, g, d of (4)–(7) are only definitions; there are no four-dimensional values.

(f) **The mass gap.** None of the examples breaks the gap. Condition (iii) of the pattern is what guarantees this, and what this chapter says goes only as far as "there is slowness that is not caught in the form of the upper-bound assumption".

07

Table of the five grades and the dependency graph

We put the rows of this chapter into one table in the format of the five grades (23-00 § 01), with the estimate label Estimate added. The symbols of the rows are the same as the nodes of the dependency graph below.

Table 5 Rows on nearly conserved quantities ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Cluster decay $\sum_z \lvert z\rvert \lvert \mathrm{Cov}(q(0), q(z))\rvert < \infty$	Assumption	Premise of (5)	Infinite volume	—
2	A reading that extends the conjecture to the slowness of topology in the periodic box	Assumption	§ 06 (a)	Periodic box	—
3	$\tau(Q)$ faster than ξ^2 in the periodic box for $SU(3)$ and above	Cited	§ 02 (d), three original sources	$SU(3), SU(4), SU(6)$, periodic	—
4	One primary source for $SU(2)$, close to a^{-2}	Cited	§ 02 (d)	HMC, small box, three points	—
5	Overall ratio of the double well $C_P/\xi^2 \approx S_0/\hbar$	Assumption	(14) of 23-04	$\hbar \to 0$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
6	ℓ_c and $\tau(Q)/\xi_{XY}^2$ of the strip	Reading	§ 02 (c)	Two-dimensional $U(1)$, $\beta = 2..6$ (also 8 for $\tau(Q)$)	—
7	Reference with the boundary gauge frozen is 2.22 to 2.53 times	Reading	§ 02 (e)	One-colour Gaussian, $\beta = 1$, side ≤ 12	—
8	$\tau_{\mathrm{int}} \leq CP$ for reversible dynamics	Proved here	(1)	Reversible semigroup	tau_int_le_CP (assumptions: weights ≥ 0, eigenvalues $\geq \lambda_1 > 0$)
9	Free-Particle-on-a-Circle Counterexample: $CP = \infty$ with $\Delta > 0$	Proved here	(23) of 23-01	Circle, all $\hbar$	rotor_ratio_unbounded (assumption: $0 < r$; only the arithmetic of the skeleton)
10	One-site ratio $= \omega_b/m$	Proved here	§ 02 (a) (ground-state transformation)	Schrödinger ψ_0	one_slice_ratio (assumptions: $0 < \varepsilon, m, \omega$; only the algebra of the ratio)
11	The subregion ratio formula (6), and ratio $\leq \chi_t/h$	Proved here	(5)(6) (on top of row 1)	Infinite volume	subregion_gap_identity, subregion_ratio_le (assumptions: positivity)
12	(3): if τ grows as ξ^p ($p > 2$), the upper-bound assumption fails	Proved here	(3)	Sequences with $\xi \to \infty$	not_hup_of_fast_tau (assumptions: $\tau_n \leq C_n$, $c\xi_n^p \leq \tau_n$, $p > 2$)
13	The upper-bound assumption reads as false in the periodic box ($SU(3)$ and above)	Assumption	Reading obtained by putting rows 3 and 2 into row 12	Periodic box	—
14	The five examples are one pattern	Estimate	§ 01– § 03	—	—

```
 1  ASM  cluster decay                                  infinite volume
 2  ASM  conjecture extended to slow topology           periodic box
         in the periodic box
 3  CIT  τ(Q) faster than ξ² for SU(3) and above        periodic box
 4  CIT  one primary source for SU(2) (near a⁻²)        HMC, small box
 5  ASM  overall ratio of the double well ≈ S₀/ħ        ħ → 0
 6  RDG  ℓ_c and τ(Q)/ξ_XY² of the strip                two-dim. U(1)
 7  RDG  factor, reference with frozen boundary gauge   one-colour Gaussian
 8  PRV  τ_int ≤ C_P                                    reversible semigroup
         Lean tau_int_le_CP
 9  PRV  Free-Particle-on-a-Circle Counterexample (23-01)  circle
10  PRV  one-site ratio = ω_b/m                         0+1
11  PRV  subregion ratio (6) ⟸ 1                         infinite volume
         Lean subregion_gap_identity, subregion_ratio_le
12  PRV  if τ ≥ cξ^p (p>2), the upper-bound assumption fails ⟸ 8
         Lean not_hup_of_fast_tau
13  ASM  upper-bound assumption reads false
         in the periodic box ⟸ 2, 3, 12                  SU(3) and up
14  EST  one form ⟸ 5, 6, 7, 9, 10, 13
```

How to check: on each "⟸" row, see by eye that the numbers on the right are smaller than the one on the left. There are no circles. The row of the primary source for $SU(2)$ is a premise of no row. That is how the unsettled reading for $SU(2)$ appears on the graph. Among the leaves of the row "the upper-bound assumption reads as false in the periodic box" is an assumption row (the reading that extends the conjecture to the slowness of topology in the periodic box), and this is why the label of that row is "Assumption".

Strictly, the statement "the upper-bound assumption reads as false in the periodic box" contains the quantity of the conclusion CP, but among its premises, the reading that extends the conjecture and the numbers of the original sources for SU(3) and above do not contain CP. The numbers of the original sources are numbers for Monte Carlo autocorrelations, and the reading that extends the conjecture is a reading about the universality class of dynamics; the only passage that reaches CP is the single one through the row τ_int ≤ C_P.

What this chapter can say

Said	Not said
Proved here $\tau_{\mathrm{int}} \leq CP$ (reversible dynamics), the static lower bound (2), the algebra of (3), the Free-Particle-on-a-Circle Counterexample, the one-site ratio ω_b/m, the subregion ratio formula (6)	A lower bound on $\tau(Q)$ satisfying the second assumption of (3) in four dimensions. Four-dimensional values of h, g, d

Said	Not said
Reading ℓ_c and $\tau(Q)$ for the two-dimensional $U(1)$ strip, the factor for the boundary gauge, the lattice ratios for the double well	All are floating-point and Monte Carlo numbers within conditions, not propositions of upper or lower bounds. The two-dimensional numbers imply nothing for four dimensions
Cited $\tau(Q)$ faster than ξ^2 in the periodic box for $SU(3)$ and above (three original sources), one primary source for $SU(2)$	Power or exponential. Whether $z > 2$ for $SU(2)$
Assumption The upper-bound assumption reads as false in the periodic box ($SU(3)$ and above, on top of a reading that extends a conjecture)	That the upper-bound assumption is false. The same reading for $SU(2)$
Estimate That the five examples are one pattern, and that the wall of the first step is a scaled-down version of this pattern	Turning the classification of the pattern into a theorem

Sources and reproduction

Item	Kind	Source / tools
Autocorrelation of the topological charge in the periodic box ($SU(3)$ and above)	Cited	L. Del Debbio, H. Panagopoulos, E. Vicari, JHEP 08 (2002) 044 § 2; S. Schaefer, R. Sommer, F. Virotta, Nucl. Phys. B 845 (2011) 93 § 4– § 5; C. Bonanno, C. Bonati, M. D'Elia, arXiv:2012.14000 § 1 (text checked)
Primary source for $SU(2)$	Cited	M. Abe, O. Morikawa, H. Suzuki, PTEP 2025 063B03 (checked in the text of § 3.2 and the wording of Figure 6 of arXiv:2501.00286; the numbers in the figure were not read)
Observations for $SU(2)$ with heat-bath algorithms, semiclassical exponents	Cited	A. Athenodorou, M. Teper, JHEP 12 (2021) 082 § 6.2; B. Lucini, M. Teper, JHEP 06 (2001) 050 § 6; B. A. Berg, D. A. Clarke, Phys. Rev. D 97 (2018) 054506 § II (text checked)
The conjecture on the universality class of HMC and Langevin, the sentence on periodic boundaries	Cited	M. Lüscher, S. Schaefer, JHEP 07 (2011) 036 (abstract, § 3.2– § 3.3 and § 5.2 checked in the text)
The algebra of (1), (3), (6)	Lean	LatticeGauge.FourDTrial (tau_int_le_CP, not_hup_of_fast_tau, tau_small_but_CP_large), LatticeGauge.LocalH (subregion_gap_identity, subregion_ratio_le). The bundles guarantee only the algebra
Numbers for the two-dimensional $U(1)$ strip	Reading	Simulation of the gauge quotient of link Langevin (width 2, outer boundary frozen, strip length ≤ 256) and fits of $A/M^2 + B$. Errors by Madras–Sokal
One-site ratio for the double well, factor for the boundary gauge	Reading	Lattice eigenvalue problem for the transfer kernel ($\varepsilon = \frac{1}{2}$), largest eigenvalue of the compression of the lattice Green function to the box (one-colour Gaussian)

Terms used in this chapter — nearly conserved global quantity, breaking of conservation, the transport coefficient d and the coefficient of breaking of conservation h, the crossing length R_*, wrapping, the topological charge of the strip, flux through a box. The notation is in 24-09.

Next chapter to read: 24-02 Choosing the ruler and the coordinates again — is the length to compare with the static correlation length, or the length at which conservation breaks? The two forms that restate the relaxation-time assumption to avoid the pattern are in 24-03.

Choosing the ruler and the coordinates again — the length at which conservation breaks

> This chapter is this book's own investigation. § 06 is a map of open questions.

What this chapter shows — The diffusive-relaxation-time assumption was a sentence comparing the relaxation time CP with the square of one length ξ. We split the rulers that could play the role of that ξ into three: the static correlation length, the length at which conservation breaks ℓ_*, and the time for the global quantity to move by one. On a hierarchical lattice we see that, even for the same equilibrium measure, whether the diffusive-relaxation-time assumption is true changes with the coordinates used to move it (the choice of Dirichlet form) ($C_P = 1/\beta$ in tree coordinates, the square of the side in link coordinates). We state that the length at which conservation breaks, measured at five points on the two-dimensional $U(1)$ strip, is not proportional to the static length; that it comes from the same wrapping as one step of the topological charge; and that splitting the relaxation rate into a "raw rate" and a "static fluctuation" reveals the factor the paper prediction had dropped. Finally we set out a candidate correction that compares with $\max(\xi^2,\ \text{the time for the global quantity to move by one})$, and that on the two-dimensional CP^{N-1} test bed the freezing of the topological charge splits with N while the ratios of local quantities did not (conditionally).

Prerequisite chapters — 24-01 (the pattern, the two accounts of (4)–(7), row (c) of Table 1), 23-01 § 05 ((13) for the relaxation-time assumption and the note on topological sectors), 23-01 § 07 (the bookkeeping length ξ_{RG} and the logarithmic factor), the detour in 2-10 (conserving dynamics), "Differences other than constants" in 2-10 (the choice of variables). The notation is collected in 24-09.

Said first — The level is [Y3]. The numbers in this chapter are numbers for the Gauss approximation on a hierarchical lattice and for two-dimensional models (the $U(1)$ strip, CP^{N-1}). Two-dimensional numbers imply nothing about four-dimensional Yang–Mills. "Correcting what to compare with" is a candidate restatement, shown in no model Estimate . This chapter shows neither the mass gap nor whether the relaxation-time assumption is true.

The order of this chapter

1. Three rulers — the static correlation length, the length at which conservation breaks, the time for the global quantity to move by one ((1)(2))
2. Choosing the coordinates again — tree coordinates and link coordinates, conservation of flux (Table 1, (3)–(5))
3. The two-dimensional $U(1)$ strip — the length at which conservation breaks at five points (Table 2, (6))
4. Two faces of the same wrapping — the flux dipole and the topological charge ((7))
5. Raw rate and static fluctuation — the factor the paper prediction had dropped ((8)(9))
6. Correcting what to compare with — $\max(\xi^2, \tau_{\mathrm{glob}})$ ((10))
7. The two-dimensional CP^{N-1} test bed — freezing splits with N, local ratios do not (Table 3, Table 4)
8. What cannot be said
9. Table of the five grades and the dependency graph — Table 5

01

Three rulers — what length is the ξ of the diffusive-relaxation-time assumption?

The relaxation-time assumption is the power statement $\lim \ln C_P / \ln \xi = 2$ ((13) of 23-01), comparing the relaxation time C_P with the square of one length ξ. Part 23 took this ξ to be the correlation length $1/m_{\mathrm{lat}}$ and used the bookkeeping length ξ_{RG} in its place (23-01 § 07). But in the pattern of 24-01, what fixes the slowness is not the correlation length but "the scale of the events that break the global quantity Q". So we split the candidates for what to compare with into three.

(a) The static correlation length ξ. The length over which two-point correlations decay, fixed by the equilibrium measure alone. In four dimensions $\xi = 1/m_{\mathrm{lat}}$. In the two-dimensional $U(1)$ strip, one row of the strip becomes a one-dimensional chain of rotors in axial gauge, and the length over which Wilson loops decay with area (the inverse string tension),

$$\xi_{XY} = -\frac{1}{\ln\bigl(I_1(\beta)/I_0(\beta)\bigr)} \ \approx\ 2\beta \qquad (\beta \text{ large}) \ (1)$$

plays this role (I_ν is a modified Bessel function. The formula is a standard computation, and this book has not checked the original source Assumption).

(b) The length at which conservation breaks ℓ_*. The global quantity (charge) Q_R in a subregion R changes by two routes: the route of transport, flowing out through the boundary (taking time of order R^2), and the route of conservation breaking inside the region (dislocations, wrapping). The length at which the two balance is ℓ_*. The crossing length R_* of (7) in 24-01 was its static version. A candidate definition written with quantities of the measure alone is

$$S := \sup_R \frac{\mathrm{Var}(Q_R)}{\mathcal{E}(Q_R)}, \qquad \ell_* := \inf\Big\{ R : \ \frac{\mathrm{Var}(Q_R)}{\mathcal{E}(Q_R)} \ \geq \ \frac{S}{2} \Big\} \quad (S = \infty \text{ implies } \ell_* = \infty) \ (2)$$

Estimate . Putting in the estimate $\mathrm{Var}/\mathcal{E} \approx \chi_t R^2/(hR^2 + d\lambda_W)$ of (7) in 24-01 gives $S = \chi_t/h$, and the ratio reaches half of it when $hR^2 = d\lambda_W$, that is, $\ell_* = R_*$ Assumption (on top of the estimate (7)). If conservation never breaks ($h = 0$), then $S = \infty$ and $\ell_* = \infty$, which agrees with the flux of the layers in the Gauss approximation of § 02 below (where conservation holds exactly). Strictly, a definition of the form "the supremum of R with $\mathrm{Var}/\mathcal{E} \leq KR^{2}$", comparing the ratio with R^2, does not fix a length: under the estimate (7) the ratio divided by R^2 decreases in R, so the set of R satisfying the condition is a half-line open upwards or empty, and the supremum is infinite or undefined. (2) avoids this by cutting at half the saturation value.

(c) The time for the global quantity to move by one τ_{glob}. A time, not a length. Its representative is the integrated autocorrelation time $\tau_{\mathrm{int}}(Q)$ of the topological charge Q in the periodic box, and for reversible dynamics $\tau_{\mathrm{int}}(Q) \leq C_P$ ((1) of 24-01).

The three can take different values within the same model. In the 0+1-dimensional double well, the slow quantity (the number of kinks) and the correlation length are born from the same kinks, and the three differ only by logarithms (24-01 Table 3). In the two-dimensional strip of § 03, (a) grows linearly in β while (b)(c) grow exponentially, and the three move apart. Fixing the model does not automatically fix which ruler the relaxation-time assumption is a sentence about.

What this section can say: (2) is a candidate definition, and the four-dimensional value of ℓ_* is not in this book. The ξ_{XY} of (1) is used only as the static length of the strip; it does not stand in for the four-dimensional correlation length.

02

Choosing the coordinates again — tree coordinates and link coordinates

The denominator of the Poincaré constant $C_P = \sup \mathrm{Var}(f)/\mathcal{E}(f)$ is a Dirichlet form, and it is not fixed until one decides the variables whose gradients it measures. We saw in Part 2 that changing variables by a constant factor changes C_P by a constant factor ("Differences other than constants" in 2-10: a factor β depending on whether one measures by link angles or by the field). In this section we give an example where the choice of coordinates changes the **type** of growth of C_P.

Stage. A hierarchical lattice replaces the lattice by nested boxes so that one step of renormalisation can be written down. For gauge fields, Ito (1984) built a three-dimensional lattice on which the recursion of the Migdal–Kadanoff approximation holds exactly, and wrote that the extension to higher dimensions is left to the reader Cited . This book defines for itself a four-dimensional version that coarsens by factors of two (each layer a two-dimensional periodic lattice of side $L = 2^N$, with the bonds of level r identified across 4^r layers), and treats only the Gauss approximation on it ($U(1)$, or linearisation to one component of the Lie algebra of $SU(2)$) Assumption (the four-dimensional construction is this book's extension).

Tree coordinates. Taking the values of the plaquettes (the holonomies at the nodes) as variables, in the Gauss approximation the configuration space is a linear subspace S, the equilibrium measure is the density $e^{-\beta|x|^2/2}$ on S, and the Hessian of the Langevin potential is βI on S.

$$\text{tree coordinates (Gauss approximation)}: \qquad \lambda_1 = \beta, \qquad C_P = \frac{1}{\beta} \qquad (\text{every number of levels } N) \ (3)$$

Proved here Proof. It is an Ornstein–Uhlenbeck process with unit metric and drift matrix βI, so the gap of the generator is β. ∎ Lean `HierLink.tree_no_decay` (bundle LatticeGaugeHierarchicalLink; assumption: $0 < \beta$; only the arithmetic that the ratio is 1). Increasing the number of levels, that is, making the lattice however large, does not slow relaxation in these coordinates. Even if the correlation length grows under the non-abelian flow, the coarse levels are expected to stay close to Haar with $C_P = O(1)$ Estimate , and **in tree coordinates the relaxation-time assumption is false or empty**.

Link coordinates. What the relaxation-time assumption of Part 23 names is Langevin in the link variables (23-01 § 04). The link generator in the Gauss approximation is $\beta\, D\Pi D^*$ (D is the exterior derivative from links to plaquettes, Π the projection of the identification), and its smallest positive eigenvalue is

$$\lambda_{\min}(N) = 4\sin^2\frac{\pi}{2^{N+1}}, \qquad 4^N \lambda_{\min}(N) \ \longrightarrow \ \pi^2 \qquad (N \to \infty) \ (4)$$

Proved here (paper: splitting functions of the layers into sectors by the hierarchical projection, the block operator for sector k becomes the Neumann Laplacian of the two-dimensional dual lattice of side 2^{k+1}, and the minimum is the value for sector $k = N -$

1 with side 2^N) Reading (agreeing with the direct full spectrum for $N \le 2$, and with a separate implementation by sectors for $N = 3$, 4). The gap of the generator is $\beta\lambda_{\min}$, which shrinks inversely with the square of the side. Lean `HierLink.lam_le` (no assumptions: $4\sin^2(\pi/2^{N+1}) \le \pi^2/4^N$; an upper bound in the direction of slowness; that $\lambda_{\min}$ is *equal* to this expression is outside Lean), `HierLink.measured_scaled` (with the interval of the measured $\lambda_{\min}(4)$ as assumption, $4^4\lambda_{\min} \in (9.83, 9.84)$).

Table 1 Coordinates and gap (Gauss approximation, one component of $U(1)$, this book's four-dimensional extension; the gap

N	Side $L = 2^N$	Tree coordinates	Link coordinates (metric of the original lattice)	$4^N\lambda_{\min}$	Link coordinates (metric per class)	Gauge quotient on the four-dimensio
1	2	1	2.000000	8.0000	2.000000	2.0000
2	4	1	0.5857864	9.3726	0.8768944	0.5857864
3	8	1	0.1522409	9.7434	0.2575235	—
4	16	1	0.0384294	9.8379	0.0667663	—

Labels of Table 1: the tree-coordinate column is Proved here by (3). Link coordinates (metric of the original lattice) are Proved here Reading , the closed formula (4) agreeing with computation. The metric per class (moving identified bonds as one variable) is computation only Reading , 1.0–1.74 times the metric of the original lattice. The gauge quotient on the four-dimensional original lattice agrees with the metric of the original lattice to 10^{-9} for $N \le 2$ Reading . The lowest eigenvector lies, for all N, in the same sector (the contrast between the four top-level groups of layers) Reading .

The slowness comes from conservation of flux. When links move, the values of the plaquettes (the flux) are only handed between neighbours.

$$\frac{d}{dt}F = d\Big(\frac{d}{dt}A\Big), \qquad \sum_{p\in\text{block}} F_p = 0 \quad \text{(when the boundary bonds are fixed; Stokes)} \tag{5}$$

Unevenness of the flux is smoothed out only by diffusion, which takes time of the square of the side (the conserving dynamics of Part 2, the model B type, the detour in 2-10). The decisive point is the last column of Table 1. If the layers are decoupled and every bond made an independent variable per layer, then statically the plaquettes are almost independent (only the constraint on the total) and the correlation length is 0, yet the smallest eigenvalue is $4\sin^2(\pi/L)$ and the same L^{-2} type remains Proved here . Hence **this slowness does not come from the correlation length**. The static correlation of plaquettes in the Gauss approximation decays masslessly, of order $-\beta^{-1}R^{-4}$; β does not flow (the Gauss-approximation face of the coupling being dimensionless in four dimensions, 24-07), and the correlation length is only cut off by the volume Proved here .

Consequence. The relaxation-time assumption is not a sentence about the measure alone but **a sentence about the pair (measure, Dirichlet form)** Estimate . Tree coordinates and link coordinates have the same equilibrium measure, yet in the former C_P does not depend on the number of levels, and in the latter it grows as the square of the side. In Part 2, "C_P changes by a constant factor with the coordinates" did not change the power of the growth, but here the power itself (N^0 versus 4^N) changes. The relaxation-time assumption of Part 23 names Langevin in the link variables, so it already meets this point. Reinterpreting heat-bath or overrelaxation numbers as C_P of link Langevin is an assumption (2-10 § "Do not identify heat bath with Langevin"). And even in link coordinates, conservation holds exactly in the Gauss approximation, so ℓ_* of (2) is infinite, and comparing "$C_P \propto L^2$" with the correlation length has no meaning. What should be compared is a quantity outside the Gauss approximation: at what length the compactness or non-abelianness of the group breaks the conservation of flux. The next section is the first measurement of it.

What this section can say: hierarchical models lack translation and rotation invariance, and imply nothing about four-dimensional Yang–Mills. The four-dimensional construction is this book's extension of the three-dimensional construction of the original source. The numbers are numbers for the Gauss approximation, with $N \le 4$.

03

The two-dimensional $U(1)$ strip — the length at which conservation breaks at five points

Model. We cut one layer of the previous section into a narrow strip. Plaquettes (x, y), $x = 0, \dots, M-1$, $y = 0, 1$ (width $W = 2$); all links on the outer boundary are fixed at angle 0, and the inner links are moved by the Langevin dynamics $d\theta_\ell = -\partial_\ell S\, dt + \sqrt{2}\, dB_\ell$ for the action $S = -\beta\sum_p \cos\varphi_p$ (unit metric). Gauge transformations at inner vertices do not change the plaquette angles φ_p, so on gauge-invariant functions the link generator reduces to a diffusion on φ, whose metric is the Neumann Laplacian of the dual graph of the plaquettes Proved here . The equilibrium measure is the product of independent $e^{\beta\cos\varphi}$ per plaquette conditioned on $\sum_p \varphi_p \equiv 0 \pmod{2\pi}$, and there is almost no static correlation between plaquettes.

In the Gauss approximation conservation holds exactly. With $\sin\varphi \to \varphi$, the smallest gap is $\beta \cdot 4\sin^2(\pi/2M) \approx \beta\pi^2/M^2$, inversely proportional to the square of the strip length Proved here (Lean `ConsLen.neumann_gap_lower` , `ConsLen.neumann_gap_upper` , bundle LatticeGaugeConservationLength; assumption: $1 \le M$; $4/M^2 \le 4\sin^2(\pi/2M) \le \pi^2/M^2$. That the gap equals this expression is outside Lean).

If the group is compact, conservation breaks. An event in which the angle of a single plaquette crosses $\pm\pi$ (a wrapping) changes by one the sum of the angles reduced to $(-\pi, \pi]$,

$$Q := \frac{1}{2\pi}\sum_p [\varphi_p] \;\in\; \mathbb{Z} \quad (6)$$

(the topological charge of the strip, an integer because the outer boundary is fixed). A wrapping is a rare event crossing a barrier of height 2β. We measure two quantities: $\lambda_X := 1/\tau_{\mathrm{int}}(X)$ for the flux dipole $X = \sum_p w_p[\varphi_p]$ ($w_p = \cos(\pi(x+\frac{1}{2})/M)$, the slowest mode of the Gauss approximation), and $\lambda_Q := 1/\tau_{\mathrm{int}}(Q)$ for the topological charge. For reversible dynamics $\tau_{\mathrm{int}} \le C_P$ ((1) of 24-01), so both are read as **upper bounds on the gap**. The length at which conservation breaks is defined by fitting $\lambda_X = A/M^2 + B$ and setting $\ell_c := \sqrt{A/B}$ (the length at which the diffusion rate A/M^2 equals the saturation value B).

Table 2 Five points for the two-dimensional $U(1)$ strip (width 2, outer boundary frozen, gauge quotient of link Langevin, strip length $M \le 256$; λ_Q is a weighted mean over M with $\mathrm{Var}\, Q \ge 0.5$; $E := \mathbb{E}\,[\varphi_p]^2$; errors by Madras–Sokal and the covariance of the fit)

β	ξ_{XY}	ℓ_c	Paper estimate	λ_Q	$\tau(Q)$	$\pi/\sqrt{E\lambda_Q}$	$\rho = \ell_c/\frac{\pi}{\sqrt{E\lambda_Q}}$	ℓ_c/ξ_{XY}
2	2.78	3.29(20)	3.53	0.769(21)	1.30	4.10	0.80(5)	1.2
3	4.75	9.88(39)	8.28	0.224(13)	4.46	10.0	0.985(50)	2.1
4	6.82	24.7(9)	19.6	0.0579(29)	17.3	23.9	1.03(5)	3.6
5	8.87	53.6(3.1)	47.7	0.0127(10)	79	58.5	0.92(6)	6.0
6	10.90	139(14)	118	0.00276(32)	362	139	1.00(11)	12.8

Every label in Table 2 is Reading . "Paper estimate" is the value, by numerical integration, of a capacity estimate with a test function (the length at which the two terms of an upper bound become equal, using a test function whose return from wrapping is optimised by capacity; for large β, $e^{\beta}\sqrt{\pi/(6\beta)}$) Assumption (an estimate on top of the independence approximation), and is of the same order as the measured ℓ_c.

Slopes. With a weighted straight line over $\beta = 2, \dots, 6$:

- The slope of $\ln \ell_c$ is $a = 0.916(21)$. Split into intervals: $0.988(34)$ for $\beta = 2$ to 4, $0.834(46)$ for 4 to 6.
- The slope of $-\ln \lambda_Q$ is $b = 1.355(18)$. Split into intervals: $1.286(28)$ and $1.520(57)$ (difference 3.7σ).
- The slope of ln of the combination $\pi/\sqrt{E\lambda_Q}$ is $0.884(9)$, and 0.879 to 0.884 on every interval.

Reading Lean `ConsExp.lc_slope_2_6` (bundle LatticeGaugeConservationExponent; assumptions: $\ell_c(2) \in [3.09, 3.49]$, $\ell_c(6) \in [125, 153]$, the measured 1σ intervals; conclusion: $a \in [3/4, 1]$), `ConsExp.lamQ_slope_2_6` (with the 1σ intervals of $\lambda_Q(2)$ and $\lambda_Q(6)$ as assumptions, $b \in [5/4, 3/2]$). The bundles guarantee only interval arithmetic.

Not proportional to the static length. The static length $\xi_{XY} \approx 2\beta$ is linear in β and grows only about fourfold over the five points. The length at which conservation breaks, $\ell_c \propto e^{0.92\beta}$, grows 42-fold, and the ratio ℓ_c/ξ_{XY} opens from 1.2 to 12.8 Reading . Of the three rulers of § 01, (a) and (b) move apart exponentially within the same model. When the strip is longer than ℓ_c, relaxation stops depending on the strip length and is fixed by the time for Q to move by one.

Conditions: two-dimensional $U(1)$, width 2, outer boundary frozen, gauge quotient of link Langevin with unit metric, $\beta = 2, \dots, 6$, strip length $M \le 256$, τ_{int} as a reading of an upper bound on the gap, fit of the form $A/M^2 + B$ (for $\beta = 2$, $\chi^2/\mathrm{dof} = 3.3$, and the form does not fit). Nothing is said about the exponent as $\beta \to \infty$.

04

Two faces of the same wrapping — the flux dipole and the topological charge

The ρ column of Table 2 shows that the two quantities are one process. The relaxation of the flux dipole X falls as M^{-2} at the rate $\mu_1(M)/E$ ($\mu_1 \approx \pi^2/M^2$), with the Gauss stiffness replaced by $1/E$, while the relaxation rate λ_Q of the topological charge Q does not depend on the strip length. The length at which the time for diffusion to cross a length ℓ equals the time for Q to move by one is

$$\frac{\pi^2}{E\,\ell^2} = \lambda_Q \quad\Longleftrightarrow\quad \ell = \frac{\pi}{\sqrt{E\,\lambda_Q}} \quad (7)$$

Proved here (algebra) Lean `ConsExp.lc_eq_iff` (assumptions: $0 < E$, $0 < \lambda$, $0 < \ell$), and its ratio ρ to the measured ℓ_c is 0.80 to 1.03 over the five points Reading . **The length at which conservation breaks reads as the distance the flux crosses by diffusion while the topological charge moves by one.** The flux dipole departing from M^{-2} and the topological charge relaxing are two faces of the same wrapping.

A single point at $\beta = 8$ supplements this reading. Up to strip length 64, X keeps M^{-2} (ratio to the Gauss stiffness 0.86 to 1.03), but at strip length 32, $\tau_{\mathrm{int}}(Q) = 5.3(1.7) \times 10^3$ (run length 10^6, $\mathrm{Var}\, Q = 0.145$), and a slow mode at about $1/380$ of the value for X appears Reading . Because $\mathrm{Var}\, Q$ is small (a sector where $Q \neq 0$ is rare), this point is not included in the slopes.

What this section can say: ℓ_c is the length of the bend in the relaxation of X, not the length of a bend in the gap itself (including Q, the upper bound on the gap at $\beta = 4$ already saturates at $M \approx 8$ to 16). We do not claim that the topological charge of the strip is the same thing as the four-dimensional topological charge.

05

Raw rate and static fluctuation — the factor the paper prediction had dropped

The paper prediction before measuring was "$\lambda_Q \propto \beta e^{-2\beta}$" (the Kramers rate for the barrier 2β), so the slope b should have been near 2. The measured $b = 1.355(18)$ is gentler, and moreover split before and after β (bottom of Table 2). To find where the split comes from, we divide the relaxation rate into two factors. Q can be regarded as a chain of ± 1 jumps with a restoring force, so from the total jump rate $WM\nu$ (ν is the rate of wrapping per plaquette) and the variance $\mathrm{Var}\, Q \approx WM\, E/(4\pi^2)$ comes

$$\lambda_Q = \kappa \cdot \frac{2\pi^2\, \nu}{E}, \qquad E = \mathbb{E}\,[\varphi_p]^2 \quad (8)$$

this form Assumption (the chain picture; κ is a correction for the raw count. It is a different quantity from the comparison constant κ of the One-Sided Bridge Proposition (Theorem 3.1 of the paper) (24-09)). The ν in the numerator is the **raw rate** (a quantity of the dynamics), and the E in the denominator is the **static fluctuation** (a quantity of the measure). Taking logarithms gives

$$\Delta \ln \lambda_Q = \Delta \ln \nu - \Delta \ln E \quad (9)$$

Proved here (algebra) Lean `ConsExp.lamQ_log_split` (assumptions: $0 < \kappa$, each quantity positive).

Counting the raw number of wrappings ν_{raw} (the number of times $\pm\pi$ is crossed), we found the following Reading .

- $\kappa = \lambda_Q E/(2\pi^2 \nu_{\mathrm{raw}})$ is constant at $0.167,\ 0.165,\ 0.158,\ 0.167$ for $\beta = 3, 4, 5, 6$ (the raw count also counts back-and-forth trips near π, so the effective crossings are about $1/6$ of it).
- The slope of the raw rate $-d \ln \nu_{\mathrm{raw}}/d\beta$ is $1.73,\ 1.75,\ 1.79$ on the intervals $3 \to 4,\ 4 \to 5,\ 5 \to 6$, agreeing within 1–3% with the slope $2 - 1/\beta$ of the Kramers rate $\beta e^{-2\beta}$ ($1.71,\ 1.78,\ 1.82$ at the midpoints of the intervals).
- The slope of the static factor $\ln(1/E)$ is $0.38,\ 0.27,\ 0.21$, decreasing with β (approaching the Gauss value $1/E \approx \beta$).

From (9), $b = (\text{slope of the raw rate}) - (\ln(1/E)\ \text{slope})$, and since the latter decreases with β, b increases with β. This is what the split really is. **The paper prediction had dropped the static factor $1/E$.** The corrected form is $\lambda_Q \propto \beta e^{-2\beta}/E$ (for large β, $\propto \beta^2 e^{-2\beta}$, $b \to 2 - 2/\beta$), in the same direction as the measured b Estimate . In the combination $\pi/\sqrt{E\lambda_Q}$, E cancels and only $\nu^{-1/2}$ remains, so the slope is half the slope of the raw rate (about 0.88) and constant over all intervals. That is the third slope at the bottom of Table 2.

Strictly, this split is a check added after seeing the split in the data; it was not among the pre-registered predictions. Also, nothing can be said from these five points about the exponent as $\beta \to \infty$ (whether $b \to 2$).

What to keep as a view Estimate : the relaxation rate of a global quantity splits into the product of "the rate of events" (dynamics, barrier) and "the static fluctuation that measures how much one event moves the global quantity" (measure). The one-loop accounting that the original sources write for the four-dimensional topological charge (24-07 § 02) also counts only the exponent of the density of events. Whether the static factor moves the reading of the exponent has not been checked in four dimensions.

06

Correcting what to compare with — $\max(\xi^2, \tau_{\mathrm{glob}})$

In the strip, measuring $\tau(Q)$ with the static length ξ_{XY} as the ruler, $\tau(Q)/\xi_{XY}^2$ is about 0.4 at $\beta = 4$ and about 24 at $\beta = 8$ (strip length 32) Reading . Since $C_P \geq \tau_{\mathrm{int}}(Q)$, C_P/ξ_{XY}^2 also grows with β. "$C_P \lesssim \xi^2$", with only the static correlation length as the ruler, tends to fail with β inside this toy (row (c) of the pattern in 24-01). So we put forward, as a candidate, a form that adds the time of the global quantity to what is compared with.

$$C_P \asymp \max\left(\xi^2,\ \tau_{\mathrm{glob}}\right), \qquad \tau_{\mathrm{glob}} := \left(\text{the time for the global quantity to move by one}\right) \quad (10)$$

Estimate . This is one direction for correcting the relaxation-time assumption. The other direction is to restrict the side of test functions to families local in time and space (24-03 § 02), where, as long as the box is smaller than ℓ_*, the time of the global quantity is expected not to enter (§ 07).

(10) has three weaknesses. (i) If τ_{glob} is defined by $\tau_{\mathrm{int}}(Q)$ of the same dynamics, (10) is close to a restatement of $\tau_{\mathrm{int}}(Q) \le C_P$ and has no content. To give it content, τ_{glob} must be fixed independently of the dynamics. Candidates are the split of § 05 (barrier rate and static fluctuation, $\tau_{\mathrm{glob}} \approx E/(2\pi^2\kappa\nu)$), or making the diffusion time ℓ_*^2/D from the static ℓ_* of (2) Estimate . (ii) The strip numbers see only the lower side of (10), $C_P \ge \tau_{\mathrm{glob}}$, and not a single number bounds C_P from above. At $\beta = 4$, $\tau(Q) \approx 17$ against $\xi_{XY}^2 \approx 46$, and whether C_P is on the side of ξ_{XY}^2 has not been measured. (iii) In four dimensions the candidate for τ_{glob} is the topological charge in the periodic box, whose measurements are the heat-bath and HMC numbers of the original sources, and reinterpreting them for Langevin is an assumption (24-01 § 06).

What this section can say: (10) is a candidate restatement, shown in no model. Nor do we propose replacing the relaxation-time assumption of Part 23 with (10). The order of the account is: because the relaxation-time assumption was set up sharply, we could name the places where a single thing to compare with is not enough.

07

The two-dimensional CP^{N-1} test bed — freezing splits with N, local ratios do not

Why this model. The two-dimensional CP^{N-1} model shares two points with four-dimensional Yang–Mills — a dimensionless coupling, and instantons with no scale (24-07) — and lets one vary N, which plays the role of the size of the group. There are primary sources reporting that in the periodic box the autocorrelation of the topological mode grows like $\exp(1.72\,\xi)$ for $\xi = 2$ to 6 ($N = 21$, Hasenbusch 2017), and worsens with N (Del Debbio–Manca–Vicari 2004, Berni–Bonanno–D'Elia 2019) Cited . It is also reported that with open boundaries the growth returns to $z \approx 1$ Cited .

Setting. The action is Hasenbusch's form $S = -\beta N \sum_{x,\mu}(\bar{z}_{x+\mu} z_x \lambda_{x,\mu} + \mathrm{c.c.} - 2)$ ($z_x \in \mathbb{C}^N$, $|z_x| = 1$, $\lambda \in U(1)$), and the topological charge is the sum of plaquette angles $Q = \frac{1}{2\pi}\sum \theta_{\mathrm{plaq}}$. The dynamics is heat bath (sites and links) mixed with n_{ov} overrelaxation steps; it is **not Langevin**. The autocorrelations measured here are in units of heat-bath sweeps, and reinterpreting them as the Langevin C_P is an assumption (2-10). The local quantities are three for a box of side R: the energy E_B, the topological charge of the box q_B, and the "magnetisation" of the box m_B.

The two-dimensional reference differs in character from four dimensions. The reference of the ratio form K_{rel} of 24-03 (the box value for the massless free field) diverges in two dimensions not only with the box size R but logarithmically in the lattice side L (the constant part of the continuum kernel $-\frac{1}{2\pi}\ln|z|$ sits on the box average) Proved here . Removing the box average makes it finite, about 0.0548 by extrapolation in $1/R^2$ Reading . Moreover, even for mean-zero quantities the free-field Langevin autocorrelation grows with $\ln L$ Reading (in a box with $R = 16$, from 36 to 119 as the lattice side goes from 64 to 1024). So the ratios of this test bed are **ratios of autocorrelations with the lattice side fixed at $L = 64$**, $K_{\mathrm{rel}}(c;\mathrm{qty}) := \tau(\mathrm{qty}_B(c\xi))/\tau^{\mathrm{free}}(D_B(c\xi))$ (D_B is the first Neumann mode of the free-field box, with the same algorithm and the same L) and $K_{\mathrm{glob}} := \tau(Q^2)/\tau^{\mathrm{free}}(D_B(4\xi))$, quantities different from the static ratios of 24-03.

Positive control. With two runs (run length 70000 sweeps) at $N = 21$, $\beta = 0.70$, $L = 44$, $n_{\mathrm{ov}} = 6$, it passed with $\tau_{\mathrm{int}}(Q^2) = 160(26)$ (0.80 times 199(18) of Hasenbusch's Table II, inside the band [139, 259] fixed in advance), an energy difference of -9×10^{-6}, and a ξ difference of -0.1% Reading .

Table 3 Core pairs and pure heat-bath pairs ($L = 64$, periodic, $\xi \approx 4$; K is three values at $c = 1, 2, 4$; one run each, some runs shortened from the registered length)

N	β	n_{ov}	ξ_{2nd}	$\tau(Q^2)$	Changes of Q / measurements	K_{glob}	$K_{\mathrm{rel}}(E_B)$	$K_{\mathrm{rel}}(m_B)$	$K_{\mathrm{rel}}(q_B)$
21	0.70	6	3.738(14)	154(46)	527 / 30600	Undefined (the reference τ^{free} is 0.33–0.54)			
2	1.140	6	4.25(8)	0.50	93123 / 99000	Same as above			
21	0.70	0	3.76(6)	3874(2810)	120 / 41400	55	0.32, 0.11, 0.038	0.68, 0.59, 0.51	1.71, 1.74, 2.27
10	0.80	0	4.62(8)	70.9(10.3)	24347 / 81000	1.01	0.35, 0.13, 0.049	0.83, 0.78, 0.57	1.51, 1.11, 0.83
2	1.140	0	4.89(24)	0.73	133493 / 144000	0.010	0.26, 0.084, 0.030	0.66, 0.41, 0.25	0.21, 0.062, 0.022

Every label in Table 3 is Reading . $\tau(Q^2)$ for $N = 21$, $n_{\mathrm{ov}} = 0$ has a run length of only about 11τ and is treated as a lower bound.

What can be read.

1. **The freezing of the topological charge splits with N.** At $\xi \approx 4$, $\tau(Q^2)$ for $N = 21$ against $N = 2$ is 154 against 0.50 with overrelaxation (about 300 times), and 3874 against 0.73 with pure heat bath (about 5000 times; the former treated as a lower bound) Reading .
2. **The ratios of local quantities are small regardless of N.** For the pure heat-bath pairs, K_{rel} for the energy and the magnetisation is almost the same at $N = 2, 10, 21$ (0.03–0.35 for E_B, 0.25–0.83 for m_B), and the ratios of one-step Rayleigh quotients (the ratio of $1/(1-\rho(1))$ to its free-field value) are all 1.1 or below Reading . The verdict at the freezing point was on the side of "freezing of topology does not break the local ratios up to box 4ξ". Conditions: $L = 64$, a single point $\xi \approx 4$, one run each, a post-hoc cell (below). There is only one freezing point, ($N = 21$, $\xi \approx 4$), and $K_{\mathrm{glob}} = 55$ has a 1σ width of 15–95 from the error of one run, so whether it exceeds the decision threshold 30 is within that error. **It is a weak reading at one point.**

3. **ℓ_* grows with N.** Table 4 lists the length at which the autocorrelation of the topological charge of the box catches up with the global value.

Table 4 Autocorrelation of the topological charge of the box $\tau(q_B(R))$ and the global $\tau(Q)$ (pure heat bath, $L = 64$)

N	$R = 2$	4	8	16	32	$\tau(Q)$	$\tau(q_B(32))/\tau(Q)$	Exponent for $R \to 2R$
21	3.08	9.82	38.9	160	738	4071	0.18	1.67, 1.98, 2.04, 2.21
10	3.01	8.72	24.7	58.4	113	222	0.51	1.54, 1.50, 1.24, 0.95
2	0.98	1.21	1.39	1.51	1.53	1.55	0.99	0.30, 0.20, 0.12, 0.02

Reading For $N = 2$ the autocorrelation of the box charge saturates at the global value at $R \approx 8$ to 16; for $N = 10$ it begins to bend at $R = 32$; for $N = 21$ it still grows as R^2 at $R = 32$. The box charge takes time R^2 through transport across the boundary (the route (b) of § 01), and the length at which it catches up with the global value is the reading of ℓ_*. **In this model and this range, freezing does not break the ratios for boxes smaller than ℓ_*, and rides only on quantities on scales beyond ℓ_*** Estimate . That in Table 3 only $K_{\mathrm{rel}}(q_B)$ of the box charge splits with N ($0.022 \to 0.83 \to 2.27$ at $c = 4$) we read as a reflection of this difference in catching-up length. In the language of 24-03, the local diffusive-relaxation-time assumption asks about the transport coefficient d and the global relaxation-time assumption about the coefficient of breaking of conservation h (24-03 § 05), so this is one two-dimensional face of the prospect that local ratios are unaffected by freezing. It also agrees in direction with the conclusion of 24-04 that what the one-sided bridge needs is the local side (24-04), but the ratios here are ratios of autocorrelations, not the ratio of the bridge's threshold (a static ratio).

What should be disclosed.

- **The registered verdict could not be defined.** The verdict registered in advance was the ratio for dynamics mixed with overrelaxation ($n_{\mathrm{ov}} > 0$), but overrelaxation makes the autocorrelation of the free-field reference oscillate, and $\tau^{\mathrm{free}}(D_B)$ came out at 0.33 to 0.54, below 1. By the rule that a ratio with a denominator below 2 is not defined (added before the main runs), all the registered ratios were "undefined". Readings 2 and 3 above are numbers for pure heat-bath pairs added as a separate cell after the registration was fixed.
- **The dynamical exponent $z_Q(N)$ cannot be read.** For each N there is only one point in ξ. The decision script printed a negative exponent for $N = 21$, but the second point was an artefact in which the script returned a small autocorrelation for a series where the topological charge never changed (only rounding error), and it was discarded.
- **The topological charge for $N = 2$ carries an ultraviolet label.** For $N = 2$ the two definitions of the topological susceptibility (the sum of plaquette angles and the triangle definition of Berg–Lüscher) differed by 13–14%, and for $N = 10, 21$ by 0.1–3% Reading . We understand that the lattice topological susceptibility for $N = 2$ (the same model as $O(3)$) is known to have an ultraviolet problem Assumption (from memory; the original source was not obtained). $N = 2$ may fail to freeze because it is on the side where the lattice scale (the ultraviolet) fixes Q Estimate . If so, the speed for $N = 2$ is not "weak freezing" but a shadow of "Q not being a topological charge in the continuum sense", and the analogy with $SU(2)$ (24-07) carries this reservation.
- Some runs were shortened from the registered length (because of an upper limit on the time of one run), and the statistics are rough. The negative control with open boundaries was not run.

What this section can say: numbers for the two-dimensional CP^{N-1}, this action and these algorithms (heat bath and overrelaxation), $L = 64$, a single point $\xi \approx 4$; nothing is said about four dimensions, $SU(N)$, the relaxation-time assumption or the mass gap.

08

What cannot be said

(a) Two dimensions and four. The numbers of § 03– § 07 are numbers for the two-dimensional $U(1)$ strip and the two-dimensional CP^{N-1}. Two-dimensional $U(1)$ is a trivial theory in the continuum limit, and neither set of numbers implies anything about exponents in four dimensions, non-abelian, $\beta \to \infty$.

(b) Readings of upper bounds. λ_X and λ_Q are readings as upper bounds on the gap (lower bounds on C_P), not the gap itself. ℓ_c is the length of the bend in the relaxation of the flux dipole and depends on the form of the fit $A/M^2 + B$.

(c) Candidates. The ℓ_* of (2) and the thing to compare with of (10) are candidates, shown in no model. The strip numbers see only the lower side of (10).

(d) The hierarchical lattice. The results are numbers for the Gauss approximation, and the four-dimensional construction is this book's extension of the three-dimensional construction of the original source. Hierarchical models lack translation and rotation invariance and imply nothing for four-dimensional Yang–Mills.

(e) Conditions of the test bed. The CP^{N-1} ratios are ratios of autocorrelations of heat-bath dynamics, conditional quantities with $L = 64$ fixed, and different from the static ratios of 24-03. For each N there is one point in ξ, one freezing point, and the verdict on local

ratios is a weak reading at one point on a cell added after registration; the registered verdict could not be defined. $z_Q(N)$ has not been read.

(f) Afterthought. The split into raw rate and static fluctuation in § 05 is a check added after seeing the split.

(g) The analogy with $SU(2)$. Using the split of freezing with N as a two-dimensional analogy of the borderline reading for $SU(2)$ is an estimate, and $N = 2$ carries an ultraviolet label (24-07).

09

Table of the five grades and the dependency graph

Table 5 Rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	The Gauss approximation (abelian linearisation), and the four-dimensional extension of the hierarchical lattice	Assumption	§ 02	$N \le 4$	—
2	$\lambda_1 = \beta$ in tree coordinates ((3))	Proved here	§ 02	Gauss approximation, all N	`HierLink.tree_no_decay` (assumption: $0 < \beta$; arithmetic of the ratio)
3	$\lambda_{\min}(N) = 4\sin^2(\pi/2^{N+1})$ in link coordinates ((4))	Proved here	Sector decomposition of § 02	Gauss approximation	`HierLink.lam_le` (no assumptions; only the direction of the upper bound)
4	Cross-check for $N \le 4$, metric per class, gauge quotient on the original lattice	Reading	Table 1	$N \le 4$ (gauge quotient $N \le 2$)	`HierLink.measured_scaled` (measured interval as assumption)
5	L^{-2} remains even with the layers decoupled (conservation of flux)	Proved here	(5), Table 1	Gauss approximation	—
6	The relaxation-time assumption is a sentence about the pair (measure, Dirichlet form)	Estimate	§ 02	—	—
7	Gauss gap of the strip $\beta \cdot 4\sin^2(\pi/2M)$	Proved here	§ 03	Width 2, outer boundary frozen	`ConsLen.neumann_gap_lower/upper` (assumption: $1 \le M$)
8	$\tau_{\mathrm{int}} \le CP$	Proved here	(1) of 24-01	Reversible dynamics	`LatticeGauge.FourDTrial.tau_int_le_CP` (assumptions: weights ≥ 0, eigenvalues $\ge \lambda_1 > 0$)
9	Five points of the strip: ℓ_c, λ_Q, slopes a, b	Reading	Table 2	Conditions of § 03	`ConsExp.lc_slope_2_6`, `lamQ_slope_2_6` (1σ intervals as assumptions)
10	Algebra of (7)	Proved here	§ 04	—	`ConsExp.lc_eq_iff` (assumptions: $0 < E, \lambda, \ell$)
11	$\rho = 0.80$ to 1.03 (two faces)	Reading	Table 2	Five points	—
12	Algebra of the split (9)	Proved here	§ 05	—	`ConsExp.lamQ_log_split` (assumptions: $0 < \kappa$, positivity)
13	The raw rate within 1–3% of the Kramers slope, κ constant (added afterwards)	Reading	§ 05	$\beta = 3..6$	—
14	The paper prediction had dropped $1/E$	Estimate	§ 05	Strip	—
15	$\tau(Q)/\xi_{XY}^2$ goes $0.4 \to 24$	Reading	§ 06	$\beta = 4, 8$	—
16	Compare with $\max(\xi^2, \tau_{\mathrm{glob}})$ ((10))	Estimate	§ 06	—	—
17	Passing the positive control for CP^{N-1}	Reading	§ 07	$N = 21$, $L = 44$	—
18	$\tau(Q^2)$ splits with N	Reading	Table 3	$L = 64$, $\xi \approx 4$	—
19	Local ratios are small regardless of N	Reading	Table 3	Post-hoc cell, one point, weak	—
20	The length at which the box charge catches up with the global one grows with N	Reading	Table 4	$L = 64$	—
21	$N = 2$ may fail to freeze because it is on the side where the ultraviolet fixes Q	Estimate	§ 07	—	—
22	Freezing rides only on quantities on scales beyond ℓ_*	Estimate	§ 07	This model, this range	—

What the machine-checking column guarantees is only arithmetic, algebra and comparisons of intervals. That eigenvalues equal closed formulas, the spectral representation of $\tau_{\mathrm{int}} \le CP$, the fits and the Monte Carlo lie outside Lean (23-13).

Dependency graph (premise numbers are smaller than conclusion numbers):

```
 1  ASM  Gauss approx., 4D extension of hier. lattice      N ≤ 4
 2  PRV  λ₁ = β in tree coordinates ⟸ 1                     Lean tree_no_decay
 3  PRV  λ_min(N) in link coordinates ⟸ 1                   Lean lam_le (upper bound)
 4  RDG  cross-check for N ≤ 4 ⟸ 3                          Lean measured_scaled
 5  PRV  L⁻² even with layers decoupled ⟸ 1
 6  EST  the relaxation-time assumption is a sentence
         about a pair (measure, form) ⟸ 2, 3, 5
 7  PRV  Gauss gap of the strip                             Lean neumann_gap_lower/upper
 8  PRV  τ_int ≤ C_P ((1) of 24-01)                         Lean tau_int_le_CP
 9  RDG  five points of the strip ℓ_c, λ_Q, a, b ⟸ 7, 8     Lean lc_slope_2_6, lamQ_slope_2_6
10  PRV  algebra of (7)                                     Lean lc_eq_iff
11  RDG  ρ = 0.80–1.03 ⟸ 9, 10
12  PRV  the split (9)                                      Lean lamQ_log_split
13  RDG  raw rate and κ (added afterwards) ⟸ 9
14  EST  the paper prediction dropped 1/E ⟸ 12, 13
15  RDG  τ(Q)/ξ_XY² goes 0.4 → 24 ⟸ 8, 9
16  EST  compare with max(ξ², τ_glob) ⟸ 6, 11, 15
17  RDG  positive control for CP^{N−1}
18  RDG  τ(Q²) splits with N ⟸ 17
19  RDG  local ratios small regardless of N ⟸ 17           post-hoc cell, one point
20  RDG  catching-up length grows with N ⟸ 17
21  EST  is N = 2 on the side where UV fixes Q? ⟸ 18
22  EST  freezing rides only on quantities beyond ℓ_* ⟸ 19, 20
```

How to check: see by eye that on every "⟸" the numbers on the right are smaller than the number on the left. There are no circles. The conclusion nodes of this chapter (the relaxation-time assumption is a sentence about a pair of measure and form; the correction of what to compare with; freezing rides only on quantities beyond the length at which conservation breaks) are all "Estimate", and there is neither a four-dimensional node nor a node for the mass gap. The two-dimensional readings (the nodes from the five points of the strip onwards) are premises of none of the four-dimensional rows.

What this chapter can say

Said	Not said
Proved here On the hierarchical lattice in the Gauss approximation, the gap in tree coordinates is β and the gap in link coordinates is $\beta \cdot 4\sin^2(\pi/2^{N+1})$, and the slowness comes from conservation of flux	The gap of non-linear, non-abelian hierarchical lattices. Implications for four-dimensional Yang–Mills
Reading The length at which conservation breaks, ℓ_c, at five points of the two-dimensional $U(1)$ strip with slope $0.916(21)$; that it is not proportional to the static length; $\ell_c \approx \pi/\sqrt{E\lambda_Q}$	The exponent as $\beta \to \infty$. The bend of the gap itself. The four-dimensional ℓ_*
Reading The raw rate follows the Kramers slope, and the split comes from the static factor $1/E$ (added afterwards)	The static factor in the rate of the four-dimensional topological charge
Estimate The three rulers, the candidate definition (2) of ℓ_*, comparing with $\max(\xi^2, \tau_{\mathrm{glob}})$	A sentence shown in any model. The right way to correct the relaxation-time assumption
Reading In two-dimensional CP^{N-1} ($L = 64$, $\xi \approx 4$), freezing splits with N, local ratios are small regardless of N (post-hoc cell, one point, weak), and the catching-up length grows with N	$z_Q(N)$. The registered verdict (could not be defined). Implications for four dimensions and $SU(N)$

Sources and reproduction

Item	Kind	Source / tools
Hierarchical gauge lattices	Cited	T. Ito, Commun. Math. Phys. 95 (1984) 247 (§ 1 checked in the text; writes that the extension to higher dimensions is left to the reader). The four-dimensional version is this book's extension
Comparison values for the hierarchical Gaussian free field	Cited	R. Bauerschmidt, D. C. Brydges, G. Slade, arXiv:1907.05474 § 4.1.2 (text checked)
Algebra of (3)(4), (5), (7)(9)	Proved here, Lean	Proofs in the text. Bundles LatticeGaugeHierarchicalLink (`HierLink.*`), LatticeGaugeConservationLength (`ConsLen.*`), LatticeGaugeConservationExponent (`ConsExp.*`). The bundles guarantee only arithmetic, algebra and comparisons of intervals
Numbers of Table 1	Reading	All eigenvalues of the dense matrix of the class model ($N \le 2$), a separate implementation per sector ($N \le 4$), the gauge quotient on the constraint surface of the four-dimensional original lattice ($N \le 2$)
Numbers of Table 2	Reading	Simulation of link Langevin on the strip (step $\beta\,dt \approx 0.04$, strip length ≤ 256). Positive control: τ of the Gauss dynamics agrees with the closed formula within 1σ. Negative control: moving the outer links too gives $O(\beta)$ independent of M. Errors by Madras–Sokal, with block averages as a separate route
The formula for ξ_{XY}, the Kramers rate	Assumption	Used as standard computations (original sources not checked)

Item	Kind	Source / tools
The action, topological charge and positive-control values for CP^{N-1}	Cited	M. Hasenbusch, Phys. Rev. D 96 (2017) 054504, arXiv:1706.04443 (equations (1), (10)–(12) and Table II checked in the text). B. Berg, M. Lüscher, Nucl. Phys. B 190 (1981) 412 (the original source for the triangle definition, bibliographic data only)
Dependence on N of the freezing of the topological mode	Cited	L. Del Debbio, G. M. Manca, E. Vicari, Phys. Lett. B 594 (2004) 315 (text), M. Berni, C. Bonanno, M. D'Elia, Phys. Rev. D 100 (2019) 114509 (text), M. Campostrini, P. Rossi, E. Vicari, Phys. Rev. D 46 (1992) 2647 (abstract)
Numbers of Tables 3 and 4	Reading	Simulation of heat bath ($O(3)$ heat bath on sites, von Mises heat bath on links) and overrelaxation, autocorrelation with Wolff's window. The free-field reference uses the same algorithm and the same L
The ultraviolet problem of the lattice topological susceptibility for $O(3)$	Assumption	From memory (original source not obtained)

Terms used in this chapter — the three rulers, the length at which conservation breaks ℓ_* (ℓ_c), the Dirichlet form (coordinates), tree coordinates and link coordinates, conservation of flux, wrapping, the topological charge of the strip, the flux dipole, raw rate and static fluctuation, the time for the global quantity to move by one, two-dimensional test beds. Their meanings are in the A-02 Glossary and 24-09.

Next chapter to read: 24-03 Two ways to repair the relaxation-time assumption — the sector assumption and the local assumption — two forms that restate the relaxation-time assumption to avoid the pattern. What is specific to four dimensions (why the two lengths split) is treated in 24-07.

2026-10-09 · chapter Yang–Mills and the mass gap Part 24 The first road, second half local Poincaré constant, ratio form, sector relaxation-time assumption · Kiichi, Shiori and Rin

Two ways to repair the relaxation-time assumption — the sector assumption and the local assumption

This chapter is this book's own investigation. The established knowledge it builds on is in 2-10 and 5-09.

What this chapter shows — We define two forms that restate the diffusive-relaxation-time assumption so as to avoid the pattern of 24-01 (a nearly conserved global quantity). The local relaxation-time assumption is a sentence about the Poincaré constant $C_P^{\mathrm{loc}}(c)$ over functionals supported in a window of length $c\ell$; it is fixed by the marginal law on the window alone, and returns unconditionally to the global C_P as $c \to \infty$. But the form "$C_P^{\mathrm{loc}} \le K(c)\ell^2$ for some $K(c)$" holds even for the massless free field and is empty, so it must be stated by the ratio K_{rel} to the massless value on the same box. The sector relaxation-time assumption is a sentence about the Poincaré constant in each sector of the smoothed topological charge; it pushes the obstruction to the relaxation-time assumption into a single quantity between sectors, C_{trans}, but it does not become a bridge to the mass gap. Seen through the charge of a subregion, the local relaxation-time assumption asks about the transport coefficient d, and the global relaxation-time assumption about the coefficient of breaking of conservation h.

Prerequisite chapters — 24-01 (the pattern, and the two accounts of (4)–(7)), 23-01 § 05 ((13)–(15) for the relaxation-time assumption and the relaxation-time upper-bound assumption, the note on topological sectors), 23-01 § 06 (the setting of the One-Sided Bridge Proposition (Theorem 3.1 of the paper)), 23-11 § 02 (the Box-Reduction Proposition and the closed box C_{box}). Notation in 24-09.

Said first — The level is [Res]; only § 01 is written to be readable at [Y3]. The core of this chapter is definitions, and **neither of the two relaxation-time assumptions has been shown at any** β. What has been shown are properties of the definitions (being a quantity of the marginal on the window, not being dense, monotonicity, the limit), an inequality for the decomposition of the variance by sectors, and values for the free-field reference. The definition on the four-dimensional Wilson measure is a copy of the definition on the path measure with continuous time, and the existence of the infinite-volume measure is a premise row.

The order of this chapter

1. Summary — a section readable at [Y3]
2. Definition of the local relaxation-time assumption — the algebra of time-averaged functionals $\mathcal{D}$, the window families $\mathcal{D}_c$, the Local Poincaré Constant Properties Proposition ((1)–(4))
3. The ratio form and the gauge-quotient reference — the emptiness of the absolute form, K_{rel}, gauge-invariant references, the note on 0+1 ((5)–(7), Table 1)
4. The sector relaxation-time assumption — cuts at regular values, the reflecting form, the Sector Poincaré Constant Proposition and the sandwich, three weaknesses ((8)–(11))
5. Open boxes and closed boxes, two coefficients — local asks d, global asks h, Table 2 (the order of volume and $a \to 0$)
6. Table of the five grades — Table 3 / dependency graph
7. What cannot be said

01

Summary

A section readable at [Y3]. The relaxation-time assumption is a sentence about "every test function", so if the relaxation of a quantity of the pattern of 24-01 (a winding number, the topological charge) grows faster than ξ^2, that alone breaks the upper half (the upper-bound assumption) ((3) of 24-01). Such a quantity is global, local neither in time nor in space. So we restrict the test functions in two ways and restate the relaxation-time assumption.

The local relaxation-time assumption. Restrict the test functions to those depending only on configurations inside a window of length $c\ell$ (in four dimensions, a spacetime box of side $c\ell$). ℓ is a ruler given from outside, independently of the gap; in four dimensions it is the bookkeeping length ξ_{RG}. The supremum $C_P^{\mathrm{loc}}(c)$ is fixed by the marginal distribution on the window alone, increases monotonically as the window widens, and returns exactly to the global C_P as $c \to \infty$. But in a four-dimensional box, C_P^{loc} is finite, of order the square of the side of the box, even for the massless free field. Merely satisfying "$C_P^{\mathrm{loc}} \le K\ell^2$" says nothing, so we ask that the ratio K_{rel} to the massless value on the same box be sufficiently smaller than 1.

The sector relaxation-time assumption. In the periodic box, divide configurations into sectors by the value of the smoothed topological charge, and measure the Poincaré constant in each sector. Splitting the variance into within sectors and between sectors, the global C_P is sandwiched: above by the sum of "the worst constant within a sector" and "the quantity between sectors C_{trans}", and below by C_{trans}. Assuming the sector relaxation-time assumption, the upper-bound assumption becomes the same as $C_{\mathrm{trans}} \lesssim \xi^2$. The obstruction does not vanish; it only moves into one quantity.

What is asked. Writing the ratio for the charge of a subregion with two coefficients ((7) of 24-01), the local relaxation-time assumption needs the **transport** coefficient of the charge d not to be too small, and the global upper-bound assumption needs the coefficient of **breaking of conservation** of the charge h not to be too small. In four dimensions, the freezing of the topological charge acts on the h side, so the local relaxation-time assumption is expected not to be affected by freezing Estimate . But the local relaxation-time assumption does not become a part of the chain that builds the upper-bound assumption; it is a separate sentence (the dependency graph of § 06). That what the one-sided bridge needs is the side local in time (under the assumptions of the one-sided bridge) is treated in the chapter on the bridge, 24-04.

02

Definition of the local diffusive-relaxation-time assumption

Setting. We use the same setting as the one-sided bridge of Part 23 ((17) of 23-01): a finite spatial lattice, the configuration space of one time $\mathcal{Q}$ (a finite-dimensional connected Riemannian manifold with metric M), $H = -\frac{\hbar^2}{2}\Delta_M + V$ with the ground-state assumption ((A1) in the paper; discrete spectrum, simple ground state $\psi_0 > 0$, $\Delta = E_1 - E_0 > 0$), μ the stationary path measure with $T = \infty$ (one-time distribution ψ_0^2, transition $e^{-\tau A}$, $A = (H - E_0)/\hbar$), and $m = \Delta/\hbar$. We take the collection of test functions to be the following **algebra of time-averaged functionals**.

$$\mathcal{D} := \Big\{ f = \Phi\Big(\int h_1(\tau) F_1(q(\tau))\, d\tau, \dots, \int h_k(\tau) F_k(q(\tau))\, d\tau \Big) \Big\}, \qquad \mathcal{E}_G(f) = \hbar \int \mathbb{E}\big\langle d_{q(\tau)} f,$$

Here $\Phi \in C_b^\infty(\mathbb{R}^k)$, $F_i \in C^\infty(\mathcal{Q})$ (up to polynomial growth for $\mathcal{Q} = \mathbb{R}^n$), and the weights $h_i \in L^\infty(\mathbb{R})$ have **bounded support**. $d_{q(\tau)} f = \sum_i \partial_i \Phi \cdot h_i(\tau)\, dF_i(q(\tau))$, and the metric G satisfies the metric comparison assumption of the one-sided bridge ((A3) in the paper). The global Poincaré constant is defined as $C_P := \sup_{f \in \mathcal{D},\, \mathcal{E}(f) > 0} \mathrm{Var}(f)/\mathcal{E}(f)$ **over** $\mathcal{D}$. The test functions of the one-sided bridge, $f_T = \int_0^T F(q)\, d\tau$, belong to this $\mathcal{D}$.

Definition of the local Poincaré constant

For a ruler of length $\ell > 0$ and $c > 0$, put the window $W_c := [0, c\ell]$ and

$$\mathcal{D}_c := \{ f \in \mathcal{D} : \text{ every weight } h_i \text{ has support in } W_c \}, \qquad C_P^{\mathrm{loc}}(c) := \sup_{f \in \mathcal{D}_c,\, \mathcal{E}(f) > 0} \frac{\mathrm{Va}}{\mathcal{E}}$$

ℓ is not the length fixed by the gap, $\xi = \hbar/\Delta$, but a length given from outside (in four dimensions, the bookkeeping length ξ_{RG}).

ℓ is not fixed by the gap because, when a lower bound on the gap is derived from the local relaxation-time assumption in 24-04, the argument would be circular if the gap entered the length of the window. Since the measure is stationary, the position of the window does not affect the value (the interval $[a, a + c\ell]$ gives the same value).

Local Poincaré Constant Properties Proposition

(i) $C_P^{\mathrm{loc}}(c)$ is fixed only by the marginal law of the path on the window, $\mu|_{W_c}$, and the Dirichlet form inside the window.

(ii) $\mathcal{D}_c$ is not dense in $L^2(\mu)$.

(iii) If $c \le c'$, then $C_P^{\mathrm{loc}}(c) \le C_P^{\mathrm{loc}}(c') \le C_P$.

(iv) $\lim_{c \to \infty} C_P^{\mathrm{loc}}(c) = C_P$. This limit needs no assumption.

Proved here Proof. (i) $f \in \mathcal{D}_c$ is fixed only by the values of the path inside the window (the path is continuous, and $\int h F(q)\, d\tau$ is the limit of Riemann sums over points inside the window). For $\tau \notin W_c$, $d_{q(\tau)} f = 0$, so $\mathcal{E}(f) = \hbar \int_{W_c} \mathbb{E}\langle \cdots \rangle\, d\tau$, and both the variance and the Dirichlet form are expectations under the marginal law on the window.

(ii) The closure of $\mathcal{D}_c$ lies in $L^2(\mathcal{F}_{W_c})$, the space of functions measurable with respect to the σ-algebra $\mathcal{F}_{W_c}$ generated by the path inside the window. The value $F(q(\tau_0))$ at a time τ_0 outside the window (F non-constant) is not measurable with respect to $\mathcal{F}_{W_c}$: with $s > 0$ the distance from the end of the window, the conditional expectation is of the form $\psi_0^{-1} e^{-sA}(F\psi_0)$, which under the ground-state assumption does not coincide with F itself.

(iii) If $c \le c'$, then $\mathcal{D}_c \subseteq \mathcal{D}_{c'} \subseteq \mathcal{D}$, and the supremum over a subset is smaller.

(iv) The weights in $\mathcal{D}$ have bounded support, so taking the window as $[-c\ell/2, c\ell/2]$ by stationarity, $\mathcal{D} = \bigcup_c \mathcal{D}_c$. If $b < C_P$, there is $f \in \mathcal{D}$ whose ratio exceeds b, and that f lies

in $\mathcal{D}_c$ for some c, so $C_P^{\mathrm{loc}}(c) > b$. Together with the monotonicity of (iii), the limit is C_P . ∎

Lean LatticeGauge.LocalH.local_le_global (assumptions: bounded above, non-empty, subset) and local_approaches_global (assumptions: $S = \bigcup_c S_c$, bounded above, non-empty; conclusion: every b smaller than $\sup S$ is exceeded by $\sup S_c$ for some c) check only the part of (iii)(iv) about suprema of sets of real numbers. The path measure, $\mathcal{D}$ and $\mathcal{E}$ are outside Lean.

(iv) holds without assumptions because C_P was defined over $\mathcal{D}$. The assumption "local functions form a core of the form" is not needed for this limit. The question of the core remains only when one identifies $C_P(\mathcal{D})$ with the gap of the generator of the equation of stochastic quantisation (the SPDE of Langevin dynamics) (the gap of the generator of the closure of the form on $\mathcal{D}$ is $1/C_P(\mathcal{D})$, and whether it coincides with the generator of the SPDE is the question of the core).

The local relaxation-time assumption. On the definition of the local Poincaré constant, we write the local relaxation-time assumption at two levels.

$$(\xi^2)_c^{\mathrm{loc}}: \ \lim \frac{\ln C_P^{\mathrm{loc}}(c)}{\ln \ell} = 2, \qquad (\xi^2)\!\uparrow_c^{\mathrm{loc}}(K): \ C_P^{\mathrm{loc}}(c) \le K\,\ell^2 \quad (3)$$

(left: the power statement for families with $\ell \to \infty$; right: the absolute form of its upper half). Two things follow in one line from the Local Poincaré Constant Properties Proposition Proved here . First, the global upper-bound assumption $C_P \le K\ell^2$ implies $(\xi^2)\!\uparrow_c^{\mathrm{loc}}(K)$ with the same constant for every c ((iii)). Second, if $(\xi^2)\!\uparrow_c^{\mathrm{loc}}(K)$ holds for every c with **a K uniform in c**, it returns by (iv) to the global upper-bound assumption. So the local relaxation-time assumption has content different from the global upper-bound assumption only when stated with **a single c** fixed.

A version local in space too. On an infinite spatial lattice (or a spatial lattice containing the box), take the product $B \times W_c$ of a spatial box B and a window, and define in the same form the family $\mathcal{D}_{B,c}$, with F_i restricted to single-time functions depending only on variables in B, and $C_P^{\mathrm{loc}}(B,c)$. (i)–(iv) of the Local Poincaré Constant Properties Proposition hold with the same proof, and the marginal in (i) is the law of the field on $B \times W_c$. The local relaxation-time assumption on the four-dimensional Wilson measure copies this definition onto a spacetime box B_c of side $c\ell$ ($\ell = \xi_{\mathrm{RG}}$), treating time and space alike, and the σ-algebra generated by the links in the box; the Dirichlet form is that of link Langevin, $\mathcal{E}(f) = \mathbb{E}\sum_\ell |\nabla_\ell f|^2$. What is presupposed at the copying stage is the existence and uniqueness of the infinite-volume Wilson measure (the infinite-volume uniqueness statement of Part 23) Assumption .

03

The ratio form and the gauge-quotient reference

The absolute form is empty. The right-hand form of (3), $(\xi^2)\uparrow_c^{\mathrm{loc}}(K)$, holds on a four-dimensional spacetime box, local in space too, even for the massless free field. Writing $C_P^{\mathrm{loc},0}(R)$ for the local Poincaré constant of the infinite-volume massless free scalar field (one component, canonical graduation) on a spacetime box B_R of side R,

$$\frac{R^2}{4\pi^2} \;\le\; C_P^{\mathrm{loc},0}(R) \;\le\; R^2 \qquad (4)$$

Proved here . The gradient Poincaré constant of a Gaussian measure equals the operator norm of the covariance compressed to the subspace spanned by the family (a transcription to subspaces of the Covariance-Compression Lemma (Lemma 2.2 of Hairer–Stuart–Voss–Wiberg) of Hairer–Stuart–Voss–Wiberg 2005) Cited , so $C_P^{\mathrm{loc},0}(R) = \|P_B(-\Delta)^{-1}P_B\|$. In four dimensions $(-\Delta)^{-1}$ is finite (transient) and its kernel is continuous, of the type $1/(4\pi^2|z|^2)$, so the Schur estimate $\sup_x \sum_{y\in B} G(x-y) \le \int_{|z|<2R} d^4z/(4\pi^2|z|^2) = R^2$ gives the upper side, and the value with the Dirichlet ground state of the box as test function gives the lower side. Hence, reading $\ell = R/c$, $(\xi^2)\uparrow_c^{\mathrm{loc}}(K)$ holds with $K = c^2$, and **the form "$C_P^{\mathrm{loc}}(c) \le K(c)\ell^2$ for some $K(c)$" says nothing about the gap**. This is the negative control of this definition (the massless free field). In numbers, $\Gamma(R) := C_P^{\mathrm{loc},0}(R)/R^2$ is $0.0802,\ 0.0768,\ 0.0756,\ 0.0750,\ 0.0747,\ 0.0744$ at $R = 4, 6, 8, 10, 12, 16$; the correction is $1/R^2$ with no logarithmic correction, and $\Gamma_\infty = 0.0740$ Reading (floating point, numerical integration of the heat kernel on the infinite lattice, extrapolation from $R \le 16$; within relative $2\cdot 10^{-3}$ of an independent route via periodic lattices). The massless field itself satisfies $K(c) = 0.0740\,c^2$.

One line of contrast with lower dimensions. For massless fields in two spacetime dimensions the infrared diverges and $C_P^{\mathrm{loc}} = \infty$; localising only to a time window on a finite spatial lattice, the constant mode gives $C_P^{\mathrm{loc}} \approx T/(2m)$ $(mT \ll 1)$, which diverges as $m \to 0$. A massless field has a finite local constant in spacetime dimension three or more when localised in space too, and the four-dimensional setting of Part 24 is exactly that case Proved here .

Definition of the ratio form

Put the ratio of the local Poincaré constant of the target measure μ (the Wilson measure, or the path measure in continuous time) on the box $B_{c\ell}$ to the value of the massless reference with a Dirichlet form on the same lattice, the same box and the same graduation as

$$K_{\mathrm{rel}}(c) := \frac{C_P^{\mathrm{loc}}(B_{c\ell};\mu)}{C_P^{\mathrm{loc},0}(c\ell)}, \qquad (\xi^2)\uparrow_c^{\mathrm{rel}}(K) : \ K_{\mathrm{rel}}(c) \le K \quad (\beta \ge \beta_0 \text{ uniformly}, \ \ell = \xi_{\mathrm{RG}}$$

The local relaxation-time assumption is this sentence at **a single** c; it is not required uniformly in c.

It corresponds to the constant of the absolute form by $K(c) = K_{\mathrm{rel}}\,\Gamma\,c^2$ Proved here . In the ratio form the constant K becomes an absolute number independent of the box size, and "sufficiently smaller than 1" has meaning.

Why it is not required uniformly in c. For the free field of mass m the ratio is a function of mR alone, $r(mR)$, and is $0.932,\ 0.825,\ 0.623,\ 0.359,\ 0.0772$ at $mR = 0.5,\ 1,\ 2,\ 4,\ 12$ Reading (agreeing within 0.3% for boxes of side 4 to 12). When the box is smaller than the correlation length ($mR \to 0$) the ratio tends to 1. An asymptotically free theory is close to a free field at short distance, so a sentence requiring $K < 1$ uniformly down to $c \to 0$ is expected to be false Estimate . For a field whose action is a convex perturbation $\frac{1}{2}\phi(-\Delta)\phi + \sum U(\phi_x)$ (U convex), the Brascamp–Lieb inequality gives $K_{\mathrm{rel}} \le 1$ Assumption (the statement of the inequality in the original source has not been checked in this part). What the bridge needs is not that the ratio is below 1 but that it is below the threshold K_*, and the threshold moves with the choice of reference: 0.342κ (scalar), 0.502κ (gauge quotient, one-colour linearisation), $\approx 0.62\kappa$ (colour singlet of $SU(2)$, leading term). The threshold, and that the band $[K_*, 1)$ is undecided, are treated in 24-04.

The reference for gauge fields is taken on the gauge quotient. For four-dimensional gauge fields, the value of the reference changes with the test space in which it is measured. The target Wilson measure is gauge invariant, and the Poincaré constant of a non-gauge-invariant function is infinite along gauge orbits, so the numerator of the ratio is finite only on gauge-invariant functions. So the reference in the denominator is also measured on gauge-invariant functions of the same box.

$$V_F := \{\text{gradients of functions of the plaquette angles of the box}\} = d^*\big(\ell^2(P_R)\big), \qquad V_Q :=$$

If the box is contractible, $V_F = V_Q$ Proved here (the cubical complex is contractible with zero first homology, so cycles in the graph of the box are boundaries of 2-chains of plaquettes). In non-contractible regions (with holes), $V_F \subsetneq V_Q$. Gauge-fixed references (Feynman gauge and so on) depend on the fixing parameter α, with $\Gamma_Q \le \Gamma_{\text{Landau}} \le \Gamma_{\alpha=1}$, and moreover $\Gamma_\alpha \ge \alpha/(\lambda_D^{\mathrm{int}} R^2)$, so they become infinite as $\alpha \to \infty$ (removing the fixing) Proved here . The numbers are in Table 1.

Table 1 References for the four-dimensional box (one-colour Gaussian approximation, $\beta = 1$, canonical graduation, floating point, extrapolation from $R \le 16$; λ_W is the Dirichlet value of the box within the test

space times R^2)

Reference	Test space	Γ_∞	$\lambda_{W,\infty}$	As a reference
Scalar (one component)	All functions on the box	0.0740 Reading	$4\pi^2 = 39.5$	Not used for gauge fields
Feynman gauge ($\alpha = 1$)	All functions of the links of the box	0.0740 Reading	39.5	Depends on α; not intrinsic
Gauge quotient	$V_Q = V_F$ (functions of plaquette angles)	0.0277 Reading	72.0 Reading	Independent of α. This one is used
Variant with the boundary gauge frozen	Divided only by gauge transformations at inner sites	2.22 to 2.53 times the gauge quotient (side 4 to 12) Reading	—	Not gauge invariant (flux through the box enters)

The last row of Table 1 is (e) of Table 1 in 24-01. Gauge transformations at the boundary sites of the box also act on links inside the box, so gauge-invariant observables of the box (Wilson loops and so on) lie in the space divided at all sites. **"Dividing by gauge transformations at all sites of the box" is part of the definition**, and leaving it out moves the numbers by a factor of 2.2 to 2.5. Every reference is a number for a free field; this book has not measured a single ratio for the target Wilson measure.

Note on 0+1. In 0+1 dimensions there is no finite massless reference (for a free particle $C_P^{\mathrm{loc}} = \infty$). The reference always has a scale, and for the double well we take the harmonic oscillator with the curvature of the bottom (angular frequency ω_b). Its ratio for the shortest window (a single time) is **exactly** ω_b/m (when ψ_0 is the Schrödinger ground state) Proved here :

$$K_{\mathrm{rel}}(\text{single time}) = \frac{\varepsilon/(2m)}{\varepsilon/(2\omega_b)} = \frac{\omega_b}{m} \quad (7)$$

(ε is the time step. The numerator is the Poincaré constant of the marginal ψ_0^2, which moves to the gap of $(H - E_0)$ by the ground-state transformation. This identity is the same as the formula for the vacuum measure in 24-05.) Lean LatticeGauge.LocalHDoubleWell.one_slice_ratio (assumptions: $0 < \varepsilon, m, \omega$; only the algebra of the ratio). In the double well, tunnelling gives $m \ll \omega_b$, so the ratio does not become 1 even for short windows; on the lattice $\varepsilon = \frac{1}{2}$ it is 4.61 ($\hbar = 0.3$) to 638 ($\hbar = 0.08$) Reading . Hence "the ratio tends to 1 at short distance" is not a fact that follows from the ratio form but **an assumption depending on a property of the theory, asymptotic freedom (being close to a free field at short distance)** Assumption . The three regimes of the ratio as the window is lengthened are treated in 24-04.

04

The sector diffusive-relaxation-time assumption

Setting. Take the periodic box $\Lambda_L = (\mathbb{Z}/L)^4$, the configuration space $\mathcal{M} = G^{\text{links}}$ (a compact connected manifold with a bi-invariant metric), the Wilson measure μ (smooth positive density), and $\mathcal{E}(f) = \int |\nabla f|^2 d\mu$. For the topological charge we use the lattice charge Q_s smoothed by the Wilson flow (flow time s) or by finitely many smoothings. The smoothing map is smooth and the charge density is a polynomial in matrix elements, so $Q_s \in C^\infty(\mathcal{M})$.

Definition of sectors, and of the form reflecting at the walls

Take the cuts $\theta_n \in (n-1, n)$ at **regular values** of Q_s (by Sard's theorem almost all values are regular, so they can be taken arbitrarily close to $n - \frac{1}{2}$).

$$I_n := \{\theta_n \le Q_s < \theta_{n+1}\}, \quad \mu_n := \mu(\,\cdot\,|I_n), \quad \mathcal{E}_n(f) := \int_{I_n} |\nabla f|^2 d\mu_n \;\; (f \in H^1(I_n)),$$

(only for n with $\mu(I_n) > 0$). No boundary condition is imposed on $\mathcal{E}_n$. Its generator is Langevin reflecting at the walls (Neumann).

Because the cut is at regular values, the closure of a sector is a compact manifold with smooth boundary, and the walls are smooth hypersurfaces of codimension 1. Cutting by a geometrically defined integer-valued charge (piecewise constant and discontinuous), the form can be defined as H^1 on open sets, but the regularity of the walls is unknown and (i) below cannot be said. This is why this chapter uses a smoothed charge.

Sector Poincaré Constant Proposition (sector Poincaré constants and the decomposition of the variance)

(i) $C_P^{\text{sec}}(n) < \infty \iff I_n$ is connected.

(ii) With N the sector index, for smooth f,

$$\mathcal{E}(f) = \sum_n \mu(I_n)\, \mathcal{E}_n(f), \qquad \operatorname{Var} f = \sum_n \mu(I_n) \operatorname{Var}_{\mu_n} f + \operatorname{Var}\big(\mathbb{E}[f \mid N]\big) \quad (9)$$

(iii) For the quantity between sectors $C_{\text{trans}} := \sup_f \operatorname{Var}(\mathbb{E}[f|N])/\mathcal{E}(f)$,

$$C_{\text{trans}} \;\le\; C_P \;\le\; \sup_n C_P^{\text{sec}}(n) + C_{\text{trans}} \quad (10)$$

(iv) $C_P^{\text{sec}}(n) \le C_P$ does not hold in general.

Proved here Proof. (i) The closure of a sector is compact with smooth boundary and the density is positive and smooth, so the resolvent of the Neumann operator is compact (Rellich's theorem Assumption : the statement in the original source has not been checked in this part). The multiplicity of the zero eigenvalue equals the number of connected components, so a positive gap is the same as connectedness. (ii) The walls have measure 0, so the Dirichlet form splits into a sum over sectors, and the variance is the sum of the expectation of the conditional variance and the variance of the conditional expectation. (iii) The left is $\mathbf{Var}(\mathbb{E}[f|N]) \le \mathbf{Var}\, f$; the right uses the Poincaré inequality in each sector on the first term of (9) and closes with $\sum_n \mu(I_n)\mathcal{E}_n = \mathcal{E}$. (iv) Functions on sectors may jump at the walls (they are not in $H^1(\mathcal{M})$), so if conditioning creates a new bottleneck, $C_P^{\mathrm{sec}}(n) > C_P$ is possible. ∎ Lean LatticeGauge.LocalH.sector_split (assumptions: weights ≥ 0, the inequality in each sector, the two formulas of (9), $\mathbf{Var}(\mathbb{E}[f|N]) \le C_{\mathrm{trans}}\mathcal{E}$; conclusion: the right of (10)), trans_le_total (assumptions: weights and sector variances ≥ 0, (9); conclusion: the inequality for the numerator on the left of (10)). The identity (9) itself, Rellich, and the measure are outside Lean.

The sector relaxation-time assumption and its corollary. The sector relaxation-time assumption is the sentence that, in a box with fixed physical side, the Poincaré constant within a sector is bounded by the square of the correlation length.

$$(\xi^2)\uparrow^{\mathrm{sec}}: \quad L = \lambda\,\xi_{\mathrm{RG}} \text{ gives } \sup_n C_P^{\mathrm{sec}}(n) \le K(\lambda)\,\xi_{\mathrm{RG}}^2 \qquad \Longrightarrow \qquad \big(\,(\xi^2)\uparrow \iff C_{\mathrm{trans}} \lesssim$$

Proved here (from (10); $\lesssim$ is up to replacing constants). The sector relaxation-time assumption pushes the obstruction to the relaxation-time assumption **exactly** into the quantity between sectors C_{trans}. It does not remove the obstruction but only moves it into one quantity, the same form as the transfer type in 23-10. The physical reading of C_{trans} is the rate of dislocations, and statically $C_{\mathrm{trans}} \ge \mathbf{Var}(\mathbb{E}[Q_s|N])/\mathcal{E}(Q_s)$.

It does not connect to the bridge. What follows from (10) and the one-sided bridge ((17) of 23-01, with corrections for finite T in periodic time) goes only as far as $\kappa/m^2 \lesssim \sup_n C_P^{\mathrm{sec}}(n) + C_{\mathrm{trans}}$, and no gap comes out unless C_{trans} is bounded Proved here . The Langevin gap within a sector, $1/C_P^{\mathrm{sec}}(n)$, is a fifth quantity of time, and the conditioned measure μ_n has no transfer operator (the constraint on the total charge breaks the Markov property in time). A "physical gap within a sector" of the type of the one-sided bridge does not stand by definition; to set it up one would need a treatment of superpositions of θ vacua, which has not been formulated Estimate .

Three weaknesses. (a) *The walls of conditioning can create artificial bistability.* In the core on the box side of Part 23, two local minima appeared on the walls of conditioning the interval to "exactly this interval", giving rise to a bistability that was

not physical metastability (23-11 § 06). The walls of sectors lie on dislocation configurations of small measure, so we expect the harm to be small, but this has not been checked Estimate . That it is unknown whether sectors are connected in (i) of the Sector Poincaré Constant Proposition is the extreme form of this weakness. (b) *When the physical volume is made large, the obstruction remains even with the sector fixed.* With β fixed and $L \to \infty$, the charge of a half box can move even within a sector I_n. With the coefficients of (5) in 24-01, the conditional variance is of order $\chi_t V/4$ and the Dirichlet form of order $hV/2 + g\,|\mathrm{face}|$, and the ratio tends to $\chi_t/(2h)$ as $V \to \infty$, the same order as the overall ratio of the periodic box χ_t/h Assumption (on the definitions of h, g and the accounting for the conditional variance of a normal distribution with the sum of the two halves fixed). Fixing the sector stops only the one mode of the total charge; the modes of the charges of subregions remain. Hence the sector relaxation-time assumption is expected to have content different from the upper-bound assumption only when the box is smaller than the crossing length (subregions within a sector are in the transport regime), that is, **to have meaning only in the order in which the physical volume is fixed** Estimate . (c) *The path measure with $T = \infty$ has no sectors.* The four-dimensional charge is a quantity integrated over time, and for $T = \infty$ it is not in L^2.

Correspondence with the splitting of the first step. (9) is the same splitting as the splitting of the variance in the first step (local Poincaré within a phase, and transport between phases, 23-05 § 01), with phase ↔ sector and kink ↔ dislocation Estimate . It is one concrete destination of "the two-scale splitting is expected to lift" in 23-03 § 06. But in the first step, the time for transport between phases (diffusion of kinks) is read to be of order $\xi^2 S_0/\hbar$ ((14) of 23-04 Assumption), and fits into the form of the double-well relaxation-time power statement. The four-dimensional quantity between sectors C_{trans} is exactly a quantity of the pattern of 24-01, and there is no prospect yet that it fits into the same form.

05

Open boxes and closed boxes, two coefficients

Local asks about the transport coefficient; global asks about the coefficient of breaking of conservation. The charge Q_w in a smooth window of (7) in 24-01 belongs to the local family of a box of side $R = c\xi$. Putting (7) into the local relaxation-time assumption (absolute form), the necessary condition is $\chi_t c^2\xi^2/(hc^2\xi^2 + d\lambda_W) \le K\xi^2$, and in the regime where h is small (the crossing length R_* much longer than $c\xi$),

$$\frac{\chi_t (c\xi)^2}{d\,\lambda_W} \le K\xi^2 \iff \chi_t c^2 \le K\,d\,\lambda_W, \qquad \text{that is} \qquad D_{\mathrm{eff}} := \frac{d}{\chi_t} \ge \frac{c^2}{K\lambda_W} \quad (12)$$

Proved here (algebra) Lean LatticeGauge.LocalH.transport_requirement (assumptions: $0 < d$, $0 < \lambda_W$, $0 < \xi$). In the ratio form, K is read as $K_{\mathrm{rel}}\Gamma c^2$. On the other hand, the necessary condition for the global upper-bound assumption is (2)(6) of 24-01 with $R \to \infty$,

$$\frac{\chi t}{h} \lesssim \xi^2 \quad (13)$$

Assumption (on top of (5) of 24-01). (12) and (13) bear on different coefficients. **The local relaxation-time assumption asks about the transport coefficient of the charge d, and the global relaxation-time assumption about the coefficient of breaking of conservation of the charge h.** The freezing of the four-dimensional topological charge can be read as the appearance of h vanishing quickly in the continuum limit (dislocation configurations becoming rare), so what it affects is the side of (13). The prospect that the local relaxation-time assumption is unaffected by freezing comes from here Estimate . There are no four-dimensional lattice values for d and h.

Closed boxes and open boxes. The Box-Reduction Proposition of Part 23 (23-11 § 02) is a reduction bounding the global C_P by a box constant C_{box}, and C_{box} was the constant of a conditioned measure **with the boundary values fixed** (with sup over the boundary values). We call this a closed box. The box of the local relaxation-time assumption is a marginal box without fixing the boundary, and we call it an open box. In four dimensions, if the boundary links are fixed, the integer part of the topological charge inside the box is expected to change only through dislocations inside the box (in the continuum, the charge of the box is the sum of the Chern–Simons value on the boundary and a winding number) Estimate . Hence for closed boxes of size $\chi t (c\xi)^4 \gtrsim 1$, C_{box} is expected to inherit the time of the dislocation barrier Estimate . In open boxes the charge flows in and out through the boundary, and relaxation is of the transport type of (12) Assumption .

The difference between closed and open boxes has the same form as the difference between periodic and open boundaries. Open boundaries remove the topological obstruction only inside a finite box with fixed physical side (the autocorrelation time of Q^2 with open boundaries grows roughly in proportion to the length in the time direction, § 5.3 of Lüscher–Schaefer 2011 Reading). This is the transport type of (12) restated in the language of boxes. A reading of the consequence Estimate : if, under the reading that extends the conjecture (§ 06 (a) of 24-01), the upper-bound assumption reads as false in four dimensions, then even when the local relaxation-time assumption holds, the four-dimensional version of the Box-Reduction Proposition fails at the stage of closed boxes. The place where it breaks is not "relaxation inside the box" but "the passage between sectors inside a closed box".

Table 2 The order of volume and $a \to 0$, and the corrections expected to work

Order	Taken first	Corrections expected to work	Corrections expected not to work	Label
$a \to 0$ with the physical side fixed	Continuum limit of a finite box	Open boundaries, the sector relaxation-time assumption	—	Cited (original sources on open boundaries), Estimate
Volume to infinity first (the order of this book)	Infinite-volume lattice measure	The local relaxation-time assumption (open boxes)	Open boundaries, the sector relaxation-time assumption	Assumption (accounting for the charge of a subregion), Estimate

The original source writes that the scaling of the autocorrelation of algorithms implementing Langevin ($1/a^2$ in lattice units) "is expected to hold on infinite lattices and on finite lattices with open boundaries", and excludes transitions changing topology with periodic boundaries from the analysis as lattice artefacts (the footnote of Lüscher–Schaefer, Non-renormalizability of the HMC algorithm, 2011) Cited . The second row of Table 2 points the same way as this wording, but the object of the original source is the autocorrelation of local quantities, not the local relaxation-time assumption itself.

06

Table of the five grades

We put the rows of this chapter into one table in the format of the five grades, with Estimate added. The symbols of the rows are the same as the nodes of the dependency graph, and one row of 24-01 (the row of cluster decay) is carried over.

Table 3 Rows on the sector relaxation-time assumption and the local relaxation-time assumption ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Existence and uniqueness of the infinite-volume Wilson measure	Assumption	The infinite-volume uniqueness statement of Part 23	$\beta \geq \beta_0$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
2	Rellich's theorem (compactness of the Neumann resolvent)	Assumption	Sector Poincaré Constant Proposition (i) (the statement in the original source has not been checked)	Smooth boundary	—
3	Poincaré constant of a Gaussian measure = norm of the compressed covariance	Cited	(4), Hairer–Stuart–Voss–Wiberg 2005	Gaussian	—
4	Local Poincaré Constant Properties Proposition (i)–(iv)	Proved here	Proof in § 02	Continuous time, finite spatial lattice, $T = \infty$	local_le_global, local_approaches_global (assumptions: bounded above, non-empty, $S = \bigcup S_c$; only the supremum part)
5	That the proof of the Local Poincaré Constant Properties Proposition goes through in the same way for boxes of the four-dimensional Wilson measure	Assumption	End of § 02 (the treatment of domains is not written out)	Four dimensions, infinite volume	—
6	Properties of the four-dimensional local Poincaré constant	Proved here	On rows 1, 4, 5	Four dimensions	—
7	The absolute form holds with $K = c^2$ for the massless free field	Proved here	(4)	Four-dimensional spacetime, local in space too	—
8	$\Gamma_\infty = 0.0740$, $\Gamma_{Q,\infty} = 0.0277$, $\lambda_W^Q = 72.0$, 2.22 to 2.53 times with the boundary frozen	Reading	§ 03, Table 1	Free field, one-colour Gaussian, $\beta = 1$, $R \le 16$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
9	$V_F = V_Q$ in contractible boxes; gauge-fixed references depend on α and diverge as $\alpha \to \infty$	Proved here	(6)	Contractible box	—
10	Ratio for the shortest window $= \omega_b/m$	Proved here	(7)	0+1, Schrödinger ψ_0	one_slice_ratio (assumptions: $0 < \varepsilon, m, \omega$)
11	The ratio tends to 1 at short distance	Assumption	The note on 0+1 in § 03 (depends on asymptotic freedom)	Four dimensions	—
12	Local relaxation-time assumption (ratio form): $K_{\mathrm{rel}}(c) \le K$ at a single c	Assumption	(5)	$\beta \ge \beta_0$	—
13	Sector Poincaré Constant Proposition (i): sector constant finite $\iff$ sector connected	Proved here	§ 04 (on top of row 2)	Periodic box, cuts at regular values	—
14	The sandwich (10)	Proved here	(9)(10)	Periodic box	sector_split, trans_le_total (assumptions: weights ≥ 0, the inequality in each sector, the two formulas of (9))
15	Corollary (11): under the sector relaxation-time assumption, the upper-bound assumption $\iff C_{\mathrm{trans}} \lesssim \xi^2$	Proved here	(11)	Box with fixed physical side	—
16	What comes out of sectors goes only as far as $\kappa/m^2 \lesssim \sup_n C^{\mathrm{sec}}_P + C_{\mathrm{trans}}$	Proved here	§ 04 (on top of the one-sided bridge)	Finite box	—
17	Cluster decay (row 1 of Table 5 in 24-01)	Assumption	(5) of 24-01	Infinite volume	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
18	(12): local asks d; (13): global asks h	Proved here	(12) (algebra), (13) on top of row 17	Four dimensions	transport_requirement (assumptions: $0 < d, \lambda_W, \xi$)
19	Ratio for half boxes within a sector $\to \chi t$ $/(2h)$	Assumption	Weakness (b) of § 04	$L \to \infty$ with β fixed	—
20	Alternating maximal-correlation assumption for four-dimensional boxes: maximal correlation < 1 between families of closed boxes, uniform in β	Assumption	Premise of the four-dimensional version of the Box-Reduction Proposition	Four dimensions, families of 2^4 colours	—
21	Gap assumption for closed four-dimensional boxes: $C_{\mathrm{box}}(c\xi_{\mathrm{RG}}) \le K' \xi_{\mathrm{RG}}^2$ for closed boxes	Assumption	§ 05 (the row where the topological barrier is expected to enter)	Four dimensions	—
22	Four-dimensional multi-family box reduction $C_P \le C(c)$ $C_{\mathrm{box}}/(1 - c_{\mathrm{alt}})$	Assumption	Four-dimensional version of the one-dimensional Box-Reduction Proposition (not written out)	Four dimensions	—
23	Relaxation-time upper-bound assumption	Proved here	Algebra on top of rows 20, 21, 22	Four dimensions	—

Dependency graph

```
 1  ASM  infinite-volume Wilson measure exists, unique        β ≥ β₀
 2  ASM  Rellich's theorem                                     smooth
 3  CIT  Gaussian C_P = norm of compressed covariance          Gaussia
 4  PRV  Local Poincaré Constant Properties Proposition
         (continuous time)                                    T = ∞
         Lean local_le_global, local_approaches_global (supremum part
 5  ASM  its proof goes through in 4D boxes                    four di
 6  PRV  properties of the 4D local C_P ⟸ 1, 4, 5
 7  PRV  vacuity of the absolute form (4) ⟸ 3                   4D spa
 8  RDG  reference numbers (Γ_∞, Γ_Q, λ_W^Q,
         factor with frozen boundary)                         free fi
 9  PRV  V_F = V_Q in contractible boxes; fixing depends on α  contra
10  PRV  ratio for the shortest window = ω_b/m                 0+1
         Lean one_slice_ratio
11  ASM  ratio → 1 at short distance (asymptotic freedom)      four di
12  ASM  local relaxation-time assumption (ratio form, one c) β ≥ β₀
13  PRV  sector constant finite ⟺ connected ⟸ 2                regu
14  PRV  sandwich C_trans ≤ C_P ≤ sup C^sec + C_trans          periodi
         Lean sector_split, trans_le_total
15  PRV  under the sector assumption, the upper-bound
         assumption ⟺ C_trans ≲ ξ² ⟸ 14
16  PRV  bridge inequality from sectors ⟸ 14 (+ one-sided bridge)
17  ASM  cluster decay (24-01)                                 infinit
18  PRV  local asks d, global asks h ⟸ 17
         Lean transport_requirement
19  ASM  ratio of half boxes in a sector → χ_t/(2h)
20  ASM  alternating max. correlation of 4D boxes < 1        four dim
21  ASM  gap of closed 4D boxes C_box ≤ K'ξ_RG²
         ← the row the topological barrier is expected to enter
22  ASM  4D multi-family box reduction (not written out)
23  PRV  upper-bound assumption ⟸ 20, 21, 22
```

How to check: on each "⟸" row, see by eye that the numbers on the right are smaller than the one on the left. There are no circles. The leaves marked "Assumption" number ten (the existence of the infinite-volume measure, Rellich's theorem, the transcription to four dimensions of the proof of the properties of the local Poincaré constant, the reading of the ratio at short distance, the local relaxation-time assumption, cluster decay, the ratio of half boxes, the alternating maximal correlation of four-dimensional boxes, the gap of closed four-dimensional boxes, the four-dimensional multi-family

box reduction); the "Cited" leaf is the row of the Gaussian Poincaré constant, and the "Reading" leaf is the row of the reference numbers.

Two things are read from this graph. First, **the local relaxation-time assumption enters nowhere in the chain that builds the upper-bound assumption.** A reduction of the type of the Box-Reduction Proposition uses closed boxes (the gap assumption for closed four-dimensional boxes), whereas the boxes of the local relaxation-time assumption are open boxes. The local relaxation-time assumption is not a part of the upper-bound assumption but a separate sentence. Where the local relaxation-time assumption is used is in the graph of the bridge in 24-04. Second, the place where the upper-bound assumption is expected to break in four dimensions is concentrated in the single row of the gap assumption for closed four-dimensional boxes. The statement of this row contains only the constant of closed boxes C_{box} and not the global C_P, so it is not of the form that puts the conclusion among the premises.

07

What cannot be said

(a) The two relaxation-time assumptions are **candidate definitions; at no β is there an inequality shown for the target gauge theory**. What has been shown are properties of the definitions (the Local Poincaré Constant Properties Proposition and the Sector Poincaré Constant Proposition) and values of the free-field reference. Nor do we claim "the right way to correct" the relaxation-time assumption.

(b) The local relaxation-time assumption on the four-dimensional Wilson measure is a copy of the definition on the path measure in continuous time, and the existence of the infinite-volume measure and the treatment of domains (that the proof of the Local Poincaré Constant Properties Proposition goes through in the same way for four-dimensional boxes) are premise rows.

(c) The accounting for the charge of a subregion (transport in open boxes, barrier in closed boxes) is an estimate on top of cluster decay and the formal coefficients h, g, d, with no four-dimensional lattice values.

(d) The four-dimensional multi-family box reduction has not been written. The harm of sector walls, and how the sector relaxation-time assumption breaks when the physical volume is made large, are estimates.

(e) There is no evidence that the local relaxation-time assumption holds in four dimensions. There is only the accounting "freezing acts on the h side, and the local relaxation-time assumption asks about d".

(f) Every number in Table 1 is a number for the free field of the reference (one-colour Gaussian, $\beta = 1$, floating point, extrapolation); not a single ratio K_{rel} for the target

measure has been measured. The threshold depends on the reference and on the family of test observables, and the bridge says nothing about fields in the band $[K_*, 1)$ (24-04).

(g) "The ratio tends to 1 at short distance" is an assumption depending on asymptotic freedom; in the 0+1 double well this reading does not hold ((7)).

What this chapter can say

Said	Not said
Proved here The definition of the local Poincaré constant and the Local Poincaré Constant Properties Proposition (a quantity of the marginal on the window, not dense, monotone, returns unconditionally to C_P as $c \to \infty$; in continuous time, on a finite spatial lattice, $T = \infty$, under the assumptions of the one-sided bridge)	The treatment of domains on the four-dimensional Wilson measure (the copy). The question of the core (identification with the generator of the SPDE)
Proved here The local relaxation-time assumption in absolute form holds even for the massless free field on a box in four-dimensional spacetime, local in space too ($K = c^2$)	That the local relaxation-time assumption in ratio form holds for the target theory
Proved here That the family of plaquette functions and the gauge quotient coincide for contractible boxes; that gauge-fixed references depend on the parameter; the ratio for the shortest window ω_b/m	References for $SU(2)$ at finite β. Values for families of large four-dimensional loops
Reading Reference numbers $\Gamma_\infty = 0.0740$, $\Gamma_{Q,\infty} = 0.0277$, $\lambda_W^Q = 72.0$, 2.22 to 2.53 times with the boundary frozen	Not propositions of upper or lower bounds. Numbers for free fields, not for the target measure
Proved here The definition of sectors (cuts at regular values, the reflecting form), finite $\Longleftrightarrow$ connected (on top of Rellich), the sandwich (10), corollary (11), the bridge inequality from sectors	Whether sectors are connected. Estimates of C_{trans}. The treatment of θ vacua
Estimate That the local relaxation-time assumption is expected not to be affected by freezing; that closed boxes are expected to inherit the barrier; that the sector relaxation-time assumption is expected to have meaning only in the order with the physical volume fixed	None of them is a theorem

Sources and reproduction

Item	Kind	Source / tools
Poincaré constant of a Gaussian measure = norm of the covariance	Cited	The Covariance-Compression Lemma of M. Hairer, A. M. Stuart, J. Voss, P. Wiberg (2005) (equality for linear functionals)

Item	Kind	Source / tools
Autocorrelation of Q^2 with open boundaries and the length in the time direction; the expected scaling of Langevin algorithms	Cited	M. Lüscher, S. Schaefer, JHEP 07 (2011) 036 § 5.3; M. Lüscher, S. Schaefer, JHEP 04 (2011) 104 § 1 and footnote (text checked)
Local Poincaré Constant Properties Proposition (iii) (iv), (10) of the Sector Poincaré Constant Proposition, the algebra of (12), the ratio (7)	Lean	LatticeGauge.LocalH (local_le_global, local_approaches_global, sector_split, trans_le_total, transport_requirement), LatticeGauge.LocalHDoubleWell (one_slice_ratio). The bundles guarantee only suprema of sets of real numbers and algebra
Reference numbers of Table 1, $\Gamma(R)$, $r(mR)$	Reading	Largest eigenvalue (Lanczos) of the compression to the box of the lattice Green function (numerical integration of the heat kernel), and an independent route via periodic lattices (removing the zero mode, extrapolation in $1/L^2$). The gauge quotient by two routes: projection via the Neumann Laplacian of the graph of the box, and projection via a dense singular value decomposition
Lattice ratios for the double well	Reading	Lattice eigenvalue problem for the transfer kernel (symmetric product with time step $\varepsilon = \frac{1}{2}$)

Terms used in this chapter — the algebra of time-averaged functionals $\mathcal{D}$, window, the local Poincaré constant $C_P^{\mathrm{loc}}(c)$, the ratio form K_{rel}, the massless reference and Γ, the gauge-quotient reference, sectors and the reflecting form, the quantity between sectors C_{trans}, open boxes and closed boxes, the transport coefficient d and the coefficient of breaking of conservation h. The notation is in 24-09.

Next chapter to read: 24-04 Which form of the relaxation-time assumption the one-sided bridge needs — the threshold K_* at which a lower bound on the gap comes out of the local relaxation-time assumption in ratio form, and the size of box at which the bridge sees the mass.

Which form of the relaxation-time assumption the one-sided bridge needs

This chapter is this book's own investigation. The established knowledge it builds on is in 5-09 and 2-10.

What this chapter shows — Which part of the diffusive-relaxation-time assumption the one-sided bridge of Part 23 uses, and nothing more. On the family of time-integrated single-time functions, the supremum of the Poincaré constant is fixed exactly by the gap, so what the bridge needs is only the part of the relaxation-time upper-bound assumption that is local in time. That a lower bound on the gap in finite spatial volume follows from the local relaxation-time assumption with a limited window; that in infinite volume the local relaxation-time assumption must be stated as a ratio to a massless reference; that its threshold changes with the choice of reference (scalar, gauge quotient, colour singlet of $SU(2)$); and that the bridge sees the mass only when the box is at least 4–5 times the correlation length. In the 0+1-dimensional double well, the ratio does not become 1 even for the shortest window.

Prerequisite chapters — 23-01 § 05 ((13) for the diffusive-relaxation-time assumption, (14)(15) for the upper-bound assumption), 23-01 § 06 ((17) of the one-sided bridge, the identity (18), the bridge (19), (20) of the time-step comparison assumption ((D) in the paper "one-sided bridge")), 23-01 § 06 (5) (the Free-Particle-on-a-Circle Counterexample), 24-03 (definitions of the local relaxation-time assumption and the ratio form). The notation is collected in 24-09.

Said first — The level is [Res]; only the summary in § 01 is [Y3]. This chapter does not show the mass gap. Neither the relaxation-time assumption, nor the upper-bound assumption, nor its local version has been shown at any β. What we show is the logical form of which part of which assumption the bridge uses, and values for free-field references. The window argument consists of sentences about the stationary path measure with $T = \infty$ on a lattice with continuous time and a finite spatial lattice; the infinite-volume version further rests on the local-excitation moment assumption and on the infinite-volume ground-state transformation identity. The values of the references are readings from floating-point computation and extrapolation.

The order of this chapter

1. Summary [Y3] — the bridge uses only the part of the upper-bound assumption local in time
2. The one-sided bridge and the family local in time — the supremum is exactly ξ^2; correspondence with the paper ((1)–(3))
3. Limiting the window — monotonicity of the window factor, the In-Window Supremum Proposition, the lower bound in finite volume ((4)–(10), Table 1)
4. "Some $K(c)$" is empty — the massless free field as a negative control, and the ratio form ((11)(12))
5. The bridge in infinite volume and the threshold — the local-excitation moment assumption, transfer to the ratio ((13)–(16))
6. Three references — scalar, gauge quotient, colour singlet (Table 2); the box at which the bridge sees the mass
7. Note on 0+1 dimensions — even for the shortest window the ratio is ω_b/m ((17), Table 3)
8. Which part of which assumption each conclusion needs — Table 4
9. What cannot be said
10. Table of the five grades and the dependency graph — Table 5

01

Summary [Y3] — the bridge uses only the part of the upper-bound assumption local in time

In 23-01 § 06 of Part 23, we showed the one-sided bridge $C_P \geq \kappa\hbar^2/\Delta^2$ for the path measure on a lattice with continuous time. The test function is the single one $f_T = \int_0^T F(q(\tau))\,d\tau$, the time integral of a function of the configuration at one time, $F = \psi_1/\psi_0$. Setting this one function beside $C_P \leq C\xi_{\mathrm{RG}}^2$ of the upper-bound assumption gives a lower bound on the gap of the lattice in physical units, $m_{\mathrm{phys}}^2 \geq \kappa/C$ ((19) of 23-01). The question of this chapter is which part of the upper-bound assumption is used there.

The answer comes in three stages.

First stage. On the family $\mathcal{F}_1$ of "functions of one time averaged with time weights", the supremum of $\mathrm{Var}\,/\mathcal{E}$ is exactly $\xi^2 = \hbar^2/\Delta^2$ in the canonical graduation Proved here . Hence the upper-bound assumption restricted to this family is the same as "the gap of the lattice in physical units is at least $C^{-1/2}$". The content of the upper-bound assumption beyond the gap lies only outside this family — in functions not local in time, such as winding numbers and the topological charge (the Free-Particle-on-a-Circle Counterexample of 23-01, the pattern of 24-01). The bridge uses only the part of the upper-bound assumption local in time.

Second stage. From the local relaxation-time assumption with the time window limited to length $c\ell$ (ℓ is a ruler given from outside, independently of the gap; in four dimensions it is the bookkeeping length ξ_{RG}), too, a lower bound on the gap in finite spatial volume follows, $m_{\mathrm{phys}} \geq \sqrt{1/c^2 + \kappa/K} - 1/c$ Proved here . As $c \to \infty$ it returns to the $\sqrt{\kappa/K}$ of Part 23.

Third stage. Going local in space too and on to infinite volume, a sentence of the form "the local Poincaré constant in the box is at most $K(c)\ell^2$" becomes empty, because the four-dimensional massless free field also satisfies it for some $K(c)$ Proved here . So we state it by the ratio K_{rel} to the value of the massless free field (the reference) on the same box with the same graduation. Under the local-excitation moment assumption, if the ratio is below the threshold $K_* = \kappa/(\lambda_W \Gamma)$, a lower bound on the gap follows Proved here (algebra on top of the assumption). The threshold changes with the choice of reference: 0.342κ for the scalar, 0.502κ for the gauge quotient (one-colour linearisation), and about 0.62κ for the colour singlet of $SU(2)$ (leading term, second chaos; a test value, so a lower value of the usable threshold) Reading . Examining free fields with mass, the bridge sees the mass only when the side of the box is at least about 4.2 times (scalar) to 5.2 times (gauge quotient) the correlation length Reading . For fields whose ratio lies in the band $[K_*, 1)$ between the threshold and 1, this bridge says nothing.

Finally we place a note on 0+1 dimensions. In the double well, even for the shortest window (a single time), the ratio is exactly ω_b/m and does not become 1 Proved here . "At short distance the ratio is close to 1" is not a fact that follows from the ratio form; in four dimensions it is an assumption depending on asymptotic freedom Assumption .

As a metaphor, the bridge looks only at "the time for the hand of the clock to go round once", not at the slowness of "the gear that counts how many times the hand has gone round". The upper-bound assumption is a sentence about the whole, gear included, and what the bridge needs is only the part about the hand.

02

The multidimensional one-sided bridge and the family local in time — the supremum is exactly ξ^2

The setting is the same as the one-sided bridge in 23-01 § 06. Space is a finite lattice, the configuration space at one time is Q, the Hamiltonian is $H = -\frac{\hbar^2}{2}\Delta_M + V$, with the ground-state assumption ((A1) in the paper; discrete spectrum, ground state $\psi_0 > 0$ non-degenerate and smooth, $\Delta = E_1 - E_0 > 0$), the stationary path-measure assumption ((A2) in the paper; the stationary path measure with $T = \infty$), and the metric comparison assumption ((A3) in the paper; comparison of the Langevin metric, $\kappa = 1/\lambda_G$). $m = \Delta/\hbar$, $\xi = 1/m$, $A = (H - E_0)/\hbar$.

We set the family local in time as follows.

$$\mathcal{F}_1 = \left\{ f_{h,G_1} = \int_{\mathbb{R}} h(\tau)\, G_1(q(\tau))\,d\tau \ :\ h \text{ bounded with bounded support},\ G_1 \text{ is a function on } Q \text{ that is smooth} \right\} \quad (1)$$

The f_T of the one-sided bridge is the case $h = \mathbf{1}_{[0,T]}$, $G_1 = \psi_1/\psi_0$.

The supremum of the one-sided bridge (best within the family local in time)

Under the assumptions of the one-sided bridge and the canonical choice $G = M^{-1}$ ($\kappa = 1$),

$$\sup_{f \in \mathcal{F}_1} \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} = \xi^2 = \frac{\hbar^2}{\Delta^2}, \quad (2)$$

and the supremum is not attained (for every $f \in \mathcal{F}_1$ the ratio is strictly smaller than ξ^2).

Proved here Proof. Let ν_φ be the spectral measure of A for $\varphi = (G1 - \langle G1 \rangle)\psi 0$; ν_φ lies on $[m, \infty)$. From the formula for correlations and Plancherel's theorem,

$$\mathrm{Var}(f) = \int d\nu_\varphi(a) \int_{\mathbb{R}} |\hat{h}(k)|^2 \frac{2a}{a^2 + k^2} \frac{dk}{2\pi}, \qquad \mathcal{E}(f) = \int d\nu_\varphi(a) \int_{\mathbb{R}} |\hat{h}(k)|^2 \, 2a \, \frac{dk}{2\pi} \quad (3)$$

(the right from the identity (18)). For $a \geq m$, $2a/(a^2 + k^2) \leq 2a/m^2$, with equality only at $k = 0$, so the ratio is smaller than ξ^2. That the supremum is ξ^2 is seen by taking $T \to \infty$ in the f_T of the one-sided bridge. ∎ For a general metric, the metric comparison assumption and the reverse comparison $G \geq \lambda'_G M^{-1}$ sandwich the supremum between $\kappa\xi^2$ and ξ^2/λ'_G. Lean `EuclideanPath.GapBridge.local_rayleigh_le` (assumptions: $0 < m$, $m \leq a_k$ for each k, $0 \leq c_k$; conclusion: $m^2 \sum_k c_k/a_k \leq \sum_k c_k a_k$) and `fourier_weight_le` (assumption: $0 < m \leq a$; conclusion: $2a/(a^2 + k^2) \leq 2a/m^2$) check only the finite sums and the pointwise inequality of this proof.

Consequence. The sentence of the upper-bound assumption restricted to $\mathcal{F}_1$, "for every $f \in \mathcal{F}_1$, $\mathrm{Var}(f) \leq C\xi_{\mathrm{RG}}^2 \, \mathcal{E}(f)$", is by (2) the same as $\xi^2 \leq C\xi_{\mathrm{RG}}^2$, that is,

$$\text{The upper-bound assumption on } \mathcal{F}_1 \iff m_{\mathrm{phys}} = m\,\xi_{\mathrm{RG}} \geq C^{-1/2} \text{ (canonical, each } \beta)$$

Proved here . Right to left also follows from (2). Everything by which the upper-bound assumption is stronger than the gap of the lattice in physical units (the conclusion of 23-01 § 06) lies outside $\mathcal{F}_1$. For a free particle on a circle (the Free-Particle-on-a-Circle Counterexample), the winding-number functional $\int w \, d\tilde{x}$ lies outside $\mathcal{F}_1$ and produces $C_P = \infty$, while the supremum over $\mathcal{F}_1$ remains ξ^2.

Correspondence with the paper. The one-sided bridge, (2) and the Free-Particle-on-a-Circle Counterexample are collected in the paper (list of papers, PDF) “The inverse squared spectral gap bounds the Langevin Poincaré constant of Euclidean path measures from below” (Kiichi, Shiori and Rin). The one sentence of its abstract is "for Euclidean path measures stationary over all time on a finite-dimensional configuration space, the Poincaré constant of the stochastic quantisation (Langevin) Dirichlet form on time-averaged smooth functionals satisfies $C_P \geq \kappa(\hbar/\Delta)^2$". The one-sided bridge corresponds to Theorem 3.1 of the paper, (2) to Proposition 4.1, and the Free-Particle-on-a-Circle Counterexample to Proposition 5.1 ($C_P = \infty$ even on the family of time-averaged smooth functionals $\mathcal{D}_0$). This book only cites the equation numbers, and leaves the proofs to 23-01 § 06 and the paper. The paper has not been checked by outside mathematicians.

03

Limiting the window — monotonicity of the window factor, the supremum within the window, the lower bound in finite volume

The definition of the local relaxation-time assumption is in (1)–(3) of 24-03 § 02. We copy only the form used here. The ruler $\ell > 0$ is a length given from outside, independently of the gap (in four dimensions, the bookkeeping length ξ_{RG}). Using the ξ fixed by the gap as the ruler would be circular when deriving a lower bound on the gap. For $c > 0$, write $\mathcal{D}_c$ for the family of time-averaged functionals whose weights are supported in a window of length $c\ell$, and put

$$C_P^{\mathrm{loc}}(c) = \sup_{f \in \mathcal{D}_c} \frac{\mathrm{Var}(f)}{\mathcal{E}(f)}, \qquad (\xi^2)\uparrow_c^{\mathrm{loc}}(K) : \;\; C_P^{\mathrm{loc}}(c) \leq K\ell^2 \quad (4)$$

$C_P^{\mathrm{loc}}(c)$ is fixed only by the marginal law of the path on the window, increases monotonically in c, and returns to the global C_P as $c \to \infty$ (24-03).

The window factor, and its monotonicity

Write $\varphi(x)$ for the bracket on the right of (17) of the one-sided bridge (the φ without argument is the vector of § 02, a different quantity; the notation follows 24-09).

$$\varphi(x) = 1 - \frac{1 - e^{-x}}{x}, \qquad g(x) = \frac{\varphi(x)}{x^2} = \frac{x - 1 + e^{-x}}{x^3} \qquad (x > 0) \ (5)$$

The ratio for a window T is $\kappa\xi^2\varphi(T/\xi) = \kappa T^2 g(mT)$. $\varphi(x) = x/2 + O(x^2)$, $\varphi(x) \to 1$ $(x \to \infty)$, and the values are $\varphi(1) = 0.368$, $\varphi(4) = 0.755$, $\varphi(10) = 0.900$ Proved here (values of the closed formula).

Window-Factor Monotonicity Proposition (the window factor decreases)

g is strictly decreasing on $(0, \infty)$.

Proved here Proof. Since $x - 1 + e^{-x} = \int_0^x \int_0^u e^{-v}\, dv\, du = x^2 \int_0^1 (1-t)e^{-xt}\, dt$, $g(x) = \frac{1}{x}\int_0^1 (1-t)e^{-xt}\, dt$. It is the product of the positive strictly decreasing function $1/x$ and the positive strictly decreasing function $\int_0^1 (1-t)e^{-xt} dt$, so it strictly decreases. ∎ Lean `LatticeGauge.LocalH.wr_strictAntiOn` (no assumptions; conclusion: g is strictly decreasing on $(0, \infty)$. In Lean it is closed via the sign of the derivative $g'(x) = h(x)/x^4$, $h(x) = 3 - 2x - (3+x)e^{-x} < 0$).

The Window-Factor Monotonicity Proposition has two uses. First, in the family where the weight is fixed to the indicator of the window $\mathbf{1}_{[0,T]}$ and only the single-time function varies, the supremum is attained exactly at the lowest excitation $a = m$ and equals $T^2 g(mT) = \xi^2 \varphi(T/\xi)$ Proved here (the ratio of weighted averages over the spectral measure does not exceed the maximum of the ratios per component; that the maximum is attained at $a = m$ is exactly the monotonicity of g). Second, it turns the window-limited upper-bound assumption $\kappa T^2 g(c\, m_{\mathrm{phys}}) \le K\ell^2$ $(T = c\ell)$ into an inequality on one number. g is a continuous strictly decreasing bijection from $(0, \infty)$ to $(0, \infty)$, so

$$\kappa\,(c\ell)^2\, g(c\, m_{\mathrm{phys}}) \le K\ell^2 \iff m_{\mathrm{phys}} \ge \frac{1}{c}\, g^{-1}\!\Big(\frac{K}{\kappa c^2}\Big) \ (6)$$

From $e^{-x} \ge 1 - x + x^2/2 - x^3/6$ $(x \ge 0)$ comes $g(x) \ge 1/(2x) - 1/6$, and the explicit form

$$m_{\mathrm{phys}} \ge \frac{3\kappa c}{6K + \kappa c^2} \ (7)$$

is obtained Proved here . Lean `LatticeGauge.LocalH.gap_lower_of_window` (assumptions: $0 < \kappa, c, K, \ell, m$, $\kappa(c\ell)^2 g(c\ell m) \le C_P$, $C_P \le K\ell^2$; conclusion: $3\kappa c/(6K + \kappa c^2) \le m\ell$). (7) is sharp for $c\, m_{\mathrm{phys}} \lesssim 1$, and for $c\, m_{\mathrm{phys}} \gtrsim 3$ the approximation on the right becomes weak, so for large c we use the next form.

Letting the weight inside the window vary too — the In-Window Supremum Proposition

In-Window Supremum Proposition (supremum of the family local in time within a window)

Under the assumptions of the one-sided bridge and the canonical choice, the supremum over elements of $\mathcal{F}_1$ whose weights are supported in $[0, T]$ is

$$\sup_{f \in \mathcal{F}_1,\ \operatorname{supp} h \subset [0,T]} \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} = \Psi_T(m) := \frac{1}{m^2 + \omega_0^2}, \qquad \omega_0 \tan\frac{\omega_0 T}{2} = m, \quad \omega_0 \in \Big(0, \frac{\pi}{T}\Big) \ (8)$$

For a general metric (the metric comparison assumption) the left-hand side is at least $\kappa\Psi_T(m)$.

Proved here Proof [Res]. Splitting φ into eigencomponents φ_k of A (eigenvalues a_k), $\mathrm{Var} = \sum_k \|\varphi_k\|^2 \langle h, e^{-a_k|s-t|} h\rangle$ and $\mathcal{E} = 2\|h\|^2 \sum_k a_k \|\varphi_k\|^2$ (the identity (18)). The ratio of weighted averages does not exceed the maximum of the ratios per component, so $\mathrm{Var}/\mathcal{E} \le \max_k \langle h, K_{a_k} h\rangle / \|h\|^2$, where K_a is the integral operator with kernel $e^{-a|s-t|}/(2a)$ restricted to $[0, T]$, the compression of $(a^2 - \partial^2)^{-1}$ on the real line. $a \mapsto (a^2 - \partial^2)^{-1}$ is decreasing as an operator, so the maximum is at $a = m$. The largest eigenvalue of K_m is fixed, since the eigenfunction satisfies $u'' = -\omega^2 u$ with the

boundary conditions $u'(0) = m\,u(0)$ and $u'(T) = -m\,u(T)$, by $\omega\tan(\omega T/2) = m$ for the first symmetric solution $u = \cos(\omega(t - T/2))$, and its value is $1/(m^2 + \omega^2)$. Equality is attained at $G_1 = \psi_1/\psi_0$, $h = \cos(\omega_0(t - T/2))$. ∎

The root ω_0 satisfies $\omega_0^2 \le 2m/T$ by $\tan z \ge z$, and also $\omega_0 < \pi/T$, so

$$\max\Big(\frac{1}{m^2 + \pi^2/T^2},\ \frac{1}{m^2 + 2m/T}\Big) \ \le\ \Psi_T(m) \ <\ \frac{1}{m^2} \quad (9)$$

Proved here . The value for the indicator weight, $T^2 g(mT)$, is strictly smaller than $\Psi_T(m)$ (cos is not constant); as $T \to 0$ both are $\approx T/(2m)$, and as $T \to \infty$, $\Psi_T = 1/m^2 - O(T^{-2})$ and $T^2 g = 1/m^2 - O(T^{-1})$.

Table 1 Two values of the window supremum ($m = 1$)

T	Indicator weight $T^2 g(T) = \varphi(T)$	Weight also varied $\Psi_T(1)$	Grade
1	0.368	0.369	The left column is Proved here (closed formula); the right column is Reading (numerical values of the root formula, agreeing within relative 2.5×10^{-5} with the largest eigenvalue of the kernel by the midpoint rule)
4	0.755	0.775	
10	0.900	0.935	

Lower bound in finite spatial volume

Combine the local relaxation-time assumption (4) with the In-Window Supremum Proposition. From $\kappa\Psi_{c\ell}(m) \le C_P^{\mathrm{loc}}(c) \le K\ell^2$ and $\Psi_{c\ell}(m) \ge 1/(m^2 + 2m/(c\ell))$ of (9), $\mu = m\ell = m_{\mathrm{phys}}$ satisfies $\mu^2 + 2\mu/c \ge \kappa/K$, and

$$m_{\mathrm{phys}} \ \ge\ \sqrt{\frac{1}{c^2} + \frac{\kappa}{K}} - \frac{1}{c} \qquad (c > 0) \quad (10)$$

Proved here (under the assumptions of the one-sided bridge). Lean `LatticeGauge.LocalH.gap_lower_of_local_window` (assumptions: $0 < \kappa, c, K, \ell, m$, $\kappa/(m^2 + w^2) \le CP$, $w^2 \le 2m/(c\ell)$, $CP \le K\ell^2$; conclusion: (10)), `local_bound_tendsto` (no assumptions; conclusion: the right-hand side of (10) tends to $\sqrt{\kappa/K}$ as $c \to \infty$). As $c \to \infty$ it returns to (19) of 23-01, and as $c \to 0$ it is $\approx c\kappa/(2K)$. (10) is stronger than (7) for every c.

For the free field, both directions. For a Gaussian path measure the Poincaré constant equals the norm of the covariance operator Cited (Hairer–Stuart–Voss–Wiberg 2005, Lemma 2.2), and $C_P^{\mathrm{loc}}(c) = \Psi_{c\ell}(m)$ holds exactly even over the whole window family $\mathcal{D}_c$ Proved here (the largest eigenvalue of the covariance compressed to the window is the value of the In-Window Supremum Proposition). Hence for the free field, besides "$(\xi^2)\uparrow_c^{\mathrm{loc}}(K) \Rightarrow$ (10)", also "$m_{\mathrm{phys}} \ge K^{-1/2} \Rightarrow (\xi^2)\uparrow_c^{\mathrm{loc}}(K)$ for every c" holds ($\Psi < 1/m^2$). One also sees that localisation is not trivial: $C_P^{\mathrm{loc}}(c) < CP$ holds strictly for every finite c, and as $c \to 0$ it is $\approx c\ell/(2m) \to 0$.

Conditions of this section. (4)–(10) are sentences on a lattice with continuous time, a finite spatial lattice, $T = \infty$, and the three assumptions of the one-sided bridge (ground state, stationary path measure, metric comparison). On Wilson's lattice, which also discretises time, the time-step comparison assumption ((20) of 23-01) remains, and the window version has not yet been written on top of it.

04

"Some $K(c)$" is empty — the massless free field as a negative control, and the ratio form

Now go local in space too. On an infinite spatial lattice, write $C_P^{\mathrm{loc}}(B_R)$ for the local Poincaré constant with only functionals supported in a spacetime box B_R of side $R = c\ell$ allowed to vary (the version of 24-03 § 03 that is local in space too; (11) below is the same inequality as (4) of 24-03). The plain local relaxation-time assumption is "$C_P^{\mathrm{loc}}(B_{c\ell}) \le K(c)\,\ell^2$". But in four dimensions this form says nothing about the gap.

Negative control (the massless free field)

For the massless free scalar field in four-dimensional spacetime and infinite volume (canonical graduation, $\hbar = 1$),

$$\frac{R^2}{4\pi^2} \ \le\ C_P^{\mathrm{loc}}(B_R) \ \le\ R^2. \quad (11)$$

Hence "$C_P^{\mathrm{loc}}(B_{c\ell}) \le K(c)\ell^2$" holds with $K(c) = c^2$, and the gap is 0.

Proved here Proof. The Gaussian Poincaré constant is the norm of the covariance compressed to the box Cited (previous section). The covariance $(-\Delta)^{-1}$ is transient in four dimensions and hence finite, with continuum kernel $1/(4\pi^2|z|^2)$. Above, by the Schur estimate, the norm is at most $\sup_x \sum_{y\in B_R} G(x-y) \le \int_{|z|<2R} \frac{d^4z}{4\pi^2|z|^2} = R^2$ (the asymptotics of the lattice kernel replaced by the continuum kernel; this step is Assumption). Below, take the Dirichlet ground state w of the box as test direction; from $\langle w, Gw\rangle\langle w, -\Delta w\rangle \ge \|w\|^4$ (Cauchy–Schwarz), $C_P^{\mathrm{loc}} \ge 1/\lambda_D(B_R) \approx R^2/(4\pi^2)$ (π^2/R^2 for each of the four directions of the spacetime box). ∎

In numbers, the ratio $\Gamma(R) := C_P^{\mathrm{loc},0}(B_R)/R^2$ decreases as $0.0802,\ 0.0768,\ 0.0756,\ 0.0750,\ 0.0747,\ 0.0744$ at $R = 4, 6, 8, 10, 12, 16$; the correction is $1/R^2$ with no logarithmic correction, and by extrapolation $\Gamma_\infty = 0.0740$ Reading (floating point; numerical integration of the lattice heat kernel agreeing with an independent route via periodic lattices within relative 2×10^{-3}; extrapolation from $R \le 16$; one component). This is inside the paper sandwich $[1/(4\pi^2),\ 0.1125]$, 2.92 times the Dirichlet value $1/(4\pi^2) = 0.0253$. Since the continuum kernel is homogeneous of degree -2, the compression being exactly proportional to R^2 is as expected.

One line of contrast with lower dimensions as well. For a massless field in two spacetime dimensions the infrared diverges and $C_P^{\mathrm{loc}} = \infty$, and even in finite spatial volume with only the time window localised, the constant mode grows as $\approx T/(2m)$. A massless field has a finite local constant only in spacetime dimension three or more when localised in space too, and the four-dimensional setting of Part 24 is exactly that case Proved here .

The ratio form

So we state the local relaxation-time assumption as a ratio to the massless free field (the reference) on the same lattice, the same box and the same graduation.

$$K_{\mathrm{rel}}(c) = \frac{C_P^{\mathrm{loc}}(B_{c\ell};\mu)}{C_P^{\mathrm{loc},0}(c\ell)}, \qquad C_P^{\mathrm{loc},0}(R) = \Gamma(R)\,R^2, \qquad (\xi^2)^{\mathrm{rel}}_{\uparrow c}(K):\ \ K_{\mathrm{rel}}(c) \le K \quad (12)$$

(uniformly in $\beta \ge \beta_0$, $\ell = \xi_{\mathrm{RG}}(\beta)$). The constant K becomes an absolute number. We set it as **a sentence at a single** c, and do not require it uniformly in c. The ratio $r(mR)$ for the free field of mass m tends to 1 as $mR \to 0$, so a sentence requiring $K < 1$ uniformly down to short distances becomes false for theories close to free fields Estimate . Requiring a K uniform in c returns to the global upper-bound assumption as $c \to \infty$ (24-03).

For a field with a convex interaction added (action $\frac12\langle\Phi, -\Delta\Phi\rangle + \sum_x U(\Phi_x)$, U convex), the Hessian is at least $-\Delta$, so the Brascamp–Lieb inequality gives $K_{\mathrm{rel}} \le 1$ Assumption (the bibliographic data of the inequality have not been checked). Convex fields lie in the band $[0, 1]$. What the bridge needs, as the next section shows, is the part of it below the threshold.

05

The bridge in infinite volume and the threshold — the local-excitation moment assumption and transfer to the ratio

In infinite volume, the single-time function $F = \psi 1/\psi 0$ is neither local nor explicit. Instead we assume that local observables in a box produce low-energy excitations.

Local-excitation moment assumption

There is a local observable O supported in a spatial box of side $c\ell$ such that the spectral measure ν_φ, with respect to the generator of the transfer A, of $\varphi = (O - \langle O\rangle)\Omega$ (Ω is the vacuum) satisfies

$$\frac{M_3}{M_1} := \frac{\int a^3\, d\nu_\varphi(a)}{\int a\, d\nu_\varphi(a)} \ \le\ m_\infty^2 + \frac{\gamma}{(c\ell)^2} \quad (13)$$

m_∞ is the infinite-volume gap in lattice units, and γ a constant independent of β.

Assumption It has not been shown for any interacting model. Assumption (13) contains the quantity of the conclusion m_∞, but it appears only in a comparison from above and does not imply $m_\infty > 0$. It is a sentence that has meaning even for $m_\infty = 0$ (a property of approximation, that low-energy excitations can be approached by local observables), and the massless free field satisfies it with $\gamma = d\pi^2$.

Proved here It holds for the free field. On one-particle states $\nu_\varphi = \sum_p |\hat{w}_p|^2/(2\omega_p)\,\delta_{\omega_p}$, $M_3/M_1 = m_0^2 + \langle w, -\Delta w\rangle/\|w\|^2 \le m_0^2 + d\pi^2/(c\ell)^2$ (w is the Dirichlet ground state of the spatial box, d the spatial dimension).

By Jensen's inequality, the ratio for a window T is bounded as $\ge \kappa/(M_3/M_1 + \pi^2/T^2)$. Treating the spacetime box isotropically and putting $\lambda_W := \gamma + \pi^2$,

$$C_P^{\mathrm{loc}}(B_R) \;\ge\; \frac{\kappa}{m_\infty^2 + \lambda_W/R^2} \quad (14)$$

and from this and the local relaxation-time assumption $C_P^{\mathrm{loc}} \le K\ell^2$, $m_{\mathrm{phys}}^2 \ge \kappa/K - \lambda_W/c^2$ Proved here (on top of (13) and the infinite-volume ground-state transformation identity). Lean `LatticeGauge.LocalH.gap_lower_dirichlet` (assumptions: $0 < \kappa, c, K, \ell$, $0 < m^2 + \lambda$, $\kappa/(m^2+\lambda) \le C_P$, $\lambda \le \gamma/(c\ell)^2$, $C_P \le K\ell^2$; conclusion: $\kappa/K - \gamma/c^2 \le (m\ell)^2$).

Transferring to the ratio form (12), from $C_P^{\mathrm{loc}} \le K_{\mathrm{rel}}\Gamma R^2$ and (14),

$$(m_\infty R)^2 \;\ge\; \lambda_W\Big(\frac{K_*}{K_{\mathrm{rel}}} - 1\Big), \qquad K_* := \frac{\kappa}{\lambda_W\,\Gamma} \quad (15)$$

Proved here (algebra on top of (13) and the infinite-volume identity). Lean `LatticeGauge.LocalHReference.ratio_bridge` (assumptions: $0 < \lambda_W, \Gamma, R, K_{\mathrm{rel}}$, $\kappa/(m^2 + \lambda_W/R^2) \le C$, $C \le K_{\mathrm{rel}}\Gamma R^2$; conclusion: (15)), `kstar_correspondence` (assumptions: $0 < \lambda_W, \Gamma, c$; conclusion: when $K(c) = K_{\mathrm{rel}}\Gamma c^2$, $K(c) < \kappa c^2/\lambda_W \iff K_{\mathrm{rel}} < K_*$), `ratio_bridge_vacuous_of_ge` (assumption: $K_* \le K_{\mathrm{rel}}$; conclusion: the right-hand side of (15) ≤ 0).

How to read it. If $K_{\mathrm{rel}} < K_*$, the right-hand side is positive, and the lower bound on the mass

$$m_\infty\,\ell \;\ge\; \frac{\sqrt{\lambda_W}}{c}\sqrt{\frac{K_*}{K_{\mathrm{rel}}} - 1} \quad (16)$$

follows. If $K_{\mathrm{rel}} \ge K_*$, (15) is empty. A reference satisfies its own Dirichlet lower bound ($\Gamma \ge 1/\lambda_W$, the same Cauchy–Schwarz as on the left of (11)), so for every reference $K_* \le \kappa$ Proved here . Lean `LatticeGauge.LocalHReference.kstar_le_kappa` (assumptions: $0 \le \kappa$, $0 < \lambda_W, \Gamma$, $1 \le \lambda_W\Gamma$; conclusion: $K_* \le \kappa$). For a field with ratio 1 ($K_{\mathrm{rel}} = 1$) the bridge does not stand (`rel_one_no_bridge`, negative control).

The λ_W in (15) needs care. λ_W is "the value of M_3/M_1 attainable within the allowed family of local observables", a number fixed once the family of test observables is chosen. The reference Γ in the denominator and the λ_W of the lower bound in the numerator must be taken over the same family. In the next section, this point moves the threshold.

06

Three references — scalar, gauge quotient, colour singlet

Table 2 sets out the values for each choice of reference. Every row is a number for a free-field reference; not a single ratio K_{rel} for the target Wilson measure has been measured.

Table 2 Three references and thresholds (four dimensions, canonical graduation; numbers are readings from floating-point computation and extrapolation)

Reference	Family of test observables	Γ_∞	λ_W (times R^2)	$\lambda_W\Gamma$	Threshold K_*/κ	Box at which the bridge sees the mass	Conditions

Reference	Family of test observables	Γ_∞	λ_W (times R^2)	$\lambda_W\Gamma$	Threshold K_*/κ	Box at which the bridge sees the mass	Conditions
Scalar	All functions of the field in the box (best is linear)	0.0740	$4\pi^2 \approx 39.5$	2.92	0.342	$mR \approx 4.2$	One component, extrapolation from $R \le 16$ Reading
Gauge quotient	Quotient by gauge transformations at **all** sites of the box = functions of the plaquette angles of the box	0.0277	72.0	1.99	0.502	$mR \approx 5.2$ (transverse Proca waves)	Gaussian for one colour component, $\beta = 1$, extrapolation from $R \le 16$ ($R \le 14$ for λ_W) Reading
Colour singlet of $SU(2)$	Colour singlets (leading term is the second chaos), a family including loops the size of the span of the box	$\Gamma_Q/2 \approx 0.0138$	115–117 (test value)	About 1.6	About 0.62 (lower value)	—	Leading term ($\beta \to \infty$), independent Gaussians for three colours, first-order test, extrapolation from $R \le 10$ Reading

Lean `LatticeGauge.LocalHReference.kstar_measured` (assumption: measured $\Gamma \in [0.0739, 0.0741]$; conclusion: $0.34 < K_* < 0.344$), `LatticeGauge.GaugeReference.kstar_gauge_measured` (assumptions: measured $\Gamma_Q \in [0.02762, 0.02772]$, $\lambda_W^Q \in [71.8, 72.1]$; conclusion: $0.5 < K_*^Q < 0.505$), `LatticeGauge.WilsonLoopReference.kstar2_measured` (assumptions: measured $\lambda^{(2)} \in [115, 118]$, $\Gamma_Q \in [0.02762, 0.02772]$; conclusion: $0.61 < K_*^{(2)} < 0.63$). All three are arithmetic with measured numbers as assumptions; the numbers themselves are outside Lean.

Why the threshold changes with the reference

Gauge-fixed values are not references. The value for the gauge-fixed free gauge field (Feynman gauge) is $\Gamma = 0.0740$, the same as the scalar, but it changes when the gauge-fixing parameter α is changed, and when the fixing is removed ($\alpha \to \infty$) it becomes infinite in the directions of gauge transformations within the box Proved here (with test functions in the directions of the image of the gauge, $\Gamma_a(\alpha) \ge \alpha/(\lambda_D^{\mathrm{int}} R^2)$). The target Wilson measure is not gauge-fixed, so the numerator of the ratio is finite only on gauge-invariant functions. The denominator is measured on the same space. Lean `LatticeGauge.GaugeReference.gauge_param_unbounded` (assumptions: $0 < \lambda_D, R$, $\alpha/(\lambda_D R^2) \le \Gamma_a(\alpha)$; conclusion: Γ_a is unbounded in α).

Gauge-invariant observables can only have high momentum. The test space of the gauge quotient is the divergence-free fields (transverse waves) on the graph of the box. If the box is contractible, this coincides exactly with the space of functions of the plaquette angles of the box Proved here (the first homology of the cubical complex is 0, so closed paths are boundaries of sums of plaquettes). Under the divergence-free constraint one cannot build a positive field pointing the same way over the whole box (the optimal test direction for the scalar). The optimal direction becomes a transverse wave of large momentum: $\lambda_W^Q = 72.0$ is 1.82 times the scalar $4\pi^2$, and Γ_Q is 0.374 times the scalar Reading. The threshold of the ratio relaxes, $0.342 \to 0.502$, but this is because Γ shrinks by more than λ_W grows ($0.502/0.342 = 1/(0.374 \times 1.82)$); the condition rewritten for the local Poincaré constant itself, $C_P^{\mathrm{loc}} < \kappa R^2/\lambda_W$, tightens from $0.0253\kappa R^2$ to $0.0139\kappa R^2$ Reading.

Divide by the gauge transformations at the boundary of the box too. If one divides only by gauge transformations at inner sites of the box and freezes the gauge at the boundary sites, flux passing uniformly through the box enters the test space, and the value returns roughly to the gauge-fixed value, 2.2–2.5 times Γ_Q Reading. Gauge-invariant observables of the box (Wilson loops) are invariant under transformations at all sites, so "dividing by gauge transformations at all sites" is part of the definition of the reference (the row "flux through a box" in 24-01).

What the one-colour approximation tacitly assumed. Linearising to one colour component, linear combinations of plaquette angles inside the box alone span the whole transverse test space, and neither large loops nor plaquettes reaching outside the box are needed Proved here (the angle of a rectangular loop is the sum of the angles of the

plaquettes inside). But the gauge invariants of $SU(2)$ (colour singlets) have no linear term in the expansion around the identity Proved here . A linear term $\sum c_\ell^a A_\ell^a$ must also be invariant under constant gauge transformations (global colour rotations $SO(3)$), and $SO(3)$ has no fixed vector in $\mathbb{R}^3$, so $c = 0$. The trace of a loop $\frac{1}{2}\operatorname{tr} U_C = \cos(|\theta_C|/2) = 1 - |\theta_C|^2/8 + \cdots$ starts at second order in the angles. Lean `LatticeGauge.WilsonLoopReference.no_linear_singlet` (assumption: a linear form invariant under the two rotations $\operatorname{diag}(-1,-1,1)$, $\operatorname{diag}(1,-1,-1)$; conclusion: the coefficients are 0).

The reference is exactly half. In the leading term ($\beta \to \infty$, independent Gaussians for three colours), functions that are colour singlets lie in the second chaos and above. The eigenvalues of the Ornstein–Uhlenbeck generator of a Gaussian measure are sums of one-particle eigenvalues in the eigencoordinates of the covariance, so the minimum on the second chaos is twice the one-particle minimum, and the Poincaré constant is $\Gamma_Q/2$ Proved here (equality at $f_w = \sum_a (\langle w, A^a\rangle^2 - \mathbb{E})$, with w the optimal transverse direction). The value of the first-order test is $\lambda_W(f_w) = \lambda(w) + 1/C(w)$, and by Cauchy–Schwarz $\lambda(w) < \lambda_W(f_w) \le 2\lambda(w)$ Proved here . In the four-dimensional box $\lambda^{(2)}R^2 \to 115$ to 117 (about 1.6 times λ_W^Q), and the threshold is $K_*^{(2)} = 2/(\lambda^{(2)}\Gamma_Q) \approx 0.62\kappa$ Reading . $\lambda^{(2)}$ is a test value (a value from above), so the threshold obtained is a lower value of the usable threshold. Lean `chaos2_gamma`, `chaos2_lambda` (assumptions: $0 < n, g$ and the Wick moments as formulas; conclusion: algebra of the ratio), `chaos2_bracket` (assumptions: $0 < C$, $1 \le \lambda C$; conclusion: $\lambda < \lambda + 1/C \le 2\lambda$).

Large loops are needed. To get this value, loops the size of the span of the box are needed. With the family of plaquettes only, $\{\operatorname{tr} U_p\}$, the reference in the leading term vanishes as $1/R^2$ (in two dimensions, from 0.29 times to 0.035 times $\Gamma_Q/2$ for $R = 4 \to 12$), and even adding loops up to 2×2 it is 0.12 times ($R = 12$), so the bridge speaks only of the lattice scale Reading . Including all rectangles spanning the box, it agrees with $\Gamma_Q/2$ to all digits in two dimensions, while flat rectangles alone give 0.73 times in three dimensions Reading . There are no values for large loops in four dimensions (including non-flat ones and products of lassos).

Box at which the bridge sees the mass

The ratio for the free field of mass m, $r(mR) = C_P^{\mathrm{loc}}(R;m)/C_P^{\mathrm{loc}}(R;0)$, collapses within 0.3% for $R = 4$ to 12 onto a function of mR alone Reading . For the scalar $r = 0.932,\ 0.825,\ 0.623,\ 0.359,\ 0.0772$ ($mR = 0.5, 1, 2, 4, 12$); for the transverse waves of the gauge quotient (Proca) $r_Q = 0.950,\ 0.842,\ 0.721,\ 0.610,\ 0.514,\ 0.434$ ($mR = 1, \ldots, 6$). The ratio falls to the threshold at $mR \approx 4.2$ for the scalar and $mR \approx 5.2$ for transverse waves (both by linear interpolation) Reading . **The bridge sees the mass only when the side of the box is at least 4–5 times the correlation length.** Transverse waves have large momentum, so they notice the mass later.

Put the other way, for fields whose ratio lies in the band $[K_*, 1)$, this bridge says nothing. Free fields of mass $mR < 4.2$ (with the scalar reference) also fall in this band, so the bridge misses fields that do have mass. Whether there is another argument (outside Jensen's inequality and Dirichlet test functions) that sees the mass for fields in the band is not settled.

07

Note on 0+1 dimensions — even for the shortest window the ratio is ω_b/m

We try the ratio form on the 0+1-dimensional double well $V(x) = (x^2-1)^2/8$ ($V''(\pm 1) = 1 = \omega_b^2$, $S_0 = 2/3$). In 0+1 dimensions there is no finite massless reference (a free particle has no stationary measure and $C_P^{\mathrm{loc}} = \infty$). There is no choice but to take as reference the harmonic oscillator ω_b with the curvature of the bottom, and the ratio sentence becomes "if $K_{\mathrm{rel}}(T) < \kappa$ then $m > \omega_b$" Proved here (from the lower bound of the In-Window Supremum Proposition and the decrease of Ψ_T in m. The threshold is κ, and its difference from the four-dimensional 0.342κ is exactly the Dirichlet loss of the spatial box). Lean `LatticeGauge.LocalHReference.gap_gt_ref_of_ratio` (assumptions: Ψ is monotone non-increasing, $0 < \kappa$, $0 < \Psi(\omega_b)$, $\kappa\Psi(m) \le C \le K\Psi(\omega_b)$, $K < \kappa$; conclusion: $\omega_b < m$).

The window of a single time can be computed in closed form. On a lattice with time step ε, the Langevin generator for the single-time marginal ψ_0^2 and $\mathcal{E}_1(f) = (\hbar/\varepsilon)\int \psi_0^2 |f'|^2$ is, by the ground-state transformation,

$$L = -\frac{2}{\varepsilon\hbar}\,\psi_0^{-1}(H - E_0)\,\psi_0, \qquad C_P^{\mathrm{loc}}(1) = \frac{\varepsilon}{2m}, \qquad K_{\mathrm{rel}}(1) = \frac{\omega_b}{m} \quad (17)$$

Proved here (when ψ_0 is the Schrödinger ground state; the reference is $\varepsilon/(2\omega_b)$). This is exactly the identity of 24-05. Lean `LatticeGauge.LocalHDoubleWell.one_slice_ratio` (assumptions: $0<\varepsilon, m, \omega$; conclusion: $(\varepsilon/(2m))/(\varepsilon/(2\omega)) = \omega/m$).

The behaviour as the window is lengthened divides, on top of the In-Window Supremum Proposition, into three regimes Proved here .

Table 3 The ratio $K_{\mathrm{rel}}(T)$ for the 0+1-dimensional double well (three regimes of the value $\Psi_T(m)/\Psi_T(\omega_b)$ of the $\mathcal{F}_1$ part)

Window	Ratio	Order for $m \propto e^{-S_0/\hbar}$
$T \to 0$ (a single time)	ω_b/m	$e^{S_0/\hbar}$
$1/\omega_b \ll T \ll \xi$	$\omega_b^2 T/(2m)$	$T\, e^{S_0/\hbar}$
$T \gg \xi$	$(\omega_b/m)^2$	$e^{2S_0/\hbar}$

The crossover is the window at which $\Psi_T(m)$ becomes half of $1/m^2$: $\omega_0 = m$, that is $\tan(mT/2) = 1$, $T_\times = \pi/(2m)$ Proved here . Lean `LatticeGauge.LocalHDoubleWell.crossover_root`, `crossover_half` (assumption: $0<m$; conclusion: at $T = \pi/(2m)$, $m\tan(mT/2) = m$ and the value is half. Uniqueness of the root is outside Lean).

The lattice values (time step $\varepsilon = \frac{1}{2}$, transfer kernel as Trotter's symmetric product, $\hbar = 0.3,\ 0.15,\ 0.08$) are $K_{\mathrm{rel}}(1) = 4.61,\ 21.6,\ 638$ and $K_{\mathrm{rel}}(1)\cdot m = 1.04,\ 1.04,\ 1.05$ for the window of a single time Reading (the deviation from (17) comes from ψ_0 of the time-discretised transfer and the discretisation of the reference). The crossover on the lattice is $k_\times m\varepsilon = 1.571$ to 1.574 ($\pi/2 = 1.5708$) Reading . Extending the functionals in the window from sums of one-point functions to multi-point functions raises the ratio further: the direct value for a window of two times is 1.02, 1.09, 1.34 times the $\mathcal{F}_1$ value, and the lower bound for a window of eight times with a family up to three-point products is at least 1.07, 1.27, 2.12 times Reading (lower bounds; the smaller $\hbar$, the more they rise with each widening of the family, and they have not saturated).

The ratio sentence "if $K_{\mathrm{rel}} < \kappa$ then $m > \omega_b$" correctly fails for the double well, whose gap is small because of tunnelling ($K_{\mathrm{rel}} \geq 4.6 > \kappa = 1$ for every window). 0+1 is not an analogue of the four-dimensional ratio but a negative control showing that the ratio form responds correctly to non-convexity.

What to take from this note to four dimensions. The non-convexity of the double well does not depend on the length of the window; it shows up in the ratio from the shortest window, as the bimodality of the single-time distribution. Hence "at short distance the ratio is close to 1" (the reason § 04 did not require uniformity in c) is not a fact that follows from the ratio form. Expecting it in four dimensions is an assumption depending on asymptotic freedom, that the theory is close to a free field at short distance Assumption .

08

Which part of which assumption each conclusion needs

Table 4 The part of the relaxation-time assumption each conclusion needs, and the other assumptions needed

Conclusion	Part of the relaxation-time assumption needed	Other assumptions needed	Grade
Lower bound on the gap of the lattice in physical units, in finite spatial volume, at each β	The local relaxation-time assumption on a window $c\ell$ ((4)), or the upper-bound assumption restricted to $\mathcal{F}_1$	The assumptions of the one-sided bridge (continuous time, finite spatial lattice, $T = \infty$). On Wilson's lattice with discretised time, the time-step comparison assumption remains, and the window version has not been written	Proved here ((6)(10)), Assumption (the time-step comparison assumption)
Lower bound on the gap of the lattice in physical units in infinite volume	$K_{\mathrm{rel}}(c) < K_*$ at a single c (ratio form (12))	the local-excitation moment assumption, the infinite-volume ground-state transformation identity, κ uniform in β, test observables from the family of colour singlets (including loops spanning the box)	Assumption
Speed of passage between sectors	The sector diffusive-relaxation-time assumption and the quantity between sectors C_{trans} (24-03)	Unrelated to the mass gap	Estimate

Conclusion	Part of the relaxation-time assumption needed	Other assumptions needed	Grade
The upper-bound assumption itself	All f	The stage of closed boxes (24-03)	Assumption (in the four-dimensional periodic box, on the side that reads as false on the numbers of original sources for $SU(3)$ and above and a reading that extends a conjecture)

The point of Table 4 is that **the conclusion of no row needs the global upper-bound assumption**. What the one-sided bridge uses is only the part of the upper-bound assumption local in time (finite volume), or the ratio form of the local relaxation-time assumption (infinite volume). The rest of the upper-bound assumption — the sentence about functions not local in time — is where quantities like the topological charge can obstruct the upper-bound assumption (24-01), and the bridge does not pass through there Proved here (the first row of Table 4), Estimate (the reading of the second row onwards).

The Jaffe–Witten exposition of the problem writes that "new ideas are needed to prove a mass gap uniform in the volume of spacetime" Cited . The local relaxation-time assumption ties the size of the box to the bookkeeping length and stands as a sentence independent of the volume, so it is one form of stating "uniform in the volume" in the language of Poincaré Estimate . But there are no tools to show it.

09

What cannot be said

- The local-excitation moment assumption has not been shown for any interacting model. The infinite-volume ground-state transformation identity is also an assumption. The infinite-volume bridge (15)(16) is algebra on top of these two.
- The values of Table 2 are readings from floating-point computation and extrapolation ($R \le 16$ for the scalar and transverse waves, $R \le 10$ for the colour singlet). 0.62κ is a threshold obtained from the value of a first-order test, a lower value of the usable threshold. The colour singlet is in the approximation of the leading term ($\beta \to \infty$) with independent Gaussians for three colours; there are no values for $SU(2)$ at finite β. Nor are there values for large loops in four dimensions.
- "The bridge sees the mass only when the box is at least 4–5 times the correlation length" is a linear interpolation of the ratios for massive **free fields**. There are no values of the ratio for interacting fields.
- For fields in the band $[K_*, 1)$, the bridge says nothing. Another argument that sees the mass within the band is undecided.
- "At short distance the ratio is close to 1" is an assumption depending on asymptotic freedom; it does not follow from the ratio form (it is false for the 0+1 double well).
- The window argument consists of sentences on a lattice with continuous time; on Wilson's lattice with discretised time, the time-step comparison assumption remains.
- Every conclusion goes only as far as a lower bound on the gap of the lattice in physical units at each β. The limit of $m_{\mathrm{phys}}(\beta)$, the continuum limit, and the existence of the infinite-volume measure do not follow. None of the mass gap, the relaxation-time assumption, the upper-bound assumption or the local relaxation-time assumption has been shown.

10

Table of the five grades and the dependency graph

We set out the rows of this chapter in the format of the five grades (23-00 § 01) with Estimate added. The symbols of the rows are the node numbers of the dependency graph below.

Table 5 Rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Existence and uniqueness of the infinite-volume measure	Assumption	23-01 § 02	$\beta \ge \beta_0$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
2	The ground-state assumption, the stationary path-measure assumption, the metric comparison assumption: discrete spectrum, stationary measure with $T = \infty$, comparison of metrics	Assumption	23-01 Table 6	Continuous time, finite spatial lattice	—
3	Properties of the local C_P (fixed by the window marginal alone, monotone in c, C_P as $c \to \infty$)	Proved here	24-03	Continuous time	`local_le_global`, `local_approaches_global` (assumptions: bounded, non-empty, union of the families is the whole)
4	$(\xi^2)\uparrow_c^{\mathrm{rel}}(K)$: $K_{\mathrm{rel}}(c) \le K$ at a single c	Assumption	(12)	—	—
5	$(\xi^2)\uparrow_c^{\mathrm{loc}}(K)$: $C_P^{\mathrm{loc}}(c) \le K\ell^2$ (finite spatial volume)	Assumption	(4)	—	—
6	The local-excitation moment assumption: moment ratio of excitations by local observables	Assumption	(13)	—	—
7	Infinite-volume ground-state transformation identity	Assumption	§ 05	—	—
8	κ uniform in β	Assumption	23-01 § 06	Expected to hold in the field graduation	—
9	Values of the three references Γ_∞, λ_W (Table 2)	Reading	§ 04, § 06	Free field, extrapolation	—
10	Identity of the ground-state transformation	Proved here	23-01 (18)	Finite spatial lattice	—
11	The one-sided bridge and (2): the supremum over $\mathcal{F}_1$ is exactly ξ^2	Proved here	23-01 (17), (2) (3)	Continuous time, canonical	`poincare_lower_of_gap` (assumptions: $0 < p, m, \kappa, E$, $2p/m \le V$, $E \le 2mp/\kappa$, $V/E \le C_P$), `local_rayleigh_le`
12	Window-Factor Monotonicity Proposition: $g(x) = \varphi(x)/x^2$ is strictly decreasing	Proved here	§ 03	$x > 0$	`wr_strictAntiOn` (no assumptions)
13	In-Window Supremum Proposition: the window supremum $\Psi_T(m)$	Proved here	(8)	Continuous time, canonical	— (existence of the root is outside Lean)
14	Window lower bound (7)	Proved here	(6)(7)	Finite spatial volume	`gap_lower_of_window` (assumptions beside (7))
15	Window lower bound (10)	Proved here	(10)	Finite spatial volume	`gap_lower_of_local_window` (assumptions beside (10))
16	The massless free field satisfies the "some $K(c)$" sentence ((11))	Proved here	§ 04	Four-dimensional spacetime, infinite volume (asymptotics of the lattice kernel is an assumption)	`dirichlet_bound_vacuous` (assumptions: $0 < \ell$, $\kappa c^2/\gamma \le K$; conclusion: the assumptions of this argument are compatible with $m = 0$)
17	The ratio bridge (15)	Proved here	§ 05	On top of rows 6, 7	`ratio_bridge`, `kstar_le_kappa`
18	Thresholds 0.342κ, 0.502κ, about 0.62κ (lower value)	Reading	Table 2	Free field, extrapolation	`kstar_measured`, `kstar_gauge_measured`, `kstar2_measured` (measured intervals assumed)
19	Colour singlets have no linear term; the reference is half	Proved here	§ 06	Leading term, independent Gaussians for three colours	`no_linear_singlet`, `chaos2_gamma`
20	The ratio sentence in 0+1 and the single-time ratio (17)	Proved here	§ 07	0+1, Schrödinger ψ_0	`gap_gt_ref_of_ratio`, `one_slice_ratio`
21	"At short distance the ratio is close to 1"	Assumption	§ 07	Depends on asymptotic freedom	—
22	Lower bound on the gap in finite spatial volume	Proved here	First row of Table 4	On top of rows 2, 5	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
23	Lower bound on the gap of the lattice in physical units in infinite volume	Proved here	Second row of Table 4	On top of rows 1, 4, 6, 7, 8	—

The namespaces of the Lean theorems are `LatticeGauge.LocalH` (rows 3, 12, 14, 15, 16), `LatticeGauge.LocalHReference` (row 17, the first of row 18, the first of row 20), `LatticeGauge.GaugeReference` (the second of row 18), `LatticeGauge.WilsonLoopReference` (the third of row 18, row 19), `LatticeGauge.LocalHDoubleWell` (the second of row 20), `EuclideanPath.GapBridge` (row 11). Every bundle guarantees only the algebra, arithmetic and one-variable analysis by which the conclusion follows from the stated assumptions; the path measure, the identities, the In-Window Supremum Proposition and the numerical values of the references are outside Lean (23-13).

Dependency graph (premise numbers are smaller than conclusion numbers; 1, 3, 4, 16 correspond to 1 (existence of the infinite-volume measure), 6 (properties of the four-dimensional local Poincaré constant), 12 (the local relaxation-time assumption) and 7 (vacuity of the absolute form) of the dependency graph of 24-03):

```
 1  ASM  existence and uniqueness of the infinite-volume measure      β ≥ β₀
 2  ASM  the three assumptions of the one-sided bridge
         (ground state, stationary path measure, metric comparison)
         continuous time, finite spatial lattice, T=∞
 3  PRV  local C_P: fixed by the window marginal, monotone in c,
         C_P as c→∞                                          Lean local_le_global
 4  ASM  local relaxation-time assumption (ratio form):
         K_rel(c) ≤ K at a single c
 5  ASM  local relaxation-time assumption (finite-volume form):
         C_P^loc(c) ≤ K ℓ²
 6  ASM  local-excitation moment assumption (13)
 7  ASM  infinite-volume ground-state transformation identity
 8  ASM  κ uniform in β
 9  RDG  reference values Γ∞, λ_W (Table 2)
10  PRV  ground-state transformation identity ((18) of 23-01)
11  PRV  one-sided bridge: supremum over F₁ is exactly ξ² ⟸ 2, 10     Lean poincare_lower_of_gap
12  PRV  window-factor monotonicity: g = φ/x² strictly decreasing     Lean wr_strictAntiOn
13  PRV  supremum within the window: Ψ_T(m) ⟸ 2, 10
14  PRV  window lower bound (7) ⟸ 5, 11, 12                          Lean gap_lower_of_window
15  PRV  window lower bound (10) ⟸ 5, 13                             Lean gap_lower_of_local_window
16  PRV  negative control: the massless free field satisfies "some K(c)" (11)
17  PRV  ratio bridge (15) ⟸ 3, 4, 6, 7                              Lean ratio_bridge
18  RDG  threshold K* = 0.342κ / 0.502κ / about 0.62κ ⟸ 9            Lean kstar_*_measured
                                                          (measured intervals assumed)
19  PRV  singlet: no linear term, reference is half                 Lean no_linear_singlet, chaos2_gamma
20  PRV  0+1: K_rel < κ ⇒ m > ω_b, single-time ratio ω_b/m ⟸ 10, 13
21  ASM  short distance: "the ratio is close to 1"
         (depends on asymptotic freedom; false for row 20 in 0+1)
22  PRV  lower bound on the lattice gap in physical units,
         finite volume, at each β ⟸ 14, 15
23  PRV  lower bound on the lattice gap in physical units,
         infinite volume (conditional) ⟸ 1, 8, 17, 18, 19
```

The check for circles is only to see that on every "⟸" the numbers on the right are smaller than the number on the left. Among the assumption rows, only the local-excitation moment assumption contains the quantity of the conclusion (the gap m_∞), and there m_∞ appears only in a comparison from above and does not imply $m_\infty > 0$, as checked in § 05. The node of the massless free field is a negative control that does not enter any conclusion; it records why the form "some $K(c)$" was discarded. "At short distance the ratio is close to 1" is an assumption, and the two gap lower bounds of the conclusions do not depend on it (because c was fixed to a single value).

What this chapter can say

Said	Not said
That the supremum over the family local in time $\mathcal{F}_1$ is exactly ξ^2 in the canonical graduation, and that the upper-bound assumption on $\mathcal{F}_1$ is the same as a lower bound on the gap of the lattice in physical units (continuous time, finite spatial lattice, $T = \infty$) Proved here	The same sentence on Wilson's lattice with discretised time (the time-step comparison assumption remains). Whether the part of the upper-bound assumption outside $\mathcal{F}_1$ is true
Monotonicity of the window factor, the In-Window Supremum Proposition, and the lower bounds (7)(10) on the gap in finite spatial volume from the local relaxation-time assumption on a window Proved here	In which model and at which β the local relaxation-time assumption holds
That the form "some $K(c)$" is satisfied by the four-dimensional massless free field and is empty; transfer to the ratio form and $K_* \le \kappa$ Proved here	The local-excitation moment assumption and the infinite-volume identity (both assumptions)
The values of the three references and the thresholds 0.342κ, 0.502κ, about 0.62κ; the boxes at which the bridge sees the mass, $mR \approx 4.2$ and 5.2 Reading	The ratio K_{rel} for the target Wilson measure. The threshold for $SU(2)$ at finite β. Values for large loops in four dimensions. Fields in the band $[K_*, 1)$
That for the 0+1 double well the ratio for the shortest window is exactly ω_b/m, and the three regimes by window Proved here ; lattice values Reading	That in four dimensions the ratio at short distance is close to 1 (an assumption depending on asymptotic freedom)

Sources and reproduction

Item	Kind	Source / tools
The one-sided bridge, the identity, the Free-Particle-on-a-Circle Counterexample	This book	(17)(18) and the Free-Particle-on-a-Circle Counterexample in 23-01 § 06. Theorem 3.1, Proposition 4.1 and Proposition 5.1 of the paper "one-sided bridge" (list of papers)
(2)(3), the Window-Factor Monotonicity Proposition, the In-Window Supremum Proposition, (6)–(10), (15)(16)	Proved here	Proofs in the text. The algebraic steps are in the Lean bundles (the machine-checking column of Table 5)
Gaussian Poincaré constant = norm of the covariance	Cited	M. Hairer, A. M. Stuart, J. Voss, P. Wiberg, Analysis of SPDEs arising in path sampling. Part I: The Gaussian case, Commun. Math. Sci. 3 (2005) 587–603, arXiv:math/0601095, Lemma 2.2
The right column of Table 1, $\Gamma(R)$ in (11), Table 2, $r(mR)$, the lattice values of Table 3	Reading	Largest eigenvalue of the free-field covariance compressed to the box (two routes, numerical integration of the heat kernel and periodic lattices; Lanczos); for 0+1, the lattice eigenvalue problem of the transfer kernel and Rayleigh–Ritz lower bounds. All floating point, with extrapolation in the ranges written in the text
New ideas are needed for a gap uniform in the volume	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000), § 6
The Brascamp–Lieb inequality ($K_{\mathrm{rel}} \le 1$ for convex perturbations)	Assumption	Bibliographic data not checked

Terms used in this chapter — the family local in time $\mathcal{F}_1$, window, the window factor $\varphi(x)$, the In-Window Supremum Proposition, the local Poincaré constant C_P^{loc}, the local relaxation-time assumption ($(\xi^2)\uparrow_c^{\mathrm{loc}}$, $(\xi^2)\uparrow_c^{\mathrm{rel}}$), the ratio K_{rel}, the massless reference and Γ, the gauge-quotient reference, the threshold K_*, the local-excitation moment assumption, the band $[K_*, 1)$, colour singlets and the second chaos. Their meanings are in the A-02 Glossary and 24-09.

Next chapter to read: 24-05 Making time continuous — the Hamiltonian lattice and the vacuum measure — stating the identity (17) on a general lattice, and reading it as a Poincaré sentence that does not pass through the relaxation-time assumption.

2026-10-09 · chapter Yang–Mills and the mass gap Part 24 The first road, second half the vacuum measure, the ground-state transformation, Kogut–Susskind · Kiichi, Shiori and Rin

Making time continuous — the Hamiltonian lattice and the vacuum measure

> This chapter is this book's own investigation. § 04 and § 06 are a map of open questions.

What this chapter shows — That on a lattice with continuous time, the gap of the Hamiltonian Δ is tied to the Poincaré constant of the vacuum distribution at one time $\nu = \psi_0^2$ exactly by $\Delta = \hbar^2/(2C_P(\nu))$ (with a proof). That this is the reverse side of $T = \infty$ for the one-sided bridge of Part 23, and the same formula as the ratio ω_b/m for the shortest window in 24-04. The road that treats the gap directly on the Hamiltonian lattice of Kogut–Susskind, and why local-to-global gap theorems do not apply there. That for the $SU(2)$ colour singlet, at leading order the Poincaré constant of the vacuum measure is halved and the lowest gauge-invariant test space is two-particle. What is needed to measure the constant of the vacuum measure.

Prerequisite chapters — 23-01 § 04 (the three CP of Table 3), 23-01 § 06 ((17) of the One-Sided Bridge Proposition (Theorem 3.1 of the paper), the identity (18)), 24-04 (the supremum (2) over the family local in time, (17) for the 0+1 window, the colour singlet), 2-05 § 08 (the one-dimensional ground-state transformation). The notation is collected in 24-09.

Said first — The level is [Y3]. The identity of this chapter only restates the gap as "the Poincaré constant of a certain measure"; it does not make the problem easier. The vacuum measure cannot be written explicitly, and there is no tool that bounds its Poincaré constant uniformly in the coupling. This chapter does not show the mass gap. It sets out one view that does not pass through the diffusive-relaxation-time assumption, and lists what that view needs.

The order of this chapter

1. The gap is the Poincaré constant of the vacuum measure — the Gap and Vacuum Measure Proposition ((1)–(4)), an example checked by hand
2. Three Poincaré constants — the path measure, the family local in time, the vacuum measure (Table 1, (5)(6))
3. The lattice version in 0+1 — the single-time window, and the discrepancy when time is discretised
4. The Hamiltonian lattice of Kogut–Susskind — the road that treats the gap directly, the Electric–Magnetic Splitting Frustration Proposition, where the tools do not apply (Table 2)
5. The $SU(2)$ colour singlet and two particles — the constant is halved at leading order ((7)(8))
6. The road to measuring the constant of the vacuum measure — what is needed
7. What cannot be said
8. Table of the five grades and the dependency graph — Table 3

01

The gap is the Poincaré constant of the vacuum measure

The setting is the same as for the one-sided bridge of Part 23. Space is a finite lattice, the space of configurations at one time is Q ($\mathbb{R}^n$, or a compact connected Riemannian manifold), the metric of the kinetic term is M, the Hamiltonian is $H = -\frac{\hbar^2}{2}\Delta_M + V$, and we place the ground-state assumption ((A1) in the paper: the spectrum is discrete, the ground state ψ_0 is non-degenerate, smooth and positive, $\Delta = E_1 - E_0 > 0$). We set the vacuum distribution at one time, and the Dirichlet form and Poincaré constant on it, as

$$\nu = \psi_0^2 \, d\mathrm{vol}, \qquad \mathcal{E}_\nu(F) = \int_Q \psi_0^2 \, |dF|^2_{M^{-1}} \, d\mathrm{vol}, \qquad C_P(\nu) = \sup_F \frac{\mathrm{Var}_\nu(F)}{\mathcal{E}_\nu(F)} \quad (1)$$

(F is a smooth, non-constant function on Q). In gauge theory we restrict F to gauge-invariant functions, and Δ is the gap on the gauge-invariant subspace.

Gap and Vacuum Measure Proposition

Under the ground-state assumption,

$$\Delta = \frac{\hbar^2}{2\, C_P(\nu)} \quad (2)$$

holds, and the supremum in (1) is attained at $F = \psi_1/\psi_0$.

Proved here Proof. The map $U : F \mapsto F\psi_0$ is an isometry from $L^2(\nu)$ to $L^2(d\mathrm{vol})$, and takes the constant 1 to ψ_0. Hence $\langle F \rangle_\nu = 0 \iff F\psi_0 \perp \psi_0$, and then $\|F\psi_0\|^2 = \mathrm{Var}_\nu(F)$. For smooth F, the identity of the ground-state transformation ((18) of 23-01)

$$\big\langle F\psi_0, \, (H - E_0)\, F\psi_0 \big\rangle = \frac{\hbar^2}{2}\, \mathcal{E}_\nu(F) \quad (3)$$

holds. Since $H - E_0$ is at least Δ on the orthogonal complement of ψ_0 (min-max), for smooth F with $\langle F \rangle_\nu = 0$ we have

$$\Delta \; \mathrm{Var}_\nu(F) \;\le\; \frac{\hbar^2}{2}\, \mathcal{E}_\nu(F), \qquad \text{that is} \qquad \frac{\mathrm{Var}_\nu(F)}{\mathcal{E}_\nu(F)} \le \frac{\hbar^2}{2\Delta} \quad (4)$$

Adding a constant changes neither side, so (4) holds for every smooth F. Equality is attained at $F = \psi_1/\psi_0$ (smooth by the ground-state assumption, with $F\psi_0 = \psi_1$, and the left-hand side of (3) is $\Delta\|\psi_1\|^2$). For $Q = \mathbb{R}^n$, the finiteness of the Dirichlet integral of ψ_1/ψ_0 is checked by the truncation lemma (Lemma 3.3) of the paper "one-sided bridge" (list of papers). ∎

In other words, the generator of Langevin dynamics on the vacuum distribution is unitarily equivalent, by the ground-state transformation, to $-(2/\hbar^2)\, \psi_0^{-1}(H - E_0)\, \psi_0$. Restatements of this type are well known (the restatements of stochastic quantisation and of supersymmetric quantum mechanics). The Gap and Vacuum Measure Proposition writes the gap part of it in this book's convention (1).

Calibration (one harmonic oscillator). For $Q = \mathbb{R}$, $M = 1$, $V = \frac{1}{2}\omega^2 x^2$, $\hbar = 1$, the gap is $\Delta = \omega$, so (2) gives $C_P(\nu) = 1/(2\omega)$. ψ_0^2 is the normal distribution with variance $1/(2\omega)$, and this agrees with the fact that the Poincaré constant of a one-dimensional normal distribution equals its variance (with equality for linear functions). The row for the vacuum C_P in Table 3 of 23-01 ($1/(2m)$ for the free field) is this value Proved here .

Example 1 (checked by hand on two sites). On the two-site lattice of Example 6 of 23-01 ($\hbar = M = 1$, $V = \frac{1}{2}x^{\mathsf{T}}Kx$, the normal mode $u = (x_1 + x_2)/\sqrt{2}$ with angular frequency m_0, $v = (x_1 - x_2)/\sqrt{2}$ with angular frequency $\sqrt{m_0^2 + 2}$), ν is the independent normal distribution with variance $1/(2m_0)$ for u and variance $1/(2\sqrt{m_0^2 + 2})$ for v. The Poincaré constant is the larger of the two variances, $1/(2m_0)$, and (2) gives $\Delta = m_0$, which agrees with the gap of Example 6 of 23-01 Proved here . If $m_0 = \frac{1}{2}$, then $C_P(\nu) = 1$, the path measure has $C_P = \xi^2 = 4$, and $4 = 4\,C_P(\nu)^2$ ((5) in the next section).

02

Three Poincaré constants

Three quantities are called by the same symbol C_P. With the Gap and Vacuum Measure Proposition, the relations among the three line up in one table.

Table 1 Three Poincaré constants (lattice with continuous time, canonical graduation. The free-field column has $\hbar = 1$ and mass m)

Quantity	Which measure, which functions	Relation to the gap	Free field	Grade
C_P	Path measure, all functionals	One-sided $C_P \geq \kappa\hbar^2/\Delta^2$	$1/m^2$	Proved here (the one-sided bridge of 23-01)
$\sup_{\mathcal{F}_1}$	Path measure, time-integrated single-time functions	$= \hbar^2/\Delta^2 = (2C_P(\nu)/\hbar)^2$	$1/m^2$	Proved here (combining (2) of 24-04 with (2) of this chapter)
$C_P(\nu)$	The vacuum distribution at one time	$= \hbar^2/(2\Delta)$	$1/(2m)$	Proved here (the Gap and Vacuum Measure Proposition)

The equality in the second row

$$\sup_{f \in \mathcal{F}_1} \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} = \left(\frac{2\,C_P(\nu)}{\hbar}\right)^2 \quad (5)$$

makes exact, on the family local in time and with interaction, the statement of 23-01 § 04 that "for the free field $C_P^{(\beta)} = 4(C_P^{(\alpha)})^2$" Proved here . In 23-01 this rereading was written as an analogy from the form of the free field. Under the assumptions of the one-sided bridge, restricted to the family local in time, it is an equality, not an analogy. The amount by which the path C_P stated by the diffusive-relaxation-time assumption exceeds $(2C_P(\nu)/\hbar)^2$ comes from functions that are not local in time (winding numbers and the topological charge).

Restated in physical units. Setting $\hbar = 1$ in lattice units and converting to physical units by the bookkeeping length ξ_{RG}, (2) becomes

$$m_{\mathrm{phys}} = m\,\xi_{\mathrm{RG}} = \frac{\xi_{\mathrm{RG}}}{2\,C_P(\nu)}, \qquad m_{\mathrm{phys}} \geq c \iff C_P(\nu) \leq \frac{\xi_{\mathrm{RG}}}{2c} \quad (6)$$

Proved here . The mass gap (that the gap in physical units of the lattice stays positive) is the same as the Poincaré sentence "the Poincaré constant of the vacuum measure is bounded by the first power of the bookkeeping length". It is a Poincaré sentence that does not pass through the relaxation-time

assumption, and the growth is the first power, not the square (in terms of the dynamical exponent, the type $z = 1$). But this is a restatement, and there is no new information on either side. An upper bound on $C_P(\nu)$ needs every F, and the difficulty is exactly that of a lower bound on the gap.

The topological problem does not arise for the vacuum measure. On ν we vary only single-valued functions at one time. For a free particle on the circle, ν is the uniform distribution on the circle and $C_P(\nu) = 1$ is finite (Wirtinger's inequality for the circle of circumference 2π), and (2) gives $\Delta = \hbar^2/2$ of the Free-Particle-on-a-Circle Counterexample of 23-01. And the winding-number functional that made $C_P = \infty$ for the path measure (the angle of the lift, integrated in time) is not a function on ν Proved here . The four-dimensional topological charge is also a quantity fixed by an integral in time, so it is not among the functions on the vacuum distribution at one time Estimate . The boundary is the same as the inside and outside of $\mathcal{F}_1$ in 24-04.

03

The lattice version in 0+1 — the single-time window

For the 0+1-dimensional double well of 24-04 § 07, the local Poincaré constant of the single-time window of the chain with time discretised at width ε was exactly $C_P^{\mathrm{loc}}(1) = \varepsilon/(2m)$ when ψ_0 is the Schrödinger ground state ((17) of 24-04). The Dirichlet form of the window $(\hbar/\varepsilon)\int \psi_0^2 |F'|^2$ is $\hbar/\varepsilon$ times $\mathcal{E}_\nu$ of (1), so this is exactly the value $C_P(\nu) = \hbar^2/(2\Delta)$ of the Gap and Vacuum Measure Proposition multiplied by $\varepsilon/\hbar$ Proved here . The ratio ω_b/m for the shortest window is this identity for the target (the double well) divided by that for the reference (the harmonic oscillator).

When time is discretised, the single-time distribution of the chain is the square of the top eigenvector of the transfer kernel (the symmetric Trotter product), and differs slightly from the Schrödinger ψ_0^2. On lattices with $V = (x^2-1)^2/8$, $\varepsilon = \tfrac{1}{2}$, $\hbar = 0.3,\ 0.15,\ 0.08$, the ratio of the direct value of the single-time window to $\varepsilon/(2m_{\mathrm{tr}})$ (m_{tr} from the second eigenvalue of the transfer) is $1.006,\ 1.013,\ 1.019$, and the ratio with the reference discretised too, $K_{\mathrm{rel}}(1)\cdot m_{\mathrm{tr}}$, is $1.037,\ 1.044,\ 1.050$ Reading . The identity is a statement in continuous time, and on a lattice with discretised time it is an approximation. This difference is the smallest form of the reason why the time-step comparison assumption of Part 23 ((D) in the paper "one-sided bridge", (20) of 23-01) remains.

04

The Hamiltonian lattice of Kogut–Susskind

This section summarises existing knowledge (the Hamiltonian lattice of Kogut–Susskind). The introduction is in 5-03 § 07.

The Hamiltonian of gauge theory with time continuous and only space made a lattice is the Kogut–Susskind lattice. The configuration space is the product of copies of the group, one per spatial link, $Q = G^{\{\text{spatial links}\}}$ (a compact connected manifold), and setting the metric of the kinetic term to $M = g^{-2}\cdot$ (the bi-invariant metric) to match the form of the one-sided bridge, we have

$$H_{\mathrm{KS}} = \frac{g^2}{2}\sum_{\ell} E_\ell^2 + \frac{1}{g^2}\sum_{p}\left(2 - \operatorname{tr} U_p\right) \qquad (\hbar = 1,\ G = SU(2)) \tag{7}$$

E_ℓ^2 is the Laplace–Beltrami operator on the group of the link ℓ with its sign changed (the square of the electric field), U_p is the product around a plaquette, and the second term is the magnetic term. The

conventions for the coefficients differ in the literature; here they are matched to the form of the one-sided bridge. Physical states are gauge-invariant functions (the Gauss law), and H_{KS} commutes with gauge transformations, so ψ_0 is automatically gauge invariant. On this lattice, the quantity in the problem statement is exactly the gap of H_{KS} on the gauge-invariant subspace, and by the Gap and Vacuum Measure Proposition it can be written as the Poincaré constant of the vacuum distribution at one time $\nu = \psi_0^2 \cdot \mathrm{Haar}$.

The road that treats the gap directly on this lattice is one of the rows of the "table of other roads" in the table of materials of 24-00. Table 2 sets out how far it has reached and where it stopped.

Table 2 Roads that treat the gap of the Hamiltonian lattice directly (the grades of the primary sources are in the source table)

Tool	Statement (summary)	Does it apply to the Kogut–Susskind lattice?	Grade
Local-to-global gap theorems (of Knabe's type, Gosset–Mozgunov)	For a frustration-free, translation-invariant local Hamiltonian, if the local gap of the open chain of length n exceeds a threshold $(6/(n(n+1))$ in Gosset–Mozgunov), the gap survives uniformly in the volume in the thermodynamic limit (one dimension; extended to the two-dimensional square lattice)	Does not apply. The splitting into the electric term and the magnetic term is not frustration-free (the Electric–Magnetic Splitting Frustration Proposition)	Cited (statement checked in the abstract), Proved here (the Electric–Magnetic Splitting Frustration Proposition)
Perturbation (Yarotsky)	For a classical lattice system with a relatively bounded quantum perturbation added, the gap and exponential decay of correlations	Strong coupling (the electric term dominating) is expected to fit the perturbative type. Not checked against the assumptions	Cited (abstract), Estimate
Results on the gap at weak coupling (the continuum-limit side)	—	Not found within the range searched (two searches with short terms)	—

Electric–Magnetic Splitting Frustration Proposition (the splitting into electric and magnetic parts is not frustration-free)

Let $g > 0$ and let the spatial lattice have at least one plaquette. When (7) is regarded as the sum of the local terms $h_\ell = \frac{g^2}{2} E_\ell^2$ and $h_p = \frac{1}{g^2}(2 - \operatorname{tr} U_p)$, $h_p - c_p$ has no eigenvector whatever constant c_p is subtracted. Hence there is no state that realises the ground states of all the local terms at once, and this splitting is not frustration-free.

Proved here Proof. h_p is a multiplication operator on $L^2(Q, \mathrm{Haar})$, and the multiplying function $2 - \operatorname{tr} U_p$ is a non-constant real-analytic function on Q. If $(h_p - c_p)\psi = 0$, then the support of ψ lies in the level set $\{2 - \operatorname{tr} U_p = c_p g^2\}$, but a level set of a non-constant real-analytic function has Haar measure 0, so $\psi = 0$. ∎ That the electric term and the magnetic term do not commute is equally elementary (E_ℓ^2 is a differential operator and does not commute with multiplication by $\operatorname{tr} U_p$). Whether a splitting that groups the electric and magnetic terms by sector also fails to be frustration-free is an expectation from their not commuting, and has not been shown Estimate . Nor have we checked the fact that the local Hilbert space $L^2(SU(2))$ is infinite-dimensional against the setting of the theorems.

Tools on the side of the vacuum measure. Moving the gap to $CP(\nu)$ by the Gap and Vacuum Measure Proposition does not increase the tools available. We list three reasons.

- ν cannot be written explicitly. $-2\ln\psi_0$ is the "action" of ν, and it is not local either. A criterion of Bakry–Émery type, which bounds the Poincaré constant by convexity, needs a lower bound on $\mathrm{Hess}(-\ln\psi_0)$. The result of Brascamp–Lieb type, that the ground state of a convex potential is log-concave, is from memory, and its bibliographic data have not been checked Assumption . Moreover, the magnetic term of (7) is a non-constant smooth function on a compact group, so its Hessian is at most 0 at a maximum point, and it cannot be uniformly convex Proved here . Whether a criterion with the Ricci curvature of the group added works uniformly as $g \to 0$ is expected to meet an obstacle of the same type as the convexity-ceiling lemma of 24-06 Estimate .
- To set up on ν an argument of the local-to-global type for Glauber dynamics (from the Poincaré constants of local boxes and strong spatial mixing to a gap uniform in the volume, of the type in Martinelli's lectures), spatial mixing with respect to the boundary values of ν is needed. Since ν is not explicit, this is expected to be harder than for the Wilson measure Estimate (the statements of theorems of that type are from memory, not checked in the originals Assumption).
- Taking the temporal lattice spacing to 0 first (the Hamiltonian limit) is a different order of limits from Wilson's isotropic lattice. That both orders lead to the same continuum limit is an assumption Assumption (the background in 23-02).

05

The $SU(2)$ colour singlet and two particles

In 24-04 § 06 we saw, for the reference on a four-dimensional spacetime box, that "the $SU(2)$ colour singlet has no linear term, and the reference is exactly halved". We state the same thing on the side of the vacuum distribution at one time.

At leading order ($g \to 0$, the three colour components independent Gaussians), the vacuum distribution ν is a Gaussian measure on the transverse fields of the spatial links; writing ω_k for the one-particle angular frequencies of the free Hamiltonian, the Langevin generator of the Dirichlet form (1) is unitarily equivalent to $-2(H - E_0)$ by the ground-state transformation of the Gap and Vacuum Measure Proposition ($\hbar = 1$). The eigenvalues of $H - E_0$ are sums of one-particle angular frequencies $\sum_k n_k\omega_k$ (n_k is the number of particles), and the eigenspaces correspond to the Wiener chaos ($n = \sum n_k$ gives the n-th chaos). On a finite spatial lattice, let the smallest angular frequency be $\omega_{\min} > 0$ (modes with $\omega = 0$, such as the constant mode of a periodic box, are treated separately).

A gauge-invariant function must be invariant under constant gauge transformations (global colour rotations $SO(3)$). A function in the first chaos $\sum c_k^a A_k^a$ is a vector in the colour index a, and $SO(3)$ has no fixed vector in $\mathbb{R}^3$, so the only invariant one is 0 Proved here . Hence the lowest test space of gauge-invariant functions is the second chaos (two particles), the smallest eigenvalue on it is $2\omega_{\min}$, and equality is attained at $F_w = \sum_a(\langle w, A^a\rangle^2 - \mathbb{E})$, with the two particles placed in the same lowest mode w and contracted in colour Proved here . Reread through the Gap and Vacuum Measure Proposition, this is

$$C_P(\nu \,|\, \text{colour singlet}) = \frac{C_P(\nu)}{2}, \qquad \Delta_{\text{singlet}} = 2\,\omega_{\min} \qquad \text{(leading order)} \quad (8)$$

Proved here (at leading order, three independent Gaussian colours, on a finite spatial lattice with $\omega_{\min} > 0$). The lowest gauge-invariant excitation is at the two-particle threshold, and the Poincaré constant is half the one-particle constant. The reference $\Gamma_Q/2$ for the spacetime box in 24-04 is the same mechanism seen through the local Poincaré constant of the spacetime box. Lean

`LatticeGauge.WilsonLoopReference.no_linear_singlet` (assumption: a linear form invariant under two rotations; conclusion: the coefficients are 0), `chaos2_gamma` (assumption: $0 < n, g$ and the Wick moments as formulas; conclusion: the algebra by which the ratio is halved).

In physics it is said that "the lightest glueball is a bound state of two gluons". One can read (8) as the local Poincaré version of its leading order Estimate . However, a physical mechanism is not a proof (the table of physical mechanisms in 24-07), and (8) is a statement in the form of the leading order. In infinite volume $\omega_{\min} \to 0$, so the right-hand side of (8) also goes to 0. The mass gap does not appear at leading order; what the leading order fixes is only the form "the gauge-invariant test space begins at two particles". How (8) changes for $SU(2)$ at finite g has not been computed at all.

06

The road to measuring the constant of the vacuum measure — what is needed

By the Gap and Vacuum Measure Proposition, measuring the Poincaré constant of the vacuum distribution is the same as measuring the gap. We set out the roads to measuring it by how far the machinery of the first step of Part 23 can be used.

- **0+1 dimensions.** The machinery of the first step (the lattice eigenvalue problem of the transfer kernel, and the generalised eigenvalues of the weighted lattice Laplacian) computes the local Poincaré constant of the single-time window directly, and agreed with $\varepsilon/(2m_{\mathrm{tr}})$ within a factor of 1.006–1.019 (§ 03) Reading . In 0+1, the machinery measures $C_P(\nu)$ as it is.
- **Samples.** ν is not explicit, so samples are needed. The marginal distribution on a single-time slice of an anisotropic Wilson lattice with small temporal lattice spacing is expected to approach ν in the Hamiltonian limit Estimate . If the time step is finite, that slice is ψ_0^2 of the transfer kernel, and the discrepancy of § 03 (1–2% at $\varepsilon = \frac{1}{2}$ in 0+1) enters.
- **Test functions.** Since we restrict to gauge-invariant functions, we take them from polynomials in Wilson loops. As in 24-04 § 06, loops of the size of the span of the box are needed.
- **Graduation.** $\mathcal{E}_\nu$ of (1) is measured in the metric of the kinetic term $M^{-1} = g^2 \cdot$ (the bi-invariant metric). Unless the Langevin graduation is matched to this, the coefficient in (2) is off by a factor κ.
- **Direction.** The Rayleigh quotient on samples gives only a lower bound on $C_P(\nu)$, that is, an **upper bound** on the gap (the side obtained by placing a single test function). A lower bound on the gap needs every F, and the asymmetry between the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption of Part 23 ((15) of 23-01) appears as it is. The road to measuring is a road that measures the easy side of the gap.

This chapter places no computation. A computation that bounds the Kogut–Susskind gap of a small periodic lattice in intervals, truncating the representations of the electric field, stays at the numbers of a finite box, and does not become a bridge to a gap uniform in the volume Estimate .

07

What cannot be said

- The Gap and Vacuum Measure Proposition is an identity, and we do not claim it has made the problem easier. There is no tool that bounds $C_P(\nu)$ uniformly in β (or g).

- Local-to-global gap theorems do not apply to the splitting of the Kogut–Susskind lattice into electric and magnetic parts (the Electric–Magnetic Splitting Frustration Proposition). That they do not apply to a splitting grouped by sector is an expectation and has not been shown.
- We have not checked the application of strong-coupling perturbative results to the Kogut–Susskind lattice. Results on the weak-coupling side were not found within the range searched.
- That the order taking the Hamiltonian limit first gives the same continuum limit as Wilson's lattice is an assumption.
- (8) is a statement at leading order (three independent Gaussian colours), and there is no value at finite g. The correspondence with glueballs is an estimate.
- This chapter does not show the mass gap. Nor does it show anything about the relaxation-time assumption.

08

Table of the five grades and the dependency graph

Table 3 The rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Ground-state assumption: discrete spectrum, $\psi_0 > 0$ smooth and non-degenerate, $\Delta > 0$	Assumption	23-01 Table 6	Finite spatial lattice	—
2	The identity (3) of the ground-state transformation	Proved here	23-01 (18)	Finite spatial lattice	—
3	Gap and Vacuum Measure Proposition: $\Delta = \hbar^2/(2C_P(\nu))$	Proved here	(2)–(4)	Continuous time, finite spatial lattice	—
4	The relation (5) among the three constants	Proved here	Table 1	Canonical, assumptions of the one-sided bridge	—
5	Restatement in physical units (6)	Proved here	§ 02	Each β	—
6	The 0+1 single-time window is $\varepsilon/\hbar$ times the Gap and Vacuum Measure Proposition	Proved here	§ 03	0+1, Schrödinger ψ_0	`LatticeGauge.LocalHDoubleWell.one_slice_ratio` (assumption: $0 < \varepsilon, m, \omega$)
7	Discrepancy on lattices with discretised time 1.006–1.019, 1.037–1.050	Reading	§ 03	$\varepsilon = \frac{1}{2}$, three values of $\hbar$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
8	Electric–Magnetic Splitting Frustration Proposition: the splitting into electric and magnetic parts is not frustration-free	Proved here	§ 04	$g > 0$, finite spatial lattice	—
9	Local-to-global gap theorems assuming frustration-freeness	Cited	Table 2 (statement checked in the abstract)	Setting of the original	—
10	Strong-coupling perturbative type (Yarotsky)	Cited	Table 2 (abstract)	Setting of the original	—
11	The ground state of a convex potential is log-concave	Assumption	§ 04 (from memory)	—	—
12	Arguments of the local-to-global type (of the type in Martinelli's lectures)	Assumption	§ 04 (from memory)	—	—
13	The Hamiltonian limit and Wilson's lattice lead to the same continuum limit	Assumption	§ 04	—	—
14	The colour singlet has no linear term	Proved here	§ 05	Leading order	`LatticeGauge.WilsonLoopReference.no_linear_singlet` (assumption: invariant under two rotations)
15	(8): for the singlet $CP(\nu)$ is halved, the gap is $2\omega_{\mathrm{min}}$	Proved here	§ 05	Leading order, three independent Gaussian colours, $\omega_{\mathrm{min}} > 0$	`chaos2_gamma` (assumption: the Wick moments as formulas)
16	Local-to-global theorems do not apply to the Kogut–Susskind lattice	Estimate	§ 04	Including splittings grouped by sector	—
17	Strong coupling fits the perturbative type	Estimate	Table 2	Not yet checked	—
18	(8) is the local Poincaré version of the glueball	Estimate	§ 05	—	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
19	There is no tool that bounds the constant of the vacuum measure uniformly	Estimate	§ 04, § 06	—	—

The machine-checking column covers only algebra. The identity (3), the Gap and Vacuum Measure Proposition, the Electric–Magnetic Splitting Frustration Proposition and the measure steps of (8) are outside Lean (23-13).

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  ASM  ground-state assumption: discrete spectrum, ψ₀ > 0, Δ > 0      finite spatial lattic
 2  PRV  identity of the ground-state transformation ((18) of 23-01)
 3  PRV  Gap and Vacuum Measure Proposition: Δ = ħ²/(2 C_P(ν)) ⟸ 1, 2
 4  PRV  three constants: sup over F₁ = (2C_P(ν)/ħ)² ⟸ 3 (+ the one-sided bridge of 23-01,
 5  PRV  physical units: m_phys ≥ c ⟺ C_P(ν) ≤ ξ_RG/(2c) ⟸ 3
 6  PRV  0+1 window: the single-time window is ε/ħ times the gap proposition ⟸ 3    Lean on
 7  RDG  lattice discrepancy: 1.006–1.019 (ψ₀ of the transfer with discretised time)
 8  PRV  Electric–Magnetic Splitting Frustration Proposition: not frustration-free
 9  CIT  local→global: gap theorems assuming frustration-freeness
10  CIT  perturbation: gap under relatively bounded quantum perturbations (abstract)
11  ASM  log-concave: ground state of a convex potential (from memory)
12  ASM  mixing: uniform gap from local boxes and spatial mixing (from memory)
13  ASM  order of limits: the Hamiltonian limit and Wilson's lattice
14  PRV  no linear term: the colour singlet has no linear term              Lean no_linear_s
15  PRV  two particles: (8) ⟸ 3, 14 (leading order)                          Lean chaos2_ga
16  EST  no fit: local→global does not apply to KS ⟸ 8, 9
17  EST  strong coupling: fits the perturbative type ⟸ 10
18  EST  glueball: the reading of (8) ⟸ 15
19  EST  no tool: no tool bounds C_P(ν) uniformly ⟸ 11, 12, 13, 16
```

The check for circles is only to see that on every "⟸" the numbers on the right are smaller than the number on the left. The conclusions of this chapter are the Gap and Vacuum Measure Proposition and the three relations that follow from it (the three constants, physical units, the 0+1 window), and the leading-order statement for two particles; there is no node for the mass gap. None of the assumption rows (the ground-state assumption, log-concavity, mixing, the order of limits) contains a lower bound on the gap.

What this chapter can say

What is said	What is not said
That on a finite spatial lattice with continuous time, the gap is tied to the Poincaré constant of the vacuum distribution at one time by $\Delta = \hbar^2/(2C_P(\nu))$ (under the ground-state assumption) Proved here	A way to bound $C_P(\nu)$ uniformly in β. Any claim that the problem has become easier
The relation among the three Poincaré constants, and that on the family local in time the free-field rereading of 23-01 becomes an equality Proved here	The size of the amount by which the path C_P exceeds $(2C_P(\nu)/\hbar)^2$
That the 0+1 single-time window is this identity Proved here ; the discrepancy when time is discretised Reading	Whether the time-step comparison assumption holds on the four-dimensional Wilson lattice

What is said	What is not said
That the splitting of the Kogut–Susskind lattice into electric and magnetic parts is not frustration-free Proved here	Whether it holds for splittings grouped by sector. The application of strong-coupling perturbation. Weak-coupling results
That at leading order the lowest test space of the colour singlet is two-particle, and $CP(\nu)$ is halved Proved here	Values at finite g. The correspondence with glueballs (an estimate)

Sources and reproduction

Item	Kind	Source / tools
The identity (3), the one-sided bridge	This book	(17)(18) of 23-01 § 06. The truncation for $\mathbb{R}^n$ is Lemma 3.3 of the paper "one-sided bridge" (Kiichi, Shiori and Rin)
Gap and Vacuum Measure Proposition, (5)(6), Electric–Magnetic Splitting Frustration Proposition, (8)	Proved here	Proofs in the text
The Kogut–Susskind Hamiltonian	Setting	J. Kogut, L. Susskind, Hamiltonian formulation of Wilson's lattice gauge theories, Phys. Rev. D 11 (1975) 395–408 (bibliographic data checked, text not obtained. The coefficients of (7) are this book's convention)
Local-to-global gap theorems	Cited	D. Gosset, E. Mozgunov, arXiv:1512.00088 (statement checked in the abstract). S. Knabe, J. Stat. Phys. 52 (1988) 627–638 (bibliographic data checked; the value of the threshold is restated from the abstract of the former)
Strong-coupling perturbation	Cited	D. A. Yarotsky, arXiv:math-ph/0412040 (Commun. Math. Phys. 261 (2006)) (abstract)
Search for weak-coupling results on the gap	—	Nothing relevant in two arXiv searches with short terms (the search is shallow)
The lattice values of § 03	Reading	The lattice eigenvalue problem of the transfer kernel of the 0+1 double well (the same as the source table of 24-04)
Log-concavity of Brascamp–Lieb type, local-to-global of Martinelli's type	Assumption	From memory. The statements have not been checked in the originals

Terms used in this chapter — the vacuum measure ν, the ground-state transformation, the Gap and Vacuum Measure Proposition, the three Poincaré constants, the Hamiltonian lattice (Kogut–Susskind), frustration-free, local-to-global gap theorems, the colour singlet and the second chaos, the Hamiltonian limit. Their meanings are in A-02 Glossary and 24-09.

Next chapter to read: 24-06 Paying with rarity instead of convexity — that convexity is not enough on groups, as a story of tools that pay with rarity.

Paying with rarity instead of convexity — renormalisation on groups and log-Sobolev

> This chapter is this book's own investigation. § 02 and § 08 are established knowledge; § 05 and § 07 are a map of open questions.

What this chapter shows — From here on the subject is tools. 24-01 to 24-05 dealt with "what to compare, and which diffusive-relaxation-time assumption to restate". To show any statement, a tool that bounds the relaxation constant from above is needed. On flat space the centre of such tools is convexity (Bakry–Émery, Brascamp–Lieb), and for scalar hierarchical models a theorem of the same type as the diffusive-relaxation-time assumption has been shown, with uniform convexity of the conditional measure at each scale as the entrance. In this chapter: (1) we look at what the convexity tools need and where they break, in three places: the summit in 0+1 dimensions, the antipode on the group, and the induction hypothesis of the hierarchical model. On a group, by the Convexity Ceiling Lemma on closed manifolds (with a proof), convexity cannot hold in principle at weak-coupling scales. (2) Between flat ϕ^4 and the group, the convex and non-convex places are reversed, and the wall does not vanish but is pushed into a place of probability $e^{-c\beta}$. (3) We then list tools that pay with rarity instead of convexity (the currency of entropy, Lyapunov conditions, field-dependent constants), and see that the margin of the action barrier in Part 23 is "an example where rarity paid". (4) We place as a candidate "the class of measures that pay with rarity", rewritten in the form of rules, and check that the "coupling window" left between the ultraviolet side and the strong-coupling side is a restatement of the second difficulty of the third step. (5) We add one section on the road of stochastic quantisation.

Prerequisite chapters — 23-09 § 06 (the one-site curvature at the summit, $\frac{15}{4}$ and 4), 23-10 currency-mix-up type, 23-11 § 06 (the margin of the action barrier), 24-02 § 02 (tree coordinates and link coordinates), 24-07 § 05 (the three roots). From Part 1, 1-05 (manifolds) and 1-04 ($SU(2)$). Notation is in 24-09.

Said first — The level is [Res]; the summary in § 01 is [Y3]. Proofs are placed only for the lemma, corollary and proposition of § 03; the rest is copied from the originals (Cited) or estimates (Estimate). The results for hierarchical models imply nothing for four-dimensional Yang–Mills. The "rules" of § 07 are candidates, and none of them promises a prospect of closing the third step (the continuum limit). This chapter shows neither the mass gap nor the relaxation-time assumption. A note on notation: in this chapter the lower bound on the Bakry–Émery curvature is written κ_{BE}, a different quantity from the comparison constant κ of the One-Sided Bridge Proposition (Theorem 3.1 of the paper) (24-09). The rate of rarity is written s ($\mu(B) \le e^{-s}$). It is the quantity written ℓ in 23-09 and 23-10 of Part 23; in Part 24 we changed the letter because ℓ is used for the ruler of length.

The order of this chapter

1. Summary [Y3] — the convexity ceiling, the reversal of places, the currency of rarity
2. What the convexity tools need — Bakry–Émery, Brascamp–Lieb, the induction hypothesis of the hierarchical model
3. The convexity ceiling on closed manifolds — the Convexity Ceiling on Groups Lemma, the No Concentration at the Identity Corollary, the One-Link Wilson Weight Proposition (with proofs)
4. The convex and non-convex places are reversed — ϕ^4 and the group; the wall goes to where the probability is small (Figure 1)
5. Carrying the hierarchical induction over to the group — applies, changes, does not apply (Table 1)
6. Tools that pay with rarity — currency, Lyapunov, field-dependent constants, the example of the action barrier (Table 2)
7. If it were made into rules — the class of measures that pay with rarity — objects, rules, candidate first theorems, the coupling window
8. The road of stochastic quantisation — subcriticality $d < 4$, and the direction of rebuilding where the objects live
9. What cannot be said
10. Table of the five grades and the dependency graph — Table 3

01

Summary [Y3] — the convexity ceiling, the reversal of places, the currency of rarity

This section is a ten-paragraph summary for readers at [Y3]. The proofs and the conditions of the originals are in § 02 onwards.

1. A hierarchical model replaces the lattice by nested boxes so that one step of renormalisation can be written down. For the four-dimensional hierarchical scalar $|\phi|^4$, both the equilibrium properties (the book by Bauerschmidt–Brydges–Slade) and the slowness of relaxation of the stochastic dynamics (Bauerschmidt–Bodineau: the gap vanishes with the same power as for the free field) are theorems Cited . The latter is a statement of the same type as the relaxation-time assumption.
2. The method of Bauerschmidt–Bodineau takes as its entrance the hypothesis that at each scale "the measure of the fine fluctuations, with the coarse field held fixed, is uniformly convex" Cited . The original itself names convexity at large fields as the main issue in extending the method beyond hierarchical models.
3. The group $SU(2)$ is a closed space (the three-dimensional sphere). On a closed space, imposing convexity bounds the oscillation of the potential by the square of the diameter, and the measure cannot leave the range of a constant multiple of the uniform distribution (the Convexity Ceiling Lemma of § 03) Proved here . At weak coupling the measure needs to gather near the identity, so the convexity hypothesis cannot hold in principle at weak-coupling scales.

4. For ϕ^4 on flat space, large fields are convex by nature, and what is not convex is small fields (between the two wells). On the group it is the reverse: near the identity (small fields) is convex, and the far hemisphere (large fields) is necessarily non-convex (§ 04) Proved here .
5. However, the probability of being on the non-convex side is small, of the order of $e^{-\beta}$ (the One-Link Wilson Weight Proposition of § 03) Proved here . The wall does not vanish; it is pushed into a place of small probability.
6. The wall of the first step of Part 23 (that the one-site diagonal at the summit $\frac{15}{4}$ falls short of the sum of couplings 4 by $\frac{1}{4}$) also took the form of paying for the lack of convexity with the rarity of the plateau. The Poincaré inequality is a currency that measures the speed of exit, and what can pay with rarity is the currency of entropy (23-10 currency-mix-up type). A step that pays with rarity needs to pass through an inequality of log-Sobolev type.
7. Tools that pay with rarity instead of convexity include Lyapunov conditions (a drift that returns from the bad set) and field-dependent constants. The estimate of the action barrier in Part 23 was an example where the cost of the bistability of the core could be paid by the rarity of the local term (a margin of about 95 times, on the $\hbar = 0$ reading).
8. The one-link measure has a single well, and its relaxation constant does not get exponentially worse in β. A wall of the $\frac{1}{4}$ type of the double well does not come back for one link Estimate .
9. Rewritten in the form of rules, this becomes "the class of measures that pay with rarity" (§ 07). On the ultraviolet side it seems possible to write a form in which the class closes at each step of coarse-graining, but at the confinement scale the rate of rarity runs out, and a "coupling window" remains between it and the curvature tools of the strong-coupling side. This is a restatement of the second difficulty of the third step (24-07 § 06) Estimate .
10. The road of stochastic quantisation (SPDE) uses the same Langevin dynamics, but its aim is the construction of the measure, and the solution theory is subcritical only for $d < 4$ Cited . It gives tools neither for the four-dimensional relaxation-time assumption nor for the local diffusive-relaxation-time assumption.

02

What the convexity tools need — Bakry–Émery, Brascamp–Lieb, the induction hypothesis of the hierarchical model

This section summarises existing knowledge (Bakry–Émery, Brascamp–Lieb, the inductive hypothesis of hierarchical models). The introduction is in 5-11 § 02.

Most tools that bound the relaxation constants (the Poincaré constant, the log-Sobolev constant) **from above** take as their entrance a lower bound on the second derivative of the potential, that is, convexity. We list three.

Bakry–Émery. For a probability measure $\mu = e^{-V} d\mathrm{vol}/Z$ on a Riemannian manifold M,

$$\mathrm{Ric} + \mathrm{Hess}\, V \;\geq\; \kappa_{\mathrm{BE}}\, g \quad (\kappa_{\mathrm{BE}} > 0) \qquad \Longrightarrow \qquad \mathrm{Ent}_\mu(f^2) \leq \frac{2}{\kappa_{\mathrm{BE}}} \int |\nabla f|^2 d\mu, \quad CP \leq \frac{1}{\kappa_{\mathrm{BE}}} \;(1)$$

Cited (the single-scale case of Theorem 3.6 of the review by Bauerschmidt–Bodineau–Dagallier). The condition is a lower bound **at every point**.

Brascamp–Lieb. On $\mathbb{R}^n$, if $\operatorname{Hess} V > 0$ then $\operatorname{Var}_\mu(f) \le \mathbb{E}_\mu \langle \nabla f, (\operatorname{Hess} V)^{-1} \nabla f \rangle$ Cited (the form used by Lemma 2.7 of Bauerschmidt–Bodineau). This too uses convexity at each point.

The hierarchical induction. Bauerschmidt–Bodineau split the Gaussian covariance by scale as $C = C_0 + \cdots + C_N$, form the renormalised potential $e^{-V_{j+1}(\phi)} = \mathbb{E}_{C_j} e^{-V_j(\phi+\zeta)}$ (**addition** of fields $\phi + \zeta$), and place the following hypothesis Cited (arXiv:1809.02075 § 2.3).

$$\text{(A1)} \qquad C^{1/2} \big(\operatorname{Hess} V_j(\phi)\big) C^{1/2} \;\ge\; -\varepsilon_j \, Q \qquad (\text{for all } \phi,\ \varepsilon_j < 1) \ (2)$$

Here Q is the projection onto the space of coarse fields, and ε_j is the notation of the original (different from the lattice spacing ε in 0+1 dimensions). Under (A1) and two structural hypotheses, the Brascamp–Lieb quadratic form is handed on at each step as $D \le C/(1-\varepsilon) + D_+/(1-\varepsilon)^2$, and the product over all steps is bounded by $\exp\big(2 \sum_l (\varepsilon_l + O(\varepsilon_l^2))\big)$ (Theorem 2.1, Corollary 2.2) Cited . The conclusion is a spectral gap (of Poincaré type), not a log-Sobolev result. The form that flows log-Sobolev over many scales is later work by the same authors (the Polchinski flow, § 05). The original writes that "to extend this method beyond hierarchical models, convexity rather than probabilistic estimates is needed, and convexity at large fields is the main issue" Cited .

Three places where it breaks. (a) The summit of the 0+1-dimensional double well. In one row of the Hessian of the action of the core, at the summit the diagonal $4 - \frac{1}{4}$ falls short of the sum of the couplings to the neighbours 4 ((11) of 23-09), and on the face where every site is at the summit, the smallest eigenvalue of a lump of length $k \ge 8$ is negative Proved here . Every convexity tool had to pay for this $\frac{1}{4}$ with the rarity of the plateau. (b) The opposite hemisphere of the group. In the One-Link Wilson Weight Proposition of § 03 we see that the Bakry–Émery curvature of the one-link Wilson weight is negative on the far hemisphere. (c) (A1) of the hierarchical induction. For group-valued fields, the natural reading of (A1) is "Bakry–Émery for the measure on the fibre with the coarse field held fixed" Estimate , and by the Convexity Ceiling Lemma of § 03 it cannot hold at weak-coupling scales.

What this section can say: (1)(2) and Theorem 2.1 are copies of the statements of the originals. The reading in (c) (reading (A1) on the group as Bakry–Émery on the fibre) is an estimate, and this book has no exact formulation of the induction on the group.

03

The convexity ceiling on closed manifolds — the ceiling lemma, no concentration at the identity, the one-link weight

Convexity Ceiling Lemma

Let M be a compact connected Riemannian manifold without boundary of diameter D, $V \in C^2(M)$, $K \ge 0$. If $\operatorname{Hess} V \ge -K\, g$, then

$$\operatorname{osc} V := \max_M V - \min_M V \;\le\; \frac{K D^2}{2}. \ (3)$$

Proved here Proof. Let p be a maximum point of V (M is compact). For any $q \in M$, take a minimising geodesic $\gamma : [0, L] \to M$ from p to q (unit speed, $L = d(p, q) \le D$) (it exists by the Hopf–Rinow theorem, since a compact manifold is complete). Setting $h(t) := V(\gamma(t))$, since p is a maximum point on a manifold without boundary, $h'(0) = dV_p(\gamma'(0)) = 0$. Along the geodesic, $h''(t) = \operatorname{Hess} V(\gamma'(t), \gamma'(t)) \ge -K$. Integrating gives $h'(t) \ge -Kt$, and integrating once more gives $h(t) \ge h(0) - Kt^2/2$. At $t = L$, $V(q) \ge \max V - KL^2/2 \ge \max V - KD^2/2$. Since q is arbitrary, (3) follows. ∎

There is no Lean check (the one-dimensional version "if $h'' \ge -K$ and $h'(0) = 0$ then $h(0) - h(x) \le Kx^2/2$" is kept as a candidate). The key of the proof is that "the first derivative vanishes at a maximum point", and here the **absence of boundary** (being closed) is what works. Flat $\mathbb{R}^n$ has no maximum point, so the same estimate does not follow.

No Concentration at the Identity Corollary (on $SU(2)$, a convex measure cannot gather at the identity)

Take $SU(2)$ as the unit three-dimensional sphere S^3 (bi-invariant metric, $\mathrm{Ric} = 2g$, diameter π), and suppose $\mu = e^{-V}\, d\mathrm{Haar}/Z$ satisfies $\mathrm{Ric} + \operatorname{Hess} V \ge 0$. Then $\operatorname{osc} V \le \pi^2$, and for every Borel set A

$$\mu(A) \;\le\; e^{\pi^2}\,\mathrm{Haar}(A), \qquad \mu\big(\{\theta \le r\}\big) \;\le\; e^{\pi^2}\,\frac{2}{3\pi}\,r^3 \quad (4)$$

(θ is the geodesic distance from the identity, $e^{\pi^2} \approx 1.9 \times 10^4$). In particular the one-link Wilson weight ($\operatorname{osc} V = 2\beta$) does not satisfy $\mathrm{Ric} + \operatorname{Hess} V \ge 0$ for $\beta > \pi^2/2$.

Proved here Proof. Since $\operatorname{Hess} V \ge -\mathrm{Ric} = -2g$, using the Convexity Ceiling Lemma with $K = 2$ and $D = \pi$ gives $\operatorname{osc} V \le \pi^2$. Since $Z = \int e^{-V}\, d\mathrm{Haar} \ge e^{-\max V}$, the density is $e^{-V}/Z \le e^{\max V - \min V} \le e^{\pi^2}$. The Haar distribution of class functions is $\frac{2}{\pi}\sin^2\theta\, d\theta$ ($\theta \in [0, \pi]$), and from $\sin\theta \le \theta$, $\mathrm{Haar}(\theta \le r) \le \frac{2}{\pi} \cdot \frac{r^3}{3}$. The Wilson weight $V = \beta(1 - \cos\theta)$ has $\operatorname{osc} V = 2\beta$, so if $2\beta > \pi^2$ it contradicts the inequality stated before (4). ∎

How to read (4). At weak coupling, the Wilson measure gathers most of its probability in a ball of radius of order $\beta^{-1/2}$. The right-hand side of (4) gets small like r^3, so when β is large, a measure with convexity of Bakry–Émery type cannot do that. The content of the lemma is that once convexity is imposed, the measure cannot move further from Haar than a constant factor e^{π^2}.

One-Link Wilson Weight Proposition

On the unit $S^3 = SU(2)$, let $V_\beta = \beta(1 - \cos\theta)$ and $\mu_\beta = e^{-V_\beta}\, d\mathrm{Haar}/Z_\beta$.

(i) $\mathrm{Ric} + \operatorname{Hess} V_\beta = (2 + \beta\cos\theta)\, g$. Hence the Bakry–Émery condition holds only when $\beta < 2$, and then $\kappa\mathrm{BE} = 2 - \beta$.

(ii) When $\beta > 2$, the set where the condition fails, $N_\beta := \{\cos\theta < -2/\beta\}$, lies in the hemisphere far from the identity, and on it $V_\beta \ge \beta + 2$.

(iii) For $\beta \ge 2$,

$$\mu_\beta(N_\beta) \;\le\; \frac{54\pi}{25}\, e^{-3/2}\, \beta^{3/2} e^{-\beta} \;<\; 1.52\, \beta^{3/2} e^{-\beta}. \tag{5}$$

Proved here Proof. (i) A function f of θ alone on the unit sphere has, in geodesic polar coordinates $g = d\theta^2 + \sin^2\theta\, g_{S^2}$, $\mathrm{Hess}\, f = f''\, d\theta^2 + f' \cot\theta\, (g - d\theta^2)$ (the radial direction from the second derivative along geodesics, the spherical directions from the second fundamental form $\cot\theta$ of the sphere of constant θ). For $f = \beta(1-\cos\theta)$, $f'' = \beta\cos\theta$ and $f' \cot\theta = \beta \sin\theta \cot\theta = \beta\cos\theta$, so $\mathrm{Hess}\, V_\beta = \beta\cos\theta\, g$. On the unit S^3, $\mathrm{Ric} = 2g$. The minimum is $2-\beta$ at $\theta = \pi$. (ii) If $\cos\theta < -2/\beta < 0$ then $\theta > \pi/2$, and $V_\beta = \beta(1-\cos\theta) > \beta + 2$. (iii) On N_β, $e^{-V_\beta} \le e^{-\beta-2}$, and the Haar measure is at most 1, so $\int_{N_\beta} e^{-V_\beta} \le e^{-\beta-2}$. For the denominator, on the ball of $r = \beta^{-1/2} \le 1$, $1 - \cos\theta \le \theta^2/2$ gives $V_\beta \le \frac{1}{2}$, and for $\theta \le 1$, $\sin\theta \ge \theta - \theta^3/6 \ge \frac{5}{6}\theta$ gives $\mathrm{Haar}(\theta \le r) \ge \frac{2}{\pi} \cdot \frac{25}{36} \cdot \frac{r^3}{3} = \frac{25}{54\pi} r^3$. Hence $Z_\beta \ge e^{-1/2} \frac{25}{54\pi} \beta^{-3/2}$, and dividing gives (5) ($\frac{54\pi}{25} e^{-3/2} = 1.514\cdots$). ∎

(5) is a crude upper bound; near the antipode ($\theta \approx \pi$, $V_\beta \approx 2\beta$) the probability is smaller still, of order $e^{-2\beta}$. $\beta < 2$ in (i) is sharper than the necessary condition $\beta \le \pi^2/2$ that follows from the Convexity Ceiling Lemma. The role of the Convexity Ceiling Lemma is that it is not limited to the Wilson weight: **any** weight, once convexity is imposed, is confined to the range of (4).

What this section can say: the Convexity Ceiling Lemma, the No Concentration at the Identity Corollary and the One-Link Wilson Weight Proposition are statements with proofs placed. What they speak of is the convexity of one measure on one manifold; about the convexity of the conditional measure on the fibre (a measure on the product of many links) they speak only through the estimate of § 05.

04

The convex and non-convex places are reversed — the wall goes to where the probability is small

For ϕ^4 on flat space (of the type of the one-site $V(\phi) = \lambda(\phi^2 - v^2)^2$), $V''(\phi) = 4\lambda(3\phi^2 - v^2)$ is negative only for $|\phi| < v/\sqrt{3}$. What is not convex is **small fields** (near the summit between the two wells), and at **large fields** V grows like ϕ^4 and is convex. Bauerschmidt–Bodineau wrote "convexity at large fields" as the issue because one needs to show uniformly that the effective potential after renormalisation stays convex at large fields too.

On the group it is the reverse. As in (i) of the One-Link Wilson Weight Proposition, near the identity (small fields, $\theta < \pi/2$) $\mathrm{Hess}\, V_\beta > 0$ and it is close to Gaussian, while on the far hemisphere (large fields) $\mathrm{Hess}\, V_\beta < 0$. This is not peculiar to the Wilson weight. A smooth function on a closed manifold has a maximum point, and at a maximum point $\mathrm{Hess}\, V \le 0$. **The non-convex place at large fields always exists, for topological reasons** Proved here . On the other hand, as in (iii) of the One-Link Wilson Weight Proposition, the probability of being there is of order $e^{-\beta}$.

```
Figure 1  Convex and non-convex places
          (horizontal axis: size of the field; |φ| for φ⁴, angle θ from the identity for

  φ⁴ (flat space):
     |φ| small ———————— |φ| ≈ v ———————————— |φ| large
     non-convex (summit)   bottom of well (convex)   convex (grows like φ⁴)
     → the non-convex place is "small fields".
       Its probability is set by the frequency of crossing between the wells

  SU(2) (closed sphere):
     θ small ———————————— θ = π/2 ———————————— θ = π (antipode)
     convex (near Gaussian)  boundary             non-convex (always, at the maximum)
     → the non-convex place is "large fields".
       Its probability is of order e^(−β) (the One-Link Wilson Weight Proposition)
```

The wall does not vanish; it is pushed into a place of small probability. On flat space, tools that avoid the non-convex summit were conceivable (imposing convexity at large fields, treating small fields separately). On the group the non-convex place cannot be avoided (the Convexity Ceiling Lemma), but in exchange its probability is exponentially small (the One-Link Wilson Weight Proposition). So tools on the group must push with probability, not with convexity Estimate .

An example where the rate of rarity was measured. On the width-2 strip of two-dimensional $U(1)$, the slope of the raw rate of wrapping of a single plaquette (a rare event that crosses a barrier of height 2β) matched the slope $2 - 1/\beta$ of the Kramers rate $\beta e^{-2\beta}$ to 1–3% ($\beta = 3..6$, outer boundary frozen, a post-hoc check, 24-02 § 05) Reading . The relation between this number and renormalisation on the group is an estimate.

The smallest example where rarity cannot be paid with local quantities. For the 0+1-dimensional double well, even for the shortest window (a single time) the local ratio is not 1 but exactly $\omega b/m$ ((7) of 24-03, 24-04 § 07) Proved here (Lean LatticeGauge.LocalHDoubleWell.one_slice_ratio, assumptions: $0 < \varepsilon$, $0 < m$, $0 < \omega$. The bundle guarantees only the algebra). The influence of the non-convex summit remains in the ratio however short the window is made. The implication for the group is an estimate Estimate .

What this section can say: that the non-convex place always exists for topological reasons, and the upper bound (5) on its probability, have proofs. "Must be tools that push with probability" is an estimate.

05

Carrying the hierarchical induction over to the group — applies, changes, does not apply

The model. As a four-dimensional hierarchical gauge model, we consider the lattice on which the recursion of the Migdal–Kadanoff approximation holds exactly (Ito 1984 Cited). Setting the couplings of the vertical plaquettes to 0 or ∞ place by place, the recursion of the plaquette weight g can be written down in the form $g \mapsto (g^{*4})^4$ (a convolution gluing four together, and a power stacking four). It is stated that for four-dimensional $SU(N)$ every coupling flows to strong coupling and confines (Ito 1985 Cited (abstract)). This model lacks translation and rotation invariance, and its theorems are not part of the statement of the four-dimensional Yang–Mills

problem. That the type of relaxation changes with the choice of coordinates to move was seen in 24-02 § 02 (in tree coordinates the gap of the Gaussian approximation is exactly β, independent of the number of steps; in link coordinates it gets smaller in inverse proportion to the square of the side).

Table 1 carries the parts of the Bauerschmidt–Bodineau induction over, row by row, to this model with values in the group $SU(2)$. "Applies" means the form holds as it is, "changes" means there is a corresponding form but its content is replaced, and "does not apply" means it does not hold; apart from the row of the Convexity Ceiling Lemma, these are estimates.

Table 1 Carrying the parts of the hierarchical induction over to $SU(2)$ (verdict labels: Proved here = from the Convexity Ceiling Lemma, Estimate = an estimate)

Part in the original	Carried over to the group	Verdict
The decomposition of the covariance $C = \sum C_j$ and the addition of fields $\phi + \zeta$	Addition is replaced by the group product, and the reference Gaussian by Haar. The role of "the covariance at scale j" is played by the stiffness of the weight (the supremum of the Hessian of $-\ln g$), and there is an inequality of the type in which the inverses of the stiffnesses roughly add under convolution (a paper record; no proof is placed in this book)	Changes Estimate
(A1) uniform convexity at each point	Read as Bakry–Émery for the measure on the fibre, it does not hold at weak-coupling scales by the Convexity Ceiling Lemma	Does not apply Proved here (the Convexity Ceiling Lemma) + the reading is Estimate
Convexity at large fields (the main issue of the original)	On the group, large fields (the far hemisphere) are necessarily non-convex (§ 04). But the probability is $e^{-c\beta}$	Changes (the wall becomes unavoidable, and goes to where the probability is small) Estimate
(A2)(A3) coupling and commuting of scales	In tree (plaquette) coordinates, automatic from the hierarchical structure. In link coordinates the Dirichlet form is not diagonal in the hierarchy, and commuting is expected to break	Tree: applies / link: expected not to apply Estimate
Lemma 2.7 (Brascamp–Lieb for the conditional measure from convexity)	For typical coarse values, the minimal broken line on the fibre is unique and the gap is of order β. When the coarse value is near -1, the geodesics from the identity to -1 form a family of two-dimensional spheres, and the gap in that direction is expected to drop to $O(1)$. Long arcs are expected to be saddle points, not a second well	Changes (convexity cannot be used, but the loss is expected to be polynomial) Estimate
Lemma 2.8 (an identity that turns the derivative in the direction of the coarse field into a covariance by parallel translation of the fluctuation and integration by parts)	Translating the last factor of each copy from the right moves the coarse holonomy by the same amount, and by the invariance of Haar an identity of the same type holds. The difference is that the score is bounded rather than linear in the field	Applies (in form) Estimate
Lemma 2.9 (the Helffer–Sjöstrand representation and an operator lower bound on the Witten Laplacian)	On a manifold, $\mathrm{Ric} + \mathrm{Hess}\, V$ enters the Witten Laplacian on one-forms. What is needed is a lower bound at each point, and for the same reason as (A1) it does not hold at weak coupling	Does not apply Estimate

Part in the original	Carried over to the group	Verdict
Multiscale Bakry–Émery (the Polchinski flow; the coefficient $\dot\lambda_t$ may be negative)	The natural semigroup on the group is convolution with the heat kernel, $V_t = -\ln(e^{-V} * K_t)$. The convolution step of Migdal–Kadanoff is its discrete version. What is needed is $\inf \mathrm{Hess}\, V_t$ (from below), and a formulation on the group was not found within the range searched	Changes (not yet formulated) Estimate

In sum. The decomposition of the variance and identities of the type of Lemma 2.8 hold on the group too. What does not hold is the two places that use convexity **at each point** (Lemmas 2.7 and 2.9), and by the Convexity Ceiling Lemma they do not hold at weak-coupling scales however the constants are arranged. The wall of "convexity at large fields" in the original does not vanish on the group but becomes unavoidable, while the probability on the wall drops to $e^{-c\beta}$ Estimate .

The Polchinski flow on the group. Theorem 3.6 of the review by Bauerschmidt–Bodineau–Dagallier, along a flow C_t of Gaussian covariances, derives from the condition

$$\dot C_t\ \mathrm{Hess}\, V_t(\phi)\, \dot C_t - \tfrac{1}{2}\ddot C_t\ \geq\ \dot\lambda_t\, \dot C_t \quad (\text{for all } \phi), \qquad \frac{1}{\gamma} = \int_0^\infty e^{-2\lambda t}\, dt \ \text{(6)}$$

the log-Sobolev constant $2/\gamma$ Cited . $\dot\lambda_t$ may be negative, and non-convex time intervals are paid for as the swelling $e^{-2\lambda t}$. A version for curved spaces (a form including the flow of the metric and Ric, Theorem 3.20) is also stated, and the original writes that "we currently do not know interesting applications of this generalisation" Cited . It is a candidate home for group-valued fields Estimate . Written on the group, the natural semigroup for this flow is convolution with the heat kernel, $V_t = -\ln(e^{-V} * K_t)$, and what is needed is an estimate of $\inf \mathrm{Hess}\, V_t$ from below. The estimates of the Migdal–Kadanoff recursion so far (the recursion of the stiffness; the paper record that the number of steps until strong coupling is entered is at most $\beta + 4\log\beta + 20$ Assumption (no proof placed in this book)) are all estimates from above ($\sup \mathrm{Hess}$), and the quantity that connects the two is the flow of $\inf \mathrm{Hess}\, V_t$. In weak-coupling time intervals $\inf \mathrm{Hess}\, V_t \approx -\beta$ (the antipode), and how much $e^{-2\lambda t}$ swells has not been estimated Estimate . The first number is one step of the one-link weight (the question "Hessian of the one-link weight", 24-08).

The $\frac{1}{4}$ type does not come back for one link. $e^{(\beta/2)\,\mathrm{tr}\, U}\, d\mathrm{Haar} = e^{\beta\cos\theta}\, d\mathrm{Haar}$ has a single well ($\theta = \pi$ is the minimum of the density; there is no second local minimum), and the gap is expected to approach the Hessian β at the identity for large β, without getting exponentially worse Estimate (general semiclassical theory. The original on uniform log-Sobolev for Boltzmann measures with a field on spheres is known only bibliographically, and has not been checked). The wall of the $\frac{1}{4}$ type at the summit of the double well (two wells) does not appear for one link. On the fibre, the degenerate family where the coarse value is near -1 is expected to produce a polynomial loss (Table 1).

What this section can say: the verdicts of Table 1, apart from the row of the Convexity Ceiling Lemma, are estimates. The results for hierarchical models have no implication for four-dimensional Yang–Mills under any verdict.

> 06

> **Tools that pay with rarity — currency, Lyapunov, field-dependent constants, the example of the action barrier**

Currency. The currency-mix-up type of 23-10 in Part 23 said that what the Poincaré inequality measures is the speed of exit, and that what can pay with rarity is the currency of entropy (23-10 currency-mix-up type). As a formula, when a set B satisfies $\mu(B) \le e^{-s}$,

$$s \int_B g^2 \, d\mu \;\le\; \mathrm{Ent}_\mu(g^2) + \ln 2 \int g^2 \, d\mu \quad (7)$$

Proved here ((3) of 23-09). The ratio of the variance to the Dirichlet form does not change when the function is multiplied by a constant, so if a function supported on a rare set is put in, the rarity cancels between numerator and denominator, and at the Poincaré step rarity has no effect. **A step that pays with rarity needs to pass through an inequality of log-Sobolev (entropy) type**. This is why the candidate tools on the group are not the Poincaré induction (Brascamp–Lieb at discrete scales) but the type that flows Bakry–Émery at each scale in log-Sobolev form ((6) of § 05) Estimate .

The Holley–Stroock perturbation. If $\mu' = e^{-U}\mu/Z$ with U bounded, the Poincaré constant (and the log-Sobolev constant) of μ' is at most $e^{\operatorname{osc} U}$ times that of μ Assumption (the standard form as remembered; the original has not been checked in this chapter). It is a tool that pays for the non-convex place with the "oscillation", and the cost is exponential in the oscillation, independent of rarity. Regarding the one-link Wilson weight as a perturbation of Haar, $\operatorname{osc} U = 2\beta$, and the loss is $e^{2\beta}$. In Part 23, the plan to pay only for the summit band with this perturbation (the detour plan "perturbing only the summit band") put the one-site oscillation $\operatorname{osc}/\hbar = 3.7/11/37$ into the exponent, multiplied by the length along the chain, and came out 2000–4900 times outside the allowance (23-10 § 03, type of the address of 1/ħ) Reading .

Lyapunov conditions. For a generator L, if there are $W \ge 1$ and a good set G with

$$LW \;\le\; -a\,W + b\,\mathbf{1}_G \qquad (a > 0,\ b \ge 0) \quad (8)$$

(a drift that returns from the bad set to the good set), and a local Poincaré inequality (constant C_G) holds on G, then the global Poincaré constant is at most $(1 + bC_G)/a$ Cited (of the type of Bakry–Barthe–Cattiaux–Guillin 2008. The statement was checked through a restatement in a secondary source, 23-14 § 03. The positive control on the one-dimensional Gaussian is row 20 of 23-14 § 04). Convexity is used only on the good set G, and the bad set is paid for by "the speed of coming back". On the group it is natural to take the good set near the identity and the bad set on the far hemisphere, and the loss is expected to be polynomial Estimate . However, (8) is a Poincaré conclusion, not a form that uses rarity itself. To use rarity, a step that moves the type of (8) into log-Sobolev form (the currency of entropy) is needed Estimate .

Field-dependent constants. Bauerschmidt–Bodineau write in Remark 2.4 that ε in (A1) may be made to depend on the coarse field, $\varepsilon = \varepsilon(\phi)$ (but uniformly less than 1), and add that it is "hard to use" Cited . It is a direction that allows ε to be large at bad coarse fields, but as long as it must be uniformly less than 1, it does not get past the ceiling of the Convexity Ceiling Lemma. To get past it, the condition must be replaced by one in the form of an **average weighted by the probability** of bad fields Estimate .

An example where rarity paid — the margin of the action barrier. In the first step of Part 23, when the end of the core on the box side of the Gap-Inside-the-Box Proposition crossed the summit, a second minimum point appeared on the wall of conditioning, and the Poincaré constant of the core grew like $e^{\Delta S/\hbar}$ (23-11 § 06). This is the cost of the non-convex place inside the core. That cost is paid by the rarity $D/(C\xi)^2 \le d e^{-2S_0/\hbar}$ carried by the local term of the Gap-Inside-the-Box Proposition (the rarity of creating a pair of kinks): under the core action-barrier assumption $C_P \le c_* \hbar^{-p} e^{\Delta S_*/\hbar}$ and the reading of ξ, the local term vanishes like $e^{-(2S_0-\Delta S_*-\delta)/\hbar}$ Proved here ((12) of 23-11. Lean LatticeGauge.CoreActionGap.cloc_term_poly, cloc_term_vanishes, assumptions: the core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$). On the $\hbar = 0$ reading, $\Delta S_*^0 \le 1.4 \times 10^{-2}$, its ratio to $2S_0 = 4/3$ is at most 1.05×10^{-2}, and the margin was about 95 times Reading (Lean LatticeGauge.CoreActionGap.measured_ratio, assumption: the measured upper bound $\Delta S^0 \le 0.01401$; the conclusion is in the form of the ratio $\le 1.06 \times 10^{-2}$). Rather than removing the non-convex place by convexity, it is paid by a stronger factor of rarity: this is a small real example of the type of this chapter. However, the core action-barrier assumption itself is an assumption row, supported only by the $\hbar = 0$ reading and the lower bound on $\hbar \ln C_P$.

Table 2 How to pay for the non-convex place (what pays, and what it costs)

Way of paying	What pays	Form of the cost	For one-link $SU(2)$	Label
Convexity (Bakry–Émery, Brascamp–Lieb)	A lower bound on the curvature at every point	The inverse of the lower bound	Usable only for $\beta < 2$ (the One-Link Wilson Weight Proposition). Impossible in principle at weak coupling by the Convexity Ceiling Lemma	Proved here
The Holley–Stroock perturbation	The oscillation of the potential	e^{osc}	$e^{2\beta}$: an exponential loss	Assumption (from memory)
Lyapunov condition (8)	A drift returning from the bad set, and a local constant on the good set	$(1 + bCG)/a$	If the good set is taken near the identity, the loss is expected to be polynomial	Cited (type), Estimate (application to the group)
The currency of entropy (7)	The rarity s of the bad set	Works in the form $1/s$	$s \approx \beta$ ((5) of the One-Link Wilson Weight Proposition). At the confinement scale it becomes $s = O(1)$ and runs out	Proved here (inequality), Estimate (running out)
Paying with a strong factor of rarity (the margin of the action barrier)	A rarity $e^{-2S_0/\hbar}$ stronger than the cost $e^{\Delta S_*/\hbar}$	The margin $\Delta S_* < 2S_0$	The example of the first step (0+1 dimensions). The counterpart on the group is undecided	Proved here (algebra), Reading (margin), Assumption (the core action-barrier assumption)

What this section can say: (7) and the algebra of the action barrier have proofs; the Lyapunov condition is a type whose statement was checked in a secondary source; Holley–Stroock is the standard form from memory. Which of them works on the group, and with what loss, is an estimate.

07

If it were made into rules — the class of measures that pay with rarity

Let us try to rewrite what we have seen in this chapter in the form of rules. It is an attempt that follows the phrasing of the question "if new mathematics is to be made, what should be made the rules of the mathematics?", and the rules written here are **candidates**. Neither consistency nor the ability to restate existing results has been checked. None of them promises a prospect of closing the third step (the continuum limit and the OS axioms) Estimate . Other candidates (axiomatising scale, the algebra of loops, conserved quantities and sectors, an axiom of asymptotic freedom, rules for curvature) and a list of combinations are placed in 24-08 § 02.

Candidate: the class of measures that pay with rarity

Objects. A pair of a probability measure $\mu = e^{-V}\,d\mathrm{vol}/Z$ on a compact Riemannian manifold M (a product of groups) and a good set $G \subset M$. The parameters $(\kappa_{\mathrm{BE}}, s, a, b)$ define the class $\mathcal{C}(\kappa_{\mathrm{BE}}, s, a, b)$.

Candidate rules. Rule of local convexity — locally convex on the good set: $\mathrm{Ric} + \mathrm{Hess}\, V \geq \kappa_{\mathrm{BE}} > 0$ on G. Rule of rarity of bad sets — $\mu(M \setminus G) \leq e^{-s}$. Rule of return drift — a drift returning from the bad set: the type (8), $LW \leq -aW + b\mathbf{1}_G$. Rule of paying for entropy — rarity is paid only through the entropy inequality (7) (the class is defined by quantities of log-Sobolev type, not Poincaré). Rule of closing under coarse-graining — convolution with the heat kernel on the group, $V_t = -\ln(e^{-V} * K_t)$, maps the parameters as $(\kappa_{\mathrm{BE}}, s, a, b) \mapsto (\kappa'_{\mathrm{BE}}, s', a', b')$ by an explicit recursion.

A constraint, not a rule. By the Convexity Ceiling Lemma, a class that places the rule of local convexity on the whole of M is empty for weights whose $\operatorname{osc} V$ exceeds $KD^2/2$ (π^2 for one-link $SU(2)$). For the one-link Wilson weight it is empty for $\beta > \pi^2/2$. **The bad set always exists.**

The rule of paying for entropy makes the currency type of Part 23 into a rule (23-10 currency-mix-up type: Poincaré is the speed of exit, rarity is the currency of entropy).

Candidate first theorems. (a) For one measure: if $\mu \in \mathcal{C}$, the log-Sobolev constant is at most $F(\kappa_{\mathrm{BE}}, s, a, b)$, and F is not exponential in s. On flat space the type that combines a Lyapunov condition with a local inequality corresponds to this (§ 06), and the Holley–Stroock perturbation pays e^{osc}. (b) Closure: applying one step of the heat kernel to the one-link Wilson weight of $SU(2)$ stays in the class, with $s' \geq s - O(1)$. This is the first step of the question "Hessian of the one-link weight" (24-08 § 01). For class functions (functions of the single variable, the angle θ from the identity), convolution with the heat kernel only multiplies the coefficients of the character expansion by exponentials of the Casimir values, so how the lower bound on the curvature on the good set and the weight of the bad set are carried over can be followed by one-variable paper calculation and small numerics Estimate .

Parts that already exist. On flat fields, the weight (regulator) $\tilde{G}(X, \phi)$ of the appendix of the book by Bauerschmidt–Brydges–Slade has four properties — product factorisation (it factorises over disjoint sets), submultiplicativity in sums of fields $\tilde{G}(X, \phi + \phi') \leq \tilde{G}^2(X, \phi)\, \tilde{G}^2(X, \phi')$, closure under the fluctuation expectation $\mathbb{E}_+ \tilde{G}^t(X) \leq 2^{|B(X)|}$, monotonicity to the next step $\tilde{G}^t \leq \tilde{G}_+ -$

and from them the one-step estimate $\|\mathbb{E}_+\theta K(X)\|_{\tilde{G}_+} \le 2^{|B(X)|}\|K(X)\|_{\tilde{G}}$ follows (the One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS)) Cited . It is a real example of **writing the rarity of large fields as a weighted norm and closing it under the expectation of one step of coarse-graining**. What is needed on the group is the group version of this "closing in one step", and the place where submultiplicativity in sums of fields ((ii) of BBS) uses **addition** of fields is replaced by the group product u_1u_2. The multiscale Bakry–Émery of Bauerschmidt–Bodineau–Dagallier ((6) of § 05, $\dot{\lambda}_t$ may be negative) and its version for curved spaces (Theorem 3.20) are also candidate parts Cited . The entropy stability lemma (a form that takes an upper bound on the covariance as its condition instead of the pointwise Hessian, Lemma 3.21) uses a linear tilt $e^{(h,\zeta)}$, so it does not carry over directly to the group Estimate .

Combining with one step of coarse-graining — the coupling window. Combining this class with axiomatising scale (a semigroup of coarse-grainings and a flow on it), one can write a form in which the coarse-graining kernel maps the class at each step, $\mathcal{C}_j = \mathcal{C}(\kappa_j, s_j, a_j, b_j)$, as $\mathcal{C}_j \to \mathcal{C}_{j+1}$. A candidate first theorem is "in a finite box, on the ultraviolet side, as long as the effective coupling β_j is at least β_*, the class closes in the form $s_{j+1} \ge s_j - c$, $\kappa_{j+1} \ge \kappa_j(1 - c/\beta_j)$" Estimate . Where it has closed, the effective measure has a uniform log-Sobolev inequality, and from there it can be handed over to the tools of the strong-coupling side (the type that pays with the Ricci curvature of the group; the result of Shen–Zhu–Zhu giving log-Sobolev and the mass gap at small β Cited (abstract)). But the rate of rarity s_j is proportional to the effective coupling ($s \approx \beta$ by (5) of the One-Link Wilson Weight Proposition), and it runs out at the confinement scale when $\beta_j = O(1)$. The currency of curvature works only in a narrow range. **Between the two, a coupling window remains where neither currency is enough**. This window is a restatement in the language of rules of the second difficulty of the third step — that there is no expansion connecting the ultraviolet and the infrared — and is not a new obstacle (24-07 § 06, 24-07 Table 3) Estimate .

In the hierarchical gauge model (§ 05), the number of steps before the static flow enters strong coupling is finite, and the three — the ultraviolet side, the window and the strong-coupling side — come together within one model. If the candidate rules are to be tried first, it is on this model. However, the static flow there is a one-variable recursion with no many-body remainder, so crossing the window is easy because of the structure of the model, and the results imply nothing for four-dimensional Yang–Mills Estimate .

Relation to the roots Estimate . Of the three roots of the third step (24-07 § 05), the one this class addresses head-on is the third (there is no global gauge fixing, and large fields cannot be controlled by convexity). This is because it places an alternative class where the Convexity Ceiling Lemma says "on the group the convexity class is empty". It addresses the first (uniformity in the number of steps) only when combined with axiomatising scale, and the key is whether the recursion of the rule of closing under coarse-graining gives constants uniform in the number of steps. It does not address the second (connecting the ultraviolet and the infrared) — the coupling window above is the shape of that failure.

What this section can say: the rules are candidates, and the candidate theorems are candidates. The existing parts (the four properties of the weight and the One-Step Bound on the Activity Norm, multiscale Bakry–Émery) hold only within the conditions of their originals (flat fields, hierarchical, flows of Gaussian covariances).

08

The road of stochastic quantisation — subcriticality, and the direction of rebuilding where the objects live

Stochastic quantisation is the road that makes the Yang–Mills measure as the long-time limit of a Langevin dynamics (a stochastic partial differential equation) that has it as its invariant measure (7-09). The dynamics is of the same kind as the Langevin dynamics measured by this book's CP, but the aim is the **construction of the measure**, and the speed of relaxation (the relaxation-time assumption) is not asked about.

The key condition of the solution theory of regularity structures is subcriticality, and it holds only when $d < 4$ (§ 1.2 of Chevyrev's review Cited). In two dimensions, a Polish state space on the space of gauge orbits and a Markov process have been made, and the Yang–Mills measure has been shown to be its invariant measure (Chandra–Chevyrev–Hairer–Shen, Chevyrev–Shen Cited (abstract)). In three dimensions, it has reached a new state space, the space of one-forms smoothed by the heat flow, local-in-time solutions and gauge-covariant renormalisation; the three-dimensional Yang–Mills measure is not yet constructed even on the periodic box $\mathbb{T}^3$ (§ 1 and § 4 of Chevyrev's review Cited).

This road goes not in the direction of extending the lattice renormalisation group, but in the direction of **rebuilding where the objects live (the state space) and the means of construction (stochastic dynamics)** (24-07 § 05) Estimate . However, the condition of subcriticality shows that this new tool too becomes exactly critical in four dimensions. It is one manifestation of the coupling being dimensionless ((c) of 24-07 § 01). This road gives tools neither for the four-dimensional relaxation-time assumption nor for the local relaxation-time assumption.

What this section can say: the condition of subcriticality and the three-dimensional non-construction are statements in the text of the review; the individual two- and three-dimensional results are copies from abstracts. This book does not use regularity structures as a tool.

09

What cannot be said

(a) Hierarchical models. The results for hierarchical models imply nothing for four-dimensional Yang–Mills (they lack translation and rotation invariance). The verdicts of Table 1 are also estimates within the hierarchical model.

(b) The Polchinski flow on the group. No source formulating the flow by convolution with the heat kernel up to the multiscale Bakry–Émery condition was found within the range searched (this is not "there is none").

(c) The fibre. That the gap in the direction of the degenerate family near coarse value -1 drops to $O(1)$, and that long arcs are saddle points, are estimates (by analogy from the Morse index of continuous geodesics).

(d) Dynamics. Statements on the gap or log-Sobolev for Glauber dynamics or Langevin dynamics of hierarchical gauge models were not found within the range searched.

(e) The kind of the original. The induction of Bauerschmidt–Bodineau 2019 is a result of Poincaré type (a spectral gap), not a log-Sobolev result. The form that flows log-Sobolev is later work by the same authors and a theorem of the review.

(f) Candidate rules. The class of § 07 is a candidate, and none of consistency, restating existing results, or the candidate theorems has been checked. "The coupling window is a restatement of the source "no expansion linking scales"" is an estimate, and the width of the window has not been measured in any model.

(g) The example of the action barrier. The margin of 95 times is a $\hbar = 0$ reading in 0+1 dimensions, and the core action-barrier assumption is an assumption row. We say nothing about whether renormalisation on the group has a margin of the same form.

(h) Stochastic quantisation. That four dimensions is critical is not a proof that tools cannot be made in four dimensions. It only copies the statement of the conditions of the original.

10

Table of the five grades and the dependency graph

Table 3 The rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Bakry–Émery (1)	Cited	Theorem 3.6 of the review	Lower bound on the curvature at each point	—
2	(A1) of the hierarchical induction, Theorem 2.1 and Corollary 2.2 (2)	Cited	§ 02	Scalar, hierarchical	—
3	The one-site diagonal at the summit $\frac{15}{4}$ and the sum of couplings 4	Proved here	(11) of 23-09	0+1, $\varepsilon = \frac{1}{2}$	—
4	Convexity Ceiling Lemma: $\operatorname{osc} V \le K D^2/2$ (3)	Proved here	Proof in § 03	Compact manifold without boundary	— (the one-dimensional version is a candidate)
5	No Concentration at the Identity Corollary: if convex on $SU(2)$ then $\mu \le e^{\pi^2}$ Haar (4)	Proved here	Proof in § 03	Unit S^3	—
6	One-Link Wilson Weight Proposition: for one link $(2 + \beta \cos\theta) g$, $\beta < 2$, (5)	Proved here	Proof in § 03	$\beta \ge 2$ ((5))	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
7	At a maximum point of a closed manifold $\operatorname{Hess} V \leq 0$	Proved here	§ 04	—	—
8	The raw rate of wrapping on the strip matches the Kramers slope to 1–3%	Reading	24-02 § 05	Two-dimensional $U(1)$, $\beta = 3..6$, post hoc	—
9	The 0+1 one-site ratio ω_b/m	Proved here	(7) of 24-03, 24-04 § 07	0+1	`LatticeGauge.LocalHDoubleWell.one_slice_ratio` (assumption: $0 < \varepsilon, m, \omega$)
10	Read as Bakry–Émery on the fibre, (A1) on the group does not hold at weak coupling	Estimate	§ 02, § 05	Weak-coupling scales	—
11	The verdicts of Table 1	Estimate	Table 1	Hierarchical model	—
12	Multiscale Bakry–Émery (6), the version for curved spaces	Cited	Theorems 3.6 and 3.20 of the review	Flows of Gaussian covariances	—
13	The number of steps until the hierarchical model enters strong coupling $\leq \beta + 4 \log \beta + 20$	Assumption	Paper record (no proof in the book)	Migdal–Kadanoff, $SU(2)$	—
14	A wall of the $\frac{1}{4}$ type does not come back for one link	Estimate	§ 05	One link	—
15	The currency inequality (7)	Proved here	(3) of 23-09	—	—
16	The cost e^{osc} of the Holley–Stroock perturbation	Assumption	Standard form from memory	—	—
17	$(1 + bCG)/a$ from the Lyapunov condition (8)	Cited	Secondary source (23-14 § 03)	—	Positive control `gauss1d_bbcg` (no assumptions, 23-14)
18	Field-dependent $\varepsilon(\phi)$ (uniformly less than 1)	Cited	Remark 2.4 of the original	—	—
19	Core action-barrier assumption	Assumption	23-11 § 06	0+1, the core on the box side	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
20	$\Delta S_*^0 \le 1.4 \times 10^{-2}$, a margin of about 95 times	Reading	23-11 § 06	The $\hbar = 0$ reading	`LatticeGauge.CoreActionGap.measured_ratio` (assumption: the measured upper bound)
21	The local term vanishes like $e^{-(2S_0 - \Delta S_* - \delta)/\hbar}$	Proved here	(12) of 23-11	$\hbar \to 0$	`LatticeGauge.CoreActionGap.cloc_term_poly`, `cloc_term_vanishes` (assumptions: the core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$)
22	Tools on the group must push with probability	Estimate	§ 04	—	—
23	The four properties of the weight and the One-Step Bound on the Activity Norm	Cited	Appendix of BBS	Flat fields	—
24	The class of measures that pay with rarity (candidate)	Estimate	§ 07	Candidate	—
25	Log-Sobolev and the mass gap at strong coupling	Cited	Shen–Zhu–Zhu (abstract)	Small β	—
26	The coupling window is a restatement of the source "no expansion linking scales"	Estimate	§ 07, 24-07 § 06	—	—
27	The solution theory is subcritical for $d < 4$; the three-dimensional measure is also not yet constructed	Cited	Chevyrev's review	—	—
28	The road of stochastic quantisation gives no tools for the four-dimensional relaxation-time assumption	Estimate	§ 08	—	—

What the machine-checking column guarantees is only real algebra and arithmetic; that ΔS is the action barrier of the core, and that CP is the Poincaré constant of the path measure, are outside Lean (23-13). The Convexity Ceiling Lemma, the No Concentration at the Identity Corollary and the One-Link Wilson Weight Proposition have no machine checking.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  Bakry–Émery: lower bound on curvature ⇒ log-Sobolev, Poincaré
 2  CIT  (A1), Theorem 2.1, Corollary 2.2 of the hierarchical induction (Bauerschmidt–Boc
         scalar, hierarchical
 3  PRV  one-site diagonal at the summit 15/4 and sum of couplings 4         0+1 (23-09)
 4  PRV  Convexity Ceiling Lemma: osc V ≤ KD²/2                              closed manif
 5  PRV  No Concentration at the Identity Corollary: if convex on SU(2), μ ≤ e^(π²) Haar
 6  PRV  One-Link Wilson Weight Proposition: curvature (2 + β cos θ), probability (5)
 7  PRV  Hess ≤ 0 at a maximum point of a closed manifold
 8  RDG  raw rate on the strip matches the Kramers slope to 1–3%             two-dim. U(1
 9  PRV  0+1 one-site ratio ω_b/m                                            Lean one_sli
10  EST  (A1) on the group does not hold at weak coupling ⟸ 2, 4
11  EST  verdicts of Table 1 ⟸ 10
12  CIT  multiscale Bakry–Émery, version for curved spaces
13  ASM  number of steps until strong coupling (paper record)
14  EST  the 1/4 type does not come back for one link ⟸ 6
15  PRV  currency inequality (7)                                            (3) of 23-09
16  ASM  Holley–Stroock cost e^osc                                           from memory
17  CIT  Lyapunov condition (8)                                              secondary sc
18  CIT  field-dependent ε (uniformly less than 1)
19  ASM  core action-barrier assumption                                      0+1
20  RDG  ΔS⁰ ≤ 1.4·10⁻², margin about 95 times                              ħ = 0 readin
21  PRV  the local term vanishes (12) ⟸ 19                                    Lean cloc_
22  EST  tools on the group push with probability ⟸ 5, 6, 7, 15
23  CIT  four properties of the weight, One-Step Bound on the Activity Norm (BBS)   flat
24  EST  class of measures that pay with rarity (candidate) ⟸ 4, 15, 17, 23
25  CIT  log-Sobolev and gap at strong coupling                              abstract
26  EST  the coupling window restates the lack of an expansion linking scales ⟸ 6, 24,
27  CIT  subcriticality d < 4; three dimensions also not yet constructed
28  EST  stochastic quantisation gives no tools for the four-dimensional relaxation-time
```

The way to check is to see by eye that on every "⟸" the numbers on the right are smaller than the number on the left. There are no circles. The nodes with proofs placed are four — the Convexity Ceiling Lemma, the No Concentration at the Identity Corollary, the One-Link Wilson Weight Proposition, and the maximum point on a closed manifold — all statements about one measure on one manifold. The nodes on the conclusion side of this chapter (tools on the group push with probability, the class of measures that pay with rarity, the restatement as the coupling window, the note on stochastic quantisation) are all Estimate, and there is neither a node for the mass gap nor a node for the relaxation-time assumption. The node where the local term vanishes (an example where rarity paid) is algebra on top of the core action-barrier assumption, and does not enter the premises of the nodes on the conclusion side. This is because the 0+1-dimensional example is an example of the type, not grounds for a statement on the group.

What this chapter can say

What is said	What is not said
Proved here The convexity ceiling on closed manifolds (the Convexity Ceiling Lemma); a convex measure on $SU(2)$ does not exceed e^{π^2} times Haar (the No Concentration at the Identity Corollary); the curvature of the one-link Wilson weight, and the upper bound on the probability of the non-convex set (the One-Link Wilson Weight Proposition)	The convexity and gap of the conditional measure on the fibre (a product of many links). The value of the gap in the direction of the degenerate family
Cited Bakry–Émery; the hypotheses and conclusions of the hierarchical induction; multiscale Bakry–Émery; the four properties of the weight and the One-Step Bound on the Activity Norm; the subcriticality of stochastic quantisation	A formulation of the Polchinski flow on the group. The gap of the dynamics of hierarchical gauge models
Proved here + Reading The margin of the action barrier in Part 23 is an example where the non-convex place was paid with rarity (on the core action-barrier assumption and the $\hbar = 0$ reading)	The core action-barrier assumption itself. That the same margin exists on the group
Estimate That tools on the group must push with probability; the verdicts of Table 1; the class of measures that pay with rarity (candidate); that the coupling window is a restatement of the lack of an expansion linking scales	None of these is a theorem. The implication of hierarchical-model results for four dimensions. The consistency of the candidate rules

Sources and reproduction

Item	Kind	Source / tools
The hierarchical induction ((A1), Theorem 2.1, Corollary 2.2, Remark 2.4, Lemmas 2.7–2.9, the self-assessing sentence)	Cited	R. Bauerschmidt, T. Bodineau, arXiv:1809.02075 (Commun. Math. Phys. 373 (2020) 1167). § 1– § 2 checked in the text
(1), (6), the version for curved spaces, entropy stability	Cited	R. Bauerschmidt, T. Bodineau, B. Dagallier, arXiv:2307.07619 (review). Theorems 3.6 and 3.20 and Lemma 3.21 checked in the text
The four properties of the weight, the One-Step Bound on the Activity Norm	Cited	R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474. Appendix A.6 checked in the text
Hierarchical gauge models	Cited	K. R. Ito, Commun. Math. Phys. 95 (1984) 247 (text). K. R. Ito, Phys. Rev. Lett. 55 (1985) 558 (abstract)
Lyapunov condition (8)	Cited	D. Bakry, F. Barthe, P. Cattiaux, A. Guillin, Electron. Commun. Probab. 13 (2008) 60 (bibliographic data from memory. The statement was checked through a restatement in a secondary source, 23-14 § 03)
The Holley–Stroock perturbation	Assumption	R. Holley, D. Stroock, J. Stat. Phys. 46 (1987) 1159 (both bibliographic data and statement from memory)
Log-Sobolev and the mass gap at strong coupling	Cited	H. Shen, R. Zhu, X. Zhu, arXiv:2204.12737 (abstract). The numbers for the range are not written, since they depend on the normalisation convention
Stochastic quantisation	Cited	I. Chevyrev, Stochastic quantisation of Yang–Mills, arXiv:2202.13359 (§ 1, § 1.2, § 4 in the text). A. Chandra, I. Chevyrev, M. Hairer, H. Shen, arXiv:2006.04987, arXiv:2201.03487 (abstract). I. Chevyrev, H. Shen, arXiv:2302.12160 (abstract)

Item	Kind	Source / tools
Convexity Ceiling Lemma, No Concentration at the Identity Corollary, One-Link Wilson Weight Proposition	Proved here	Proofs in § 03 of the text. The only numbers are the values of $\frac{54\pi}{25}e^{-3/2}$ and e^{π^2}
The examples from Part 23 (the summit, currency, the Holley–Stroock plan, the action barrier)	This book	23-09 § 06, 23-10 § 04, 23-11 § 06
Lean	Lean	LatticeGauge.LocalHDoubleWell (`one_slice_ratio`), LatticeGauge.CoreActionGap (`cloc_term_poly`, `cloc_term_vanishes`, `measured_ratio`). The bundles guarantee only algebra and arithmetic

Terms used in this chapter — the convexity ceiling, the lower bound on the Bakry–Émery curvature κ_{BE}, Brascamp–Lieb, fibre, the reversal of convex and non-convex places, the rate of rarity s, currency (entropy), the Holley–Stroock perturbation, Lyapunov conditions, field-dependent constants, the Polchinski flow on the group, the class of measures that pay with rarity, the coupling window, subcriticality. Their meanings are in A-02 Glossary and 24-09.

Next chapter to read: 24-07 Because it is four dimensions — what works in four dimensions, limited to two points: the dimensionless coupling and topology.

Because it is four dimensions — the dimensionless coupling and topology

> This chapter organises established knowledge. § 03, § 04, § 06 and § 07 are a map of open questions.

What this chapter shows — We state what works in four dimensions, limited to two points: that the coupling is dimensionless (the critical dimension, dimensional transmutation, the opposite directions of asymptotic freedom for ϕ^4_4 and for Yang–Mills), and that there are topological configurations with no scale (instantons and the topological charge). Putting the two points together gives the estimate that the slow quantity and the correlation length split into different origins. Next, for each of the four steps of the staircase of Part 23, we put the hurdle, the known reach, whether the tools of this book reach it, and the kind of cost into one table, and state why the third step (the continuum limit) stayed open so long, through the three sources of difficulty and the descriptions in primary sources (for the existence part of the third step, the committee's assessment is that the construction published on 2026-11-10 has shown it; we read it in Volume 2). On topology, we set the range in which the relaxation-time upper-bound assumption reads as false on a periodic box, and the fact that for this book's group $SU(2)$ it is undecided whether freezing breaks the diffusive-relaxation-time assumption, beside the two-dimensional test bed of 24-02. We add, one line each, why the non-perturbative mechanisms of physics are not proofs.

Prerequisite chapters — 23-03 § 02 (the definition of the four steps) and 23-03 § 06 (the prospect of lifting and the reasons it does not lift), 23-15 § 07 (the distance from the second step on), 23-16 (what is not claimed), 23-01 § 05 (the note on the relaxation-time assumption and topological sectors), 4-07 (dimensional transmutation), 7-07 (Balaban and MRS), 24-01, 24-02. Notation is in 24-09.

Said first — The level is [Y3]. We state the two points not as "things that happen only in four dimensions" but as properties of a theory being at its critical dimension. The reading that puts the two points together is an estimate Estimate . "Does it reach" and the costs in the table of the four steps are estimates and guesses. § 05 is an arrangement of the descriptions in the primary sources read, and the text of Balaban's papers was not obtained (we go through the bibliographic data and the descriptions in other sources). This chapter shows neither the mass gap nor the construction of the continuum limit.

The order of this chapter

1. That the coupling is dimensionless — the critical dimension, dimensional transmutation, asymptotic freedom in the opposite direction ((1)(2), Figure 1)
2. Topological configurations with no scale — instantons, and two ways of counting the weight of dislocations ((3), Table 1)
3. An estimate that puts the two points together — the slow quantity and the correlation length split into different origins
4. The four hurdles — step, hurdle, known reach, does it reach, cost (Table 2)

5. Why the third step stayed open so long — the three sources of difficulty, the object, tools, theory, history (Table 3)
6. A paragraph on candidate rules — the "coupling window" is a restatement of the source "no expansion linking scales"
7. Topology and the box — undecided for $SU(2)$ — the periodic box, open boundaries, correspondence with the two-dimensional test bed
8. Physical mechanisms, and why they are not proofs — Table 4
9. What cannot be said
10. Table of the five grades and the dependency graph — Table 5

01

That the coupling is dimensionless — the critical dimension, dimensional transmutation, asymptotic freedom in the opposite direction

This section summarises existing knowledge (the critical dimension, dimensional transmutation, asymptotic freedom). The introduction is in 4-07 § 02.

Writing the action of d-dimensional Yang–Mills as

$$S = \frac{1}{4g^2}\int d^d x \ \operatorname{tr} F_{\mu\nu}F^{\mu\nu}, \qquad [g^2] = (\text{length})^{\,d-4} \quad (1)$$

the coupling g^2 is exactly dimensionless in four dimensions Cited (dimension counting. On the lattice, by Wilson's heuristic, $\beta \propto \varepsilon^{4-d}$, and in four dimensions β does not depend on the lattice spacing ε; § 3 of Chatterjee's review). In three dimensions or fewer, g^2 has the dimension of a negative power of length, and the interaction weakens like a power at short distances (super-renormalisable). In five dimensions or more it strengthens instead (non-renormalisable). In exactly four dimensions, the correction from each change of scale does not shrink like a power, and remains at the size of g^2. In the language of power counting, a quartic monomial in four dimensions is neither relevant nor irrelevant but marginal (Definition 5.2.1 of the book by Bauerschmidt–Brydges–Slade Cited).

Dimensional transmutation. In the non-abelian case, the sign of that correction points to the side "weaker at shorter distances" (asymptotic freedom, 4-06). The action has no ruler of length, yet on quantisation, at one loop,

$$\Lambda = \mu\,\exp\!\Big(-\frac{1}{2b_0\,g^2(\mu)}\Big), \qquad b_0 = \frac{11N}{48\pi^2} \quad (SU(N)\ \text{pure gauge}) \quad (2)$$

an inverse length is born Cited (the standard one-loop form, 4-07). The mass gap and the confinement scale are expected to be constant multiples of Λ. All derivatives of $\exp(-1/g^2)$ vanish at $g = 0$, so Λ appears at no order of the perturbative expansion in g ("invisible to perturbation theory" in 4-07).

How it shows in the relaxation-time assumption. (a) The correlation length grows exponentially in β ($\xi \propto e^{3\pi^2\beta/11}$ by the one-loop bookkeeping for $SU(2)$, 23-03 Table 2), and quantities measured in ξ

carry a factor $\ln \xi \sim \beta$. This is why Part 23 placed the relaxation-time assumption as a power statement rather than a band statement (23-01 § 07). (b) In the Gaussian approximation the coupling does not flow. In the Gaussian approximation on the hierarchical lattice, the static correlation of plaquettes decayed like the massless R^{-4}, and the correlation length was only cut off by the volume (24-02 § 02). A scale is born only from the non-abelian correction. (c) The solution theory of the equations of stochastic quantisation has subcriticality ($d < 4$) as its key condition, and four dimensions is exactly critical (§ 1.2 of Chevyrev's review Cited , 7-09).

The same marginal, but in the opposite direction. Four-dimensional ϕ^4 also has a dimensionless (marginal) coupling, but under the renormalisation flow the coupling approaches 0 on the side of long distances ("infrared asymptotic freedom", the Infrared Asymptotic-Freedom Proposition (Proposition 6.1.7 of BBS) of Bauerschmidt–Brydges–Slade Cited). In the direction one wants to control (long distances) the theory approaches a Gaussian, so the tools reach, and the limit they reached was trivial (the scaling limit is Gaussian) (Aizenman–Duminil-Copin 2021 Cited (abstract), 7-08). In Yang–Mills the coupling approaches 0 on the side of short distances, and grows towards long distances.

```
Figure 1  Two arrows (the flow of the coupling g;
          left is short distance, right is long distance)

            short distance (UV)                long distance (IR)
  φ⁴₄ :  g toward the large side   ←──────────   g → 0 (approaches a Gaussian)
         → near a Gaussian in the direction to be controlled.
           The tools reach; the limit is trivial

  YM₄ :  g → 0 (approaches a Gaussian)  ────→  g grows (g = O(1) at the scale Λ)
         → near a Gaussian only in the UV.
           No reference for an expansion on the IR side
```

We do not write "only in four dimensions". The two-dimensional $O(N)$ ($N \geq 3$) model and the CP^{N-1} model also have the same pair, a dimensionless coupling and asymptotic freedom, and CP^{N-1} also has instantons with no scale. There too, the construction of the ultraviolet limit is not complete (Rivasseau 2000, Douglas 2004 Cited). Conversely, even at the critical dimension, an abelian ($U(1)$) or non-gauge (ϕ^4_4) theory is equivalent to a free field or trivial. The "because it is four dimensions" of this chapter refers to **being at the critical dimension (a dimensionless coupling) and being non-abelian (the direction of asymptotic freedom, topology, the geometry of gauge orbits) happening at the same time** Estimate .

02

Topological configurations with no scale — instantons and the weight of dislocations

By $\pi_3(SU(N)) = \mathbb{Z}$, four-dimensional gauge fields have an integer topological charge Q, and the action of a self-dual solution (an instanton) is $8\pi^2/g^2$, independent of its size ρ Cited (the abstract of Belavin–Polyakov–Schwartz–Tyupkin 1975; the value of the action from the texts of Lucini–Teper 2001 and Athenodorou–Teper 2021). That the action does not depend on ρ is a manifestation of classical Yang–Mills having no scale in four dimensions.

On the lattice, the smoothed topological charge changes only when a small instanton the size of the lattice spacing passes through the lattice ("disappears into a hypercube") (§ 6.1 of Lucini–Teper 2001

Cited). Its weight is estimated from the one-loop density of small instantons $D(\rho) \propto \rho^{11N/3-5}$, and for the time for the topological charge to move by one unit,

$$\ln \tau_Q = b + \left(\frac{11N}{3} - 5\right) \ln \frac{1}{a\Lambda} \quad (3)$$

a primary source gives this count ((32)(34) of Athenodorou–Teper 2021 Cited). For $SU(2)$ the coefficient is $7/3$, and the original itself writes that it is "rather weak". The comparison with fitted slopes is Table 1.

Table 1 The coefficient of (3) and the fitted slopes (Table 43 of Athenodorou–Teper 2021. Heat bath and overrelaxation, periodic box, normalised to a standard physical volume)

N	**Coefficient of (3) $11N/3-5$**	**Fitted slope**
2	$7/3 \approx 2.33$	Excluded from the table (τ_Q does not become large enough compared with the measurement interval)
3	6.0	6.82(7)
4	9.67	9.27(12)
5	13.3	11.13(20)
6	17.0	13.50(30)
8	24.3	17.58(43)

The label of Table 1 is Cited (numbers from the original). There is one independent route. The figure of the original suggests that the probability that the field smoothed by the Wilson flow is between topological sectors (the probability of exceeding the smoothness threshold $h \geq 0.067$) vanishes in proportion to a^6 on lattices at fixed physical size (§ 4.3 of Lüscher 2010, $SU(3)$, the word "suggests" Cited), and this is the same number as the coefficient 6 of (3) for $SU(3)$. For $SU(2)$, no measurement of the same type was found within the range searched.

Two ways of counting. In the crude count that sets the action of a dislocation to c times the one-loop instanton action, the dynamical exponent is of the type $z = (11N/3)c - 4$ Assumption . It differs by 1 from the counting of (3), in how the volume factor at $\rho \sim a$ and the width $d\rho$ are counted. Fitting c to the numbers for $SU(3)$ (the fitted 6.82, or $z \approx 5$ from another original), **in either counting $SU(2)$ falls near the boundary $z = 2$** Estimate . This book does not judge which of the two counts is right. Nor has it checked that c does not depend on the group.

03

An estimate that puts the two points together — the slow quantity and the correlation length split into different origins

Setting the two points side by side gives the following estimate Estimate . The time of the slow quantity (the topological charge) and the correlation length ξ are both born from the same dimensionless coupling, in the form $e^{(\text{constant})\cdot\beta}$. But the exponents are different numbers. The correlation length is $\xi \propto \Lambda^{-1}$ from (2), and the time of the topological charge is $\tau_Q \propto (a\Lambda)^{-(11N/3-5)}$ from (3); in lattice units τ_Q is of the type $\xi^{11N/3-5}$, and its exponent does not match that of ξ^2, the thing it is compared with.

- For the 0+1-dimensional double well, the slow quantity (the number of kinks) and ξ are born from the same kinks, the exponents match, and only a logarithmic difference remains (24-01 Table 3).
- On the two-dimensional $U(1)$ strip, the static length is linear in β, the length at which conservation breaks is $e^{0.92\beta}$, and the two separate exponentially (24-02 § 03, a reading of a two-dimensional toy).
- For two-dimensional CP^{N-1}, varying N split the strength of freezing and the length at which conservation breaks, while the local ratio did not split (24-02 § 07, a conditional one-point reading).

The reading is that the candidate for the true nature of "even if the mass gap is born, it is not caught in the form of the relaxation-time assumption (the form that ties C_P and ξ^2 with the same power)" is **the slow quantity and the correlation length splitting into different origins**, and that the split comes from two things: the coupling being dimensionless, and topology. On this reading, the form that catches the mass gap is not the whole of the relaxation-time assumption but lies on the side of the part local in time (24-04) and its reverse side, the Poincaré constant of the vacuum measure (24-05).

What this section can say: it is a reading, not a theorem. The numbers for the exponents are fits from the originals and semiclassical counts, and for $SU(2)$ they fall right near the boundary, so whether they "split" for $SU(2)$ is undecided (§ 07). The autocorrelations of the originals are numbers for heat-bath, overrelaxation and HMC dynamics, and rereading them as C_P for the link Langevin dynamics is an assumption (24-01 § 06).

04

The four hurdles

Part 23 split the diffusive-relaxation-time assumption into four steps (23-03 § 02): first step = the path measure of the 0+1-dimensional double well; second step = non-abelian gauge theory on the four-dimensional lattice (with the lattice spacing held fixed); third step = the continuum limit and the OS axioms; fourth step = the mass gap in the limit. Table 2 puts on one sheet, for each step, the hurdle, the known reach, whether the tools of this book reach it, and the kind of cost. There are three kinds of cost. "Weeks" = expected to be reached by accumulating work combining existing tools; "years" = a large extension of existing tools, or a reconstruction of a large body of work, is needed; "new mathematics" = no description of a route in the sources read (this phrase borrows as it is the wording of the question "why is there no mathematics for the third step?").

Table 2 The four hurdles ("does it reach" and "kind of cost" are estimates. The labels for the reach are inside the rows)

Step	Hurdle	Known reach [sources and labels]	Do the tools of this book reach it?	Kind of cost

Step	Hurdle	Known reach [sources and labels]	Do the tools of this book reach it?	Kind of cost
First step: the 0+1-dimensional double well	The double-well relaxation-time power statement uniformly as $\hbar \to 0$. Paying for the lack of curvature at the summit, $\frac{1}{4}$, with rarity	The static side (tunnelling splitting, correlation length) is known semiclassically Cited . The upper side of the relaxation-time assumption goes as far as the Double-Well Relaxation-Time Upper-Bound Proposition (algebra on four conditions), with Proved here, Cited, Computer-assisted and Assumption mixed row by row (23-12)	Partly reached. What remains is the rows for each condition (23-15 Table 1)	Weeks (many rows). But the rows for $\hbar < 0.005$ are "another method", and the one proposition at the summit is "method undecided", with no estimate
Second step: four-dimensional lattice, non-abelian	Exponential decay of correlations (the lattice mass gap) at weak coupling (large β), uniformly in the volume. The upper-bound assumption and the relaxation-time lower-bound assumption	At strong coupling (small β) a theorem (Osterwalder–Seiler 1978 Cited (bibliographic data and secondary sources)). In hierarchical models, confinement at every β (Ito 1985 Cited (abstract)). On the real lattice at weak coupling, unsolved (the lattice correlation-length problem, Chatterjee's Problem 5.1)	Within hierarchical models the static side is known, and the dynamic side (coordinates and conservation, 24-02 § 02) is within reach. Weak coupling on the real lattice is out of reach (it is crossing the confinement scale itself, the source "no expansion linking scales" of § 05)	Weeks (the dynamic side of hierarchical models) / "new mathematics" (weak coupling on the real lattice)

Step	Hurdle	Known reach [sources and labels]	Do the tools of this book reach it?	Kind of cost
Third step: the continuum limit and the OS axioms	Existence and uniqueness of the limit as the lattice spacing $\to 0$, infinite volume, the OS axioms, non-triviality	The construction published on 2026-11-10: a Lean formalisation that builds, for any compact simple group, a theory satisfying the Wightman axioms as a limit of the four-dimensional Wilson lattice (the committee's assessment is that it is proved in Lean Cited). The continuum limit on a periodic cubic box, and an infinite-volume limit along a subsequence of volumes. Convergence and uniqueness of the infinite-volume limit are not shown (6-04). The reach before that: ultraviolet stability of the lattice in finite volume (three and four dimensions, Balaban's 13 papers in Commun. Math. Phys. 1983–89 Cited (bibliographic data and descriptions in the texts of other sources)). Schwinger functions with an infrared cutoff in axial gauge (Magnen–Rivasseau–Sénéor 1993 Cited (introduction in the text)). Even the three-dimensional measure was not yet constructed in a finite box (Chevyrev 2022 Cited)	The tools of this book do not reach it. The existence part is cited from the published construction (Volume 2). The remaining items (convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory) are in 6-04	(The existence part is the published construction. The remaining items are expected to need "new mathematics")
Fourth step: the mass gap in the limit	A gap uniform in the volume $\Delta > 0$	None. The problem statement writes that "new ideas are needed for a mass gap uniform in the volume" (§ 6.5 of Jaffe–Witten Cited). Numerically, the lightest glueball of $SU(3)$ is $3.405(21)$ times the square root of the string tension (Athenodorou–Teper 2021 Cited), an extrapolation from finitely many lattice spacings and finite boxes. The published construction has no uniform mass gap either (what it has is polynomial bounds, 6-04)	Does not reach. The one-sided bridge (23-01 § 06) is expected to lift in form only Estimate	"New mathematics"

Before the published construction, the cost of the third step in Table 2 was estimated as "years" for the source "dimensionless coupling" and the source "no global gauge fixing". This estimate depends on descriptions in the sources Estimate : Rivasseau 2000's "simplify and rewrite Balaban's results; since they run to hundreds of pages, this is not a small task", and that Dimock's reconstruction, starting from three-dimensional ϕ^4, reached the stability of the partition function of three-dimensional QED in a little over ten years (abstract). For the source "no expansion linking scales" there are no sources to support an estimate of the cost. The costs of the rows of the first step are as in the estimates of 23-15 § 03, and not a single estimate has been run even for the rows marked "weeks".

What this section can say: the rows of known reach are as their source labels say. "Does it reach" and the kind of cost are estimates, and for the steps marked as not reached we do not say "impossible".

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05
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Why the third step stayed open so long

This section arranges descriptions in the primary sources read before the construction published on 2026-11-10. On 2026-11-10, a Lean formalisation that builds a theory satisfying the Wightman axioms as a limit of the four-dimensional Wilson lattice, together with the committee's assessment report, was published, and the existence part of the third step reached the level of "the committee's assessment is that it is proved in Lean" (Volume 2, 6-00). What is not shown is convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory, a uniform mass gap, Wilson loops and the area law, and one-particle states (6-04). The summary below and Table 3 are kept as the reach before that construction and as a reading of the sources of difficulty.

First a summary in thirteen lines [Y3].

1. The third step is that "there is a single probability measure in the limit of finer and finer lattices, and it satisfies the OS axioms". The theorems in the primary sources before the published construction went only as far as a uniform estimate that, inside a finite box, however fine the lattice is made, the coarse-grained theory does not run wild (Balaban's ultraviolet stability).
2. Beyond that, three things were needed separately — estimates of correlation functions, uniqueness of the limit, and enlarging the box to infinity. The published construction gives the continuum limit on a periodic box and an infinite-volume limit along a subsequence of volumes, but does not show uniqueness of the infinite-volume limit. For three-dimensional non-abelian gauge theory, the measure inside a finite box has not been constructed within the sources read.
3. The first difficulty is that in four dimensions the coupling is dimensionless. In three dimensions or fewer, the interaction weakens like a power as the scale is made finer, and only finitely many divergences had to be dealt with. In four dimensions the correction from each step does not shrink like a power, and the corrections from infinitely many steps must all be controlled in a form independent of the number of steps.
4. The second is that asymptotic freedom works "only at short distances". At short distances the coupling is small and expansions around the free field can be used. At long distances the coupling becomes large, and the confinement scale Λ appears.
5. This Λ appears at no order of the expansion in the coupling, so extending the expansion from short distances does not reach it. The strong-coupling expansion, which converges on the side where the coupling is large, can be used only on the side of large lattice spacing, and does not reach the continuum limit. There is no tool that crosses the scales between the two expansions.
6. The third is non-abelian gauge invariance. No continuous gauge fixing that works for all configurations at once exists (Gribov, Singer), and Balaban chose the gauge afresh scale by scale and place by place.
7. Where the field is large, expansions cannot be used, and the only way to control it is by its being "unlikely". On a group, large fields are necessarily non-convex, so methods that rely on convexity cannot be used, and methods that pay with rarity are needed (24-06).
8. Magnen–Rivasseau–Sénéor (1993) constructed the Schwinger functions in axial gauge, keeping a cutoff at long distances. They wrote in their introduction that the reasons for not removing the cutoff were that the coupling becomes large and enters the non-perturbative effects of confinement, and that there is no easy solvable model around which to expand.
9. In lattice numerics, the mass of the lightest glueball of $SU(3)$ has been extrapolated to the continuum limit as about 3.41 times the square root of the string tension.

10. But the numbers are extrapolations from finitely many lattice spacings and finite boxes, and do not show that the limit exists, that it is unique, or uniformity in the size of the box.
11. Field theories with non-trivial interaction in four dimensions are thought to be limited to asymptotically free ones. The ϕ^4 theory has been shown to be trivial in four dimensions (its scaling limit is Gaussian).
12. As a result, before the published construction there was not a single "interacting theory already constructed" in four dimensions to compare with, and Yang–Mills was sought as its first example.
13. The walls this book met in the first step (the double well) and the second step (the hierarchical gauge model) — paying for the lack of convexity at the summit with rarity, the convexity hypothesis not holding on the group, global quantities not reflected in the gap — are small versions of the third difficulty of the third step. In the third step, that "rarity" ceases to be rare at long distances Estimate .

Three sources of difficulty

Table 3 The sources of difficulty (the column of facts is as its source labels say; "what stops" is an estimate)

Source	Fact (descriptions in primary sources)	What stops	Correspondence with the walls of this book
Source: dimensionless coupling — the coupling is dimensionless, and every scale works with the same weight	In four dimensions g is dimensionless and the quartic term is marginal (§ 01). The models constructed are only super-renormalisable ones in two and three dimensions and the massive Gross–Neveu model in two dimensions (Rivasseau 2000 Cited). $d = 2, 3$ is considerably easier than $d = 4$ (Chevyrev 2022 Cited)	The corrections of infinitely many steps need to be summed uniformly in the number of steps. Asymptotic freedom gives $g_j^2 \sim 1/(bj)$, and the sum converges only logarithmically	Placing the relaxation-time assumption as a power statement (the factor $\ln \xi \sim \beta$)
Source: no expansion linking scales — there is no expansion connecting the ultraviolet and the infrared	The sentence in the introduction of MRS 1993 (below). "The weak-coupling expansion of the ultraviolet limit is to some extent in contradiction with the low-temperature expansion"; "techniques for gluing together different expansions and different regions of the renormalisation group should be developed" (Rivasseau 2000 Cited). The strong-coupling expansion converges only on the side of large lattice spacing (Chatterjee Cited)	The control of Balaban and MRS works only at steps where g_j is small, and there is no theorem usable at the steps that cross the scale Λ^{-1}	In the first step the summit configurations were rare, and there was a currency called rarity. In the infrared, configurations where the Gaussian approximation does not work become typical, and that currency is absent

Source	Fact (descriptions in primary sources)	What stops	Correspondence with the walls of this book
Source: no global gauge fixing — non-abelian gauge invariance	There is no continuous global gauge fixing (Gribov 1978, Singer 1978 Cited (bibliographic data only; the content from memory)). Balaban fixed the gauge locally and scale by scale on regular configurations, and defined averages of group-valued fields separately (titles of the papers). Convexity at large fields is the main issue in going beyond hierarchical models (Bauerschmidt–Bodineau Cited)	Large fields cannot be controlled by convexity, and can only be paid for with rarity	The convexity ceiling on the group (the lemma of 24-06). The convex and non-convex places are the reverse of flat ϕ^4

The relation among the three sources Estimate : the source "dimensionless coupling" is "how many steps there are" (infinitely many, each with the same weight), the source "no expansion linking scales" is "at which step the expansion breaks off" (the confinement scale), and the source "no global gauge fixing" is "what is lacking within one step" (convexity and a global gauge). Balaban handled the lack of global gauge fixing at the ultraviolet steps in finite volume, and handled the source "dimensionless coupling" in the range where g_j is small. None of the sources read has touched the lack of an expansion linking scales.

Where Balaban and MRS stopped

Balaban. The gauge-theory series is 13 papers in Commun. Math. Phys. (1983–1989, including one on estimates of lattice Green's functions), about 426 pages when the pages are added up Cited (counted from the bibliographic data; the texts were not obtained). Combining the descriptions in the texts of other sources, the reach is as follows. In finite volume on the lattice, the effective action on the unit lattice stays bounded after arbitrarily many repetitions of the block-spin transformation (ultraviolet stability). From this it should follow that limits of gauge-invariant quantities such as smoothed Wilson loops exist along **subsequences** by compactness, but **the limit is not necessarily unique** (the summary in the introduction of MRS 1993 Cited). No source includes estimates of correlation functions, uniqueness of the limit, infinite volume, the OS axioms or the mass gap in Balaban's reach.

MRS 1993. What they showed is the construction of the Schwinger functions of pure $SU(2)$ four-dimensional Yang–Mills with a fixed infrared cutoff and no ultraviolet cutoff, in a regularised axial gauge, in the trivial topological sector, and that the Slavnov identities hold non-perturbatively. The authors themselves write that they do not claim to have given the proof of convergence in every detail, and that it is a detailed sketch Cited (introduction checked in the text; the body of the proof not read). The reason they stopped is the following sentence of the introduction (this book's translation).

> We do not investigate invariance under large gauge transformations, or non-trivial topological effects such as instantons. Nor do we try to remove the infrared cutoff, because that leads to large values of the coupling constant, to the so-called non-perturbative effects, which probably correspond to confinement. These problems are for the moment beyond the reach of our constructive methods, in particular because there is no easy solvable model of these phenomena around which to expand.

The "we" of the quotation refers to the authors of MRS. Since the infrared cutoff is not removed, the introduction goes on, the OS axioms as a whole cannot be examined either.

The object, tools, theory, history

The object is undefined. The problem statement is "prove that a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap"; for the existence part it requires axioms as strong as Wightman's or OS, and in a footnote it says that weak existence by compactness alone is not counted as a solution (Jaffe–Witten Cited). The construction itself is the content of the problem; it is not in the form of asking about properties of an object already constructed. Douglas's report of 2004 also writes that "there is no satisfactory mathematical definition of quantum Yang–Mills theory" Cited . Moreover, the space on which to place the measure is not fixed. The natural space is the quotient of everything by gauge orbits, which for the non-abelian case is a non-linear space, and even the construction of the state space is not trivial. Chevyrev's review writes that already in three dimensions the Gaussian free field is too singular to be restricted to lines, and the space of holonomies that could be used in two dimensions cannot be used Cited . The objects up to the second step (finite-dimensional probability measures on the lattice) are defined, and there is mathematics that asks about their properties. Before the published construction, for the object of the third step, even on which space and with which topology to place it was undecided. One step before "there are no numbers" there was "there is no object" Estimate . The published construction gives the object as a table of correlations (Schwinger functions) and an OS space, and does not give a probability measure for the continuum theory (6-04).

The tools are perturbation around a Gaussian reference. Most of the models constructed are in the weak-coupling region, made with cluster expansions, Mayer expansions and their multiscale versions (Rivasseau 2000 Cited). The analysis of one step of the renormalisation group relies on perturbation in powers around the field being 0 and on Taylor approximation, which is a good approximation only when the field is small (the book by Bauerschmidt–Brydges–Slade Cited). The solution theory of stochastic quantisation uses the fact that, under subcriticality ($d < 4$), the solution is a perturbation of the stochastic heat equation, which has the Gaussian free field as its invariant measure (Chevyrev Cited). All of them take as input "close to a Gaussian measure at each scale". The asymptotic freedom of Yang–Mills gives this input **only at short distances**. At long distances, the grounds for starting from a Gaussian reference run out. For four-dimensional ϕ^4, on the contrary, it approaches a Gaussian at long distances, so the same tools work, and the conclusion was triviality. **In four dimensions, the theories the tools reach become trivial, and the theory that should be non-trivial gives the tools no input** Estimate .

There is no theory that crosses infinitely many scales with neither a small parameter nor convexity. The multiscale theorems within the range read all place a small parameter or convexity at each step: a small coupling (hierarchical and Euclidean ϕ^4_4), uniform convexity of the conditional measure at each step (Bauerschmidt–Bodineau), a convergent perturbative expansion (two-dimensional Gross–Neveu, bibliographic data only), a small g_j and the rarity of large fields (Balaban, MRS) Cited . For group-valued fields, the convexity hypothesis cannot hold in principle at weak-coupling scales (the lemma of 24-06: on a compact manifold of diameter D, if $\mathrm{Hess}\, V \geq -K$ then $\mathrm{osc}\, V \leq KD^2/2$). **Gauge invariance forbids this control** Estimate : a gauge-invariant function is constant along orbits, so it cannot have convexity ($\mathrm{Hess}\, V > 0$) in the direction of the orbits. If one tries to remove the orbit directions by fixing the gauge, there is no continuous global fixing. Balaban's local fixing and the seam between the two gauges of MRS both work only in the region of small fields. The Riemannian geometry of the orbit space has been described (Singer 1981, bibliographic data only),

but a general theory handling measures on it that are neither Gaussian nor perturbations of one is not in the sources read.

History. The introduction of MRS writes that Balaban's work is "very impressive but hard to access; the results are scattered over many publications, and it is very difficult to check the consistency of all the arguments", and Rivasseau 2000 lists rewriting it among the open problems Cited . Dimock's reconstruction writes that "others have been slow to adopt this method", starts from three-dimensional ϕ^4 (text), and has reached an estimate of the partition function of three-dimensional QED uniform in the lattice spacing (abstract) Cited . A reconstruction for the four-dimensional non-abelian case was not in the sources read before the published construction. Of the intermediate goals, the committee's assessment is that construction on a finite box T^4 was given by the published construction as the continuum limit on a periodic box. The three-dimensional measure is open within the sources read. There are also moves to make different mathematics: the formulation in the language of Wilson loops and probability (Chatterjee), and the road that makes a Markov process on gauge orbits by stochastic quantisation and poses the construction of the invariant measure as an open problem (Chandra–Chevyrev–Hairer–Shen, Chevyrev's review) Cited . Both go in the direction not of extending the lattice renormalisation group but of rebuilding where the objects live (the state space) or the means of construction (stochastic dynamics). However, the key condition $d < 4$ of the latter shows that this new tool too becomes exactly critical in four dimensions.

What this section can say: an arrangement of descriptions in primary sources, and the matching to sources of difficulty (an estimate). Balaban's texts were not obtained, so the description of his reach goes through the texts of other sources. The contents of Gribov and Singer are from memory; only the bibliographic data were checked.

06

A paragraph on candidate rules

The list of candidates for what to place as rules if the "new mathematics" in the table of § 04 were to be made is dealt with in 24-08. Here we write only one Estimate . Combining axiomatising scale (a semigroup of coarse-grainings and a flow on it) with the class of measures that pay with rarity (locally convex on the good set, the bad set rare, a drift returning from the bad set, rarity paid only through the entropy inequality), it seems possible on the ultraviolet side to write a form in which the class closes at each step of coarse-graining. On the strong-coupling side, tools of the type that pay with the curvature of the group work in a narrow range. **Between the two, a "coupling window" remains where neither currency is enough.** It is the range where the rate of rarity, proportional to the effective coupling, runs out at the confinement scale, and the currency of curvature is not yet enough. This window is a restatement in the language of rules of the lack of an expansion linking scales (that there is no expansion connecting the ultraviolet and the infrared), and is not a new obstacle. The hierarchical gauge model is a place where the three — the ultraviolet side, the window, and the strong-coupling side — come together within one model, but the results there imply nothing for four-dimensional Yang–Mills.

07

Topology and the box — undecided for this book's group $SU(2)$

The periodic box. For reversible dynamics $\tau_{\mathrm{int}}(Q) \le C_P$ ((1) of 24-01), so if the autocorrelation of the topological charge grows faster than ξ^2, the upper-bound assumption breaks ((3) of 24-01. Lean

`LatticeGauge.FourDTrial.not_hup_of_fast_tau`, assumptions: $\tau_n \le C_n$, $c\,\xi_n^p \le \tau_n$, $p > 2$, $\xi_n \to \infty$. Negative control `z_two_compatible_with_hup`, no assumptions: $\tau \propto \xi^2$ is compatible with the upper-bound assumption). On the periodic box, originals for $SU(3)$ and above report this, so under the reading that extends the conjecture that HMC and Langevin are in the same dynamic universality class (in the original, a phrase attached to observations with open boundaries) to the topological slowness on the periodic box, **the upper-bound assumption reads as false on the periodic box** Assumption (the note in 23-01 § 05, 24-01 § 06).

Open boundaries rescue only finite boxes. With open boundaries, inside a box with fixed physical side, the growth of the autocorrelation goes back to a^{-2}. But if the spatial side of the box is held fixed and only the length in the time direction is extended, the autocorrelation time of Q^2 grows nearly in proportion to the length: $42.1(2.5)$, $113(6)$, $148(10)$ (lattices with time-direction lengths 24, 48, 80) (§ 5.3 of Lüscher–Schaefer 2011 Cited). This is because the charge enters and leaves only by transport through the boundary, and in this book's order, which takes the volume to infinity first, it is expected not to be a rescue Assumption (24-03 § 05 and Table 2).

For $SU(2)$ there is one primary source, on the a^{-2} side. Within the range searched, there is one primary source that measured the dependence on the lattice spacing of the autocorrelation of the topological charge in pure $SU(2)$ gauge theory (Abe–Morikawa–Suzuki 2025: HMC, periodic, a small box of physical side about $2.2/\sqrt{\sigma}$, three lattice spacings, reporting that it fits a^{-2} fairly well Cited). The heat-bath originals are all observations on the side where freezing in $SU(2)$ does not become strong enough to measure, and $SU(2)$ was excluded in Table 1 as well. Both counts of § 02 place $SU(2)$ near the boundary $z = 2$. **Hence, for this book's group $SU(2)$, it is undecided whether the freezing of the topological charge breaks the relaxation-time assumption.** The primary source leans to the boundary or below it Reading .

Correspondence with the two-dimensional test bed of 24-02 Estimate . For two-dimensional CP^{N-1} at $\xi \approx 4$, freezing split by N (strongly frozen at $N = 21$, not frozen at $N = 2$), while the ratio of local quantities did not split (24-02 § 07, a conditional one-point reading). That the strength of freezing splits with N, which plays the role of the size of the group, can be read as a two-dimensional analogue of $SU(2)$ falling on the boundary in four dimensions while $SU(3)$ and above clearly cross it. The two-dimensional count in the primary source is also of the type "$z \sim N$", and it writes that the same argument applies to four-dimensional $SU(N)$ (Del Debbio–Manca–Vicari 2004 Cited). However, $N = 2$ carries an ultraviolet flag (the two definitions of the topological susceptibility differ by 13–14%), and the reason $N = 2$ does not freeze may be that it is on the side where the lattice scale determines the topological charge. The analogy carries this reservation, and **the two-dimensional results imply nothing for four dimensions**.

What this section can say: "the upper-bound assumption reads as false on the periodic box" is a statement on top of the originals for $SU(3)$ and above and a reading that extends a conjecture; we do not claim that the upper-bound assumption is false. For $SU(2)$ it is undecided.

08

Physical mechanisms, and why they are not proofs

There are many physical mechanisms that explain the mass gap and confinement. This book uses none of them as tools of proof. Table 4 lists, one line each, what each mechanism says and why it is

not a proof. The last row is an estimate that brings the leading-order statement of 24-05 of this book close to the language of physics, and its character differs from the other rows.

Table 4 Physical mechanisms, and why they are not proofs (labels: Cited = statement checked in the text or abstract, Assumption = from memory)

Mechanism	What it says	Why it is not a proof (one line)	Origin and label
The area law at strong coupling	On the strong-coupling lattice, Wilson loops decrease with area, and there are no free quarks	The strong-coupling limit has no rotation (Euclidean) invariance, and does not extend to the side of the continuum limit (weak coupling)	Wilson 1974 Cited (abstract). The strong-coupling theorem (Osterwalder–Seiler 1978) is Assumption (from memory)
Dual superconductivity (abelian projection, condensation of monopoles)	Gauge-fixing the non-abelian part first leaves $N-1$ abelian gauge fields and monopoles, and if the monopoles condense, there is confinement	Condensation has not been shown, and the projection depends on the choice of gauge. The examples where confinement is described by condensation are supersymmetric theories with matter fields plus a perturbation, and do not apply to pure gauge theory	't Hooft 1981, Seiberg–Witten 1994 Cited (abstract), Jaffe–Witten § 6.6 Cited
Centre vortices	Vortices with values in the elements of the centre carry the string tension	Lattice numerics on the maximal centre gauge and projection (the word "suggest"), with neither inequalities nor statements about the limit. The algebra of 't Hooft 1978 classifies phases, but does not decide which phase one is in	Del Debbio–Faber–Greensite–Olejník 1997, 't Hooft 1978 Cited (abstract)
The dilute instanton gas	Approximates the vacuum by a semiclassical sum around the topological solutions of action $8\pi^2/g^2$	Plausible only at small ρ; as ρ grows, (anti-)instantons overlap and the effective action changes. The premise of the approximation breaks down at the scale where the gap is born	BPST 1975, 't Hooft 1976, Callan–Dashen–Gross 1978 Cited (abstract), Lucini–Teper 2001 § 6 Cited (text). The infrared divergence of the integral over sizes is Assumption (from memory)
Large N and duality with strings	Planar diagrams dominate, and $1/N$ plays the role of the string coupling constant	A rearrangement of the perturbation series, or a conjecture, with no way back to finite N. The problem statement too, while saying "for sufficiently large N it may be a starting point", writes that there is no effective road to gauge theories without fermions	't Hooft 1974, Witten 1979 Cited (abstract), Jaffe–Witten § 6.6 Cited
Curvature of the orbit space	From the curvature of the space of gauge orbits and the gauge invariance of the wave functional, excitations have finite energy	Feynman himself writes that it is a qualitative, suggestive argument, and does not discuss 3+1 dimensions. Singer's is a geometric description, with no statement of a lower bound on the Hamiltonian. The language is closest to Bakry–Émery (a gap from a lower bound on curvature) Estimate	Singer 1981, Feynman 1981 Cited (abstract), Jaffe–Witten § 6.6 Cited

Mechanism	What it says	Why it is not a proof (one line)	Origin and label
(This book's estimate) The glueball is the two-particle colour singlet	On the side of the vacuum measure, the lowest gauge-invariant test space for $SU(2)$ is not in one particle but in two particles (the second chaos) (24-05 § 05)	A statement at leading order ($g \to 0$, three independent Gaussian colours), saying nothing about values at finite g. In infinite volume the right-hand side at leading order also goes to 0. The correspondence with glueballs is itself an estimate	(7)(8) of 24-05 Proved here (leading order), the correspondence Estimate

What this section can say: "what it says" in each row is a copy from the abstract or text of the original, and we judge none of the mechanisms right or wrong.

09

What cannot be said

(a) The candidate for the true nature. "The slow quantity and the correlation length split into different origins" of § 03 is an estimate. Neither the value of the ratio c for the action of a dislocation nor its independence of the group has been checked. $SU(2)$ falls near the boundary in either of the two counts.

(b) Mechanisms. This book uses none of the mechanisms of Table 4 as tools of proof.

(c) Two dimensions. We do not say what the results for two-dimensional models (the $U(1)$ strip, CP^{N-1}) imply in four dimensions.

(d) We do not say "only in four dimensions". The two points were stated as properties of being at the critical dimension.

(e) Table 2. "Does it reach" and the kind of cost are estimates and guesses; "new mathematics" refers to the absence of a description of a route in the sources read, and does not mean impossible.

(f) Sources. § 05 is an arrangement within the range of primary sources read, and the texts of Balaban, Gribov, Singer and Osterwalder–Seiler were not obtained. A theory written as "there is none" means "there is none in the sources read", and § 05 is an arrangement of sources from before the publication of 2026-11-10.

(g) $SU(2)$. For this book's group $SU(2)$, it is undecided whether the freezing of the topological charge breaks the relaxation-time assumption. Nor do we claim that the upper-bound assumption is false.

10

Table of the five grades and the dependency graph

Table 5 The rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	In four dimensions g^2 is dimensionless and the quartic term is marginal ((1))	Cited	§ 01	—	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
2	Dimensional transmutation (2) (one loop)	Cited	4-07	One loop	—
3	The scaling limit of ϕ^4_4 is Gaussian	Cited	Aizenman–Duminil-Copin (abstract)	Nearest neighbour, lattice cutoff	—
4	Balaban's ultraviolet stability (finite volume, lattice, subsequences)	Cited	Bibliographic data and texts of other sources	Three and four dimensions, finite volume	—
5	The MRS construction (infrared cutoff, axial gauge, $SU(2)$, trivial sector, sketch)	Cited	Introduction in the text	Fixed infrared cutoff	—
6	The three-dimensional measure is also not yet constructed; $d < 4$ is the key for the solution theory	Cited	Chevyrev's review	—	—
7	$8\pi^2/g^2$, $D(\rho) \propto \rho^{11N/3-5}$, (3)	Cited	§ 02	One loop	—
8	The fits of Table 1 ($6.82(7)$ for $SU(3)$)	Cited	Table 1	Heat bath and overrelaxation, periodic	—
9	One primary source for $SU(2)$ (close to a^{-2})	Cited	§ 07	HMC, small box, three points	—
10	With open boundaries the autocorrelation of Q^2 is proportional to the length in the time direction	Cited	§ 07	$SU(3)$, box with fixed physical side	—
11	If $\tau \geq c\xi^p$ ($p > 2$), the upper-bound assumption breaks	Proved here	(3) of 24-01	Sequences with $\xi \to \infty$	`LatticeGauge.FourDTrial.not_hup_of_fast_tau` (assumptions: $\tau_n \leq C_n$, $c\xi_n^p \leq \tau_n$, $p > 2$), negative control `z_two_compatible_with_hup` (no assumptions)
12	A reading that extends the conjecture to topological slowness on the periodic box	Assumption	24-01 § 06	Periodic box	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
13	The upper-bound assumption reads as false on the periodic box ($SU(3)$ and above)	Assumption	§ 07	Periodic box	—
14	In the order that takes the volume to infinity first, open boundaries are no rescue	Assumption	24-03 § 05	Infinite volume	—
15	In the two counts, $SU(2)$ is near the boundary $z = 2$	Estimate	§ 02	Semiclassical	—
16	Whether freezing breaks the relaxation-time assumption for $SU(2)$ is undecided	Estimate	§ 07	—	—
17	The slow quantity and the correlation length split into different origins (the estimate of the two points)	Estimate	§ 03	—	—
18	The two points are properties of the critical dimension (we do not say only in four dimensions)	Estimate	§ 01	—	—
19	The three sources of difficulty and their relation	Estimate	§ 05, Table 3	Within the sources read	—
20	The "coupling window" is a restatement of the lack of an expansion linking scales	Estimate	§ 06	—	—
21	"Does it reach" and the costs in the table of the four steps	Estimate	Table 2	—	—
22	The split in N in two dimensions is an analogue of the boundary for $SU(2)$	Estimate	§ 07	Two dimensions; an ultraviolet flag on $N = 2$	—

What the machine-checking column guarantees is only the algebra of sequences of reals; that τ_n is the autocorrelation time of a reversible dynamics, and that C_n is the Poincaré constant of the Wilson measure, are outside Lean (23-13).

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  in four dimensions g² is dimensionless, marginal
 2  CIT  dimensional transmutation Λ = μ e^(−1/(2b₀g²))          one loop
 3  CIT  the limit of φ⁴₄ is Gaussian
 4  CIT  Balaban's UV stability (subsequences, no uniqueness)    finite volume
 5  CIT  MRS 1993 (IR cutoff, axial gauge, SU(2), sketch)
 6  CIT  3D measure also not yet constructed, d < 4 is the key
 7  CIT  8π²/g², D(ρ) ∝ ρ^(11N/3−5), (3)
 8  CIT  SU(3) fit 6.82(7)                                      heat bath, periodic
 9  CIT  one primary source for SU(2) (close to a⁻²)            HMC, small box
10  CIT  open boundaries: τ(Q²) proportional to time length     box with fixed physical sid
11  PRV  τ ≥ cξ^p (p > 2) ⇒ the upper-bound assumption breaks   Lean not_hup_of_fast_tau
12  ASM  reading that extends the conjecture to the periodic box
13  ASM  the upper-bound assumption reads as false on the periodic box ⟸ 8, 11, 12   SU(3)
14  ASM  infinite volume first: open boundaries no rescue ⟸ 10
15  EST  SU(2) is near the boundary z = 2 ⟸ 7, 8
16  EST  for SU(2), whether freezing breaks the relaxation-time assumption is undecided ⟸
17  EST  slow quantity and correlation length split into different origins ⟸ 1, 2, 7
18  EST  properties of the critical dimension ⟸ 1, 3
19  EST  three sources of difficulty ⟸ 1, 2, 4, 5, 6
20  EST  the coupling window restates the lack of an expansion linking scales ⟸ 19
21  EST  "does it reach" and costs in the four-step table ⟸ 4, 5, 6, 19
22  EST  the split in N in 2D is an analogue of the SU(2) boundary ⟸ 15, 16
```

The way to check is to see by eye that on every "⟸" the numbers on the right are smaller than the number on the left. There are no circles. The conclusion nodes of this chapter (from "the upper-bound assumption reads as false on the periodic box" onwards) are Assumption or Estimate, and there is neither a node for the mass gap nor a node for the continuum limit. The node for the primary source for $SU(2)$ enters only the node "whether freezing breaks the relaxation-time assumption is undecided", not the premises of the node for the periodic box. This is how the undecided reading for $SU(2)$ looks on the graph (the same form as the dependency graph of 24-01).

What this chapter can say

What is said	What is not said
Cited That the coupling is dimensionless in four dimensions; dimensional transmutation; the triviality of ϕ^4_4; the instanton action and the one-loop density; the fits of the originals	A mathematical proof of the relation between β and the lattice spacing. That a constant multiple of Λ is the mass gap
Cited The reach of the third step (the continuum limit on a periodic box of the construction published on 2026-11-10, at the level of the committee's assessment; before that, Balaban's ultraviolet stability and the MRS construction with an infrared cutoff), and the sentence in the introduction giving why MRS stopped	A check of Balaban's texts. Convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory, a uniform mass gap (absent from the published construction too)
Assumption The upper-bound assumption reads as false on the periodic box ($SU(3)$ and above, a reading that extends a conjecture)	That the upper-bound assumption is false. The same reading for $SU(2)$

What is said	What is not said
Estimate The estimate of the two points, the three sources of difficulty, the coupling window, "does it reach" and the costs in the table of the four steps, the two-dimensional analogue	None of these is a theorem. The implication of two-dimensional results for four dimensions

Sources and reproduction

Item	Kind	Source / tools
The problem statement and the status report	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000) (§ 6, § 6.5, § 6.6 and footnote 2 checked in the text). M. R. Douglas, Report on the Status of the Yang–Mills Millenium Prize Problem (2004) (text)
The current state of constructive field theory	Cited	V. Rivasseau, Constructive field theory and applications: perspectives and open problems, arXiv:math-ph/0006017 (II and VI checked in the text. The journal volume and pages from memory). S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950 (§ 3, § 5, § 6 in the text). I. Chevyrev, Stochastic quantisation of Yang–Mills, arXiv:2202.13359 (§ 1, § 3, § 4 in the text)
Balaban's series	Cited (bibliographic data)	T. Balaban, Commun. Math. Phys. 89 (1983) 571; 95 (1984) 17; 96 (1984) 223; 98 (1985) 17; 99 (1985) 75; 99 (1985) 389; 102 (1985) 255; 102 (1985) 277; 109 (1987) 249; 116 (1988) 1; 119 (1988) 243; 122 (1989) 175; 122 (1989) 355 (bibliographic data checked, texts not obtained. Pages counted from the bibliographic data)
MRS 1993	Cited	J. Magnen, V. Rivasseau, R. Sénéor, Construction of YM_4 with an infrared cutoff, Commun. Math. Phys. 155 (1993) 325–383 (introduction checked in the text; the quotation is this book's translation)
Reconstruction	Cited	J. Dimock, The renormalization group according to Balaban I, arXiv:1108.1335 (introduction in the text). The same author's overview of three-dimensional QED, arXiv:2204.07201 (abstract)
Power counting, infrared asymptotic freedom, large fields	Cited	R. Bauerschmidt, D. C. Brydges, G. Slade, arXiv:1907.05474 (Definition 5.2.1, the Infrared Asymptotic-Freedom Proposition, § 7 in the text). R. Bauerschmidt, T. Bodineau, arXiv:1809.02075 (introduction and § 2 in the text)
The triviality of ϕ^4_4	Cited	M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163–235 (abstract. The source table of 7-08)
The one-loop form of (2)	Cited	The source tables of 4-06 and 4-07
Instantons, (3), Table 1	Cited	A. A. Belavin, A. M. Polyakov, A. S. Schwartz, Yu. S. Tyupkin, Phys. Lett. B 59 (1975) 85 (abstract). A. Athenodorou, M. Teper, JHEP 12 (2021) 082 (§ 6.2, (32)(34), Tables 42 and 43 in the text). B. Lucini, M. Teper, JHEP 06 (2001) 050 (§ 6 in the text). M. Lüscher, JHEP 08 (2010) 071 (§ 4.3 in the text)
Autocorrelation of the topological charge, open boundaries	Cited	M. Lüscher, S. Schaefer, JHEP 07 (2011) 036 (§ 3, § 5.3 in the text). M. Abe, O. Morikawa, H. Suzuki, PTEP 2025 063B03 (the text of § 3.2 and of Figure 6 in the text; the numbers in the figure were not read). L. Del Debbio, G. M. Manca, E. Vicari, Phys. Lett. B 594 (2004) 315 (text)
Global obstruction to gauge fixing	Cited (bibliographic data only)	V. N. Gribov, Nucl. Phys. B 139 (1978) 1. I. M. Singer, Commun. Math. Phys. 60 (1978) 7. Contents from memory

Item	Kind	Source / tools
The mechanisms of Table 4	Cited	K. G. Wilson, Phys. Rev. D 10 (1974) 2445. G. 't Hooft, Nucl. Phys. B 190 (1981) 455. N. Seiberg, E. Witten, Nucl. Phys. B 426 (1994) 19. L. Del Debbio, M. Faber, J. Greensite, S.˘Olejník, Phys. Rev. D 55 (1997) 2298. G. 't Hooft, Nucl. Phys. B 138 (1978) 1. G. 't Hooft, Phys. Rev. D 14 (1976) 3432. C. G. Callan, R. F. Dashen, D. J. Gross, Phys. Rev. D 17 (1978) 2717. G. 't Hooft, Nucl. Phys. B 72 (1974) 461. E. Witten, Nucl. Phys. B 160 (1979) 57. I. M. Singer, Phys. Scripta 24 (1981) 817. R. P. Feynman, Nucl. Phys. B 188 (1981) 479 (all checked in the abstract). K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (bibliographic data only)
Glueball numbers	Cited	A. Athenodorou, M. Teper, JHEP 12 (2021) 082 (Tables 24 and 33)
The one-line algebra by which the upper-bound assumption breaks	Lean	LatticeGauge.FourDTrial (`not_hup_of_fast_tau`, `z_two_compatible_with_hup`). The bundle guarantees only the algebra of sequences of reals

Terms used in this chapter — critical dimension, marginal, dimensional transmutation, infrared asymptotic freedom, instanton, dislocation, the four hurdles, the three sources of difficulty, ultraviolet stability, the coupling window, dual superconductivity, centre vortex. Their meanings are in A-02 Glossary and 24-09.

Next chapter to read: 24-08 Questions, and what is not claimed — questions that can be settled at the size of an undergraduate thesis, the list of candidate rules, and what Part 24 does not claim.

Questions, and what is not claimed

> This chapter is a map of open questions. It marks the boundary between the parts that close and the parts that do not.

What this chapter shows — We put the questions left by the chapters of Part 24 into one table (done, partly done, not yet, new; with what moves if it is settled and what does not move even if it is settled), list candidate rules in answer to the question "if new mathematics is to be made, what should be placed as rules?", and write what Part 24 does not claim as sixteen rows, matching each to the sentence of which chapter of the text supports it.

Prerequisite chapters — None. This chapter can be read from anywhere. The three meanings (not shown, not read strongly, not called false) are used as in 23-16 § 01. Notation is in 24-09.

Said first — The level is [Y2]–[Y3]. This chapter adds no new results. The table of questions is questions, and contains no claims. The candidate rules are candidates, and none of them promises a prospect of closing the third step (the continuum limit and the OS axioms). The "what is not claimed" of Part 23 (23-16) applies to this part as it is.

The order of this chapter

1. The table of questions — nineteen questions, their status, what does not move even if settled (Table 1)
2. The list of candidate rules — six candidates and three combinations, relation to the sources of difficulty (Tables 2 and 3)
3. What is not claimed — sixteen items and the sentences of the text that support them (Table 4)
4. Relation to Part 23

01

The table of questions — questions that can be settled at the size of an undergraduate thesis

Table 1 lists the questions left by the chapters of Part 24. "Size" is the kind of work needed to settle it (paper, reading, computation, Lean), not an estimate of time. In each row we always wrote "what does not move even if it is settled". Even if a question is settled, the mass gap problem does not move. "Done" in the status means that the text of this part has answered the question (with conditions), and the label of the answer is in the chapter on the right.

Table 1 Questions of Part 24 (status: done, partly done, not yet, new [a question added while writing this part])

No.	Question	Size	Status	What moves if settled	What does not move even if settled	From which chapter
1	**Comparing local excitations of the free field**: for the free field, the comparison of excitation energies of observables supported in the box (computing the local-excitation moment assumption)	Paper	Done: in the moment form, it holds for the free field with $\gamma = d\pi^2$	The form of the coefficient of the local bridge	The local-excitation moment assumption for interacting models	24-04 § 05
2	**Monotonicity of the window factor**: monotonicity of the window factor $\varphi(x)/x^2$	Elementary, Lean	Done: strictly decreasing for $x > 0$ (Lean `wr_strictAntiOn`, no assumptions)	The explicit form of the window threshold	Nothing (only the form of the constant)	24-04 § 03
3	**Sign of the Polyakov loop**: whether the static ratio $\mathbf{Var}/\mathcal{E}$ catches the slowness, for the sign of the Polyakov loop on a small periodic box (a quantity of a slow sector whose answer is known)	Small computation	Not yet	A positive control for the method of static test functions	The verdict on the topological charge	24-01 § 02

No.	Question	Size	Status	What moves if settled	What does not move even if settled	From which chapter
4	**U(1) gap on two-dimensional layers**: for compact $U(1)$ on a two-dimensional layer, whether the gap keeps decreasing like M^{-2} or saturates as the strip is lengthened (the non-linear value of the length at which conservation breaks)	Computation	Done: on the width-2 strip, outer boundary frozen, five points $\beta = 2..6$, $\ell_c \propto e^{0.916\beta}$ (a reading)	The non-linear value of the length at which conservation breaks	Four-dimensional ℓ_*	24-02 § 03
5	**Data on topological-charge autocorrelation**: a primary source on the autocorrelation of the topological charge in pure $SU(2)$ gauge theory (if none, a small measurement at fixed physical volume)	Reading, computation	Partly done: one primary source (HMC, small box, close to a^{-2}). The small measurement is not yet done	The label of the $SU(2)$ row (at the boundary, or clearly faster than ξ^2)	The conjecture on the relation between HMC and Langevin	24-07 § 07
6	**Many-family version of the Two-Projection Lemma**: a many-family version of the Two-Projection Lemma (the family of 2^d colours for a d-dimensional box), and its condition on maximal correlation	Paper	Not yet	The node of reduction of the four-dimensional box	An estimate of maximal correlation uniform in β	24-03 § 06
7	**Hessian of the one-link weight**: on $SU(2)$, the flow of $\inf \operatorname{Hess} V_t$ for the one-link Wilson weight convolved with the heat kernel, and of the weight of the bad set (one variable, class functions)	Paper + small computation	Not yet (the first step of the class of measures that pay with rarity)	The first numbers of the Polchinski flow on the group; whether the class closes in one step	Lattices that are not hierarchical; closing uniformly over many steps	24-06 § 07
8	**Time-step comparison for the transfer operator**: for Wilson's transfer operator, show the time-step comparison assumption ((D) in the paper "one-sided bridge": the comparison of increments at neighbouring times with gradients, uniform in β) restricted to functions cut off at low energy	Paper	Not yet	Carrying the finite-volume window bridge to Wilson's lattice with discretised time	Infinite volume	24-04 § 08
9	**Lean for τ_{int} and C_P**: for finite-state reversible Markov chains, close in Lean $\tau_{\mathrm{int}} \le C_P$ and the identity of the ground-state transformation (the algebra of the Gap and Vacuum Measure Proposition), defined from probabilistic objects	Lean	Not yet	An example of formalisation with the objects themselves as types (a check of a different form from 23-13)	Path measures, continuum fields	24-05 § 01
10	**Rereading the modes of $\mathbf{CP^{N-1}}$**: reread the autocorrelations of local modes and topological modes in two-dimensional CP^{N-1} through the eyes of the local diffusive-relaxation-time assumption (index in A-06)	Reading, computation	Partly done: the literature has been read. The test-bed measurement goes only as far as a conditional reading at one volume and one correlation length	Material for the test bed	Four dimensions	24-02 § 07

No.	Question	Size	Status	What moves if settled	What does not move even if settled	From which chapter
11	**Loop families of two-dimensional SU(2)**: for two-dimensional $SU(2)$ (plaquettes independent), find λ_W and Γ of families of loops at finite β	Computation	New	The β-dependence of the threshold for the colour singlet	Four-dimensional values	24-04 § 06
12	**Values of large four-dimensional loops**: on a four-dimensional box, the values for families of loops the size of the span (including non-planar ones and products of lassos)	Computation	New	The four-dimensional value of the colour-singlet reference	The ratio for the target measure	24-04 § 06
13	**Reduction to kink positions**: for the 0+1 double well, sandwich the local ratio by a reduction to one dimension with the position of the kink as coordinate	Paper + small computation	New	The constant of the motion that crosses the window	The four-dimensional ratio	24-04 § 07
14	**The single point β=7 of the band**: add one point $\beta = 7$ for the strip, and see whether the slope of the raw rate keeps matching $2 - 1/\beta$ and whether the slope of ℓ_c settles	Computation	New	The reliability of the slopes for the two-dimensional strip	Four dimensions	24-02 § 05
15	**Mass in the band between the threshold and 1**: for fields whose ratio lies in the band $[K_*, 1)$ between the threshold and 1, whether there is another argument that sees the mass while staying in the ratio form (outside Jensen and Dirichlet test functions)	Paper	New	Handling of the undecided band	Values of the ratio for interacting fields	24-04 § 06
16	**Window bridge on a time-step lattice**: on Wilson's lattice with discretised time, write the window version of the bridge (the In-Window Supremum Proposition and the lower bound in finite volume) under the time-step comparison assumption	Paper	New	The lattice version of the window bridge	Infinite volume	24-04 § 03
17	**Frustration-free sector splitting**: whether the Kogut–Susskind Hamiltonian becomes frustration-free when it is split with terms grouped by sector	Paper	New	Whether local-to-global gap theorems apply	The gap at weak coupling	24-05 § 04
18	**Extending the CP^{N-1} test bed**: on the CP^{N-1} test bed, add other random seeds and the remaining pairs of N, and read the exponent of the slowness of the topological mode at a second correlation length	Computation	New	The reliability of the reading that freezing splits by N	Four dimensions	24-02 § 07

No.	Question	Size	Status	What moves if settled	What does not move even if settled	From which chapter
19	**Conservation length of subregion charge**: measure the definition (2) of the length at which conservation breaks (the smallest R at which the ratio reaches half its saturation value) with the charge of a four-dimensional subregion	Computation	New	The first value of the four-dimensional length at which conservation breaks	Whether the diffusive-relaxation-time assumption is true	24-02 § 01

If any question is settled, the label of one row of one chapter moves. Whether the mass gap, the diffusive-relaxation-time assumption or the relaxation-time upper-bound assumption is true does not move when any question is settled.

02

The list of candidate rules — "if new mathematics is to be made, what should be placed as rules?"

The considerations of Part 24 also ran into the question "if new mathematics is to be made, what should be made the rules of the mathematics?" (24-07 § 06). Like Boolean algebra, which threw away the meaning of propositions and kept only the rules of equality, how could one assemble a system that places properties of renormalisation and the continuum limit as rules? Table 2 is the list of candidates. **Every row is a candidate**, and neither consistency nor the ability to restate existing results has been checked Estimate . The comparison with Boolean algebra is a metaphor for how to assemble, not a mathematical claim.

Every candidate can be written in the same form: remove one assumption of an existing framework, and place a replacement rule instead. We condensed what is removed and what is placed into one line per row, and added one question and what does not move even if that question is settled.

Table 2 Candidate rules (one line each, one question) Estimate

Rule	Candidate rule (one line)	Question (one)	What does not move even if settled
Rule of axiomatising scale	A semigroup of coarse-grainings, locality, gauge covariance, contracting and marginal directions, weights for large fields	On the hierarchical $SU(2)$ model, which of the five rules apply and which do not	Non-hierarchical lattices, the continuum limit
Rule of the loop algebra	Bounded, the Mandelstam identities, positivity and reflection positivity, the loop equations (throw away the connection, keep the values of loops)	On the two-dimensional $SU(2)$ lattice, whether the state satisfying the four rules is determined uniquely at finite β	Definition of continuum loops in three dimensions and above
Rule of conserved quantities and sectors	Decomposition of the variance, the sandwich $C_{\mathrm{trans}} \le CP \le \sup C_P^{\mathrm{sec}} + C_{\mathrm{trans}}$, local windows, two lengths (24-03 § 04)	Whether C_{trans} in the sandwich can be written as a quantity of the reduced chain between sectors	Construction (the third step)
Rule of the rarity class	Local convexity on the good set, rarity of the bad set, a returning drift, the currency of entropy (24-06 § 07)	How the parameters of the class are carried over when the one-link Wilson weight is convolved once with the heat kernel (the question "Hessian of the one-link weight")	Closing uniformly over many steps
Rule of an asymptotic-freedom axiom	Changes of scale, a monotone coupling, a free field in the ultraviolet, dimensional transmutation (nothing is placed in the infrared)	Whether it follows from the rule of approaching a free field in the ultraviolet that the local ratio approaches 1 at short distances	The infrared, the mass gap
Curvature rule	Bakry–Émery on the space of gauge orbits	On the orbit space of one plaquette, whether the curvature term of the quotient widens the range of strong coupling	Weak coupling
Rule of axiomatising scale + rule of the rarity class	A rarity class closed under one step of coarse-graining	On the hierarchical $SU(2)$ model, how wide the coupling window is that remains between how the class closes on the ultraviolet side and the strong-coupling tools	The four-dimensional window
Rule of conserved quantities and sectors + rule of the rarity class	A rarity class with sectors (of Eyring–Kramers type, sandwiching by the larger of the constant within sectors and the rate of transitions between sectors)	On the two-dimensional $U(1)$ strip, whether the upper bound on the gap can be sandwiched by the larger of "within sectors" and "the rate of transitions"	The four-dimensional topological charge
Rule of the loop algebra + rule of conserved quantities and sectors	Putting sectors into the algebra of loops (reading the decomposition of the set of states into extreme points as sectors)	Whether the non-uniqueness of solutions of the loop rules corresponds to there being several phases (sectors)	Uniqueness at weak coupling

Table 3 gives the relation to the three sources of difficulty (24-07 Table 3: controlling infinitely many steps uniformly in the number of steps [source: dimensionless coupling]; that there is no expansion connecting the ultraviolet and the infrared [source: no expansion linking scales]; that there is no global gauge fixing and large fields cannot be controlled by convexity [source: no global gauge fixing]).

Table 3 Candidate rules and the three sources of difficulty ("works" means the language of the rule can handle that source, not that it gets past the source)

Candidate	Source: dimensionless coupling (uniform in the number of steps)	Source: no expansion linking scales (ultraviolet and infrared)	Source: no global gauge fixi
Rule of axiomatising scale	Works (it is the language itself)	Does not work (leaves the region when the coupling is $O(1)$)	Half (the weights on the gr formulated)
Rule of the loop algebra	Does not work (goes back to the definition of continuum loops)	Works in form (the same rules at every β). Whether the solution is unique is undecided	Works best (no gauge fixin
Rule of conserved quantities and sectors	Does not work	Does not work	Only gives a name
Rule of the rarity class	Only when combined with the rule of axiomatising scale	Does not work (the rate of rarity runs out)	Works head-on
Rule of an asymptotic-freedom axiom	Only copies it out	Does not work (leaves the infrared empty)	Does not work
Curvature rule	Does not work	Does not work (only on the strong-coupling side)	Gives a meaning

Reading the table Estimate . There is not a single "works" in the column of the source "no expansion linking scales". Only the rule of the loop algebra has rules of the same form at every coupling, but whether the solution of the rules is unique has been replaced by the very content of the lack of an expansion linking scales. The coupling window that appears with the rule of axiomatising scale + the rule of the rarity class (24-06 § 07) is this lack of an expansion linking scales restated in the language of rules.

Where the Boolean-algebra type is possible Estimate . Replacing continuous objects by discrete rules is already half there at the level of the **algebra** of observables: the lattice (replacing a continuous field by group elements on edges), the Mandelstam identities and loop equations for loops, the integers of sectors, the one-variable recursion of hierarchical models. But the rules that decide the third step are **inequalities**, not equalities: positivity, rarity, convexity, estimates uniform in the number of steps. The loop equations (equalities) do not determine a unique solution; what narrows the solutions is positivity (infinitely many inequalities). Put as a metaphor, what the third step needs is closer to the type of measure theory (rules of inequality such as σ-additivity, and rules of limits) than to Boolean algebra.

What is not claimed about the candidate rules. (i) None of the candidates promises a prospect of closing the third step. (ii) The comparison with Boolean algebra is a metaphor, and "the rules of the third step are inequalities" is also an estimate. We have not shown that the rules of Yang–Mills cannot be written as equalities. (iii) The correspondences between candidates and existing results (for example, between the form of the stable manifold of the rule of axiomatising scale and ultraviolet stability in finite volume) are all restatements not checked in the texts of the originals. (iv) "The hierarchical model is a place where the three — the ultraviolet side, the window and the strong-coupling side — come together" is a way of putting it within the range searched, with no implication for four dimensions. (v) A candidate written as not reaching is not evidence of being false.

03

What is not claimed — sixteen items and the sentences of the text that support them

Table 4 writes, one row each, what Part 24 does not claim, and lists the places in the text that support each row. "Support" means that the sentence of that chapter already writes the conditions, assumptions and scope, and this row merely copies it into one line. The Kind column is the three meanings of 23-16 (not shown, not read strongly, not called false).

Table 4 What Part 24 does not claim, and the sentences of the text that support it

Item	What is not claimed	Kind	Supporting sentence of the text (summary)
1	Solution of the problem of the existence of Yang–Mills and the mass gap. The continuum limit, recovery of rotations, the OS axioms and non-triviality are not touched (for the existence part, the construction published on 2026-11-10 is read in Volume 2)	Not shown	24-00 § 05: what is not dealt with. 24-07 § 04: the third and fourth steps are not reached by the tools of this book
2	That the relaxation-time assumption is false. The statement that the upper-bound assumption does not hold on the four-dimensional periodic box is a statement on top of the numbers of the originals for $SU(3)$ and above, and a reading that extends a conjecture attached to observations with open boundaries (that HMC and Langevin are in the same dynamic universality class) to the topological slowness on the periodic box	Not called false	(a)(b) of 24-01 § 06, 24-07 § 07
3	The sector diffusive-relaxation-time assumption and the local relaxation-time assumption are candidate definitions, not shown at any β. Nor do we claim "the right way to fix" the relaxation-time assumption	Not shown	24-03 § 07
4	The one-sided bridge (including the local version) is an inequality on a lattice with continuous time, and on Wilson's lattice with discretised time the time-step comparison assumption remains. The local version further rests on the local-excitation moment assumption	Not read strongly	24-04 § 08 (Table 4), 24-04 § 09

Item	What is not claimed	Kind	Supporting sentence of the text (summary)
5	"The gap = the Poincaré constant of the vacuum measure" is an identity, and we do not claim it has made the problem easier	Not read strongly	24-05 § 01, 24-05 § 07
6	The definition of the length ℓ_* at which conservation breaks, and the restatement of what to compare with, $CP \asymp \max(\xi^2, \tau_{\mathrm{glob}})$, are candidates, not shown in any model	Not read strongly	24-02 § 06, 24-02 § 08
7	The results for hierarchical models and two-dimensional models imply nothing for four-dimensional Yang–Mills	Not read strongly	(a) of 24-06 § 09, 24-02 § 08
8	The non-perturbative mechanisms of physics (confinement, instantons, centre vortices, duality) are not used as tools of proof in this part	Not shown	24-07 § 08 (Table 4)
9	The two points of "because it is four dimensions" are estimates as properties of the critical dimension, and we do not claim they happen only in four dimensions	Not read strongly	24-07 § 01, (d) of 24-07 § 09
10	The "what is not claimed" of Part 23 applies to this part as it is	—	23-16 § 02
11	The local relaxation-time assumption goes only as far as the definition and the ratio form. At no β has the local relaxation-time assumption been shown for the target gauge theory. What has been shown is properties of the definition and values for the free-field reference	Not shown	24-03 § 03, 24-04 § 04
12	The thresholds of the local bridge (0.342κ, 0.502κ, about 0.62κ) depend on the choice of reference and on the family of test observables. For fields whose ratio lies between the threshold and 1, the bridge says nothing	Not read strongly	24-04 § 06 (Table 2)
13	For the 0+1 double well, the local ratio is not 1 even for the shortest window. Taking "the ratio approaches 1 at short distances" in four dimensions is an assumption depending on asymptotic freedom	Not read strongly	24-04 § 07, (7) of 24-03
14	The length at which conservation breaks was measured only on the width-2 strip of two-dimensional $U(1)$ (outer boundary frozen, $\beta = 2..6$). Nothing is said about four dimensions, the non-abelian case, or the exponent as $\beta \to \infty$	Not read strongly	24-02 § 03 (Table 2)
15	For $SU(2)$, it is undecided whether the freezing of the topological charge breaks the relaxation-time assumption. There is one primary source, and its numbers are on the side close to a^{-2}	Not called false	24-07 § 07, (b) of 24-01 § 06
16	The results of the two-dimensional CP^{N-1} test bed imply nothing for four-dimensional Yang–Mills (the ratio is a ratio of conditional autocorrelations at one fixed volume, a different quantity from the static ratio)	Not read strongly	24-02 § 07

The word "conjecture" in item 2 is, in the original, attached to observations with open boundaries, so its application to the periodic box is written as "an extended reading". What is not claimed about the candidate rules (§ 02) is placed at the end of § 02.

04

Relation to Part 23

The fourteen items of "what is not claimed" of Part 23 (23-16) apply to this part as they are. We do not list them again here. In particular, two of them — that not reaching is not evidence of being false (items 8 and 12 of 23-16), and that conditions fall away in summaries (23-16 § 06) — apply equally when reading the estimate sentences of this part (label "Estimate").

What this chapter can say

What is said	What is not said
The list of questions of Part 24, and what does not move even if each is settled (Table 1)	The answer to any of the questions. The "size" of a question is the kind of work, not an estimate of time
The list of candidate rules, and the estimate of their relation to the three sources of difficulty (Tables 2 and 3)	The consistency of any candidate, or its prospect for the third step
The sixteen items that Part 24 does not claim, and the places in the text that support them (Table 4)	Whether the proposition of each item is true

Sources and reproduction

Item	Kind	Source / tools
Each row of Table 1	This book	The chapter and section in the right-hand column

Item	Kind	Source / tools
The candidates and sources of difficulty of Tables 2 and 3	Estimate	24-06 § 07, 24-07 § 05, § 06. The sources of the existing results the candidates draw on are in the source table of each chapter (the four properties of the weight from the book by Bauerschmidt–Brydges–Slade, the two-dimensional loop equations from the summary in § 6 of Chatterjee's review, log-Sobolev at strong coupling from the abstract of Shen–Zhu–Zhu)
Each row of Table 4	This book	The chapter and section in the right-hand column
The three meanings	This book	23-16 § 01

Terms used in this chapter — the status of a question (done, partly done, not yet, new), what does not move even if settled, candidate rules, combinations, the three sources of difficulty, the coupling window, not shown / not read strongly / not called false. Their meanings are in A-02 Glossary and 24-09.

Next page to read: 24-09 Table of notation for the first road, second half. To go back to the map of Part 24, 24-00.

Table of notation for the first road, second half

This chapter is a guide to notation and terms.

What this page shows — A list of the symbols used in Part 24. For each symbol, its meaning, where it is defined, its unit, and whether it is the same quantity as the symbol with the same letter in Part 23 or a different one.

Prerequisite chapters — None. The notation of Part 23 is in 23-01 § 01 and A-01 Table of notation.

Said first — Quantities are counted in lattice units (the lattice spacing is 1). Langevin time has the dimension of length squared (diffusion), and the value of C_P measured in physical units is $a^2 C_P$. Numbers are values under the conditions of the chapter of definition, and their labels are in that chapter.

01

List of symbols

Table 1 Symbols of Part 24 ("Same" = the same quantity as the same letter in Part 23; "New" = a symbol added in Part 24)

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
ℓ	A ruler of length given from outside (independent of the gap). In four dimensions the bookkeeping length; for the 0+1 double well $1/\omega_b = 1$	24-03 § 02	Length / $a\ell$	New (in four dimensions the same quantity as the bookkeeping length of 23-01 § 07)
ξ	Correlation length $1/m_{\mathrm{lat}}$. In 0+1, $\hbar/\Delta$	(1) and (17) of 23-01	Length / $a\xi$	Same

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
ξ_{RG}	The bookkeeping length (the length fixed by counting the steps that coarsen the lattice) $= 1/a$	23-01 § 07	Length / 1	Same
ξ_{XY}	The static length of the two-dimensional $U(1)$ strip, $-1/\ln(I_1(\beta)/I_0(\beta))$ (the inverse of the string tension)	24-01 § 02, 24-02	Length	New
$R = c\ell$	Side of the local window or box	24-03 § 03	Length	New
c	The ratio of the window length to the ruler	(2) of 24-03	Dimensionless	New
$\mathcal{D}$	The algebra of time-averaged functionals (weights with bounded support)	(1) of 24-03	—	New (fixes the collection over which the supremum in (7) of 23-01 is taken)
$\mathcal{D}_c$	The subfamily of $\mathcal{D}$ whose weights have support in the window $[0, c\ell]$	(2) of 24-03	—	New
$\mathcal{F}_1$	The family of time-integrated single-time functions	24-04	—	A name given to the family of test functions of the One-Sided Bridge Proposition (Theorem 3.1 of the paper) of 23-01
C_P	The Poincaré constant $\sup \mathrm{Var}/\mathcal{E}$ (path measure)	(7) of 23-01. In Part 24 the supremum is taken over $\mathcal{D}$	Length squared (Langevin time) / $a^2 C_P$	Same (the path C_P of 23-01 Table 3)

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
$C_P^{\mathrm{loc}}(c)$	The local Poincaré constant (the supremum over $\mathcal{D}_c$)	(2) of 24-03	Length squared	New
$C_P^{\mathrm{loc},0}(R)$	The value of the massless reference in the same box with the same graduation, $= \Gamma(R)R^2$	(4)(5) of 24-03	Length squared	New
$K(c)$	The constant of the local diffusive-relaxation-time assumption in absolute form, $C_P^{\mathrm{loc}}(c) \le K(c)\ell^2$	(3) of 24-03	Dimensionless	The local version of C in (14) of 23-01 (K in Lean)
K_{rel}	The ratio $C_P^{\mathrm{loc}}/C_P^{\mathrm{loc},0}$	(5) of 24-03	Dimensionless	New
Γ	The coefficient of the value for the reference box, $C_P^{\mathrm{loc},0}(R)/R^2$. $\Gamma_\infty = 0.0740$ for the scalar, $\Gamma_Q = 0.0277$ for the gauge quotient	24-03 § 03, Table 1	Dimensionless	New
λ_W	The Dirichlet value of the box within the test space multiplied by R^2 ($4\pi^2$ for the scalar, 72.0 for the gauge quotient). For a window function W, $\|\nabla W\|^2/\|W\|^2$	(7) of 24-01, 24-03 Table 1	Dimensionless	New

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
κ	The constant comparing the Langevin metric with the metric of the kinetic term (the metric comparison assumption ((A3) in the paper)). A different quantity from the correction κ to the raw rate on the strip in 24-02 § 05 (which is neither κ_BE nor this κ)	The one-sided bridge of 23-01	Dimensionless	Same
κ_{BE}	The lower bound on the Bakry–Émery curvature. A different quantity from the comparison constant κ of the one-sided bridge	24-06 § 01	Inverse length squared	New
s	The rate of rarity (the exponent with which the measure of the bad set B falls, as $\mu(B) \le e^{-s}$). The same quantity as ℓ of Part 23	24-06 § 01 (formulas: (7) of § 06, the rule of rarity of bad sets of § 07)	Dimensionless ($-\ln \mu(B)$; in the 0+1-dimensional example, the order of the action difference divided by $\hbar$; for one link of the group, $s \approx \beta$)	The rarity exponent ℓ of 23-09
K_*	The threshold of the bridge $\kappa/(\lambda_W \Gamma)$	24-04	Dimensionless	New

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
ℓ_c	The length at which conservation breaks on the two-dimensional $U(1)$ strip (the bend in the relaxation of the flux dipole)	24-01 § 02, 24-02	Length	New
ℓ_*	The length at which conservation breaks (the general name. The candidate definition is (2) of 24-02: the smallest R at which the ratio reaches half its saturation value)	24-02	Length	New
R_*	The crossing length $\sqrt{d\lambda_W/\hbar}$ (the boundary between the regimes of transport and of the barrier)	(7) of 24-01	Length	New
λ_Q	The relaxation rate of the topological charge of the strip, $1/\tau_{\mathrm{int}}(Q)$	24-02	Inverse Langevin time	New
τ_{int}	Integrated autocorrelation time	(1) of 24-01, 23-02 § 05	Langevin time (in the originals, numbers of sweeps or molecular-dynamics time)	Same
m	The gap as an inverse time, $m = \Delta/\hbar$. In four dimensions m_{lat}	(1) and (17) of 23-01	Inverse length	Same (m_{lat})

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
m_{phys}	The mass in physical units $m\ell$ ($m_{\mathrm{lat}}\xi_{\mathrm{RG}}$ in four dimensions)	(12)(16) of 23-01	Dimensionless (physical units)	Same
Δ	The gap of the Hamiltonian $E_1 - E_0$	The one-sided bridge of 23-01	Energy	**Restricted to one meaning** (§ 02 below)
ω_b	The angular frequency at the bottom of the double well $\sqrt{V''(\pm 1)} = 1$	(7) of 24-03, 24-04	Inverse time	New (a quantity of the model of 23-04)
S_0	The kink action $2/3$	23-04 § 03	Action	Same
$\varphi(x)$	The window factor $1 - (1 - e^{-x})/x$	The bracket in (17) of 23-01, 24-04	Dimensionless	Same (different from the plaquette angle φ_p)
φ_p	The plaquette angle of the two-dimensional strip	24-01 § 02, 24-02	Angle	New
Q, Q_s	The topological charge, the smoothed topological charge. On the two-dimensional strip $Q = \sum_p [\varphi_p]/2\pi$	24-01 § 02, 24-03 § 04	Integer (Q_s is real)	The same as Q in the note of 23-01 § 05
χ_t, h, d, g	The topological susceptibility, the density of breaking of conservation, the transport coefficient, the coefficient of the boundary layer	(4)(5) of 24-01	Lattice units (h, d, g depend on the graduation of $\mathcal{E}$)	New

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
$C_P^{\mathrm{sec}}(n)$, C^{trans}	The Poincaré constant of a sector, the quantity between sectors	(8)(10) of 24-03	Length squared	New

02

Symbols with the same letter but different meanings

Δ. In 23-01 of Part 23, Δ was used in two meanings: $\Delta(\beta) = m_{\mathrm{lat}}/a$ of (1) (the mass in physical units) and $\Delta = E_1 - E_0$ of the one-sided bridge (the gap of the Hamiltonian). In Part 24, Δ is **restricted to the meaning $E_1 - E_0$ of the one-sided bridge**, and the mass in physical units is written m_{phys}.

K. (14) of 23-01 writes the constant of the relaxation-time upper-bound assumption as C, and the Lean theorems write the same constant as K. $K(c)$ in Part 24 is its local version, K_{rel} is the constant of the ratio form, and K_* is the threshold of the bridge. The three are different quantities.

φ. The window factor $\varphi(x)$ carries an argument, and the plaquette angle of the two-dimensional strip is φ_p with a subscript.

Q. The four-dimensional topological charge and the topological charge of the two-dimensional strip are written with the same letter. Both are "global integers that do not show in the gap", examples of the Q of the type of 24-01. The two-dimensional numbers do not imply anything for four dimensions.

Symbols with λ. λ_1 (the Langevin gap $= 1/C_P$, 23-01), λ_W (the Dirichlet value of the box) and λ_Q (the relaxation rate of the topological charge of the strip) are different quantities.

Next chapter to read: 25-00 Entrance — space, not time.

Entrance — space, not time

> This chapter is a guide (the entrance to this part).

This part is a record of this book's study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — Part 23 was a road towards the mass gap from the direction of time (the time over which fluctuations are smoothed out, and the transfer matrix). Part 25 is the second road, towards confinement from the direction of space. We shrink two of the four directions to a small twisted two-dimensional torus T^2 and consider the area law of Wilson loops on the remaining plane $\mathbb{R}^2$. In this chapter we state (1) why we go once by way of a small twisted T^2, (2) how the problem then splits in two, (3) which tool works at which scale (Figure 2, "the ladder of scales"), and (4) the map of the ten chapters of Part 25.

Prerequisite chapters — 5-01 (Wilson's lattice), 5-02 (strong coupling), 17-02 centre vortices, 4-06 (asymptotic freedom), 4-07 (dimensional transmutation), 7-07 (what the four-dimensional construction has reached). The entrance of Part 23, 23-00.

Said first — This chapter is a map, and it contains no newly proved statements. "On a small twisted T^2 the coupling stays weak up to the size of the T^2" is a physical estimate made on the assumption of ultraviolet control (a bound, uniform in a, from the lattice spacing a up to the size of the T^2). What the construction published on 2026-11-10 showed is the continuum limit with periodic boundary conditions on a cubic box; construction with twisted boundary conditions or on elongated boxes is not in it (within the range searched). The "ultraviolet control" of this part remains as a still-open condition: "that the same construction holds under twisted boundary conditions". The road of Part 25 heads first for confinement (the area law), not for the mass gap itself. That the two are different statements is as in 17-02 and 16-08.

The order of this chapter

1. The second road — what we look at, and from which direction
2. Two walls — the ultraviolet and the crossover band
3. Why a small twisted T^2 — two reasons

4. The ladder of scales — which tool works at which scale (Figure 2, Table 1)
5. Map of the ten chapters of Part 25 (Table 2)
6. The conventions of notation and words in this part
7. Guide to Part 23 and Part 26
8. What cannot be said
9. The table of five grades and the dependency graph (Table 3)

01

The second road — what we look at, and from which direction

The road of Part 23 regarded one direction of the lattice as "time", and approached the mass gap from the gap of the transfer matrix and from the time over which fluctuations are smoothed out. Where it stopped is summarised in 23-15.

In Part 25 we change direction. Of the four directions of the four-dimensional lattice, we wind two (directions 1 and 2) into a torus T^2 with small period L, and regard the other two (directions 3 and 4) as a large plane. We ask whether a Wilson loop $W(C)$ drawn in the plane decreases as $\langle W(C)\rangle \le e^{-\sigma A}$ in the area A it encloses (the area law). Where the time road looked at "the gap", what this road looks at is "how vortices are counted in the plane".

We put 't Hooft's twist (25-01) on the wound T^2. It is known in the physics literature that if one shrinks to a periodic T^2 without twist, a family of flat connections (zero modes) remains, and there is no prospect of joining the small T^2 to large four dimensions without a phase transition Cited (the title of § 2.1 of Tanizaki–Ünsal 2022 is "compactification to periodic T^2 and the absence of adiabatic continuity"). For a twisted T^2, the same paper states the conjecture that the structure of the weakly coupled vacuum on a small $T^2 \times \mathbb{R}^2$ is joined without a phase transition to the strongly coupled region of $\mathbb{R}^4$ Cited . This conjecture remains a conjecture, and this book also treats it as one.

02

Two walls — the ultraviolet and the crossover band

If one starts from the lattice, builds the four-dimensional theory and tries to show confinement or a mass gap, one hits two walls of different nature.

1. **The ultraviolet wall.** One needs uniform bounds at all scales that do not break down as the lattice spacing a approaches 0. Thanks to asymptotic freedom (4-06) the

coupling becomes weak at short distances, and one can consider a multiscale analysis that controls each scale in perturbative language. For four-dimensional Yang–Mills, on 2026-11-10 a Lean formalisation that builds a theory satisfying the Wightman axioms as a limit of Wilson's lattice was published together with the committee's assessment report, and the committee's assessment is that it is proved in Lean (Volume 2, 6-00). However, what that construction showed is the continuum limit with periodic boundary conditions on a cubic box. Construction on the small T^2 with twisted boundary conditions used in this part, and on elongated boxes (the product of a small T^2 and a large plane), is not in the published construction (within the range searched). In this book we treat the ultraviolet stage of this part as an assumption named "ultraviolet control", and read its content as the open condition "that the same construction holds under twisted boundary conditions" Assumption . The reach before that (Bałaban's series) is in 7-07.

2. **The crossover band.** One must cross, without relying on perturbation theory, the intermediate scale at which the coupling passes from weak (short distances) to strong (long distances), around $1/\Lambda$ in physical length (Λ is the scale born of dimensional transmutation, 4-07). On each side there are tools (perturbation theory and semiclassics on the weak-coupling side, the strong-coupling expansion on the strong-coupling side, 5-02). That no tool works inside the band is where both the road of Part 23 of this book and this road stop (Part 26).

If one heads straight for four dimensions, the two walls appear at the same time. This is because one has to manage, within a single bound, both making the lattice spacing small (the ultraviolet) and looking at long distances in a large box (crossing the band).

03

Why a small twisted T² — two reasons

The skeleton of this road is that going by way of a small twisted T^2 lets us separate the two walls into different stages. There are two reasons.

Reason 1: the twist removes the zero modes, so one can proceed at weak coupling up to the size of the T^2 (an estimate). On a twisted T^2, the configuration with action 0 is fixed to be unique up to gauge (25-01 Uniqueness of the Twisted Vacuum), and the lattice Laplacian of the adjoint fluctuations around it has a positive gap $4\sin^2(\pi/2N)$ (25-01 Twisted Adjoint Gap Lemma, $N = L/a$) Proved here . In physical units, when N is large this gap is of order $(\pi/L)^2$, and all classical fluctuations have masses of order $1/L$. The running of the coupling then stops at the size L of the T^2, and if $L\Lambda \ll 1$ the effective coupling $g^2(L)$ is expected to stay small Reading (a physical estimate, on the assumption of ultraviolet control). The band does not appear in this stage.

Reason 2: the surface of a vortex winding the T^2 closes locally in the plane. A centre vortex (17-02) is a two-dimensional surface in four dimensions, and in four dimensions the vortex surface linked with a Wilson loop in the plane is an extended object whose cost grows with area. In the work of this book, every road that tried to cross the band while staying in four dimensions stopped at one point: "a topological quantity (whether a vortex links a loop) cannot be compared monotonically by local operations" (map in 26-08) Reading . In $\mathbb{R}^2 \times T^2$, the surface $\{p\} \times T^2$ above a point p of the plane is a closed surface of area L^2, and in the plane it behaves like a point. Adding or removing one vortex sheet becomes an operation inside a bounded region of the plane Proved here (that the surface lies in a bounded region is a fact from the definition). This locality supports all of the containers (25-03), slab confinement (25-04) and the insertion identities (25-05).

In this way the problem splits in two.

- **The intermediate goal (on a small T^2):** on $\mathbb{R}^2\times$ a small twisted T^2, show the gap and the area law uniformly in the lattice spacing a. The only wall that remains is the ultraviolet (construction under twisted boundary conditions) (though this is a conditional research plan, placing assumptions named ultraviolet control, uniform acceptance of insertion and frame decorrelation, and not a theorem) Assumption .
- **The last stage (widening the T^2):** that the strength of the area law stays uniform even as L is made larger, beyond $1/\Lambda$. The only wall that remains is the band (25-09) Assumption .

Splitting does not make it easier. The band remains whole in the last stage. What the splitting gains is that all the stages before the band can be arranged in a form that can be checked independently of the band, and that the form of the inequality the band demands can be narrowed to one (Part 26).

04

The ladder of scales — which tool works at which scale

Figure 2 takes the physical length L (the side of the small T^2, or the scale being looked at) vertically, and lays out, from the lattice spacing a at the bottom to infinity at the top, which tool works at which scale. It is a figure for seeing in one picture the shape of the problem: there are tools at the bottom (weak coupling) and at the top (strong coupling), and only the band in the middle is empty. The number and title of the figure match the entrance of Part 26 (Figure 2 "the ladder of scales" of 26-00). Figure 1 (the flow of the logic) is in 16-09.

Figure 2 The ladder of scales. The vertical axis is physical length (shorter towards the bottom). The labels of each stage are in Table 1; the text in the figure summarises the labelled rows and is not itself a claim

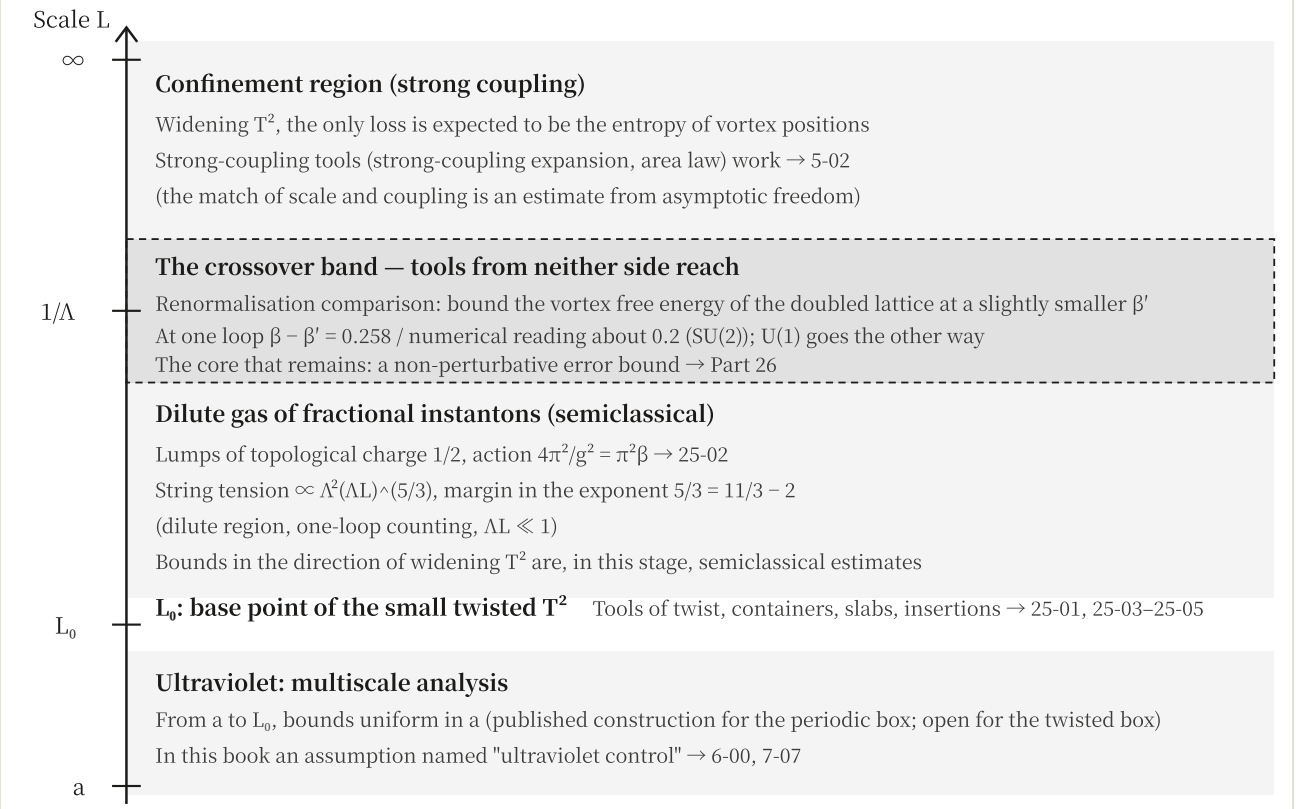

Figure 2 in words is as follows. From the bottom: from the lattice spacing a up to the base-point length L_0 is the ultraviolet stage (construction under twisted boundary conditions; an assumption in this book); L_0 is the base point of the small twisted T^2 (where the tools of Part 25 work directly); from L_0 to $1/\Lambda$ is the stage of the dilute gas of fractional instantons (a semiclassical estimate, with string tension of order $\Lambda^2(\Lambda L)^{5/3}$); around $1/\Lambda$ is the crossover band (where the renormalisation comparison is known only at the one-loop level and numerically); and above that is the confinement region (where strong-coupling tools are expected to work).

Table 1 Content and labels of each stage of Figure 2 (conditions in the same row)

Stage	Content	Label	Conditions, scope	Chapter
$a - L_0$	Multiscale bounds uniform in a (locality of the effective action, rarity of large fields, flow of the coupling)	Assumption	An assumption named ultraviolet control (that the same construction holds under twisted boundary conditions). The published construction covers only the periodic cubic box. In Bałaban's earlier series, stages remain that the original sources call unpublished	6-00, 7-07
L_0	Uniqueness of the vacuum that absorbs the twist, the gap $4\sin^2(\pi/2N)$ of the adjoint Laplacian, positive minimal action in the (1,1) sector	Proved here	SU(2), lattice, N fixed, $n_{12} = 1$	25-01
L_0	With the lattice spacing and the T^2 fixed, an area law at every β	Proved here	N fixed, lattice spacing fixed. In the continuum limit the tension becomes 0. Comparison with earlier work not yet done	25-04

Stage	Content	Label	Conditions, scope	Chapter
L_0	The gap and the area law on a small T^2, uniformly in a	Assumption	A conditional research plan under control of the ultraviolet, uniform acceptance of insertions and frame decorrelation	25-09, 27-01
L_0 – $1/\Lambda$	The lower bound $4\pi^2/g^2$ on the action of a self-dual lump of topological charge $\frac{1}{2}$, and the string tension $\Lambda^2(\Lambda L)^{5/3}$ of the dilute gas	Cited + Reading	The former is the BPS bound (a formula of the original source). The latter is a physical estimate from the dilute region and one-loop counting (equations (2.32) and (2.37) of Tanizaki–Ünsal)	25-02
Around $1/\Lambda$	Renormalisation comparison: at the one-loop level $\beta - \beta' = 0.258$ (per doubling of the lattice)	Proved here	One-loop level, SU(2), $\beta = 4/g^2$. Not a claim outside perturbation theory	26-06
Around $1/\Lambda$	Numerical reading: for SU(2) $\beta - \beta' \approx 0.2$; for U(1) the other way	Reading	Lattices $4^4 \to 8^4$ and others; the number of seeds and the errors are in 26-09	26-06
Around $1/\Lambda$	An inequality that crosses the band without relying on perturbation theory	Assumption	Unsolved. Its form is narrowed to one inequality in Part 26	26-02
$1/\Lambda$ – ∞	Area law on the strong-coupling side	Cited	A lattice statement at small β. The match with the scale of physical length is an estimate from asymptotic freedom	5-02

Of the rows of Table 1 at L_0, the upper two are statements with the lattice spacing fixed, and differ in strength from the third row, "uniform in a". This difference (making things uniform in the lattice spacing) is the ultraviolet wall, and each chapter of Part 25 writes in the same row on which side each statement lies.

05

Map of the ten chapters of Part 25

Part 25 consists of the following ten chapters. The labels are the main labels of the rows in each chapter; the finer division is in each chapter's table of five grades. The "Conditions" column gives the conditions that always attach to the chapter's main statements.

Table 2 The ten chapters of Part 25 (labels and conditions in the same row)

Chapter	Title	Content in one line	Main labels	Conditions

Chapter	Title	Content in one line	Main labels	Conditions
25-00	Entrance — space, not time	This chapter	Reading + Cited	"Stays weakly coupled" is an estimate under the assumption of control of the ultraviolet
25-01	The twisted T^2 and 't Hooft flux	$n_{12} = 1$, sectors, the flux free energy, the twist-summed ensemble. Three elementary facts	Cited (definitions) / Proved here (three facts)	SU(2), lattice, N fixed, $n_{12} = 1$
25-02	Vortices and fractional instantons	Lumps of topological charge $\frac{1}{2}$, action $4\pi^2/g^2$, the dilute gas, string tension $\Lambda^2(\Lambda L)^{5/3}$, and that the lattice action approaches $\pi^2\beta$ from below	Cited / Reading / Computer-assisted / Assumption	Dilute region, one loop, SU(2). The cooled action is an upper bound
25-03	Containers — conditioning on the frame makes things independent	The container identity (Mack–Petkova), nesting in the directions of the plane, limits in four dimensions	Cited / Proved here	The identity is known. Lean covers only the finite algebra
25-04	Slab confinement	With the lattice spacing and the T^2 fixed, an area law at every β	Proved here	N fixed, lattice spacing fixed, tension 0 in the continuum limit. Comparison with earlier work not yet done
25-05	Vortex-insertion identities	Expectation 1 of the insertion ratio, correlation with the sign of the sector, averaged bounds, second moments	Proved here	Finite configuration space, positive weights, a bijection that flips the sector
25-06	Insertion driven by dynamics, and the flow lemma	A Jarzynski-type identity, bounds on simultaneous flips	Proved here / Cited	Bounds on the kernel of one step cannot be used to test the band
25-07	Why fixed maps fail	A map that does not change the shape of fluctuations cannot see the 4 of antiscreening. $11/3 = -1/3 + 4$	Proved here / Reading	One-loop level

Chapter	Title	Content in one line	Main labels	Conditions
25-08	The measured record	Work of driven insertion, vortex free energy, string tension of the slab	Reading	Group, β, lattice size, number of measurements and errors in each row
25-09	Where this road stops	Widening the T^2 = the crossover band	Assumption + Reading	—

We recommend reading in numerical order. A reader in a hurry can follow the skeleton of the road by reading 25-01 § 01– § 02 (the setting), 25-02 § 01– § 03 (fractional instantons and the dilute gas), 25-04 (slab confinement) and 25-09 (where it stops), in that order.

06

The conventions of notation and words in this part

- **Group and coupling.** Throughout Part 25 the group is fixed to $SU(2)$. The Wilson action is $\beta \sum_p (1 - \frac{1}{2} \operatorname{tr} U_p)$ in the convention of 5-01, with $\beta = 4/g^2$.
- **N and M.** The N of Part 25 is not the degree of the group but the number of lattice points along a side of the small T^2 ($L = Na$). We write M for the number of lattice points along a side of the plane. The lattice is $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$.
- **Band and window.** The subject of this part and of Part 26 is "the crossover band" (the band for short). It is a different thing from the "windows" of Parts 23 and 24 (the site window, the window in $\hbar$, the time window), and that the words differ carries no meaning beyond that (note in 26-00).
- **Names of assumptions.** We call assumptions by names, not symbols: control of the ultraviolet, uniform acceptance of insertions, frame decorrelation, the uniform slab tension, the uniform slab gap. The index is collected in A-06.
- **Labels.** The same five as in Part 23: Proved here (a paper proof placed in the text), Cited (known, checked in the original source), Computer-assisted (a computer-assisted bound), Reading (a numerical reading; physical estimates are marked "physical estimate"), Assumption (an assumption, or a row for which only a strand of proof exists; the evidence column says where the strand is). How to read them is in 23-00.

07

Guide to Part 23 and Part 26

To Part 23 (the first road). Part 23 is the road towards the mass gap from the direction of time, and it built the diffusive-relaxation-time assumption in the model of the first step. The entrance is 23-00, where it stopped is 23-15, and the considerations beyond

are in Part 24 (24-00). The two roads aim to show different statements (Part 23 the gap, Part 25 first the area law).

To Part 26 (the band). Both roads stop just before a statement of the type "the same constant at all scales". Part 26 (26-00) treats the correct form of that statement (the correct form of the band, 26-01), its reduction to one inequality (26-02), and the same band seen from the side of time (26-07). The last chapter of Part 25 (25-09) is the bridge to Part 26.

To the table of stages. The table of the stages a proof must contain (the container, the ultraviolet, the gap on a small T^2, confinement on a small T^2, the band, $\mathbb{R}^4$, general groups) is in 16-09. Part 25 fills the two "small T^2" stages of that table and the entrance to the "band" stage.

08

What cannot be said

(a) Weak coupling. "The coupling stays weak up to the size of the T^2" is a physical estimate on the assumption of ultraviolet control, not a theorem. Bounds uniform in the lattice spacing in the setting of this part (the small T^2 with twisted boundary conditions and elongated boxes) appear neither in this book nor in the published construction (the published construction is for the periodic cubic box).

(b) Adiabatic continuity. That the small T^2 and $\mathbb{R}^4$ are joined without a phase transition is a conjecture, and this book does not use it. What it uses is only the form "if they are joined, what must be shown".

(c) Confinement and the gap. The goal of this road is the area law, and there is no known road from the area law to the mass gap. The two are different statements (16-09, 27-01).

(d) Figure 2. Figure 2 is a map of where the tools sit, and the match of scale and coupling (which L corresponds to which β) is an estimate from asymptotic freedom. The numbers in the figure summarise the labelled rows of Table 1.

09

The table of five grades and the dependency graph

Table 3 The rows of this chapter ("—" is a row without machine checking)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	On a periodic T^2 adiabatic continuity does not hold (as stated in the physics literature)	Cited	Tanizaki–Ünsal § 2.1	Physical argument	—
2	The conjecture of continuity from the twisted $T^2 \times \mathbb{R}^2$ to $\mathbb{R}^4$	Cited	Same, abstract	As a conjecture	—
3	Uniqueness of the vacuum that absorbs the twist, and the gap of the adjoint Laplacian	Proved here	Uniqueness of the Twisted Vacuum and the Twisted Adjoint Gap Lemma of 25-01	SU(2), lattice, N fixed, $n_{12} = 1$	—
4	The surface $\{p\} \times T^2$ lies in a bounded region of the plane	Proved here	Definition	—	—
5	Ultraviolet control (that the same construction as the published one holds under twisted boundary conditions)	Assumption	6-00, 7-07	From a to L_0	—
6	Weak coupling up to the size of the T^2 $\Longleftarrow$ 3, 5	Reading	Physical estimate	$L\Lambda \ll 1$	—
7	The one point at which the roads that tried to cross the band in four dimensions stopped (topological quantities cannot be compared locally)	Reading	Map of 26-08	Within the roads tried	—
8	The problem splits into an intermediate goal and a last stage $\Longleftarrow$ 4, 6, 7	Assumption	This chapter § 03	The form of a research plan	—
9	Figure 2 and each stage of Table 1 $\Longleftarrow$ 5, 6, 8	The label of each row	Table 1	Conditions of Table 1	—

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
1  CIT  no continuity on a periodic T²                          physical argumen
2  CIT  continuity from twisted T² × R² to R⁴ (conjecture)   conjecture
3  PRV  unique vacuum absorbing the twist, gap 4sin²(π/2N)    SU(2), N fixed,
4  PRV  surface {p} × T² lies in a bounded region of the plane
5  ASM  ultraviolet control (construction under twisted boundary conditions)
6  RDG  weak coupling up to the size of T² ⟸ 3, 5             physical estima
7  RDG  the one point where roads crossing in 4D stopped     within the roads
8  ASM  splits into intermediate goal and last stage ⟸ 4, 6, 7
9  MIX  Figure 2, Table 1 ⟸ 5, 6, 8
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The only nodes with proofs placed are the uniqueness of the twisted vacuum and the gap (proved in 25-01) and that the surface lies in a bounded region of the plane (definition). The conjecture of continuity from the twisted T^2 to R^4 enters as a premise of no node. The assumption row (ultraviolet control) does not contain the quantities of the conclusion (gap, area law) and states only the form of a multiscale bound. The nodes on the conclusion side of this chapter (weak coupling persists, the problem splits in two) are "Reading" and "Assumption", and there is neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Proved here Uniqueness of the configuration with action 0 on the twisted T^2 and the gap of the adjoint Laplacian (proved in 25-01); that the vortex surface is local in the plane	A gap uniform in the lattice spacing. The form of the area law on a small T^2 uniform in a
Cited What the physics literature says about the difference between a periodic T^2 and a twisted T^2; the conjecture of continuity (as a conjecture)	The conjecture of continuity itself
Reading "Weak coupling up to the size of the T^2" (a physical estimate under the assumption of control of the ultraviolet); the arrangement of Figure 2	The quantitative form of the match of scale and coupling. The width of the band
Assumption The form of the research plan in which the problem splits into two stages	Neither stage is closed. An inequality that crosses the band

Sources and reproduction

Item	Kind	Source / tools
Compactification on periodic T^2 and on twisted T^2, the conjecture of continuity	Cited	Y. Tanizaki, M. Ünsal, Center vortex and confinement in Yang–Mills theory and QCD with anomaly-preserving compactifications, arXiv:2201.06166 (v2, 2022). Abstract, the title of § 2.1 and § 2.2–2.3 checked in the text
Numerics of the gas of fractional instantons on the lattice	Cited	G. Bergner, A. González-Arroyo, I. Soler, Fractional instantons and confinement: first results on a $T_2 \times R^2$ roadmap, arXiv:2502.09463 (2025). Abstract checked
Twisted boundary conditions	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 / G. 't Hooft, Commun. Math. Phys. 81 (1981) 267 (bibliographic data as in 3-05)
What the ultraviolet has reached	This book	6-00 (the published construction), 7-07
One-loop $\beta - \beta' = 0.258$	Proved here	$\beta = 4/g^2$, and from the one-loop coefficient $b_0 = 22/(48\pi^2)$, $8b_0 \ln 2 \approx 0.2575$. The derivation is in 26-06
Numerical $\beta - \beta' \approx 0.2$	Reading	The numerical record of this book's work (26-09)
Titles and labels of the ten chapters	This book	Table 2. Each chapter's table of five grades is authoritative

Words used in this chapter — the second road, the small twisted T^2, the ultraviolet wall, the crossover band, control of the ultraviolet, the intermediate goal, the ladder of scales, the base point L_0, the dilute gas of fractional instantons, the renormalisation comparison. Their meanings are in A-02 Glossary.

Next chapter to read: 25-01 The twisted T² and 't Hooft flux — the setting, sectors, the flux free energy, and three elementary facts.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space twist, sectors, the flux free energy · Kiichi, Shiori and Rin

The twisted T² and 't Hooft flux

> In this chapter, § 01 and § 02 are established knowledge and § 03 onward are this book's own investigation.

What this chapter shows — We define the stage of Part 25. To the $SU(2)$ Wilson action on the lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ we add 't Hooft's twist $n_{12} = 1$ in the plane of the small T^2, and set up the two sectors of the flux n_{34} in the directions of the plane, the ratio of their partition functions (the flux free energy), and the twist-summed ensemble that combines the two sectors into one measure. We then state with proofs three elementary facts that form the foundation of this road: (1) uniqueness of the configuration with action 0 (the vacuum that absorbs the twist), (2) the antiperiodicity of the adjoint field around it and the gap $4\sin^2(\pi/2N)$ of the two-dimensional lattice Laplacian, and (3) that the $(1,1)$ sector has no flat configuration and its minimal action is positive.

Prerequisite chapters — 25-00 (the entrance and notation of this part), 5-01 (link variables, plaquettes, the Wilson action), 1-04 ($SU(2)$ and the Pauli matrices), 3-05 (topological charge), 17-02 centre vortices.

Said first — Every statement of this chapter is about the group $SU(2)$, a finite lattice, N fixed and the twist $n_{12} = 1$. It says nothing about the continuum limit, bounds uniform in the lattice spacing, or confinement. Most of the definitions are known ones of the type of 't Hooft (1979, 1981) and Mack–Petkova (1980) Cited ; the three properties are elementary, and their proofs are placed in the text Proved here . All three are lattice versions of things known in the physics literature, and we claim nothing new about them.

The order of this chapter

1. The setting — lattice, action, the sign field z
2. Sectors and the flux free energy — the Moving-Surface Lemma and the definition
3. The twist-summed ensemble — the Parity Representation of the Bias

4. Plaquettes with central values, and holonomy — configurations with action zero, and the commutator of the rounds
5. Uniqueness of the Twisted Vacuum
6. The twisted adjoint gap — antiperiodicity and $4\sin^2(\pi/2N)$
7. Positivity of the Minimal Action in the Twisted Sector — the (1,1) sector has no flat configuration
8. What the three properties of the twisted T^2 say
9. What cannot be said
10. The table of five grades and the dependency graph (Table 1)

01

The setting — lattice, action, the sign field z

This section summarises existing knowledge ('t Hooft's twisted boundary conditions written with a sign field on the lattice). The introduction is in 3-05 § 03.

Let the lattice be $\Lambda = (\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ ($N, M \geq 2$). Directions 1 and 2 form the small T^2, and directions 3 and 4 the plane. On each link (x, μ) we place $U_{x,\mu} \in SU(2)$, and we write the value of a plaquette $p = (x; \mu, \nu)$ ($\mu < \nu$) as

$$U_p = U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^{-1}_{x+\hat{\nu},\mu}\, U^{-1}_{x,\nu} \quad (1)$$

(as in 5-01). Given a sign $z_p \in \{+1, -1\}$ for each plaquette, we set the action and the partition function to be

$$S^z(U) = \sum_p \Big(1 - \tfrac{1}{2}\, z_p \operatorname{tr} U_p\Big), \qquad Z[z] = \int e^{-\beta S^z(U)} \prod_\ell dU_\ell, \qquad \beta = \frac{4}{g^2} \quad (2)$$

dU_ℓ is the normalised Haar measure. If $z \equiv +1$, (2) is exactly the Wilson action of 5-01, and each term lies in $[0, 2]$ because $|\operatorname{tr} U| \leq 2$. A plaquette with $z_p = -1$ is "a plaquette with one central element -1 inserted", and there the action is 0 when $U_p = -1$.

The signs of the two sectors. We put the twist of the small T^2 in by placing $z_p = -1$ on exactly one plaquette of each $(1, 2)$ plane (the $N \times N$ plane with x_3, x_4 held fixed), namely the $(1, 2)$ plaquette at $(x_1, x_2) = (N-1, N-1)$. The position is the same for every (x_3, x_4). We write this as $z^{(0)}$ (all others are $+1$). The sign $z^{(1)}$ with one unit of flux added in the directions of the plane is obtained from $z^{(0)}$ by taking, above a unit square $q = (q_3, q_4)$ of the plane, the surface

$$\Pi_q = \{\, ((x_1, x_2, q_3, q_4); 3, 4) \;:\; x_1, x_2 \in \mathbb{Z}/N \,\} \quad (3)$$

and placing $z_p = -1$ on its N^2 $(3,4)$ plaquettes. Π_q is a "surface winding the T^2" standing above the point q of the plane, and in the plane it behaves like a point (reason 2 of 25-00 § 03).

02

Sectors and the flux free energy

Where the surface is placed, q, has no effect on the partition function. The next lemma shows this.

Moving-Surface Lemma Proved here For a link ℓ, write $\delta_\ell z$ for the field in which the signs of all plaquettes containing ℓ are flipped. Then $Z[\delta_\ell z] = Z[z]$. In particular, (a) the surface Π_q can be moved to the surface $\Pi_{q'}$ of a neighbouring square, and the partition function of $z^{(1)}$ does not depend on q. (b) The partition function of the sign $z^{(0)}$ $\prod_{q \in Q}(\text{surface } \Pi_q)$, with surfaces placed over a set Q, is determined only by the parity of $|Q|$.

Proof. The change of variables $U_\ell \mapsto -U_\ell$ preserves the Haar measure. Since -1 is a central element, in a plaquette that contains ℓ exactly once, U_p changes to $-U_p$, and other plaquettes do not change (a plaquette of the lattice contains a given link at most once). Hence $S^{\delta_\ell z}(-U_\ell, \ldots) = S^z(U)$, and the integrals are equal. (a): apply the lemma in turn to all the links in direction 3 above the point $(a, b+1)$ of the plane (the N^2 links obtained by running over x_1, x_2). The only $(3,4)$ plaquettes containing these links are those on $\Pi_{(a,b)}$ and $\Pi_{(a,b+1)}$, each containing exactly one. A $(\mu, 3)$ plaquette ($\mu \le 2$) contains two links in direction 3, both above the same point of the plane, so its sign is flipped twice and returns to what it was. Other plaquettes contain no link in direction 3. So the signs on the two surfaces $\Pi_{(a,b)}$ and $\Pi_{(a,b+1)}$ are flipped, and the surface moves one step over. Direction 4 is the same. (b): moving surfaces so that two overlap makes them cancel, $(-1)^2 = +1$, so this follows from (a) and the connectedness of the squares of the plane. ∎

The Moving-Surface Lemma is a basic property of the lattice version of 't Hooft's twist: it says that it is not the field z itself but only the parity of the number of -1 piercing each plane (the flux $n_{\mu\nu} \in \{0,1\}$) that affects the partition function Cited (the general form is 't Hooft 1979; the form with a lattice $\mathbb{Z}_2$ sign field is Mack–Petkova 1980). In this chapter we use only cases (a) and (b).

Definition of the sector and flux free energies We write $Z(0) = Z[z^{(0)}]$, $Z(1) = Z[z^{(1)}]$, and call the flux pairs $(n_{12}, n_{34}) = (1,0)$ and $(1,1)$ sectors. The flux in the mixed planes ($(1,3)$ and so on) is taken to be 0. We then set

$$\Delta F = \ln \frac{Z(0)}{Z(1)}, \qquad \theta(N, M, \beta) = \frac{Z(0) - Z(1)}{Z(0) + Z(1)} = \tanh \frac{\Delta F}{2} \tag{4}$$

ΔF is the difference of free energy for putting in one unit of flux n_{34} in the directions of the plane ('t Hooft's flux free energy), and θ is the "bias" between the two sectors. The right-hand equality of (4) comes from $(1 - e^{-\Delta F})/(1 + e^{-\Delta F}) = \tanh(\Delta F/2)$. Since $Z(0), Z(1) > 0$, $|\theta| < 1$. $\theta \geq 0$ (that is, $Z(1) \leq Z(0)$) needs a separate argument through the transfer matrix, and is treated in 26-04.

The reason this quantity matters for confinement is that the Tomboulis–Yaffe inequality bounds the expectation of a Wilson loop in the plane from above by a ratio of twisted partition functions Cited . If θ decreases in the size M of the plane as $-\ln \theta \geq cM^2$, the area law follows through that inequality. The exact form of the inequality, and that it is not enough for θ merely to move away from 1, are treated in 26-01.

03 The twist-summed ensemble

Instead of keeping the two sectors as separate measures, we combine them into one measure in which the placement of the surfaces is also a random variable. Let T be the set of all unit squares of the plane (M^2 of them), and for each subset $Q \subset T$ form the sign $z^Q = z^{(0)} \prod_{q \in Q} (\Pi_q \text{: sign } -1)$.

Definition (the twist-summed ensemble) Cited We call the probability measure on pairs (U, Q) of a configuration and a set of surfaces

$$\tilde{\mu}(dU, Q) = \frac{1}{\Xi} e^{-\beta S^{z^Q}(U)} \prod_{\ell} dU_\ell, \qquad \Xi = \sum_{Q \subset T} Z[z^Q] \tag{5}$$

the twist-summed ensemble. The form that sums the $\mathbb{Z}_2$ signs of plaquettes as random variables is of the type of the representation of Mack–Petkova (1980), who sum over all closed sign fields (where the product of the six on each cube is $+1$). Here we use a small form that sums only the surfaces involved in the flux n_{34} of the plane.

Parity Representation of the Bias (the bias as one expectation) Proved here $\theta = \mathbb{E}_{\tilde{\mu}}\left[(-1)^{|Q|}\right]$.

Proof. By the Moving-Surface Lemma (b), $Z[z^Q]$ is $Z(0)$ if $|Q|$ is even and $Z(1)$ if it is odd. Of the 2^{M^2} subsets of T, exactly half have an even number of elements. Hence $\Xi = 2^{M^2-1}(Z(0)+Z(1))$ and $\sum_Q (-1)^{|Q|} Z[z^Q] = 2^{M^2-1}(Z(0)-Z(1))$, and the ratio is the θ of (4). ∎

In the Parity Representation of the Bias, the placement Q of the surfaces is a record of "whether a vortex is there" above the points of the plane, and $(-1)^{|Q|}$ is the parity of the number of vortices in the whole plane. That the bias θ is small means that the parities are mixed almost fairly, that is, vortices well up freely in the plane. In $\mathbb{R}^4$, the vortex surface linked with a loop is an extended surface and cannot be written in this form of "a record above points" (25-00 § 03). This representation is the starting point both for rewriting things as a product of biases per column by dividing the plane into blocks (the Container Identity, 25-03) and for the identities of the map that inserts one vortex (25-05).

04

Plaquettes with central values, and holonomy

The proofs of the three properties reduce to properties of configurations in which every plaquette takes a central value (± 1). First we check that action 0 is the same thing.

Zero-Action Configuration Lemma (configurations with action 0) Proved here A necessary and sufficient condition for $S^z(U) = 0$ is that $U_p = z_p \cdot \mathbf{1}$ for every plaquette.

Proof. An element of $SU(2)$ can be written $U = \cos\alpha\,\mathbf{1} + i\sin\alpha\,(n\cdot\sigma)$ ($|n|=1$), and $\operatorname{tr} U = 2\cos\alpha$ (1-04). Each term $1 - \frac{1}{2} z_p \operatorname{tr} U_p = 1 - z_p \cos\alpha_p$ of (2) is at least 0, and it is 0 only when $z_p \cos\alpha_p = 1$, that is, when $\alpha_p = 0$ ($U_p = \mathbf{1}$) if $z_p = 1$, and $\alpha_p = \pi$ ($U_p = -\mathbf{1}$) if $z_p = -1$. ∎

Next, if all plaquettes take central values, the commutators of the holonomies around a plane are determined by the product of the signs of that plane. We write $P_\mu(x)$ for the product of links along the straight path that goes once around in direction μ from the base point x (of length N if $\mu = 1,2$, of length M if $\mu = 3,4$).

Lemma on the Sign of the Holonomy Commutator (the commutator is the product of the signs of the plane) Proved here If $U_p = z_p \cdot \mathbf{1}$ for every plaquette, then for $\mu \neq \nu$ and a base point x

$$P_\mu(x)\,P_\nu(x)\,P_\mu(x)^{-1}\,P_\nu(x)^{-1} = \varepsilon_{\mu\nu}\,\mathbf{1}, \qquad \varepsilon_{\mu\nu} = \prod_{p \subset (\mu,\nu)\text{ plane } \ni x} z_p \tag{6}$$

holds. The product on the right runs over all plaquettes of the (μ,ν) plane (a two-dimensional torus) through x.

Proof. Cut the (μ,ν) plane through x open into a square R with x at a corner. The path that goes once anticlockwise around the boundary of R from x consists, in this order, of one round in direction μ, one round in direction ν, one round backwards in direction μ and one round backwards in direction ν, and its holonomy is the left-hand side of (6). On the other hand, on a disc of the square lattice, the holonomy of the boundary equals a product, in some order, of the values of the plaquettes conjugated by parallel transport along paths to the base point (the lattice non-abelian Stokes theorem: an induction that peels plaquettes off one at a time and shrinks the boundary path). If all plaquette values are central elements $z_p\mathbf{1}$, conjugation does not change the values and the order does not matter, so the product is $\prod_{p\subset R} z_p$. ∎

For $z^{(0)}$, every $(1,2)$ plane has exactly one -1, so $\varepsilon_{12}=-1$, and the other planes have $\varepsilon=+1$. For $z^{(1)}$, in addition, the $(3,4)$ planes have exactly one -1 (the one meeting the surface Π_q), so $\varepsilon_{34}=-1$. This is the lattice version of the relation, in 't Hooft's twisted boundary conditions of the continuum theory, that the consistency of the gluing in two directions is off by the central element $e^{\pi i n_{\mu\nu}}$ Cited ('t Hooft 1979).

Finally, we state how far the holonomies determine a configuration.

Determination-by-Holonomy Lemma (determined by holonomy) Proved here If two configurations U, U' satisfying $U_p = z_p\mathbf{1}$ (for every p) for the same z have the same four holonomies at the base point 0 ($P_\mu = P'_\mu$, $\mu=1,\dots,4$), then U' is a gauge transform of U.

Proof. For each lattice point x choose a fixed path γ_x from 0 to x (the path that goes in the positive sense in direction 1, direction 2, direction 3 and direction 4, in that order), and write $U(\gamma_x)$ for the product along it. For a link $\ell=(x\to y)$, consider the closed path $c_\ell=\gamma_x\,\ell\,\gamma_y^{-1}$. In the two-dimensional skeleton of the toroidal lattice, a closed path based at 0 can be deformed, by sweeping finitely many plaquettes, into a word in the four straight rounds (the lattice version of $\pi_1(T^4)=\mathbb{Z}^4$, a standard topological fact Cited). The plaquettes swept in the deformation take central values, so the holonomy of c_ℓ is the product of "the product of z over the swept plaquettes" and "a word in the P_μ", both of which are the same for U and U'. Hence $U(\gamma_x)U_\ell U(\gamma_y)^{-1} = U'(\gamma_x)U'_\ell U'(\gamma_y)^{-1}$, that is, setting $g(x)=U'(\gamma_x)^{-1}U(\gamma_x)$ gives $U'_\ell = g(x)\,U_\ell\,g(y)^{-1}$. ∎

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05
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Uniqueness of the Twisted Vacuum

The configuration that absorbs the twist. Set $\Omega_1 = i\sigma_1$, $\Omega_2 = i\sigma_2$, and define the configuration U^0 as follows. Of the links in direction 1, only those at $x_1 = N-1$ (those crossing the seam of the T^2) are Ω_1; of the links in direction 2, only those at $x_2 = N-1$ are Ω_2; all other links (including those in directions 3 and 4) are $\mathbf{1}$.

Lemma on the Configuration that Absorbs the Twist Proved here $\Omega_1\Omega_2\Omega_1^{-1}\Omega_2^{-1} = -\mathbf{1}$. Moreover, among the plaquettes of U^0, only the $(1,2)$ plaquette at $(x_1, x_2) = (N-1, N-1)$ is $-\mathbf{1}$, and all others are $\mathbf{1}$. Hence $S^{z^{(0)}}(U^0) = 0$.

Proof. From $\sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1$ we get $\Omega_1\Omega_2 = -\sigma_1\sigma_2 = -i\sigma_3$ and $\Omega_1^{-1}\Omega_2^{-1} = (-i\sigma_1)(-i\sigma_2) = -i\sigma_3$, and the product is $(-i\sigma_3)^2 = -\mathbf{1}$. Looking at the plaquette (1), all $(1,2)$ plaquettes not containing the seam are $\mathbf{1}$. At $x_1 = N-1$, $x_2 \neq N-1$ we get $\Omega_1 \cdot \mathbf{1} \cdot \Omega_1^{-1} \cdot \mathbf{1} = \mathbf{1}$; at $x_2 = N-1$, $x_1 \neq N-1$ we get $\mathbf{1} \cdot \Omega_2 \cdot \mathbf{1} \cdot \Omega_2^{-1} = \mathbf{1}$; at the corner $(N-1, N-1)$ we get $\Omega_1\Omega_2\Omega_1^{-1}\Omega_2^{-1} = -\mathbf{1}$. The links in directions 1 and 2 do not depend on x_3, x_4, so all plaquettes involving directions 3 and 4 are $\mathbf{1}$. By the Zero-Action Configuration Lemma the action is 0. ∎

Rigidity Lemma for Anticommuting Pairs (an anticommuting pair) Proved here If A, $B \in SU(2)$ satisfy $ABA^{-1}B^{-1} = -\mathbf{1}$, then for some $g \in SU(2)$ we have $gAg^{-1} = i\sigma_1$, $gBg^{-1} = i\sigma_2$. Moreover, the only elements of $SU(2)$ that commute with both $i\sigma_1$ and $i\sigma_2$ are $\pm\mathbf{1}$.

Proof. Taking the trace of $ABA^{-1} = -B$ gives $\operatorname{tr} B = -\operatorname{tr} B$, hence $\operatorname{tr} B = 0$ and $B = i\,b \cdot \sigma$ ($|b| = 1$). Likewise $BAB^{-1} = -A$ gives $A = i\,a \cdot \sigma$. Since $(a \cdot \sigma)(b \cdot \sigma) = (a \cdot b)\mathbf{1} + i(a \times b) \cdot \sigma$, $AB = -BA$ is equivalent to $a \cdot b = 0$. A pair (a, b) of orthogonal unit vectors can be moved to (e_1, e_2) by a rotation $R \in SO(3)$ ($(a, b, a \times b)$ is a positively oriented orthonormal basis). $SU(2) \to SO(3)$ is surjective, and taking g with $g(v \cdot \sigma)g^{-1} = (Rv) \cdot \sigma$ gives the first half of the claim. Second half: if $C = c_0\mathbf{1} + i\,c \cdot \sigma$ commutes with σ_1 then $c \times e_1 = 0$, and if it commutes with σ_2 then $c \times e_2 = 0$; hence $c = 0$ and $C = \pm\mathbf{1}$. ∎

Uniqueness of the Twisted Vacuum (uniqueness of the vacuum that absorbs the twist) Proved here In the sector $(1, 0)$, the configuration with action 0 ($S^{z^{(0)}}(U) = 0$) is U^0 alone, up to gauge transformations and central transformations in directions 3 and 4 (the operation of multiplying by -1 all the links in direction 3 of some direction-3 cut $x_3 = c$; likewise for direction 4). On the small T^2 alone (the two-dimensional lattice $(\mathbb{Z}/N)^2$, twist $n_{12} = 1$), it is unique up to gauge transformations only.

Proof. By the Zero-Action Configuration Lemma, $U_p = z_p^{(0)}\mathbf{1}$. By the Lemma on the Sign of the Holonomy Commutator, the holonomies at the base point 0 satisfy $P_1P_2P_1^{-1}P$

$_2^{-1} = -\mathbf{1}$, and the other pairs commute. Re-choosing the gauge at the base point by the Rigidity Lemma for Anticommuting Pairs, $P_1 = i\sigma_1$, $P_2 = i\sigma_2$. P_3, P_4 commute with both, so they are $\pm\mathbf{1}$. The holonomies of U^0 are $(i\sigma_1, i\sigma_2, \mathbf{1}, \mathbf{1})$. If one multiplies the direction-3 links of the cut $x_3 = c$ by -1, every plaquette containing them contains two (the two direction-3 sides of the $(\mu, 3)$ and $(3, 4)$ plaquettes lie at the same x_3), so the action does not change and only P_3 changes sign. Direction 4 is the same. So all four tuples $(i\sigma_1, i\sigma_2, \pm\mathbf{1}, \pm\mathbf{1})$ are obtained from U^0 by central transformations, and by the Determination-by-Holonomy Lemma configurations with action 0 whose holonomies agree are related by gauge transformations. In two dimensions there are no P_3, P_4, so gauge transformations alone suffice. ∎

As the proof of the Rigidity Lemma for Anticommuting Pairs shows, $(-i\sigma_1, i\sigma_2)$ is moved to $(i\sigma_1, i\sigma_2)$ by conjugation with $g = i\sigma_2$. Central transformations in the directions of the small T^2 are absorbed into gauge transformations. Central transformations in the directions of the plane are not absorbed, and there are four configurations with action 0 as gauge classes.

06

The twisted adjoint gap — antiperiodicity and 4sin²(π/2N)

In the quadratic part of the fluctuations around the vacuum U^0, the covariant Laplacian of the adjoint field appears. On the small T^2 (the two-dimensional lattice $(\mathbb{Z}/N)^2$), for a field ϕ with values in $\mathfrak{su}(2)$ we set

$$(-\Delta_U \phi)(x) = \sum_{\mu=1,2} \Big[2\phi(x) - \mathrm{Ad}(U_{x,\mu})\, \phi(x+\hat{\mu}) - \mathrm{Ad}(U_{x-\hat{\mu},\mu})^{-1} \phi(x-\hat{\mu}) \Big], \qquad \mathrm{Ad}(g)X$$

Since Ad is an orthogonal transformation, $-\Delta_U$ is symmetric and non-negative with respect to the inner product of $\mathfrak{su}(2)$.

Twisted Adjoint Gap Lemma Proved here When $U = U^0$, for each component ϕ^1, ϕ^2, ϕ^3 in the basis $i\sigma_1, i\sigma_2, i\sigma_3$ of $\mathfrak{su}(2)$, $-\Delta_{U^0}$ splits into scalar Laplacians under the following signed boundary conditions.

Component	Direction 1	Direction 2	Smallest eigenvalue
ϕ^1 $(i\sigma_1)$	Periodic	Antiperiodic	$4\sin^2(\pi/2N)$
ϕ^2 $(i\sigma_2)$	Antiperiodic	Periodic	$4\sin^2(\pi/2N)$
ϕ^3 $(i\sigma_3)$	Antiperiodic	Antiperiodic	$8\sin^2(\pi/2N)$

Every component is antiperiodic in at least one direction. Hence the smallest eigenvalue of $-\Delta_{U^0}$ is $4\sin^2(\pi/2N) > 0$, and there are no zero modes.

Proof. $\mathrm{Ad}(i\sigma_1)$ preserves $i\sigma_1$ and changes the signs of $i\sigma_2, i\sigma_3$ ($\sigma_1\sigma_2\sigma_1 = -\sigma_2$ and so on), and $\mathrm{Ad}(i\sigma_2)$ preserves $i\sigma_2$ and changes the signs of $i\sigma_1, i\sigma_3$. Only the links of U^0 that cross the seam are Ω_μ, so (7) becomes, component by component, a scalar Laplacian with a sign ± 1 attached only to the bonds crossing the seam. On a ring of length N with a sign s on one bond, the difference operator $2f(x) - f(x+1) - f(x-1)$ has eigenfunctions $f(x) = e^{ikx}$ for k satisfying $e^{ikN} = s$, with eigenvalues $2 - 2\cos k = 4\sin^2(k/2)$. If $s = +1$ then $k \in \frac{2\pi}{N}\mathbb{Z}$ (minimum 0); if $s = -1$ then $k \in \frac{2\pi}{N}(\mathbb{Z} + \frac{1}{2})$ (minimum $4\sin^2(\pi/2N)$, at $k = \pi/N$). In two dimensions it is the sum over the two directions, which gives the values in the table. ∎

For the vacuum $U \equiv \mathbf{1}$ of a periodic T^2 without twist, every component is periodic in both directions, and constant ϕ give three dimensions' worth of zero modes. The role of the twist is to remove these zero modes and give the fluctuations the lower bound $4\sin^2(\pi/2N)$. When N is large, $4\sin^2(\pi/2N) \approx \pi^2/N^2$, and restoring the unit of the lattice spacing a gives $(\pi/L)^2$ ($L = Na$).

Reading the same gap as a mass, the lattice mass m_N is given by $4\sinh^2(m_N/2) = 4\sin^2(\pi/2N)$, that is,

$$m_N = 2\ \mathrm{arsinh}\sin\frac{\pi}{2N} \tag{8}$$

(the relation between the decay rate of the lattice propagator and the gap, $2\cosh m - 2 = 4\sinh^2(m/2)$) Proved here . This m_N appears in 25-02 § 05 as the rate of convergence of the action of a fractional instanton confined to a column.

07

Positivity of the Minimal Action in the Twisted Sector — the (1,1) sector has no flat configuration

Positivity of the Minimal Action in the Twisted Sector Proved here In the sector $(1,1)$ there is no configuration with action 0. Hence the minimal action

$$S_{\min}(N,M) = \min_U S^{z^{(1)}}(U) \tag{9}$$

is positive. Moreover, $S_{\min}(N,M) \le 2N^2$.

Proof. If there were a configuration with action 0, then by the Zero-Action Configuration Lemma and the Lemma on the Sign of the Holonomy Commutator the holonomies at the base point would satisfy $P_1P_2P_1^{-1}P_2^{-1} = -\mathbf{1}$, $P_3P_4P_3^{-1}P_4^{-1} = -\mathbf{1}$ ($\varepsilon_{34} = -1$), with the mixed pairs commuting. By the Rigidity Lemma for

Anticommuting Pairs, P_3 commutes with both P_1, P_2 and so is $\pm\mathbf{1}$, and the same holds for P_4. Then $P_3 P_4 P_3^{-1} P_4^{-1} = \mathbf{1}$, a contradiction. The configuration space $SU(2)^{\#\text{links}}$ is compact and the action is continuous, so the minimum is attained, and since it is not 0 it is positive. Upper bound: evaluating U^0 in the sector $(1,1)$, on the N^2 plaquettes of the surface Π_q we have $U_p = \mathbf{1}$, $z_p = -1$, giving $1 - \frac{1}{2}(-1)(2) = 2$ per plaquette, and 0 elsewhere. ∎

The configuration of the upper bound is a "thin vortex surface", which pays action $2N^2$ on a surface one plaquette thick. The Positivity of the Minimal Action in the Twisted Sector says only that the minimal action is not 0; it does not say how small it is (whether there is a smooth lump far cheaper than the thin surface). That is the story of fractional instantons in the next chapter; lattice numerics read $S_{\min}$ as of order π^2, much smaller than $2N^2$ (25-02 § 05).

In the continuum theory, the topological charge of a bundle with twist $n_{12} = n_{34} = 1$ is restricted to half-integers, and the action is bounded below by $8\pi^2|Q|/g^2 \geq 4\pi^2/g^2$ Cited ('t Hooft 1981, equations (2.24) and (2.25) of Tanizaki–Ünsal). The Positivity of the Minimal Action in the Twisted Sector is the weakest form (positivity) of the lattice version of this lower bound. That $S_{\min}$ is close to π^2 on the lattice has not been shown as a lattice version of the continuum lower bound.

08

What the three properties of the twisted T² say

- **The Uniqueness of the Twisted Vacuum** says that the classical vacuum of the small T^2 is isolated. On a periodic T^2, the whole set of commuting tuples of holonomies (the family of flat connections) is vacuum, and fluctuations move freely over it, so semiclassical counting cannot begin. The twist shrinks it to one point (and its central images).
- **The Twisted Adjoint Gap Lemma** says that fluctuations around the isolated vacuum have the lower bound $4\sin^2(\pi/2N)$. This is the classical side of the grounds for the estimate of 25-00 § 03, "weak coupling up to the size of the T^2". The quantum side (ultraviolet control: that the same construction as the one published on 2026-11-10 stands under twisted boundary conditions) remains an open assumption.
- **The Positivity of the Minimal Action in the Twisted Sector** says that putting in a flux n_{34} in the directions of the plane always costs action, that is, vortices cannot be placed "for free". What moves the bias θ away from 1 is the competition between the cost of vortices and the entropy of their positions, and the Positivity of the Minimal Action in the Twisted Sector is the weakest form of the lower bound on that cost.

All three are statements on a finite lattice with N fixed. None of the statements of this chapter includes increasing N (widening the T^2 as a physical length). That is the problem of the crossover band (25-09).

09

What cannot be said

(a) The continuum limit. None of the three properties says anything about $a \to 0$. The gap of the Twisted Adjoint Gap Lemma is a lower bound on classical fluctuations, not a gap of the quantum effective action.

(b) Confinement. The definition of θ and the Parity Representation of the Bias are where the tools of confinement sit, and they do not say how small θ is. Bounds with the lattice spacing fixed are in 25-04, and the semiclassical rate in 25-02.

(c) Other groups. The proofs rely on computations with the Pauli matrices of $SU(2)$ (the Rigidity Lemma for Anticommuting Pairs). The case of $SU(N)$ is not treated in this chapter.

(d) The value of the minimal action. That $S_{\min}(N, M)$ is close to π^2, and that it approaches π^2 as N grows, are numerical readings and are not shown in this chapter.

10

The table of five grades and the dependency graph

Table 1 The rows of this chapter ("—" is a row without machine checking. The conditions in every row are SU(2), the finite lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, $N, M \geq 2$)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	Only the flux affects the partition function (general form); the twist relation	Cited	't Hooft 1979, Mack–Petkova 1980	—	—
2	Moving-Surface Lemma: moving the surface, determined by parity alone	Proved here	Proof in § 02	The case of the surface Π_q	—
3	Definition of the sector and flux free energies: ΔF, $\theta = \tanh(\Delta F/2)$ (4)	Definition	§ 02	$n_{12} = 1$, mixed flux 0	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
4	Tomboulis–Yaffe: a Wilson loop bounded by a ratio of twisted partition functions	Cited	Formula in 26-01	—	—
5	Parity Representation of the Bias: $\theta = \mathbb{E}_{\tilde{\mu}}[(-1)^{\lvert Q\rvert}] \Longleftarrow 2$	Proved here	Proof in § 03	The ensemble summed over surfaces (5)	—
6	Zero-Action Configuration Lemma: action $0 \Longleftrightarrow U_p = z_p \mathbf{1}$	Proved here	§ 04	—	—
7	Lemma on the Sign of the Holonomy Commutator: the commutator is the product of the signs of the plane (6)	Proved here	§ 04 (lattice non-abelian Stokes)	All plaquettes central	—
8	A closed path can be deformed into a word in straight rounds	Cited	Standard topology	Toroidal lattice	—
9	Determination-by-Holonomy Lemma: determined by holonomy $\Longleftarrow$ 7, 8	Proved here	§ 04	Same z, central values	—
10	Lemma on the Configuration that Absorbs the Twist: U^0 has action 0 in the sector (1,0)	Proved here	§ 05	—	—
11	Rigidity Lemma for Anticommuting Pairs: normal form of an anticommuting pair, and commutant $\{\pm\mathbf{1}\}$	Proved here	§ 05	$SU(2)$	—
12	Uniqueness of the Twisted Vacuum: uniqueness of the vacuum that absorbs the twist $\Longleftarrow$ 6, 7, 9, 10, 11	Proved here	§ 05	Up to gauge transformations and central transformations in the directions of the plane	—
13	Twisted Adjoint Gap Lemma: antiperiodicity of the adjoint field and the gap $4\sin^2(\pi/2N) \Longleftarrow$ 10	Proved here	§ 06	Two-dimensional lattice $(\mathbb{Z}/N)^2$, $U = U^0$	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
14	(8): $m_N = 2\,\mathrm{arsinh}\,\sin(\pi/2N) \Longleftarrow$ 13	Proved here	§ 06	Algebraic restatement	—
15	Positivity of the Minimal Action in the Twisted Sector: no flat configuration in (1,1), and $0 < S_{\min} \leq 2N^2 \Longleftarrow$ 6, 7, 11	Proved here	§ 07	Sector (1,1)	—
16	The continuum lower bound $8\pi^2\lvert Q\rvert/g^2 \geq 4\pi^2/g^2$	Cited	't Hooft 1981, Tanizaki–Ünsal (2.24)(2.25)	Continuum, twisted bundle	—

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  only the flux matters; the twist relation                  't Hooft,
 2  PRV  Moving-Surface Lemma: moving the surface                   surface Π_
 3  DEF  ΔF, θ = tanh(ΔF/2)                                         n₁₂ = 1
 4  CIT  Tomboulis–Yaffe                                            26-01
 5  PRV  Parity Representation of the Bias: θ = E[(−1)^|Q|] ⟸ 2
 6  PRV  Zero-Action Configuration Lemma: action 0 ⟺ U_p = z_p
 7  PRV  Sign of the Holonomy Commutator: commutator = product of plane signs
 8  CIT  a closed path deforms into a word of rounds
 9  PRV  Determination-by-Holonomy Lemma ⟸ 7, 8
10  PRV  Configuration that Absorbs the Twist: U⁰ has action 0
11  PRV  Rigidity Lemma for Anticommuting Pairs: commutant ±1        SU(2)
12  PRV  Uniqueness of the Twisted Vacuum ⟸ 6, 7, 9, 10, 11
13  PRV  Twisted Adjoint Gap Lemma: gap 4sin²(π/2N) ⟸ 10             (ℤ/N)²
14  PRV  m_N = 2 arsinh sin(π/2N) ⟸ 13
15  PRV  Positivity of the Minimal Action: 0 < S_min ≤ 2N² ⟸ 6, 7, 11
16  CIT  continuum lower bound 4π²/g²                               continuum
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. This chapter has no assumption rows. There are four cited rows (only the flux matters, Tomboulis–Yaffe, the deformation of closed paths, the continuum lower bound), and the only citation used by the proofs of the three properties is the deformation of closed paths (through the Determination-by-Holonomy Lemma, for the Uniqueness of the Twisted Vacuum only). The Twisted Adjoint Gap Lemma and the Positivity of the Minimal Action in the Twisted Sector close without citations.

The definition of twisted boundary conditions as an object of this book is in 19-02.

What this chapter can say

What is said	What is not said
Proved here The Moving-Surface Lemma, the representation of the bias as an expectation (the Parity Representation of the Bias), uniqueness of the vacuum that absorbs the twist (up to gauge and the centre in the directions of the plane), the gap $4\sin^2(\pi/2N)$ of the adjoint Laplacian, $0 < S_{\min} \leq 2N^2$ in the (1,1) sector. All for SU(2), a finite lattice, N fixed, $n_{12} = 1$	A gap of quantum fluctuations. Bounds uniform in the lattice spacing. The value of $S_{\min}$ and its dependence on N
Cited The general form of "only the flux matters", the Tomboulis–Yaffe inequality, the lower bound on the action in the continuum theory	$\theta \geq 0$ (treated in 26-04). Other groups

Sources and reproduction

Item	Kind	Source / tools
Twisted boundary conditions, the flux free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141. G. 't Hooft, Commun. Math. Phys. 81 (1981) 267 (bibliographic data as in 3-05)
The representation summing the $\mathbb{Z}_2$ signs of plaquettes	Cited	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) (pages not checked). The form of the formula is in 26-01
The fractional part of the topological charge, and the lower bound on the action	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166, equations (2.24) and (2.25) (checked in the text)
The lemmas and the three properties of § 02– § 07, (8)	Proved here	Proofs in the text. No numerical computation is used

Words used in this chapter — 't Hooft's twist, sector, flux $n_{\mu\nu}$, the surface Π_q, the flux free energy ΔF, the bias θ, the twist-summed ensemble, the configuration that absorbs the twist U^0, central transformation, the adjoint Laplacian, antiperiodic, the minimal action $S_{\min}$. Their meanings are in A-02 Glossary.

Next chapter to read: 25-02 Vortices and fractional instantons — what lies inside the minimal action of the $(1,1)$ sector, and the counting of the dilute gas.

Vortices and fractional instantons

This chapter organises established knowledge. § 02 is a map of open questions; § 05 and § 06 are this book's own investigation.

What this chapter shows — The Positivity of the Minimal Action in the Twisted Sector in the previous chapter said that putting in one unit of flux n_{34} in the directions of the plane always costs action. This chapter looks at what that minimal cost consists of. (1) In the continuum theory, the cheapest object carrying the flux on the twisted $T^2 \times \mathbb{R}^2$ is a self-dual lump of topological charge $\frac{1}{2}$ (a fractional instanton), and the lower bound on its action is $4\pi^2/g^2 = \pi^2\beta$. (2) Counting the dilute gas in which such lumps are scattered like points in the plane gives a string tension of order $\Lambda^2(\Lambda L)^{5/3}$ (a physical estimate). The $5/3 = 11/3 - 2$ in the exponent is the difference between the coefficient of asymptotic freedom and the two dimensions of the plane. (3) On the lattice, the minimal action of the sector with flux approaches π^2 from values smaller than π^2 (a numerical reading, and certified upper bounds). (4) We place, as an assumption row, a strand that derives the existence of the lump in the continuum theory by assuming a certain property.

Prerequisite chapters — 25-01 (sectors, θ, the three properties of the twisted T^2), 3-05 (instantons and topological charge, the BPS bound), 17-02 the model of a vortex gas, 4-06 the one-loop coefficient, 4-07 (Λ).

Said first — The counting of the dilute gas is a physical estimate in the dilute region ($\Lambda L \ll 1$), at the one-loop level, for the group $SU(2)$; it is not a theorem. The values of the action obtained by cooling on the lattice are the actions of configurations that were found, so they are **upper bounds** on the minimal action, not lower bounds. The existence of a self-dual solution in the continuum theory is placed in this chapter only as a strand with an assumption.

The order of this chapter

1. The self-dual lump of topological charge 1/2 — the lower bound on the action
2. Does the lump exist — numerics, and a strand with an assumption
3. The dilute gas — counting the string tension (equations (5)–(8))

4. How to read 5/3 = 11/3 − 2
5. The action on the lattice — approaching $\pi^2\beta$ from below (Tables 1 and 2)
6. A statement about the rate on the lattice — from the rate $2N^2$ of the thin surface to the rate of the fractional instanton
7. What cannot be said
8. The table of five grades and the dependency graph (Table 3)

01

The self-dual lump of topological charge 1/2 — the lower bound on the action

The topological charge Q of the continuum theory and the lower bound on the action of an instanton are as in 3-05: the Euclidean action satisfies

$$S_E \;\geq\; \frac{8\pi^2}{g^2}\,|Q| \qquad (1)$$

and equality is attained by solutions of the (anti-)self-duality equation $F = \pm *F$ Cited. On a periodic four-dimensional torus or on $\mathbb{R}^4$, Q is an integer. On a bundle with 't Hooft's twist, however, the fractional part of Q is fixed by the twist: for $SU(N)$ with $n_{12} = n_{34} = 1$, $Q \equiv -1/N \pmod 1$ (when this chapter writes $SU(N)$, the N is the degree of the group, different from the lattice N of this part) Cited (equation (2.24) of Tanizaki–Ünsal; the original source is 't Hooft 1981). For $SU(2)$, $Q \in \frac{1}{2} + \mathbb{Z}$, and from (1)

$$S_E \;\geq\; \frac{4\pi^2}{g^2} = \pi^2\beta \qquad (\beta = 4/g^2) \qquad (2)$$

(ibid., equation (2.25): the lower bound is $1/N$ of the ordinary instanton action $8\pi^2/g^2$) Cited.

A solution attaining this lower bound is a fractional instanton. Tanizaki–Ünsal state that on $\mathbb{R}^2 \times T^2$ (with the twist on T^2) the action density of this solution is localised in the directions of $\mathbb{R}^2$ and takes a vortex-like shape, and that when it lies inside a Wilson loop in the plane it multiplies the loop by the central element -1 ($e^{2\pi i/N}$ for $SU(N)$), that is, it behaves as a centre vortex (17-02) Cited (around equation (2.26)). In the words of the previous chapter, this lump is the cheapest inhabitant of the sector $(1,1)$, carrying the surface Π_q not as a thin surface but as a smooth lump.

A word on size. The action of the lump is fixed by (2) and does not depend on size, but on the twisted T^2 the lump is fixed to an extent of order the size L of the T^2 (a footnote

of the same paper cites earlier numerics in which, for $SU(N)$, the radius of the lump is proportional to the square root of the degree of the group) Cited . Since the size is fixed, no infrared divergence arises in the counting of the dilute gas. This contrasts with the integral over instanton sizes on a periodic $\mathbb{R}^4$, which diverges in the infrared (3-05).

02

Does the lump exist — numerics, and a strand with an assumption

The state of the literature Cited . Tanizaki–Ünsal write that almost no analytic forms of self-dual solutions with fractional topological charge are known (except for abelian constant solutions on tori of particular aspect ratios), and cite reports that numerical searches by lattice cooling have found configurations attaining the lower bound (2) within the discretisation errors of the lattice, and that the limit in which the directions of $\mathbb{R}^2$ are stretched to infinity is also well defined (ibid. § 2.3.1). Analytic existence is not shown in the literature within the range this book checked.

A strand with an assumption Assumption (strand: the "Route to Existence by the Nahm Transform" in the record of this book's study; evidence in the Nahm-transform row of Table 3). We assume the following property.

> **Assumption (equivariance of the Nahm transform).** The Nahm transform between doubly periodic instantons (charge-1 solutions on $T^2 \times \mathbb{R}^2$) and Hitchin-type data on the dual torus is equivariant under translations of physical space and central gauge transformations: translations of physical space correspond to multiplication by flat line bundles on the dual side, and the central twist corresponds to translation by a half period of the dual torus.

The strand has three steps. (i) Map a charge-1 $SU(2)$ doubly periodic instanton on $T^2_{2L\times L} \times \mathbb{R}^2$ by the Nahm transform to data on the dual side (the construction of doubly periodic instantons and the bijectivity of the transform are said to be in the literature; not yet checked). (ii) The composite τ of translation by a half period in the long direction and a central gauge transformation corresponds, by the assumption, to the composite of translation by a half period and a constant shift on the dual side, and if the singular points are placed at a quarter period, data invariant under τ can be chosen. (iii) The corresponding solution is invariant under τ up to gauge, and when brought down to the quotient $T^2_{L\times L} \times \mathbb{R}^2$, since τ contains the central twist, it becomes an anti-self-dual solution of charge $\frac{1}{2}$ on the bundle with $n_{12} = 1$, whose action equals the lower bound (2).

If this strand closes, the lower bound (2) is attained in the continuum theory. What is not closed is the assumption itself (the exact form of equivariance), the check of the literature in (i), and the confirmation that the topology of the bundle (n_{12} and the flux at infinity in the plane) comes out correctly on passing to the quotient. As another strand there is a variational strand combining the Bogomolny bound, exclusion of bubbles by Uhlenbeck compactness, and exclusion of splitting by concentration compactness, but that the minimal value equals the lower bound must itself be shown separately, so it too is Assumption . This book uses neither strand as a premise of the chapters that follow.

03

The dilute gas — counting the string tension

This section summarises existing knowledge (the dilute-gas count). The introduction is in 3-05 § 07.

We assume that the lumps are scattered in the plane far apart from one another (dilute), and count ignoring the interactions between lumps. This is the dilute-gas approximation, and the counting from here on transcribes § 2.3.2 of Tanizaki–Ünsal to $SU(2)$ Cited (a copy of the formulae). The correctness of the approximation itself is a physical estimate Reading .

Note: the original source places the small T^2 in directions 3 and 4 and the plane in directions 1 and 2, so the indices are swapped relative to this book (T^2 in 1 and 2, the plane in 3 and 4). Below we write with this book's indices.

The weight of one lump. The weight of one lump (self-dual, $Q = +\frac{1}{2}$) and of one antilump ($Q = -\frac{1}{2}$) in the plane, with the θ angle set to 0, is

$$K\, e^{-S_I/2}, \qquad S_I = \frac{8\pi^2}{g^2}, \qquad \frac{S_I}{2} = \frac{4\pi^2}{g^2} = \pi^2\beta \tag{3}$$

($N = 2$ in equation (2.27) of the original source). The prefactor K has mass dimension 2 and $K \sim 1/L^2$; its exact value is fixed by an analysis of the fluctuation determinant, which the original source does not give Cited .

Rewriting with asymptotic freedom. Writing the coupling at scale L in terms of Λ with the one-loop running coupling (4-06), for $SU(2)$ we have $8\pi^2/g^2(L) = \frac{22}{3}\ln\frac{1}{\Lambda L}$ (at the one-loop level; for general $SU(N)$, $\frac{11N}{3}$), so

$$e^{-S_I/2} = e^{-4\pi^2/g^2(L)} = (\Lambda L)^{11/3} \quad \text{(one loop)}, \qquad K\, e^{-S_I/2} \sim \frac{1}{L^2}(\Lambda L)^{11/3} = \Lambda^2(\Lambda L)$$

(the last step of equation (2.32) of the original source) Cited . The exponent $11/3$ does not depend on the degree of the group (for $SU(N)$ the action of the lump is S_I/N, and it cancels against the N of the coefficient $11N/3$).

The Wilson loop. If one lump lies inside a loop C of area A, the value of the loop is multiplied by -1 (a centre vortex). If lumps and antilumps are each scattered in the plane as independent Poisson points of density $Ke^{-\pi^2\beta}$, the number n inside follows a Poisson distribution with mean ρA ($\rho = 2Ke^{-\pi^2\beta}$), and

$$\langle W(C)\rangle = \mathbb{E}\big[(-1)^n\big] = e^{-2\rho A} = \exp\!\big(-4Ke^{-\pi^2\beta}\,A\big) \quad (5)$$

This is a one-line computation, $\mathbb{E}[(-1)^n] = e^{-\lambda}\sum_k(-\lambda)^k/k! = e^{-2\lambda}$ for the Poisson distribution Proved here (a computation within the model). The original source obtains the same answer by putting in the constraint on the total charge (the difference between the numbers of lumps and antilumps is a multiple of N) through a Fourier sum: at $\theta = 0$ the string tension of the fundamental representation is

$$T_R \sim \Lambda^2(\Lambda L)^{5/3}\sin^2\frac{\pi|R|}{N} \quad (6)$$

(equation (2.37) of the original source; N is the degree of the group, and $|R|$ is the number of boxes of the Young diagram of the representation modulo N), and for the fundamental representation of $SU(2)$, $\sin^2(\pi/2) = 1$ Cited . Combining (5) and (4),

$$\sigma = 4K\,e^{-\pi^2\beta} \;\sim\; \Lambda^2(\Lambda L)^{5/3} \qquad \text{(dilute region, one loop, } SU(2)) \quad (7)$$

Reading (a physical estimate; $\sim$ is proportionality with the constant of the prefactor omitted).

Relation to the bias θ. Applying the same counting to the parity of the number of lumps in the whole plane (area $A_{\text{plane}} = (Ma)^2$), the $\theta = \mathbb{E}[(-1)^{|Q|}]$ of the Parity Representation of the Bias of 25-01 becomes

$$\theta \approx e^{-2\rho A_{\text{plane}}}, \qquad -\ln\theta \approx 4Ke^{-\pi^2\beta}\,(Ma)^2 \quad (8)$$

and one reads both that $-\ln\theta$ is proportional to the area of the plane (the type of an area law) and that its rate is fixed by the weight $e^{-\pi^2\beta}$ of the fractional instanton Reading (the dilute-gas estimate). The statement, on the lattice, of the part of this form saying "the rate is $e^{-\beta S}$ with S close to π^2" is § 06.

Numerics on the lattice also support the same picture. Bergner–González-Arroyo–Soler (2025) report, for the $SU(2)$ lattice on a four-dimensional torus with two directions far smaller than the others and twisted boundary conditions, that at small sizes of the T^2 the topological charge and the string tension arise from a gas of vortex-like fractional

instantons, and that as the size increases the density increases and the spacing of the structures approaches a constant Cited (checked in the abstract; the numbers in the text not yet checked).

04

How to read 5/3 = 11/3 − 2

The exponent of (4) is the difference of two contributions Reading (a physical estimate).

- $11/3$**: asymptotic freedom.** The smaller the T^2, the weaker the coupling, and lumps become rare at the rate $(\Lambda L)^{11/3}$. The coefficient $11/3$ is exactly the one-loop coefficient ($11/3 = -1/3 + 4$ of 4-06).
- -2**: the two dimensions of the plane.** The size of a lump is L, so the number of places where one can be put per unit area of the plane is $1/L^2$ ($K \sim 1/L^2$). This corresponds to the entropy of the lump positions.

Since the difference $11/3 - 2 = 5/3$ is positive, if one holds Λ fixed and widens the T^2 (increases L), the string tension in physical units increases. It is the direction in which confinement strengthens as one widens, and reaching order Λ^2 at $L \sim 1/\Lambda$ is consistent with dimensional counting. Counting per square of the size of one T^2 (area L^2) gives $\sigma L^2 \sim (\Lambda L)^{11/3}$, which can be restated as: the probability that a lump is in that square increases with L.

The abelian group $U(1)$ has no localised lumps carrying fractional flux. The flux can only spread uniformly over the T^2, and the behaviour in the direction of widening the T^2 is reversed. This difference is treated as "the only place where non-abelianness enters" in 26-03, and there too this counting is an estimate at the one-loop level. Within the scope of this chapter, (4)–(8) are all statements in the dilute region $\Lambda L \ll 1$, and they do not extend into the band $L \sim 1/\Lambda$.

05

The action on the lattice — approaching π²β from below

Matching units. The action $S^z(U) = \sum_p (1 - \frac{1}{2} z_p \operatorname{tr} U_p)$ of 25-01 is the quantity before multiplying by β. In the classical continuum limit of 5-01, for $SU(2)$ and $\beta = 4/g^2$, βS^z approaches the continuum action S_E, so a lump of action $4\pi^2/g^2$ corresponds to $S^z \approx \pi^2$ (that is, $\beta S^z \approx \pi^2 \beta$). All the numbers below are values of S^z divided by π^2.

Cooling numerics Reading . Table 1 gives the result of searching for the minimal action $S_{\min}(N, M)$ of the sector $(1, 1)$ by repeatedly lowering S^z (cooling) through the

operation of replacing each link by the direction of the sum of the remaining sides of the plaquettes containing it (the staple). The action of the configuration at which cooling stopped is an **upper bound** on the minimal action, since that configuration exists (within floating-point computation). The possibility that the true minimum is lower still is set aside only as a reading, on the grounds that different initial configurations arrived at the same value.

Table 1 Cooling values of the minimal action in the sector $(1,1)$ ($S_{\min}/\pi^2$. SU(2), Wilson action, $n_{12}=n_{34}=1$, 3 runs from random initial configurations)

Lattice $T^2_N \times T^2_M$	$N \times M$	$S_{\min}$	$S_{\min}/\pi^2$
Symmetric	2×2	7.841	0.794
Symmetric	3×3	9.661	0.979
Symmetric	4×4	9.799	0.993
Symmetric	6×6	–	0.998
Elongated	2×4	–	0.683
Elongated	2×6	–	0.677
Elongated	3×6	–	0.921
Elongated	4×8	–	0.960

How to read it Reading : (i) every value is smaller than π^2 and approaches π^2 as N grows (the lattice values approach the continuum value from below; a physical reading consistent with the discretisation correction of the Wilson action being negative). (ii) On elongated lattices ($M > N$), at $N=2$ the value hardly changes when the plane is widened, $M = 4 \to 6$. The lump is localised in the directions of the plane and its action does not depend on the size of the plane; that is, the lattice supports the premise of the dilute gas (that a lump is a single localised object). (iii) That the values on symmetric lattices are larger than on elongated ones can be read as the lump being squeezed to order N in the directions of the plane as well on a symmetric lattice. In a separate cooling that measured the distribution of the action on a slab of size 16 in the directions of the plane ($4\times4\times16\times16$), $S^z/\pi^2 \approx 0.96$, about 91% of the action lay within radius 2 of the centre of the lump in the directions of the plane, and about 99.6% within radius 4 ($=N$) Reading .

A lump confined to a column, and certified upper bounds Computer-assisted . The paper on lattice fractional instantons (one of the papers that came out of this book, a draft; A-07) divides the plane into square blocks of side ℓ, puts flux into the column $T^2_N \times [0,\ell]^2$ above one block, and defines the minimal action when the outer frame of the column is fixed to the vacuum that absorbs the twist U^0 (25-01 § 05) as

$$S_*(N,\ell) = \min\{\ \text{action of the column with frame fixed to } U^0 \ :\ \text{one flux } n_{34} \text{ in the column}\ \}$$

The values obtained by normalising the saved configurations exactly, checking that the outer frame agrees with the vacuum, and re-evaluating the action in 40-digit arithmetic are upper bounds on each S_*:

$$S_*(2,12) \le 6.67348665542 \le 0.6762\,\pi^2, \quad S_*(3,12) \le 9.07284242009 \le 0.9193\,\pi^2, \quad S_*(4,8)$$

($N=4$ is a value at which cooling had not converged). Conditions: the group $SU(2)$, the Wilson action, an evaluation of the action of concrete configurations, and only in the direction of upper bounds. The label "Computer-assisted" attaches only to these upper bounds (actions of concrete configurations evaluated with guaranteed digits), not to the minimal values themselves. All are smaller than π^2, so "the minimal cost on the lattice is smaller than the continuum value π^2" can be said directly from the upper bounds for $N=2,3,4$.

Dependence on the width of the column Reading . A table from the same paper (cooling values):

Table 2 Dependence of $S_*(N,\ell)/\pi^2$ on ℓ (cooling. SU(2), outer frame fixed to the vacuum)

ℓ	3	4	5	6	7	8	10	12
$N=2$	0.7082	0.6924	0.6779	0.6771	0.6763	0.6762	0.67617	0.67617
$N=3$	1.1524	1.0185	0.9493	0.9347	0.9230	0.9213	0.9195	0.9193

Elementary properties of S_* Proved here : (a) the minimum is attained (compactness). (b) $S_*(N,\ell) \le 2N^2$ (place one thin surface in the column). (c) It does not increase in ℓ (embed the minimal configuration of a smaller column into a larger column, filling the rest with vacuum; the plaquettes on the border are vacuum against vacuum, with action 0). (d) Relation to the minimal action on a periodic lattice: $S_*(N,\ell) \ge S_{\min}(N, k\ell)$ ($k \ge 1$; extend the configuration of the column to the periodic lattice, filling the outside with vacuum). Hence $\lim_\ell S_*(N,\ell)$ exists.

The values on periodic lattices (cooling) are $S_{\min}(2,M) = 6.740117,\ 6.677345,\ 6.673724,\ 6.673486,\ 6.673485$ ($M=4,6,8,12,16$) and $S_{\min}(3,16) = 9.072494$, and $S_*(N,\ell)$ approaches them from above as ℓ grows. The difference $S_*(N,\ell) - S_{\min}(N,\infty)$ is read as decreasing like $e^{-m_N \ell}/\ell$ in terms of the twist mass $m_N = 2\,\mathrm{arsinh}\,\sin(\pi/2N)$ of (8) of 25-01 ($m_2 \approx 1.317$, $m_3 \approx 0.962$) Reading (the draft of the paper on lattice fractional instantons sets the observed slopes 1.41, 1.02 beside the predicted values 1.45, 1.06; how the predicted values are obtained has not been checked in this book). The reading is that the tail of the lump is cut off by the mass of the adjoint fluctuations on the twisted T^2.

In summary, the minimal cost on the lattice is roughly $0.68\pi^2,\ 0.92\pi^2,\ 0.96\pi^2$ for $N=2,3,4$ (values on periodic lattices; $N=4$ is cooling on the elongated lattice 4×8), and

approaches π^2 from below as N grows. "$S_{\min}(N,\infty) \to \pi^2$ as $N \to \infty$" is numerical only, with no proof Reading .

06

A statement about the rate on the lattice — from the rate 2N² of the thin surface to the rate of the fractional instanton

With N and the lattice spacing fixed, a bound of area-law type $-\ln\theta \geq c\,M^2$ follows elementarily at every β (25-04). The rate c there is fixed by the cost of the thin surface, $c \approx e^{-2\beta N^2}$ (the configuration of the upper bound $2N^2$ in the Positivity of the Minimal Action in the Twisted Sector of 25-01). $2N^2$ grows with N and has nothing to do with fractional instantons.

The main theorem of the paper on lattice fractional instantons (the Confinement Lower Bound by Fractional Instantons, Theorem A of that paper) is a statement that replaces this rate by $S_*(N,\ell)$. We transcribe the form of the draft with its conditions Assumption (this book has not yet placed the proof in the text; the evidence column is § 3– § 6 of the draft of the paper on lattice fractional instantons. The proof uses the Chessboard Estimate and a Morse–Bott lemma by citation, and only the existence of the rigidity constants is shown, with their values not made explicit).

Confinement Lower Bound by Fractional Instantons (the statement of the draft). For all $N \geq 2$, $\ell \geq 2$, $\beta \geq 1$, $M \in 2\ell\mathbb{N}$, there exist $c(N,\ell) > 0$, $a > 0$ and $\varepsilon(\beta) = O\big(\sqrt{\ln\beta/\beta}\big)$ such that

$$-\ln\theta(N,M,\beta) \;\geq\; c(N,\ell)\,\beta^{-a}\,e^{-\beta\,(S_*(N,\ell)+\varepsilon(\beta))}\,M^2. \quad (11)$$

Combined with the upper bounds of (10), the rate drops from $2N^2 = 8, 18, 32$ to about $\leq 0.68\pi^2$, $0.92\pi^2$, $0.98\pi^2$ ($N = 2, 3, 4$). However, the terms of the prefactor β^{-a} and $\varepsilon(\beta)$ are crude, so (11) beats the thin-surface bound only from some large, non-explicit β onwards. (11) is not uniform in N, and says nothing about the continuum limit or the mass gap. The bound in the reverse direction (an upper bound from the dilute gas, uniform in M) is open. Where these stand is treated in 27-01 and 27-02.

Setting the estimate (8) beside (11), the position is that, of the part of the estimate saying "the rate is the action of the fractional instanton", only the direction of confinement (the bound from below), on a lattice with fixed N, has the form of a statement.

07

What cannot be said

(a) The dilute gas. (5)–(8) are counting that ignores interactions, a physical estimate in the dilute region $\Lambda L \ll 1$ at the one-loop level. The value of the prefactor K is not known.

(b) Cooling values. The values of Tables 1 and 2 are upper bounds on the minimal action, not lower bounds. Of "approaches π^2 from below", "below π^2" can be said for $N = 2, 3, 4$ from the upper bounds of (10), and "approaches" is a numerical reading.

(c) Continuum existence. The existence of a solution attaining the lower bound (2) on the twisted $T^2 \times \mathbb{R}^2$ is a strand with an assumption in this book, and is not shown.

(d) The band. Every statement of this chapter is a statement with $\Lambda L \ll 1$ or with N fixed, and says nothing about widening the T^2 beyond $1/\Lambda$ (25-09).

(e) The paper on lattice fractional instantons. The Confinement Lower Bound by Fractional Instantons is a statement of a draft, and this book's label is "Assumption". It is not raised until the proof of the draft has been checked line by line and placed in the text.

08

The table of five grades and the dependency graph

Table 3 The rows of this chapter ("—" is a row without machine checking)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The BPS bound (1)	Cited	3-05	Continuum	—
2	On the twisted bundle $Q \in \frac{1}{2} + \mathbb{Z}$, lower bound $4\pi^2/g^2 = \pi^2\beta$ (2) $\Longleftarrow 1$	Cited	Tanizaki–Ünsal (2.24) (2.25), 't Hooft 1981	Continuum, SU(2), $n_{12} = n_{34} = 1$	—
3	The lump is localised in the plane and acts as a centre vortex; size of order L	Cited	Tanizaki–Ünsal § 2.3.1	Numerics and statements of the original source	—
4	Analytic existence is not known (as stated in the original source)	Cited	Tanizaki–Ünsal § 2.3.1	—	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
5	A strand of existence under the equivariance of the Nahm transform	Assumption	Strand: the "Route to Existence by the Nahm Transform" in the record of the study, § 02	One assumption, literature not yet checked	—
6	Weight of one lump $Ke^{-\pi^2\beta}$, $K \sim 1/L^2$ (3)	Cited	Tanizaki–Ünsal (2.27)	Dilute-gas model	—
7	$Ke^{-\pi^2\beta} \sim \Lambda^2(\Lambda L)^{5/3}$ (4)	Cited	Tanizaki–Ünsal (2.32)	One loop	—
8	In the Poisson model $\langle W\rangle = e^{-2\rho A}$ (5)	Proved here	One line in § 03	Computation within the model	—
9	$T_R \sim \Lambda^2(\Lambda L)^{5/3}\sin^2(\pi\lvert R\rvert/N)$ (6)	Cited	Tanizaki–Ünsal (2.37)	$\theta = 0$, dilute region	—
10	$\sigma = 4Ke^{-\pi^2\beta} \sim \Lambda^2(\Lambda L)^{5/3} \Longleftarrow$ 6, 7, 8, 9	Reading	Physical estimate	Dilute region, one loop, SU(2)	—
11	$-\ln\theta \approx 4Ke^{-\pi^2\beta}(Ma)^2$ (8) $\Longleftarrow$ 8, 10	Reading	Physical estimate	Same as above	—
12	Numerics of the lattice gas (Bergner–González-Arroyo–Soler 2025)	Cited	Abstract	SU(2), twisted small T^2	—
13	Reading of $5/3 = 11/3 - 2 \Longleftarrow$ 10	Reading	§ 04	One loop, dilute region	—
14	Cooling values (Table 1), fraction of localisation	Reading	Numerical record of this book's study	Upper bounds, 3 initial configurations	—
15	Properties (a)–(d) of S_*	Proved here	§ 05	SU(2), lattice	—
16	Certified upper bounds (10)	Computer-assisted	Draft of the paper on lattice fractional instantons § 7 (40-digit re-evaluation)	Upper direction only; $N = 4$ not converged	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
17	Dependence on ℓ in Table 2, decay $e^{-mN\ell/\ell}$	Reading	Same, § 7	Cooling	—
18	Approaches π^2 from below ⟸ 14, 15, 16, 17	Reading	§ 05	"Below" for $N = 2, 3, 4$ from row 16	—
19	Confinement Lower Bound by Fractional Instantons (11)	Assumption	Draft of the paper on lattice fractional instantons § 3– § 6	N fixed, $\beta \geq 1$, not uniform in N, no continuum limit	—

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  S ≥ 8π²|Q|/g²                                                  continuum
 2  CIT  twist gives Q ∈ 1/2 + ℤ, bound π²β ⟸ 1                           SU(2), n
 3  CIT  lump localised in plane, centre vortex, size ~ L
 4  CIT  analytic existence not known
 5  ASM  strand of existence under the equivariance assumption            literatur
 6  CIT  weight of one lump K e^(−π²β)
 7  CIT  K e^(−π²β) ~ Λ²(ΛL)^(5/3)                                       one loop
 8  PRV  Poisson model ⟨W⟩ = e^(−2ρA)
 9  CIT  T_R ~ Λ²(ΛL)^(5/3) sin²(π|R|/N)                                 θ = 0
10  RDG  σ = 4K e^(−π²β) ~ Λ²(ΛL)^(5/3) ⟸ 6, 7, 8, 9                     dilute, or
11  RDG  −ln θ ≈ 4K e^(−π²β)(Ma)² ⟸ 8, 10
12  CIT  numerics of the lattice gas (abstract)
13  RDG  reading of 5/3 = 11/3 − 2 ⟸ 10
14  RDG  cooling values, fraction of localisation                        upper boun
15  PRV  properties (a)-(d) of S_*
16  CMP  S_* ≤ 0.6762π², 0.9193π², 0.9707π²                              upper; N =
17  RDG  ℓ dependence, e^(−m_N ℓ)/ℓ
18  RDG  approaches π² from below ⟸ 14, 15, 16, 17
19  ASM  Confinement Lower Bound by Fractional Instantons (Theorem A of the la
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The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The only nodes with proofs placed in the text are the one line of the Poisson model and the properties of S_*, both one-line arguments within a model or on a finite lattice. The "Assumption" of the Route to Existence by the Nahm Transform and of the Confinement Lower Bound by Fractional Instantons enters as a premise of no node. The nodes on the conclusion side of this chapter (the string tension, the reading of θ, the reading of 5/3, approaching π^2 from below) are all "Reading", and there is neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Cited The lower bound $4\pi^2/g^2 = \pi^2\beta$ on the action of the twisted bundle, equations (2.27)(2.32)(2.37) of the original source for the dilute gas, the report of numerics of the lattice gas (abstract)	Existence of a continuum solution attaining (2)
Reading The string tension $\Lambda^2(\Lambda L)^{5/3}$ (an estimate in the dilute region, at one loop, for SU(2)), the reading of $5/3 = 11/3 - 2$, that the lattice action approaches π^2	The correctness of the dilute-gas approximation. The value of K. The limit $N \to \infty$
Computer-assisted $S_* \leq 0.6762\pi^2$, $0.9193\pi^2$, $0.9707\pi^2$ ($N = 2, 3, 4$, upper bounds from concrete configurations)	Lower bounds on, and the agreement of, S_* and $S_{\min}$
Proved here The one line of the Poisson model, the elementary properties of S_*	—
Assumption The strand of existence (assumption of equivariance), the Confinement Lower Bound by Fractional Instantons (draft)	Neither has a proof in the text

Sources and reproduction

Item	Kind	Source / tools
(2)(3)(4)(6), localisation of the lump, the state of analytic solutions	Cited	Y. Tanizaki, M. Ünsal, Center vortex and confinement in Yang–Mills theory and QCD with anomaly-preserving compactifications, arXiv:2201.06166 (v2, 2022). § 2.2– § 2.3 and equations (2.24)–(2.37) checked in the text
The gas of fractional instantons on the lattice	Cited	G. Bergner, A. González-Arroyo, I. Soler, arXiv:2502.09463 (2025, a Lattice 2024 report). Abstract checked
Topological charge of the twisted bundle	Cited	G. 't Hooft, Commun. Math. Phys. 81 (1981) 267 (bibliographic data in 3-05)
The strand of existence (assumption of equivariance)	Assumption	The record of this book's study (Route to Existence by the Nahm Transform). The literature cited there on the construction of doubly periodic instantons and the Nahm transform is not yet checked
Table 1, fraction of localisation	Reading	The numerical record of this book's work (cooling, 3 runs from random initial configurations)
(9)(10), Table 2, values on periodic lattices, slopes of decay, the Confinement Lower Bound by Fractional Instantons	Computer-assisted, Reading, Assumption	The paper on lattice fractional instantons (lattice confinement by fractional instantons, draft of 2026-10-14). § 2, § 7, § 8. The title and status are in A-07

Item	Kind	Source / tools
(5), properties of S_*, numerical values of m_N	Proved here	The text. m_N is the value of (8) of 25-01 with $N = 2$, 3

Words used in this chapter — fractional instanton, topological charge $\frac{1}{2}$, the BPS bound, centre vortex, dilute gas, the prefactor K, string tension, $5/3 = 11/3 - 2$, cooling, a lump confined to a column $S_*(N, \ell)$, a certified upper bound, the twist mass m_N. Their meanings are in A-02 Glossary.

Next chapter to read: 25-03 Containers — conditioning on the frame makes things independent — divide the plane into blocks, and bound the bias by a product of quantities per column.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space the container identity, nesting, limits in four dimensions · Kiichi, Shiori and Rin

Containers — conditioning on the frame makes things independent

> This chapter organises established knowledge. § 04, § 05 and § 07 are this book's own investigation.

What this chapter shows — The most naive tool for bounding a Wilson loop in the direction of confinement is to divide the surface spanned by the loop into small regions (containers) and to consider things with the outside of the containers (the frame) held fixed. With a nearest-neighbour action, fixing the frame makes the containers independent of each other, and the expectation of the Wilson loop is bounded from above by the average of the product of "the biases of the sectors of each container" (the Container Identity). This is the vortex-container identity of Mack–Petkova (1980), rewritten in the sector-summed ensemble, and this book claims nothing new about it. In this chapter we look in turn at (1) the Container Identity and its tanh form, (2) the form of "inner and outer regions" needed to pass to a fixed sector, (3) an inequality nesting containers in the directions of the plane (the directions of R^2), (4) the point made by Kovács–Tomboulis that in four dimensions, placing containers of fixed length independently gives only up to $V(R) \gtrsim R/\ln R$, and (5) that in a solvable two-dimensional model the bound of the Container Identity is sharp in its exponent.

Prerequisite chapters — 25-01 (the twisted T^2 and 't Hooft flux, sectors), 25-02 (vortices). From Part 5, 5-01 (the Wilson action and Wilson loops) and 5-08 (string tension); from Part 17, 17-02 (the picture of centre vortices).

Said first — The group is $SU(2)$, the action is the Wilson action, the lattice is finite and the lattice spacing is fixed. All statements of this chapter are statements "on the lattice, at some fixed β", and say nothing about the continuum limit. The Container Identity is known (Cited). What this book places on paper are three things: the rewriting in the sector-summed ensemble, the form of the correction when passing to a fixed sector, and the nesting inequality (Proved here). What the Lean column guarantees is only the algebra of finite sums; the Haar measure and the topology of sectors rest on the argument in the text.

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The order of this chapter
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1. Summary — fixing the frame makes things independent
2. The setting — the sector-summed ensemble, frame and containers
3. The Container Identity and its tanh form — correspondence with the identity of Mack–Petkova
4. To a fixed sector — inner and outer containers
5. The Nesting Inequality in the Plane Directions
6. Limits in four dimensions — $V(R) \gtrsim R/\ln R$ (equations (23)–(25) of Kovács–Tomboulis)
7. A test in a solvable two-dimensional model — Table 1
8. What cannot be said
9. The table of five grades and the dependency graph — Table 2

01

Summary — fixing the frame makes things independent

This section is a summary for [Y3] readers. The formulae and proofs are from § 02 on.

1. The Wilson action on the lattice is an action that couples only neighbouring links (nearest-neighbour). So the links inside a region and the links outside it talk only through the links on the border. If all of the border is held fixed, the insides of separate regions become independent of each other. This is exactly the Markov property of statistical mechanics.
2. The Wilson loop $W(C)$ can be rewritten as a product of "signs of the sector" $\sigma_j = \pm 1$ per small region (container) on the surface spanned by the loop (in the sector-summed ensemble, § 02). If the frame (the outside of the containers) is held fixed, the σ_j are independent of each other, so the expectation becomes a product, and taking absolute values and averaging gives $|\langle W \rangle| \le \mathbb{E}[\prod_j \theta_j]$, with $\theta_j = |\mathbb{E}[\sigma_j \,|\, \mathbf{frame}]|$. This is the Container Identity Cited .
3. θ_j can be written $\tanh(|\Delta F_j|/2)$. ΔF_j is the difference in free energy, inside a container with the frame fixed, between the sector with one vortex and the sector without. The more cheaply a vortex enters a container (the smaller ΔF_j), the smaller θ_j, and the smaller the bound on the Wilson loop.
4. This identity is the "vortex container" identity of Mack–Petkova (1980), written as it stands in equations (18)–(22) of Kovács–Tomboulis (2001). The only differences are that we wrote it with the sectors summed, and that we took the average over frames rather than the maximum. This book claims no novelty.

5. To pass to the Wilson loop in a fixed sector (one twisted sector), containers are needed not only inside the loop but also outside it (the Inner and Outer Container Proposition of § 04) Proved here .
6. If containers are nested in the directions of the plane, the bias of a large container is bounded by the average of the product of the biases of small containers (the Nesting Inequality in the Plane Directions of § 05) Proved here . But this concerns a slab with the size of the T^2 held fixed, and cannot be used for widening together with the T^2 (the crossover band).
7. In four-dimensional space (R^4), the length of the containers through points inside the loop grows with the loop. To keep the cost of a container constant its thickness must grow to order $\sqrt{\ln R}$, and independent containers give only up to $V(R) \gtrsim R/\ln R$ (Kovács–Tomboulis) Cited . In a slab with a small T^2, the length of a container stops at the area of the T^2, so this obstruction does not appear.
8. Two-dimensional $SU(2)$ is solvable, so one can see in numbers how sharp the bound of the Container Identity is. Tiling with containers of side ℓ, the string tension coming from the bound approaches the true value in the form $\sigma - \ln(16/(3\pi))/\ell^2$ (§ 07) Proved here + Reading .

02

The setting — the sector-summed ensemble, frame and containers

The lattice Λ is a finite hypercubic lattice, and the link variables are $U = (U_b) \in SU(2)^{\text{links}}$. On each plaquette p we place a sign $z_p \in \{\pm 1\}$, and impose $\prod_{p \in \partial c} z_p = 1$ (coclosed) on every 3-dimensional cube c. We set the weight and the measure to be

$$w(U,z) = \exp\Big(\frac{\beta}{2}\sum_p z_p \operatorname{tr} U_p\Big), \qquad d\mu \;\propto\; w(U,z)\prod_b dU_b \;\times\; (\text{counting of coclosed } z) \tag{1}$$

(dU_b is the Haar measure). z in the same cohomology class are related by central gauge transformations $U_b \mapsto \omega U_b$ ($\omega \in \{\pm 1\}$), so the sum (1) equals, up to a constant factor, the sum of the $SU(2)$ partition functions of all of 't Hooft's twisted sectors (25-01) (the summed ensemble of Mack–Petkova) Cited . Which sector one is in is fixed by the cohomology class of z, and the indicator function of a sector is a function of z. In this chapter we call (1) the **sector-summed ensemble**, and write its expectation $\mathbb{E}$.

Taking a loop C and one surface $S(C)$ spanning C (a collection of plaquettes whose boundary is C),

$$W(C) = \tfrac{1}{2}\operatorname{tr} U_C \prod_{p \in S(C)} z_p \tag{2}$$

is unchanged by central gauge transformations, and $|W(C)| \le 1$. The Wilson loop with the sector fixed to one is (2) multiplied by the indicator function of the sector, i.e. conditioned (§ 04).

Frame and containers. We take regions $R_1, \dots, R_n$ (containers) in the lattice, separated from each other, call the variables inside a container (the inner links and the z of the inner plaquettes) x_j, and call all other variables the **frame** Φ. The links on the border of a container belong to the frame. Since the action is nearest-neighbour, each plaquette term and each cube constraint depends only on the inside of at most one container and on the frame. Hence the weight splits when the frame is fixed.

$$w(\Phi, x) = a(\Phi) \prod_{j=1}^{n} f_j(\Phi, x_j), \qquad a > 0,\ f_j > 0 \tag{3}$$

If the loop C runs inside the frame, and we write $\sigma_j(x_j) \in \{\pm 1\}$ (the **sign of the sector** of the container) for the product of the z on the part of $S(C)$ inside the container R_j, then (2) splits as well.

$$W(C) = c(\Phi) \prod_{j=1}^{n} \sigma_j(x_j), \qquad |c(\Phi)| \le 1 \tag{4}$$

($c(\Phi)$ is $\frac{1}{2}\mathrm{tr}\, U_C$ times the product of the z on the part of $S(C)$ in the frame.) σ_j is unchanged by central gauge transformations inside the container (the boundary C of $S(C)$ lies in the frame), so it is well defined as a quantity on the conditional distribution with the frame fixed. Moreover, fixing the frame does not determine σ_j. This is because, in the small-T² slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, if a container is taken as "a disc of the plane $\times$ T²", then the surface $S_p = \{p\} \times T^2$ above a plaquette p of the plane inside the disc (N^2 plaquettes of type (3,4)) is coclosed and lies inside the container, and flipping all the z on it flips only σ_j without changing the frame Proved here (the check that it is coclosed is step 2 of 25-04 § 03).

03

The Container Identity and its tanh form

This section summarises existing knowledge (the container identity of Mack–Petkova). The introduction (independent vortex signs and the area law) is in 17-02 § 06.

With the frame Φ fixed, we write $q_j^{\pm}(\Phi) = \mathbb{P}(\sigma_j = \pm 1 \mid \Phi)$ for the probabilities of the sectors of container j, and $Z_j^{\pm}(\Phi) = \sum_{x_j : \sigma_j = \pm 1} f_j(\Phi, x_j)$ for the partition functions of the two sectors inside the container (Haar integrals for continuous variables), and set

$$\theta_j(\Phi) := \big|\mathbb{E}[\sigma_j \,|\, \Phi]\big| = |q_j^{+} - q_j^{-}|, \qquad \Delta F_j(\Phi) := \ln \frac{Z_j^{+}(\Phi)}{Z_j^{-}(\Phi)} = \ln \frac{q_j^{+}}{q_j^{-}} \tag{5}$$

We call θ_j the **bias** of container j, and ΔF_j the **vortex free energy** with the frame fixed.

Container Identity

Under (3)(4),

$$\mathbb{E}[W(C) \,|\, \Phi] = c(\Phi) \prod_{j=1}^{n} \mathbb{E}[\sigma_j \,|\, \Phi], \qquad |\langle W(C) \rangle| \;\le\; \mathbb{E}_{\Phi}\Big[\, |c(\Phi)| \prod_{j=1}^{n} \theta_j(\Phi) \Big] \;\le\; \mathbb{E}_{\Phi}\Big[\prod_{j=1}^{n} \theta$$

Moreover, $\theta_j(\Phi) = \tanh\big(|\Delta F_j(\Phi)|/2\big)$.

Cited (the identity) + Proved here (the rewriting in the summed ensemble). Proof. By (3), the conditional distribution with the frame fixed is the product of the distributions $f_j(\Phi, \cdot)/(Z_j^{+} + Z_j^{-})$ per container, and each σ_j is a function of the variables of a different container, so the conditional expectation splits into a product. Substituting (4) gives the first formula. Taking absolute values, using $|c| \le 1$ and averaging over the frame gives the second. Finally, from $q^{+} + q^{-} = 1$ and $q^{\pm} > 0$,

$$\frac{q^{+} - q^{-}}{q^{+} + q^{-}} = \frac{e^{\Delta F/2} - e^{-\Delta F/2}}{e^{\Delta F/2} + e^{-\Delta F/2}} = \tanh \frac{\Delta F}{2}, \qquad \Delta F = \ln \frac{q^{+}}{q^{-}} \tag{7}$$

∎

Lean FrameLemma (`FrameLemma.frame_bound`) (hypotheses: the space of frames, the number of containers and the space inside each container are finite, the inside of each container is non-empty, $a(\varphi) > 0$, $f_j(\varphi, y) > 0$. Conclusion: $|\sum_{\varphi} \sum_{x} a \prod_j f_j \cdot c \prod_j \sigma_j| \le \sum_{\varphi} a \prod_j Z_j \cdot |c| \prod_j |\sum_y f_j \sigma_j| / Z_j$, $Z_j = \sum_y f_j$), FrameLemma (`FrameLemma.sign_mean_le_one`) (hypotheses: $f > 0$, $\sigma = \pm 1$. Conclusion: the absolute value of the normalised conditional mean is at most 1), FrameLemma (`FrameLemma.ratio_eq_tanh`) (hypotheses: $q^{+}, q^{-} > 0$. Conclusion: (7)), Containers (`Containers.container_bound`) (hypotheses: finite, $a \ge 0$, $|s| \le 1$. Conclusion: $|\sum_{\varphi} \sum_{x} a\, s \prod_i f_i \sigma_i| \le \sum_{\varphi} a \prod_i |\sum_y f_i \sigma_i|$). All are algebra of finite sums; that (3)(4) follow from the Haar integral, a coclosed z, the topology of sectors and the nearest-neighbour action is outside Lean.

Correspondence with the original source. Kovács–Tomboulis (hep-lat/0108017, Phys. Rev. D 65 (2002) 074501), in a fixed sector, fix the links on the border of a container Λ_i and integrate the inside, and from the partition functions $z_{\Lambda_i}^{(\mp)}$ with the signs of the couplings on a coclosed surface V_i flipped (their equation (19)) write

$$f_{\Lambda_i}(U_{\partial\Lambda_i}) = \frac{1}{2}\Big(1 - \frac{z_{\Lambda_i}^{(-)}}{z_{\Lambda_i}^{(+)}}\Big) \;\; (18), \qquad W[C] = \Big\langle \tfrac{1}{2}\mathrm{tr}\, U[C] \prod_i f_{\Lambda_i}(U_{\partial\Lambda_i}) \Big\rangle \;\; (20), \qquad W[C] \le$$

and attribute (20) to Mack–Petkova (Ann. Phys. 125 (1980) 117, their reference [3]) Cited (the formulae of the original source checked in the text of hep-lat/0108017; the text of Mack–Petkova has not been checked in this book). There are only two differences from the Container Identity. (i) Since we wrote it with (1), with the sectors summed, the factor is $(Z^{+}-Z^{-})/(Z^{+}+Z^{-}) = \tanh(\Delta F/2)$ rather than $\tfrac{1}{2}(1-z^{-}/z^{+})$ (a difference in the normalisation of the measure). (ii) (22) takes the maximum over frames, while (6) takes the average. (ii) leaves room to use the rarity of bad frames. In the two-dimensional test of § 07, the averaged form approaches the true value faster.

04

To a fixed sector — inner and outer containers

The Container Identity is a statement about the sector-summed ensemble. What one wants to see in physics is the Wilson loop with the twisted sector fixed to one. In the slab $(\mathbb{Z}/N)^{2}\times(\mathbb{Z}/M)^{2}$, let the loop C lie in the (3,4) plane at a point y_{0} of the T^{2}. Writing $(-1)^{n_{34}}$ for the parity of the (3,4) flux among the labels of the sector (the product of z over the whole (3,4) plane at y_{0}), and $\mathbf{1}_{O}$ for the indicator function of the other labels (n_{12} and so on), the expectation with the sector fixed to $s\in\{\pm 1\}$ is

$$\langle W(C)\rangle_{s} = \frac{\mathbb{E}\big[W\cdot\tfrac{1}{2}\big(1+s(-1)^{n_{34}}\big)\mathbf{1}_{O}\big]}{\mathbb{E}\big[\tfrac{1}{2}\big(1+s(-1)^{n_{34}}\big)\mathbf{1}_{O}\big]} \tag{8}$$

If containers are placed only inside the loop, the second term of the numerator, $\mathbb{E}[W(-1)^{n_{34}}\mathbf{1}_{O}]$, is a problem. $W(-1)^{n_{34}}$ is $\tfrac{1}{2}\operatorname{tr}U_{C}$ multiplied by the product of the z **outside $S(C)$ in the plane**; it contains no signs of the inner containers, so the Container Identity does not make it small. The remedy is to place containers outside as well.

Inner and Outer Container Proposition (inner and outer regions)

In the (3,4) plane, arrange containers $R_{1},\dots,R_{n}$ inside the loop and containers $R'_{1},\dots,R'_{m}$ outside it in a grid, with the frame made of straight strips, so that in each direction there is a straight line passing only through the frame. Then $\mathbf{1}_{O}$ is a function of the frame, and one can write $(-1)^{n_{34}} = c''(\Phi)\prod_{j}\sigma_{j}\prod_{k}\sigma'_{k}$ ($|c''|=1$). Let θ_{j},θ'_{k} be the biases of (5) and set $\mathbb{E}_{O}[\,\cdot\,] := \mathbb{E}[\,\cdot\,\mathbf{1}_{O}]/\mathbb{E}[\mathbf{1}_{O}]$; then, when $\mathbb{E}_{O}[\prod_{j}\theta_{j}\prod_{k}\theta'_{k}]<1$,

$$|\langle W(C)\rangle_{s}| \;\le\; \frac{\mathbb{E}_{O}\big[\prod_{j}\theta_{j}\big]+\mathbb{E}_{O}\big[\prod_{k}\theta'_{k}\big]}{1-\mathbb{E}_{O}\big[\prod_{j}\theta_{j}\prod_{k}\theta'_{k}\big]} \tag{9}$$

Proved here Proof. $n_{12}, n_{13}, n_{14}, n_{23}, n_{24}$ are each the product of z over the respective plane, and that plane can be retaken on a straight strip passing only through the frame (since z is coclosed, the product does not depend on the position of the plane), so they are functions of the frame. Hence $\mathbf{1}_O$ is a function of the frame. The (3,4) plane splits into $S(C)$ and its complement, each of which splits into a frame part and container parts, so $W = c(\Phi)\prod_j \sigma_j$, $W(-1)^{n34} = c'(\Phi)\prod_k \sigma'_k$, $(-1)^{n34} = c''(\Phi)\prod_j \sigma_j \prod_k \sigma'_k$ ($|c|$, $|c'| \le 1$, $|c''| = 1$). Multiplying the first formula of the Container Identity by $\mathbf{1}_O$ and averaging over the frame gives $|\mathbb{E}[W\mathbf{1}_O]| \le \mathbb{E}[\mathbf{1}_O \prod_j \theta_j]$, $|\mathbb{E}[W(-1)^{n34}\mathbf{1}_O]| \le \mathbb{E}[\mathbf{1}_O \prod_k \theta'_k]$, $|\mathbb{E}[(-1)^{n34}\mathbf{1}_O]| \le \mathbb{E}[\mathbf{1}_O \prod_j \theta_j \prod_k \theta'_k]$. The numerator of (8) is at most half the sum of the first two, and the denominator is at least $\frac{1}{2}(\mathbb{E}[\mathbf{1}_O] - \mathbb{E}[\mathbf{1}_O \prod \theta \prod \theta'])$. Dividing both by $\mathbb{E}[\mathbf{1}_O]$ gives (9). ∎

The second term of (9) goes to 0 as the size of the plane $M \to \infty$ when the biases are typically smaller than 1 in the outer containers too (when the rarity of bad frames can be used outside as well). This book does not show that Assumption . In the theorem of slab confinement (25-04), the group of flips acts on all plaquettes of the plane, so the outer term automatically becomes small, $t^{M^2 - A}$, and this problem does not arise.

05

Nesting in the directions of the plane

We divide a large container R (a disc of the plane $\times$ T^2) into small containers $R_1, \dots, R_k$ separated by an inner frame Φ_{int}. The product of z over the surface spanned by R is $\sigma_R = s(\Phi_{\mathrm{int}})\prod_i \sigma_i$ ($|s| = 1$, the product of z over the part in the frame).

Nesting Inequality in the Plane Directions

Writing Φ_R for the frame outside R,

$$\left|\mathbb{E}[\sigma_R \mid \Phi_R]\right| \;\le\; \mathbb{E}\Big[\prod_{i=1}^{k} \left|\mathbb{E}[\sigma_i \mid \Phi_R, \Phi_{\mathrm{int}}]\right| \;\Big|\; \Phi_R\Big] \tag{10}$$

Proved here Proof. Apply the Container Identity with Φ_R and Φ_{int} together as the frame ($c = s$), and average over Φ_{int} with Φ_R held fixed. ∎ Lean Containers (`Containers.container_bound`) (hypotheses: finite, $a \ge 0$, $|s| \le 1$) is the finite-sum version of this algebra, the form whose description in the Lean file says it "contains the Mack–Petkova container identity and the planar doubling inequality".

How to read it. If, for containers of some size, the bias is $\theta_0 < 1$ for typical frames, one can expect that each doubling of the containers in the directions of the plane lowers

the bias to order θ_0^4 Reading (a physical estimate: the right-hand side of (10) is the average of a product, not the product of averages, so this does not follow directly from the inequality). Disorder propagates to large scales in the directions of the plane.

However, (10) concerns **a slab with the size of the T^2 held fixed**. What is needed when widening the T^2 (approaching four-dimensional space) is the behaviour of the bias $\theta(L, L)$ when the T^2 and the containers are doubled together, and (10) says nothing about that (when the T^2 changes, the very surface around which the vortex sheet winds changes). That the form needed there cannot be "$\theta(L, L)$ stays uniformly away from 1" but must be $\ln(1/\theta(L, L)) \geq cL^2$ (the contrast with $U(1)$) is treated in the chapter on the crossover band, 26-01.

06

Limits in four dimensions — independent containers reach only $R/\ln R$

The Container Identity is an identity that depends neither on the size of the lattice nor on the dimension. So, if one lines up many containers in four-dimensional space (the lattice of R^4), does an area law follow? Kovács–Tomboulis (hep-lat/0108017 § 3) show by the following counting that **independent containers do not reach it** Cited ((23)–(27) below are the equation numbers of the original source; checked in its text).

1. Let d_i be the width in the two directions transverse to the coclosed surface V_i of a container Λ_i, and $|V_i|$ its longitudinal length. **Assume** that a sufficiently thick container is in the regime where $-\ln(z^{(-)}/z^{(+)}) \sim |V_i| \exp(-c\, d_i^2)$ for all boundary values ((23) of the original source; c is the ρ of the original source).
2. To keep the free-energy cost of one container below a constant f, one needs $d_i^2 \geq c^{-1} \ln(|V_i|/f)$ ((24)). Then $\max |f_{\Lambda_i}| < \frac{1}{2}(1 - e^{-f})$ is constant.
3. For a rectangular loop with sides $T \times R$ ($T > R$), the length of the containers through points away from the perimeter is $|V_i| \sim R^{d-2}$ (R^2 in four dimensions). So a thickness of order $\sqrt{\ln R}$ is needed, and $\sim RT/\ln R$ containers fit inside the loop. From (22)

$$V(R) \;\geq\; \text{const} \cdot \frac{R}{\ln R} \qquad \text{((25) of the original source)} \quad (11)$$

is all that follows, which is not a purely linear potential (an area law). The original source goes on to state that if the boundary values of neighbouring containers are also integrated, grouping containers as thick as possible, and containers with transverse area $\sim R^2$ are used, one gets $V(R) \geq \text{const}\, R - \frac{\text{const}}{R}(\ln R + \text{const})$ ((27)), that is, **the linear potential comes from thick vortex fluctuations of the size of the loop**. It also states that the form (22) is crude, and that the inequalities coming from reflection

positivity (their (11)(13), the global form of Tomboulis–Yaffe) give an area law directly, with bounds uniform in the size of the lattice Cited .

Restated in the words of this chapter: in R^4 the length of the containers grows with the loop, so the road of placing containers of fixed length independently does not reach the area law. In a small-T^2 slab, the length of the container $\{\mathrm{disc}\} \times T^2$ (the area of the vortex sheet) stops at the area of the T^2, so this obstruction does not appear Proved here (a rereading of the fact that $|V_i| = N^2$ becomes a constant in (24) of the original counting). The obstruction returns when the T^2 is widened, and that is the crossover band (26-01, 25-09).

07

A test in a solvable two-dimensional model

Two-dimensional $SU(2)$ (a planar lattice, Wilson action $\frac{\beta}{2}\mathrm{tr}\, U_p$) is solvable because the plaquettes behave independently of each other. Here we see how close the bound of the Container Identity is to the true value. Two dimensions has no problem of "the length of containers", so this test says nothing about the four-dimensional obstruction of § 06.

Setting. Tile the inside of the loop with square containers of side ℓ, and take the links on the borders of the containers as the frame (container area $A = \ell^2$). With the frame fixed, container j sees only the holonomy $V_j \in SU(2)$ of its border. Using the characters $\chi_n(V) = \sin((n+1)\vartheta)/\sin\vartheta$ ($e^{\pm i\vartheta}$ are the eigenvalues of V, and $n = 0, 1, 2, \ldots$ is $2\times$ the spin of the representation) and $a_n = I_{n+1}(\beta)/I_1(\beta)$ (I is the modified Bessel function), the heat-kernel expansion gives

$$\mathbb{E}[\sigma_j \,|\, \mathrm{frame}] = \frac{\sum_{n\ \mathrm{odd}} (n+1)\, a_n^A\, \chi_n(V_j)}{\sum_{n\ \mathrm{even}} (n+1)\, a_n^A\, \chi_n(V_j)} \quad (12)$$

and the holonomies of the containers are independent of each other, distributed with the denominator as density. So the bound of the Container Identity becomes a product of factors per container,

$$|\langle W\rangle| \;\le\; B(A)^{\#\mathrm{containers}}, \qquad B(A) = \int_{SU(2)} dV \,\Big|\sum_{n\ \mathrm{odd}} (n+1)\, a_n^A\, \chi_n(V)\Big| \quad (13)$$

The true value is $\langle W\rangle = a_1^{\mathrm{area}}$, and the true string tension is $\sigma = -\ln a_1$. We write $\sigma_{\mathrm{L5}}(\ell) := -\ln B(\ell^2)/\ell^2$ for the string tension coming from (13), and $\sigma_{\max}(\ell) := -\ln\big(\max_V |(12)|\big)/\ell^2$ for that coming from the form of (22) of Kovács–Tomboulis (maximum over frames).

The subscript L5 is a letter for the form that comes from the Container Identity (the average over frames), used as the counterpart of the maximum form $\sigma_{\max}$.

The limit of large containers. As $A \to \infty$, the odd sum is dominated by the $n = 1$ term $2a_1^A \chi_1(V) = 4a_1^A \cos\vartheta$ and the even sum by the $n = 0$ term 1, so with the Haar distribution $\frac{2}{\pi}\sin^2\vartheta\, d\vartheta$

$$\frac{B(A)}{a_1^A} \longrightarrow \frac{2}{\pi}\int_0^{\pi} 4|\cos\vartheta|\sin^2\vartheta\, d\vartheta = \frac{16}{3\pi} \approx 1.6977, \qquad \sigma_{\mathrm{L5}}(\ell) \;=\; \sigma - \frac{\ln(16/(3\pi))}{\ell^2} + (A\text{-e}$$

Proved here . The bound of the Container Identity is **sharp in its exponent**, and the loss vanishes as $1/\ell^2$.

Table 1 The string tension $\sigma_{\mathrm{L5}}(\ell)$ from the Container Identity and the maximum form $\sigma_{\max}(\ell)$ for two-dimensional $SU(2)$. Floating point, trapezoidal rule in ϑ with 4×10^5 points, representations truncated at $n \le 200$, the check of the normalisation of the denominator is 1.000000 in every row. The numbers are numerics from this book's study (Reading)

β	True σ	$\ell = 1$	$\ell = 2$	$\ell = 4$	$\ell = 8$	$\ell = 16$	$\sigma_{\max}$: $\ell = 1, 4, 16$
1	1.4263	0.9132	1.2940	1.3932	1.4180	1.4242	0.2723, 1.3397, 1.4209
2	0.8367	0.3663	0.7044	0.8036	0.8285	0.8347	0.0366, 0.7501, 0.8313
4	0.4185	0.0624	0.2866	0.3854	0.4102	0.4164	0.00067, 0.3318, 0.4131
8	0.1994	0.0016	0.0761	0.1663	0.1911	0.1973	$< 10^{-6}$, 0.1128, 0.1940

How to read it Reading . (i) The $\ell = 16$ column agrees with the right-hand side of (14) to the fourth decimal place (for example, at $\beta = 1$, $1.426309 - \ln(1.69765)/256 = 1.424242$). (ii) For thin containers ($\ell = 1$), the bound becomes exponentially worse when β is large (at $\beta = 8$, 0.0016 against 0.1994). This is the two-dimensional form of the need for thick containers. (iii) The maximum form approaches the same limit, but lags behind. The Container Identity, which takes the average, is always closer to the true value at the same ℓ.

The maximum form for $\ell = 1$ in (ii) can be written in closed form. A container of side 1 is a single plaquette, and all four of its links belong to the frame, so $\mathbb{E}[z_p \mid V] = \tanh\!\left(\frac{\beta}{2}\operatorname{tr} V\right)$, and the maximum is $\tanh\beta$ at $\operatorname{tr} V = \pm 2$ Proved here . The $\sigma_{\max}(1)$ of the table agrees with $-\ln\tanh\beta$ (0.27234, 0.036635, 0.000671 at $\beta = 1, 2, 4$). This is the same formula as the case $N = 1$ of the slab-confinement bound $\sigma \ge -\ln\tanh(\beta N^2)$ of the next chapter (25-04 § 05).

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08
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What cannot be said

(a) Novelty. The Container Identity is the container identity of Mack–Petkova, written in (18)–(22) of Kovács–Tomboulis. The rewriting in the sector-summed ensemble, the Inner and Outer Container Proposition and the Nesting Inequality in the Plane Directions are elementary, and have not been compared with literature other than Kovács–Tomboulis. No novelty is claimed for the results of this chapter.

(b) The second term of the Inner and Outer Container Proposition. That the biases of the outer containers are typically small (that the second term goes to 0 as $M \to \infty$) is not shown.

(c) The reading of the Nesting Inequality in the Plane Directions. "θ_0^4 at each doubling" is an estimate and does not follow directly from (10). (10) says nothing about widening together with the T^2.

(d) The four-dimensional counting. The $R/\ln R$ of § 06 is counting on top of the assumption on the form of the thick-container free energy in the original source ((23) of Kovács–Tomboulis: that the free energy of a thick container has the form $|V_i| e^{-c d_i^2}$ uniformly in the boundary values), and is not a four-dimensional theorem. This book shows neither (23) nor its negation.

(e) Uniform bounds on the free energy of containers. Bounding ΔF_j uniformly over good frames and uniformly in the lattice spacing is not shown anywhere in this book. That needs a multiscale analysis of the conditional measure with the frame fixed, and the variational principle alone (which gives only lower bounds on partition functions) points the wrong way.

(f) Lean. The machine-checking column is only the algebra of finite sums. That (3)(4) follow from the Haar measure, a coclosed z, the topology of sectors and the nearest-neighbour action is outside Lean.

(g) The two-dimensional test. Table 1 is two-dimensional numerics and says nothing about the problem of the length of containers in four dimensions.

09

The table of five grades and the dependency graph

Table 2 The rows of this chapter ("—" is a row without machine checking)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The sector-summed ensemble (1) is the sum of the partition functions of all twisted sectors	Cited	Kovács–Tomboulis § 2 (moving the twist), Mack–Petkova (text not checked)	$SU(2)$, finite lattice	—
2	With the frame fixed, the weight and the Wilson loop split per container (3)(4); σ_j is not determined by the frame	Proved here	§ 02	Nearest-neighbour action, containers separated	—
3	(18)(20)(22) of Kovács–Tomboulis	Cited	Text of the original source	Fixed sector, maximum over frames	—
4	Container Identity: (6) and $\theta_j = \tanh(\lvert\Delta F_j\rvert/2)$	Cited + Proved here	§ 03 (the identity is row 3, the rewriting is in the text)	Finite lattice, any β	FrameLemma (`FrameLemma.frame_bound`) (hypotheses: finite, $a > 0$, $f > 0$, inside non-empty), `sign_mean_le_one` (hypotheses: $f > 0$, $\sigma = \pm 1$), `ratio_eq_tanh` (hypotheses: $q^{\pm} > 0$), Containers (`Containers.container_bound`) (hypotheses: finite, $a \geq 0$, $\lvert s\rvert \leq 1$)
5	Inner and Outer Container Proposition: (9) for a fixed sector	Proved here	§ 04	Slab, containers in a grid, $\mathbb{E}_O[\prod\theta\prod\theta'] < 1$	— (the algebra has the same form as the bundles of row 4)
6	The second term of (9) goes to 0 as $M \to \infty$	Assumption	§ 04 (rarity of bad frames outside)	—	—
7	Nesting Inequality in the Plane Directions (10)	Proved here	§ 05	Slab with the size of the T^2 fixed	Containers (`Containers.container_bound`) (hypotheses: finite, $a \geq 0$, $\lvert s\rvert \leq 1$)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
8	The bias drops to order θ_0^4 at each doubling	Reading (physics)	§ 05	Typical frames, T^2 fixed	—
9	Independent containers give only $V(R) \geq \text{const}\, R/\ln R$; thick containers give linear	Cited	(23)–(27) of § 3 of the original source	Four dimensions, on top of the assumption on the form of the thick-container free energy in the original source	—
10	In a slab the length of containers stops at the area of the T^2, and the obstruction of row 9 does not appear	Proved here	§ 06 (a rereading of (24))	Small-T^2 slab, N fixed	—
11	For large containers $B(A)/a_1^4 \to 16/(3\pi)$, (14)	Proved here	§ 07	Two-dimensional $SU(2)$, Wilson action	—
12	The numbers of Table 1	Reading	Numerics from this book's study	Two dimensions, floating point, $n \leq 200$	—
13	The maximum form for a container of side 1 is $\tanh \beta$	Proved here	§ 07	Two dimensions	—

The bundles in the machine-checking column are those confirmed, by re-checking on this book's side, to contain no `sorry` and to use axioms forming a subset of the three standard ones (propext, Classical.choice, Quot.sound). What is guaranteed is only the algebra of sums and products of real numbers over finite sets (23-13).

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  sector-summed ensemble (1)                                    SU(2),
 2  PRV  splits by frame (3)(4); σ_j not fixed by frame                neares
 3  CIT  Kovács–Tomboulis (18)(20)(22)                                 fixed
 4  PRV  Container Identity (6)(7) ⟸ 1, 2 (same identity as 3)          Lean
 5  PRV  Inner and Outer Container Proposition (9) ⟸ 4                  slab,
 6  ASM  second term of (9) → 0                                        rarity
 7  PRV  Nesting Inequality in the Plane Directions (10) ⟸ 4            T² fi
 8  RDG  θ₀⁴ per doubling (physics) ⟸ 7
 9  CIT  R/ln R; thick containers linear (on the assumption on the form of the
        four dimensions
10  PRV  in a slab container length stops ⟸ 9
11  PRV  limit 16/(3π) ⟸ 4                                              two c
12  RDG  Table 1 ⟸ 4, 11                                               float
13  PRV  maximum for side 1 = tanh β ⟸ 3                                two c
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The row of rarity outside (second term of (9) $\to$ 0) is an assumption row and enters as a premise of no row. The inequality (9) of the Inner and Outer Container Proposition holds without this row; the row is needed only when reading (9) as an area law. This chapter has neither a node of the continuum limit nor a node of the four-dimensional area law.

The machine-checked theorems on the vortex side are in Part 21.

What this chapter can say

What is said	What is not said
Cited + Proved here The Container Identity and its tanh form. Finite lattice, any β, the sector-summed ensemble	Novelty (the identity is Mack–Petkova's). Bounds on the free energy of containers ΔF_j uniform over good frames and in the lattice spacing
Proved here How to pass to a fixed sector (the Inner and Outer Container Proposition), and nesting in the directions of the plane (the Nesting Inequality in the Plane Directions)	That the second term of (9) goes to 0. The behaviour of the bias when the T^2 is widened together (26-01)
Cited In four dimensions independent containers give only up to $R/\ln R$ (counting on top of the assumption on the form of the thick-container free energy in the original source); why this obstruction does not appear in a slab	The four-dimensional area law. The assumption on the form of the thick-container free energy itself
Proved here + Reading That for two-dimensional $SU(2)$ the bound of the Container Identity is sharp in its exponent (loss $\ln(16/(3\pi))/\ell^2$)	The sharpness of the bound in four dimensions

Sources and reproduction

Item	Kind	Source / tools
The container identity (18)(20)(22), the four-dimensional counting (23)–(27)	Cited	T. G. Kovács, E. T. Tomboulis, Vortex waistlines and long range fluctuations, arXiv:hep-lat/0108017 (Phys. Rev. D 65 (2002) 074501). § 2 and § 3 checked in the text
Vortex containers and the summed ensemble	Cited (indirect)	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117 (reference [3] of Kovács–Tomboulis. The text has not been checked in this book)
The global form from reflection positivity	Cited (indirect)	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313 (reference [9] of Kovács–Tomboulis, and the restatement of it in their text (11)(13))
The form of the Container Identity in the summed ensemble, the Inner and Outer Container Proposition, the Nesting Inequality in the Plane Directions, (14), the maximum for side 1	Proved here	Proofs in the text, § 03– § 07
Table 1	Reading	Numerics from this book's work (two-dimensional $SU(2)$, the character expansion (12) integrated numerically; a few seconds of computation)
Lean	Lean	FrameLemma (`FrameLemma.frame_bound`, `FrameLemma.sign_mean_le_one`, `FrameLemma.ratio_eq_tanh`), Containers (`Containers.container_bound`). What the bundles guarantee is only the algebra of finite sums

Words used in this chapter — the sector-summed ensemble, frame, container, the sign of the sector, the bias θ, the vortex free energy with the frame fixed ΔF, the Container Identity, inner and outer containers, nesting, the length and thickness of containers. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-04 Slab confinement — letting the group of flips act on all plaquettes of the plane gives, in a slab with the lattice spacing and the T^2 fixed, an area law at every β.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space the Slab Confinement Theorem, group averaging, thin vortices · Kiichi, Shiori and Rin

Slab confinement

> This chapter is this book's own investigation. The established knowledge it builds on is in 25-03 and 5-01.

What this chapter shows — Consider a "slab" lattice in which two directions are wound into a small T^2 (a lattice of side N) and the remaining two directions (the plane) are wide. With the lattice spacing and the size of the T^2 fixed, the $SU(2)$ Wilson loop obeys an area law **at every** β, with string tension $\sigma \geq -\ln\tanh(\beta N^2) \geq 2e^{-2\beta N^2} > 0$ (the slab-confinement theorem). The proof is elementary: one just averages over the group that flips at once the signs on the vortex sheets $\{p\} \times T^2$ standing directly above the plaquettes of the plane. In this chapter we place the statement of the theorem (§ 02), a proof in five steps (§ 03), the container form $\sup\theta \leq \tanh(\beta N^2)^{\ell^2}$ and why this lower bound becomes 0 in the continuum limit (§ 04), a test in two dimensions (§ 05), and where it stands relative to earlier literature (§ 06).

Prerequisite chapters — 25-03 (the sector-summed ensemble, frame and containers, the Container Identity), 25-01 (twisted sectors and flux). From Part 5, 5-01 (the Wilson action) and 5-02 (the area law at strong coupling); from Part 7, 7-02 (two dimensions).

Said first — The theorem is a statement on the lattice, with the lattice spacing and the side N of the T^2 fixed. In the continuum limit ($N \to \infty$, $\beta \to \infty$) this lower bound vanishes to 0. That the theorem holds at every β without using the strong-coupling expansion is because the slab is in effect a two-dimensional system, and it says nothing about confinement in four-dimensional space. Comparison with earlier literature is not done (comparison with arguments of Tomboulis–Yaffe type and with Borgs 1985 remains).

The order of this chapter

1. Summary — flip the vortex sheets all at once and average
2. Setting and the Slab Confinement Theorem
3. Proof — five moves
4. The container form and the cost of thin vortices — in the continuum limit the lower bound on the tension is 0

5. A test in two dimensions
6. Where it stands relative to earlier literature
7. What cannot be said
8. The table of five grades and the dependency graph — Table 1

01

Summary — flip the vortex sheets all at once and average

1. Two-dimensional lattice gauge theory confines at every coupling (Wilson loops decrease exponentially in the area). This is because the plaquettes behave independently of each other (7-02).
2. Seen from afar, the lattice of a slab $T^2 \times \mathbb{R}^2$ with two small closed directions is a two-dimensional system. So, with the size of the T^2 held fixed, one can expect it too to confine at every coupling. This chapter states that on the lattice, with a proof Proved here .
3. The tool is the sector-summed ensemble of 25-03. For each plaquette p of the plane, consider the sheet S_p of the N^2 (3,4) plaquettes at all points of the T^2 standing directly above it. Flipping the signs z on S_p all at once preserves the coclosedness condition and leaves the links unchanged. Flipping at a p inside the loop changes the sign of the Wilson loop, and the weight changes by at most a factor $e^{\pm 2\beta N^2}$.
4. Averaging over all combinations of flips (2^{M^2} of them), the Wilson loop is bounded by a product of factors $(1 - r_p)/(1 + r_p)$ over the inner plaquettes, and each factor is at most $\tanh(\beta N^2)$. Hence $|\langle W \rangle| \le \tanh(\beta N^2)^A$ (A is the area of the loop).
5. This lower bound is superexponentially weak in N (at $\beta = 2.5$, $N = 4$, $\sigma \ge 2e^{-80}$), and becomes 0 in the continuum limit. This is because the cost $2\beta N^2$ of flipping a thin vortex sheet at once grows with the number of points of the T^2 lattice. The cost of a physical vortex (a fractional instanton) is of order $\pi^2\beta$ and does not depend on N, so the road of doing the same thing with thick vortices is the next two chapters (25-05, 25-06).

02

Setting and the Slab Confinement Theorem

The lattice is $\Lambda = (\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$. Directions 1 and 2 form the T^2 (side N), and directions 3 and 4 the plane (cut off with period M). The group is $SU(2)$, the action is

the Wilson action $\frac{\beta}{2}\sum_p \operatorname{tr} U_p$, and the twist of the T^2 (n_{12} and so on) is fixed arbitrarily. The loop C lies in the (3,4) plane at a point y_0 of the T^2, and encloses a region $S(C)$ of area A (number of plaquettes) in the plane.

Slab Confinement Theorem

Set $t := \tanh(\beta N^2)$. For every $\beta > 0$, every M, and any fixed twisted sector,

$$\left|\langle \tfrac{1}{2}\operatorname{tr} U_C\rangle\right| \;\le\; \frac{t^{A} + t^{M^2 - A}}{1 - t^{M^2}} \quad (1)$$

In particular, at any limit point as $M \to \infty$, $|\langle \frac{1}{2}\operatorname{tr} U_C\rangle| \le t^{A}$, and uniformly in the area of the loop,

$$\sigma \;\ge\; -\ln\tanh(\beta N^2) \;\ge\; 2e^{-2\beta N^2} \;>\; 0. \quad (2)$$

Proved here (proof in § 03). The σ of (2) is the string tension in lattice units, and the conditions are "$SU(2)$, Wilson action, N fixed, lattice spacing fixed, every β". The right-hand inequality of (2) follows from setting $u = e^{-2x}$ and $-\ln\tanh x = \ln(1+u) - \ln(1-u) = 2(u + u^3/3 + \cdots) \ge 2u$.

03

Proof — five moves

Step 1 (the sector-summed ensemble). (1) of 25-03 § 02: for coclosed $z \in \{\pm 1\}^{\text{plaquettes}}$ and U the weight is $w(U,z) = \exp(\frac{\beta}{2}\sum_p z_p \operatorname{tr} U_p)$, and the measure is Haar $\times$ counting. Every z that can be written $z = z_0 \cdot \delta\omega$ ($\omega \in \{\pm 1\}^{\text{links}}$, $(\delta\omega)_p = \prod_{b\in\partial p}\omega_b$) gives the same integral as z_0 by the invariance $U_b \mapsto \omega_b U_b$ of the Haar measure, so the sum over z in one cohomology class is a constant multiple of the $SU(2)$ partition function of the twisted sector that the class represents Proved here. The sector is fixed by the class of z, and the indicator function of a sector is a function of z. The gauge-invariant Wilson loop is $W(C) = \frac{1}{2}\operatorname{tr} U_C \prod_{p\in S(C)} z_p$; under the same substitution $\prod_{p\in S(C)}(\delta\omega)_p = \prod_{b\in C}\omega_b$ cancels against the sign on the side of $\operatorname{tr} U_C$, so within a sector it has the same expectation as $\frac{1}{2}\operatorname{tr} U_C$ (up to a sign depending on the choice of representative of the sector).

Step 2 (vortex sheets are coclosed). For each plaquette p of the plane set $S_p := \{$ p's overlying (3,4) plaquettes $\} \times T^2$ (N^2 plaquettes). Among three-dimensional cubes, only those of orientation (1,3,4) and (2,3,4) have (3,4) faces; for example, the (3,4) faces

of the (1,3,4) cube that extends in direction 1 from a point y of the T^2 and spans the plaquette p' of the plane are the two copies of p' at y and $y+e1$. If $p'=p$ both lie in S_p; if $p' \neq p$ neither does. Cubes of type (1,2,3) and (1,2,4) have no (3,4) faces. Hence every cube contains 0 or 2 faces of S_p, and flipping the z on S_p preserves the coclosedness condition Proved here . If $p \neq q$, S_p and S_q have no plaquette in common.

Step 3 (the group action). Let P be the set of all plaquettes of the plane ($|P| = M^2$), and assign to a subset $g \subseteq P$ the map $z \mapsto z \cdot \chi_{S_g}$ (flipping on $S_g = \bigcup_{p\in g} S_p$). This is an action of the group $G = \{\pm 1\}^P$, a bijection on coclosed z, and since it does not change U it preserves the Haar $\times$ counting measure.

- The Wilson loop: $S(C)$ consists of A plaquettes in the plane at y_0, and S_p contains exactly one plaquette of the plane at y_0, the one at the position of p. Hence $W(g\cdot x) = W(x)\prod_{p\in g}\varepsilon_p$, with $\varepsilon_p = -1$ ($p \in S(C)$) and $+1$ (otherwise).
- The weight: flipping the z on S_p changes the action by $-\beta\sum_{q\in S_p} z_q \operatorname{tr} U_q$, so setting $r_p(x) := \exp(-\beta\sum_{q\in S_p} z_q \operatorname{tr} U_q)$, from $|\operatorname{tr} U_q| \le 2$ we get $r_p \in [e^{-2\beta N^2}, e^{2\beta N^2}]$. r_p is determined only by the values on S_p, and the sheets do not meet, so $w(g\cdot x) = w(x)$ $\prod_{p\in g} r_p(x)$.

Step 4 (averaging). Let $\mathbb{E}$ be the expectation of the sector-summed ensemble, and $\sum_x$ the sum over z together with the Haar integral over U. For each g, relabelling the variables gives $\sum_x W(x)w(x) = \sum_x W(g\cdot x)w(g\cdot x) = \sum_x W(x)w(x)\prod_{p\in g}\varepsilon_p r_p(x)$. Averaging over g and using $\sum_{g\subseteq P}\prod_{p\in g}\varepsilon_p r_p = \prod_{p\in P}(1+\varepsilon_p r_p)$,

$$2^{M^2}\sum_x Ww = \sum_x W(x)w(x)\prod_{p\in S(C)}(1-r_p(x))\prod_{p\in\complement S(C)}(1+r_p(x)), \qquad 2^{M^2}\sum_x w = \sum_x$$

The one-sheet inequality $|1-r| \le \tanh(a)\,(1+r)$ ($r\in[e^{-2a}, e^{2a}]$, here $a=\beta N^2$) follows from $(1-r)/(1+r) = -\tanh(\tfrac12 \ln r)$ and $|\tfrac12 \ln r| \le a$. Together with $|W|\le 1$, the absolute value of the right-hand side of the left formula of (3) is at most $t^A\sum_x w\prod_P(1+r_p) = t^A\,2^{M^2}\sum_x w$. Hence

$$|\mathbb{E}[W(C)]| \;\le\; t^A. \quad (4)$$

At this step there is no need to split into orbits. We use only the relabelling (a bijection) for each g, and the exchange of the sum over g with the sum over x.

Step 5 (projection to a fixed sector). The group G changes only the parity of the (3,4) flux. S_p contains one plaquette of the (3,4) plane at y_0, so it flips $(-1)^{n34}$ (the product of z over the whole (3,4) plane at y_0); it contains no plaquettes of other orientations, so

it does not change the indicator function $\mathbf{1}_O$ of the other labels such as n_{12}. The expectation with the sector fixed to s is, as in (8) of 25-03,

$$\langle W\rangle_s = \frac{\mathbb{E}[W\mathbf{1}_O] + s\,\mathbb{E}[W(-1)^{n_{34}}\mathbf{1}_O]}{\mathbb{E}[\mathbf{1}_O] + s\,\mathbb{E}[(-1)^{n_{34}}\mathbf{1}_O]} \tag{5}$$

$W(-1)^{n_{34}} = \frac{1}{2}\mathrm{tr}\,U_C \prod_{p\in/\,S(C)} z_p$ (the complement in the plane at y_0, of area $M^2 - A$) changes sign under flips at $p \in/\ S(C)$ and does not change under flips at $p \in S(C)$. $(-1)^{n_{34}}$ changes sign under flips at every p. $\mathbf{1}_O$ is invariant under G, so by the same averaging as in step 4 (with the roles of ε_p exchanged)

$$|\mathbb{E}[W\mathbf{1}_O]| \le t^{A}\,\mathbb{E}[\mathbf{1}_O], \qquad |\mathbb{E}[W(-1)^{n_{34}}\mathbf{1}_O]| \le t^{M^2-A}\,\mathbb{E}[\mathbf{1}_O], \qquad |\mathbb{E}[(-1)^{n_{34}}\mathbf{1}_O]| \le t^{M^2}\,\mathbb{E}$$

Bounding the numerator and denominator of (5) by (6) gives (1). As $M \to \infty$, $t^{M^2-A} \to 0$ and $t^{M^2} \to 0$ ($t < 1$), so at limit points $|\langle W\rangle| \le t^{A}$, and (2) follows. ∎

Here the group acts on all plaquettes of the plane, so the problem of 25-03 § 04, "the outer term does not become small", does not arise. The outer term automatically becomes small, t^{M^2-A}.

Lean SlabConfine (`SlabConfine.group_average`) (no hypotheses: $\sum_{g\subseteq P}\prod_{p\in g}\varepsilon_p r_p = \prod_{p\in P}(1+\varepsilon_p r_p)$), SlabConfine (`SlabConfine.factor_split`) (no hypotheses: the splitting into inner and outer), SlabConfine (`SlabConfine.tanh_bound`) (hypotheses: $t \ge 0$, $r_p > 0$, $|1-r_p| \le t(1+r_p)$), SlabConfine (`SlabConfine.one_sheet`) (hypotheses: $a \ge 0$, $e^{-2a} \le r \le e^{2a}$. Conclusion: $|1-r| \le \tanh a \cdot (1+r)$), SlabAverage (`SlabAverage.average_bound`) (hypotheses: the set of configurations is finite, $w > 0$, $|W| \le 1$, the action of each $g \subseteq P$ is a bijection, $r_p > 0$, $|1-r_p| \le t(1+r_p)$, $W(gx)w(gx) = W(x)w(x)\prod_{p\in g}\varepsilon_p r_p(x)$, $w(gx) = w(x)\prod_{p\in g} r_p(x)$. Conclusion: $|\sum Ww| \le t^{|P\cap A|}\sum w$). These are the finite-sum version of step 4. Replacing the sums by Haar integrals, and the coclosed z and the topology of sectors (steps 1, 2 and 5), are outside Lean and rest on the argument in the text.

04

The container form and the cost of thin vortices

Carrying out the same averaging only inside one container, it can be written in the language of the biases of 25-03. Take as container the product of a square Q of side ℓ in the plane (ℓ^2 plaquettes) with the T^2, and let σ_Q be the product of z over the surface spanned by Q. Flips by $g \subseteq Q$ change only the z inside the container, so they do not change the frame Φ, and they multiply σ_Q by $\prod_{p\in g}(-1)$. On the orbit $\{g\cdot x : g \subseteq Q\}$ of one x, the weight has the product form $w(x)\prod_{p\in g} r_p(x)$, so conditioning on the orbit makes the components of g independent, and

$$\mathbb{E}[\sigma_Q \mid \mathrm{orbit}] = \sigma_Q(x)\prod_{p\in Q}\frac{1-r_p(x)}{1+r_p(x)} = \pm\prod_{p\in Q}\tanh X_p, \qquad X_p = -\tfrac{1}{2}\ln r_p \in [-\beta N^2, \beta N^2]$$

Proved here . Conditioning on the orbit is finer than conditioning on the frame, so

$$\sup_{\Phi}\ \theta_Q(\Phi) = \sup_{\Phi}\big|\mathbb{E}[\sigma_Q \mid \Phi]\big| \ \le\ \tanh(\beta N^2)^{\ell^2} \qquad (8)$$

Proved here . That it can be bounded by the **maximum** over frames is because the cost of flipping a thin sheet, $|\ln r_p| \le 2\beta N^2$, is uniform and independent of the frame.

Extremely weak. The lower bound of (2) becomes superexponentially small in N. At $\beta = 2.5$, $N = 4$, $2\beta N^2 = 80$ and $\sigma \ge 2e^{-80} \approx 3.6 \times 10^{-35}$. The lower bound $2e^{-2\beta N^2}/a^2$ on the physical string tension $\sigma_{\text{phys}} = \sigma/a^2$ goes to 0 if the physical size $L = Na$ of the T^2 is held fixed and $a \to 0$ ($N = L/a \to \infty$, and β increases by asymptotic freedom) Proved here . **In the continuum limit, the lower bound on the tension in this theorem is 0.**

The reason is the cost of thin vortices. The cost of a flip, $2\beta N^2$, is the action of turning over at once the N^2 plaquettes of the lattice sheet S_p, one lattice unit thick, and it grows with the number of points of the T^2 lattice. On the other hand, what physics regards as the cost of a vortex is of the order of the action $4\pi^2/g^2 = \pi^2\beta$ ($\beta = 4/g^2$) of a smooth fractional instanton winding the T^2, which does not depend on N (25-02) Reading (physics). The same group-averaging argument goes through with $t = \tanh(F/2)$ whenever the ratio of weights r_p of a flip lies in $[e^{-F}, e^{F}]$, so if the flip of a thin sheet is replaced by "a measure-preserving map that inserts a thick vortex" and F can be bounded by a quantity independent of N, a lower bound uniform in the lattice spacing can be expected Assumption (where the evidence is: the form (7) of this chapter, the insertion identities of 25-05, and the insertion driven by dynamics of 25-06). That needs a uniform bound on the cost of insertion (ultraviolet control; in the setting of this part, that the same construction as the one published on 2026-11-10 stands under twisted boundary conditions), and this book does not show it. The published construction is on a periodic cubic box and does not reach here.

(8) also shows that arguments using $\sup_\Phi$ get superexponentially worse in N. To get uniformity in the problem of widening the T^2 (the crossover band), only the averaged form over frames can be used, not the maximum (one of the reasons why the Container Identity of 25-03 takes the average).

05

A test in two dimensions

Applying the five moves of the proof directly to the two-dimensional lattice $(\mathbb{Z}/M)^2$, there are no cubes, so the coclosedness condition is empty, and the vortex sheet is $S_p = \{p\}$ (a single plaquette, the formula of the Slab Confinement Theorem with $N = 1$).

So the same proof gives $|\langle W\rangle| \le \tanh(\beta)^A$ for two-dimensional $SU(2)$. The true value in two dimensions is $\langle W\rangle = a_1^A$, $a_1 = I_2(\beta)/I_1(\beta)$, so the theorem contains $I_2(\beta)/I_1(\beta) \le \tanh\beta$ for every β.

- Leading terms at large β: from $I_\nu(\beta) \sim e^\beta(2\pi\beta)^{-1/2}(1 - \frac{4\nu^2-1}{8\beta})$ we get $a_1 \approx 1 - \frac{3}{2\beta}$, while $\tanh\beta \approx 1 - 2e^{-2\beta}$. The lower bound is exponentially smaller than the true tension $\approx \frac{3}{2\beta}$, and there is no contradiction Proved here .
- In numbers: the true $\sigma = -\ln a_1$ is $1.4263,\ 0.8367,\ 0.4185,\ 0.1994$ at $\beta = 1, 2, 4, 8$, and the lower bound of the theorem $-\ln\tanh\beta$ is $0.2723,\ 0.0366,\ 0.00067,\ 2.25\times10^{-7}$ Reading (floating point).
- This lower bound agrees exactly with "the maximum form for containers of side 1", $\sigma_{\max}(1)$, in Table 1 of 25-03. This is because in a container of side 1 only the z of one plaquette is inside, $\mathbb{E}[z_p \mid V] = \tanh(\frac{\beta}{2}\operatorname{tr} V)$, and its maximum is $\tanh\beta$ Proved here (25-03 § 07). The slab theorem has the same form as the bound "containers one unit thick, bounded by the maximum over frames".

Measurements of Wilson loops on four-dimensional slabs (Creutz ratios, comparison with and without twist) are placed in the chapter of the measured record, 25-08.

06

Where it stands relative to earlier literature

The method of the Slab Confinement Theorem (averaging, as a group, the signs on central sheets winding the compact directions) is of the same lineage as the vortex free-energy arguments of Mack–Petkova and Tomboulis–Yaffe, and amounts to bounding directly by the action the free energy of a container $\{p\}\times T^2$ one unit thick Cited (the correspondence of lineage is by comparison with the framework of Kovács–Tomboulis hep-lat/0108017 § 2– § 3). Since the idea is classical, the same bound may already exist in the literature, or as folklore. Two comparisons remain.

- Arguments of Tomboulis–Yaffe type (Commun. Math. Phys. 100 (1985) 313): inequalities that bound a Wilson loop by a ratio of twisted partition functions (as restated in Kanazawa arXiv:0808.3442). We have not checked whether a bound of the same form on a slab is written out explicitly.
- Borgs (Nucl. Phys. B 261 (1985), the area law of spatial Wilson loops in lattice gauge theory at high temperature): recorded as work that showed, by a cluster expansion, an area law for spatial Wilson loops at sufficiently high temperature. This book has not checked the text of the original source (bibliographic data from secondary records). The expected differences from the Slab Confinement Theorem are three —

non-abelian $SU(2)$, every β with N fixed, and an explicit and elementary bound — but the comparison is not done.

Within this book, the Slab Confinement Theorem reappears in the next chapter as a special case (thin insertion) of the insertion identities (25-05 § 06).

07

What cannot be said

(a) The continuum limit. The lower bound of (2) becomes 0 as $a \to 0$. The Slab Confinement Theorem says nothing about a tension uniform in the lattice spacing.

(b) Four-dimensional space. The Slab Confinement Theorem is a statement with the size N of the T^2 fixed, and the lower bound vanishes as $N \to \infty$ (approaching four-dimensional space). It says nothing about confinement in four dimensions. The stage of widening the T^2 is the crossover band (25-09, 26-01).

(c) Improvement to thick vortices. "Bounding F independently of N with a map of thick vortices" in § 04 is a provisional strand; there is no proof, including of the uniform bound on the cost of insertion.

(d) Earlier literature. The comparison with the two items of § 06 is not done. No novelty is claimed for the results of this chapter.

(e) Lean. The machine-checking column is only the algebra of finite sums (the group-average identity, the tanh bound, the one-sheet inequality). The Haar measure of $SU(2)$, the coclosed z, and the topology of sectors and the projection (steps 1, 2 and 5) are outside Lean.

(f) Existence of the limit. The existence of the limiting measure as $M \to \infty$ is not treated in this chapter; the statement is an upper bound at all limit points.

08

The table of five grades and the dependency graph

Table 1 The rows of this chapter ("—" is a row without machine checking)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The sector-summed ensemble is the sum of the sector partition functions; the correspondence of Wilson loops (move 1)	Proved here	§ 03	$SU(2)$, finite lattice	—
2	The vortex sheets S_p are coclosed and do not meet each other (move 2)	Proved here	§ 03	Slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$	—
3	Under a flip W is multiplied by ε_p and the weight by $r_p \in [e^{-2\beta N^2}, e^{2\beta N^2}]$ (move 3)	Proved here	§ 03	Wilson action	—
4	Group average (3) and $\lvert\mathbb{E}[W]\rvert \le t^A$ (4)	Proved here	§ 03 move 4	Sector-summed ensemble	SlabAverage (`SlabAverage.average_bound`), SlabConfine (`SlabConfine.group_average`, `SlabConfine.tanh_bound`, `SlabConfine.one_sheet`) (hypotheses as at the end of § 03)
5	Projection to a fixed sector (5)(6)	Proved here	§ 03 move 5	The group acts on all plaquettes of the plane	—
6	Slab Confinement Theorem: (1) and $\sigma \ge -\ln\tanh(\beta N^2) \ge 2e^{-2\beta N^2}$	Proved here	§ 02, § 03	$SU(2)$, Wilson action, N fixed, lattice spacing fixed, every β	— (only the finite-sum core, row 4)
7	Container form (7), $\sup_\Phi \theta_Q \le \tanh(\beta N^2)^{\ell^2}$ (8)	Proved here	§ 04	Container of side $\ell \times \mathrm{T}^2$	—
8	In the continuum limit ($L = Na$ fixed, $a \to 0$) the lower bound is 0	Proved here	§ 04	That β does not decrease as $a \to 0$	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
9	The cost of a physical vortex is of order $\pi^2\beta$, independent of N	Reading (physics)	25-02	Dilute region, semiclassical	—
10	A map inserting thick vortices gives a lower bound uniform in the lattice spacing	Assumption	Strand: the form (7) of § 04, and 25-05, 25-06	Under ultraviolet control	—
11	Applied to two dimensions, $I_2(\beta)/I_1(\beta) \le \tanh\beta$; agrees with the maximum form for containers of side 1	Proved here	§ 05	Two-dimensional $SU(2)$	—
12	The numbers of § 05	Reading	§ 05	Floating point	—
13	The method is of the lineage of the vortex free energy of Mack–Petkova and Tomboulis–Yaffe	Cited	Comparison with Kovács–Tomboulis § 2– § 3	—	—

The bundles in the machine-checking column are those confirmed, by re-checking on this book's side, to contain no `sorry` and to use axioms forming a subset of the three standard ones. What is guaranteed is the algebra of sums over a finite set of configurations; moves 1, 2 and 5 and the Haar integral are outside Lean (23-13).

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  summed = sum of sector partition fns; W matches   SU(2), finite latti
 2  PRV  vortex sheets S_p coclosed, disjoint              slab
 3  PRV  W times ε_p, weight times r_p, |ln r_p| ≤ 2βN²
 4  PRV  group average (3)(4) ⟸ 2, 3                       Lean average_bound
 5  PRV  to a fixed sector (5)(6) ⟸ 1, 4
 6  PRV  Slab Confinement Theorem (1)(2) ⟸ 5                N fixed, a fixed,
 7  PRV  container form (7)(8) ⟸ 3
 8  PRV  lower bound 0 in continuum limit ⟸ 6              L = Na fixed
 9  RDG  cost of physical vortex π²β (physics)             dilute, semiclassic
10  ASM  uniform bound with thick vortices ⟸ 7, 9           under ultraviolet
11  PRV  to two dimensions; matches maximum form ⟸ 6
12  RDG  numbers of §05 ⟸ 11                                floating point
13  CIT  lineage of vortex free energy
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The premises of the Slab Confinement Theorem are only the rows of the five moves of § 03, and the assumption row (uniform bound with thick vortices) enters as a premise of no row. Its conclusion is "a lower bound uniform in the lattice spacing", and does not contain the quantity of the Slab Confinement Theorem (a lower bound with N fixed).

The machine-checked theorems on the vortex side are in Part 21.

What this chapter can say

What is said	What is not said
Proved here For the $SU(2)$ Wilson action on the slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, with N and the lattice spacing fixed, $\sigma \geq -\ln \tanh(\beta N^2) \geq 2e^{-2\beta N^2}$ at every β and in every twisted sector (the Slab Confinement Theorem)	A tension uniform in the lattice spacing. The tension when the T^2 is widened (the crossover band). Novelty (comparison not done)
Proved here The container form $\sup_\Phi \theta Q \leq \tanh(\beta N^2)^{\ell^2}$; that the lower bound becomes 0 in the continuum limit	A uniform bound on the cost of inserting a thick vortex (the thick-vortex row is an assumption)
Proved here + Reading Consistency in two dimensions ($I_2/I_1 \leq \tanh\beta$, agreement with the maximum form for containers of side 1)	Measurements on four-dimensional slabs (25-08)

Sources and reproduction

Item	Kind	Source / tools
The Slab Confinement Theorem and the five moves of the proof, the container form (7)(8), application to two dimensions	Proved here	Proofs in the text, § 03– § 05
The lineage of the arguments on vortex containers and free energy	Cited	T. G. Kovács, E. T. Tomboulis, arXiv:hep-lat/0108017 (Phys. Rev. D 65 (2002) 074501), § 2– § 3 checked in the text. G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117 and E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313 through that paper's citations (texts not checked)
Restatement of the Tomboulis–Yaffe inequality	Cited	T. Kanazawa, arXiv:0808.3442
The area law of spatial Wilson loops at high temperature	Comparison not done	C. Borgs, Nucl. Phys. B 261 (1985) (bibliographic data from secondary records; text not checked)
Numbers in two dimensions	Reading	Floating-point values of I_2/I_1 and $\tanh\beta$ (the same computation as 25-03 Table 1)
Lean	Lean	SlabConfine (`SlabConfine.group_average`, `SlabConfine.factor_split`, `SlabConfine.tanh_bound`, `SlabConfine.one_sheet`), SlabAverage (`SlabAverage.average_bound`). What the bundles guarantee is only the algebra of finite sums

Words used in this chapter — slab, the vortex sheet S_p, the group of flips, group averaging, the cost of thin vortices, the bias of a container θ_Q, the Slab Confinement Theorem. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-05 Vortex-insertion identities — if the flip of a thin sheet is replaced by a general "bijection that flips the sector", what can be said unconditionally.

Vortex-insertion identities

This chapter is this book's own investigation. The established knowledge it builds on is in 25-03 and 1-08.

What this chapter shows — In 25-04 we obtained slab confinement by averaging over maps that flip thin vortex sheets. In this chapter we list what can be said **unconditionally** when that map is replaced by a general "bijection that flips the sector" T (a map that inserts one vortex). We use only three things: a finite configuration space, positive weights, and a bijection that flips the sign of the sector. From these follow (1) the mean of the insertion ratio ρ is 1, and $\mathbb{E}[\rho\sigma] = -\mathbb{E}[\sigma]$; (2) the bias of the sector is bounded by the fluctuation of the insertion ratio: $4\mathbb{E}[\sigma]^2 \le \mathbf{Var}(\bar{\rho})$; (3) passing to an involution, a bounded form $\mathbb{E}[H] = \mathbb{E}[H\,\tanh(\Delta S/2)]$ and a form averaged over positions; (4) the second-moment form and the Paley–Zygmund form that follow from $\mathbb{E}[H\sqrt{\bar{\rho}}] = 0$. All are elementary identities and Cauchy–Schwarz, and their finite-sum versions are checked in Lean.

Prerequisite chapters — 25-03 (the sector-summed ensemble, the sign of the sector, the bias), 25-04 (slab confinement). From Part 1, 1-08 (probability and measure, Cauchy–Schwarz).

Said first — Every theorem of this chapter is a statement under three conditions — a finite configuration space, positive weights, and a bijection that flips the sector (a finite version of a Haar-preserving map) — and the row of each theorem states those conditions. To pass to the continuous configuration space of $SU(2)$ one must replace the sums by Haar integrals and have the map preserve the measure (or put the Jacobian into the weight), and that part is outside Lean. The results are elementary, and comparison with earlier literature is not done. Nothing about confinement follows from these identities alone. What is missing is written in § 07 and § 08.

The order of this chapter

1. Summary — the balance of a map that inserts one vortex
2. The setting — three conditions
3. The Mean Insertion-Ratio Identity, and the bias estimate by fluctuation

4. Bounded forms — the Bounded Insertion Identity, the fluctuation relation, the average over positions
5. Second-moment forms — the square-root identity, the Paley–Zygmund form, many cells
6. Rederiving slab confinement
7. What it means, and what is missing — Table 1
8. What cannot be said
9. The table of five grades and the dependency graph — Table 2

01

Summary — the balance of a map that inserts one vortex

1. In the sector-summed ensemble (25-03), whether a vortex passes through a region is expressed by a sign $\sigma = \pm 1$. A map T that inserts (or removes) one vortex flips σ.
2. We call the ratio of weights when a vortex is inserted in a configuration x the insertion ratio $\rho(x)$. The smaller ρ is than 1, the more expensive it is to insert a vortex.
3. The mean of the insertion ratio is always exactly 1 (the Mean Insertion-Ratio Identity). This is nothing but a balance equation: "relabelling configurations by a map does not change the total sum of weights". From the same balance also follows $\mathbb{E}[\rho\sigma] = -\mathbb{E}[\sigma]$ Proved here .
4. Combining the two, the bias of the sector $|\mathbb{E}[\sigma]|$ is at most half the square root of the fluctuation (variance) of the insertion ratio (the Bias Estimate by Insertion-Ratio Fluctuation). If one inserts at many positions and averages the insertion ratio, the fluctuation can become small. Averaging over positions plays the role of entropy Proved here .
5. The insertion ratio can take large values (heavy tails), so we also prepare a bounded form that is easy to handle. $\tanh(\Delta S/2)$ always lies in $[-1, 1]$, and $\mathbb{E}[H] = \mathbb{E}[H \tanh(\Delta S/2)]$ holds (the Bounded Insertion Identity). $1 - \mathbb{E}[\tanh]$ is twice the mean acceptance rate of a Monte Carlo move that inserts a vortex (the Insertion Fluctuation Relation Lemma) Proved here .
6. The form using $\sqrt{\rho}$ has a second moment that is always at most 1, so there is no worry about tails (the Square-Root Insertion Identity, the Paley–Zygmund-Type Bias Estimate) Proved here .
7. Taking the flip of a thin vortex sheet (25-04) as the insertion, the slab-confinement theorem comes out as a special case of this chapter (§ 06).

8. These are not a proof of confinement. An area law needs the bias to be small in many regions at once (large deviations) and a bound on the cost of insertion uniform in the lattice spacing (the ultraviolet), and both are outside this chapter.

02

The setting — three conditions

Throughout this chapter we place the following three.

- **Finite configuration-space condition** X (a finite set).
- **Positive-weight condition** $w : X \to (0, \infty)$. The expectation is $\mathbb{E}[f] = \sum_x w(x) f(x) / \sum_x w(x)$.
- **Sector-flipping bijection condition**: a sign of the sector $\sigma : X \to \{\pm 1\}$ and a bijection $T : X \to X$ with $\sigma(Tx) = -\sigma(x)$.

The sector-flipping bijection condition is a finite version of "a map that preserves the Haar measure and flips the sign of the sector of some region" in the sector-summed ensemble (for example, the composite of flipping the z of a vortex sheet with a smooth left multiplication supported inside the region). Being an involution, commuting with one another, and worst-case bounds are not used in § 03. To pass from the finite configuration-space condition to the continuous configuration space $(SU(2)^{\text{links}}$ and coclosed z), one must replace the sums by Haar and counting integrals, and T must preserve the measure (if not, the Jacobian must be put into the weight). The proofs go through with the same relabelling of variables, but that version is outside Lean.

We set the insertion ratio to be

$$\rho(x) = \begin{cases} w(Tx)/w(x) & (\sigma(x) = +1) \\ w(T^{-1}x)/w(x) & (\sigma(x) = -1) \end{cases} \quad (1)$$

The orientation is chosen so that a vortex is inserted by T into configurations with $\sigma = +1$, and configurations with $\sigma = -1$ are taken back by T^{-1}. In terms of the action $S = -\ln w$, $\rho = e^{-\Delta S}$.

03

The Mean Insertion-Ratio Identity, and the bias estimate by fluctuation

Mean Insertion-Ratio Identity

Under the finite configuration-space condition, the positive-weight condition and the sector-flipping bijection condition,

$$\mathbb{E}[\rho] = 1, \qquad \mathbb{E}[\rho\,\sigma] = -\mathbb{E}[\sigma]. \tag{2}$$

Proved here Proof. Set $X_\pm = \{\sigma = \pm 1\}$. T maps X_+ to X_- and X_- to X_+, and since it is a bijection, $T(X_+) = X_-$ and $T^{-1}(X_-) = X_+$. Hence

$$\sum_{x \in X_+} w(x)\rho(x) = \sum_{x \in X_+} w(Tx) = \sum_{y \in X_-} w(y), \qquad \sum_{x \in X_-} w(x)\rho(x) = \sum_{x \in X_-} w(T^{-1}x) =$$

Adding gives $\sum w\rho = \sum w$, and multiplying by σ and adding gives $\sum w\rho\sigma = \sum_{X_-} w - \sum_{X_+} w = -\sum w\sigma$. ∎

Lean InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`) (hypotheses: X is finite, $w > 0$, T is a bijection of X, $\sigma(Tx) = -\sigma(x)$, $\sigma = \pm 1$. Conclusion: $\sum w\rho = \sum w$, $\sum w\rho\sigma = -\sum w\sigma$). No involution, commutativity or measure structure is used.

Bias Estimate by Insertion-Ratio Fluctuation (the bias of the sector is bounded by the fluctuation of the insertion ratio)

Under the finite configuration-space condition and the positive-weight condition, take a sign of the sector σ and a non-empty finite family $T_1, \dots, T_n$ of bijections that flip it (each T_k satisfying the sector-flipping bijection condition), and let $\bar\rho = \frac{1}{n}\sum_k \rho_k$ be the mean insertion ratio. Then

$$4\,\mathbb{E}[\sigma]^2 \;\le\; \mathbb{E}\big[(\bar\rho - 1)^2\big] = \mathrm{Var}(\bar\rho), \qquad \text{that is} \quad |\mathbb{E}[\sigma]| \le \tfrac{1}{2}\sqrt{\mathrm{Var}(\bar\rho)}. \tag{3}$$

Proved here Proof. Applying the Mean Insertion-Ratio Identity to each T_k and averaging gives $\mathbb{E}[\bar\rho] = 1$, $\mathbb{E}[\bar\rho\sigma] = -\mathbb{E}[\sigma]$. Hence $\mathbb{E}[(\bar\rho - 1)\sigma] = -2\mathbb{E}[\sigma]$. From Cauchy–Schwarz and $\sigma^2 = 1$, $4\mathbb{E}[\sigma]^2 \le \mathbb{E}[(\bar\rho - 1)^2]\mathbb{E}[\sigma^2] = \mathrm{Var}(\bar\rho)$. ∎

Lean InsertionIdentity (`Insertion.sector_bias_le`) (hypotheses: X is finite, the index set of the family is finite and non-empty, $w > 0$, each T_k is a bijection with $\sigma(T_k x) = -\sigma(x)$, $\sigma = \pm 1$. Conclusion: $4(\sum w\sigma)^2 \le (\sum w)\sum w(\bar\rho - 1)^2$).

To apply it to containers (25-03), use the Bias Estimate by Insertion-Ratio Fluctuation on the conditional measure with the frame fixed. A bijection that moves only the inside of a container does not change the frame, so $\theta_j(\Phi) = |\mathbb{E}[\sigma_j \mid \Phi]| \le \frac{1}{2}\sqrt{\mathrm{Var}(\bar\rho \mid \Phi)}$ Proved here . The reading is that the factor $\tanh(|\Delta F_j|/2)$ of the Container Identity is bounded from above by the fluctuation of the mean insertion ratio of insertions at many positions inside the container.

04

Bounded forms — tanh and the average over positions

The insertion ratio $\rho = e^{-\Delta S}$ can take large values, so $\mathrm{Var}(\bar{\rho})$ can be large. We therefore switch to a bounded factor. From the T of the sector-flipping bijection condition we build the map oriented by the sector

$$\tau(x) = \begin{cases} Tx & (\sigma(x) = +1) \\ T^{-1}x & (\sigma(x) = -1) \end{cases} \quad (4)$$

and τ is an **involution** ($\tau \circ \tau = \mathrm{id}$) (if $\sigma(x) = +1$, then $\tau x = Tx \in X_-$, so $\tau(\tau x) = T^{-1}Tx = x$; the converse is the same) Proved here . We set $\Delta S = S \circ \tau - S$, $\rho = w \circ \tau / w = e^{-\Delta S}$, and

$$t(x) := \frac{w(x) - w(\tau x)}{w(x) + w(\tau x)} = \tanh \frac{\Delta S(x)}{2} \quad \in (-1, 1) \quad (5)$$

with these definitions.

Bounded Insertion Identity

Under the finite configuration-space condition and the positive-weight condition, if τ is an involution of X and H is a function odd under τ ($H \circ \tau = -H$), then $\mathbb{E}[H] = \mathbb{E}[H\, t]$.

Proved here Proof. Since $1 - t = 2w \circ \tau / (w + w \circ \tau)$, $\mathbb{E}[H] - \mathbb{E}[Ht]$ is proportional to $2 \sum_x g(x) / \sum w$ with $g(x) = w(x)H(x)w(\tau x)/(w(x) + w(\tau x))$. $g(\tau x) = -g(x)$ (H is odd and the rest is symmetric under $x \leftrightarrow \tau x$), so relabelling the sum by τ gives $\sum g = -\sum g$, hence 0. ∎

Lean BoundedInsertion (`Bounded.single`) (hypotheses: X is finite, τ is a bijection with $\tau \circ \tau = \mathrm{id}$, $w > 0$, $H \circ \tau = -H$. Conclusion: $\sum wHt = \sum wH$).

Insertion Fluctuation Relation Lemma (the fluctuation relation and the acceptance rate)

In the setting of the Bounded Insertion Identity, for every real s, $\mathbb{P}(\Delta S = s) = e^{s}\, \mathbb{P}(\Delta S = -s)$. In particular $\mathbb{E}[t] \geq 0$, and

$$1 - \mathbb{E}[t] = 2\,\mathbb{E}\Big[\frac{1}{1 + e^{\Delta S}}\Big] = 2 \times (\text{mean Barker acceptance rate of an insertion move}). \quad (6$$

Proved here Proof. Relabelling gives $\mathbb{E}[g] = \mathbb{E}[(g \circ \tau)\, \rho]$ for any g. Since $\Delta S \circ \tau = -\Delta S$, taking $g = \mathbf{1}\{\Delta S = s\}$ gives $g \circ \tau = \mathbf{1}\{\Delta S = -s\}$, and on the set $\{\Delta S = -s\}$ we have $\rho =$

$e^{-\Delta S} = e^{s}$, which gives the first formula. Pairing s with $-s$ gives $\mathbb{P}(s)\tanh\frac{s}{2} + \mathbb{P}(-s)\tanh\frac{-s}{2} = \mathbb{P}(-s)(e^{s}-1)\tanh\frac{s}{2} \geq 0$, so $\mathbb{E}[t] \geq 0$. Finally $1 - \tanh(\Delta S/2) = 2/(1+e^{\Delta S})$. ∎ This lemma has no Lean check.

How to read (6). A Barker-type move is a Monte Carlo move that "accepts the configuration with a vortex inserted with probability $w(\tau x)/(w(x)+w(\tau x))$". If insertions are almost always refused (the cost is high), $\mathbb{E}[t] \approx 1$; if they pass often, $\mathbb{E}[t]$ moves away from 1. The first formula has the same form as what non-equilibrium statistical mechanics calls Crooks' fluctuation relation (comparison with the original source, and the version driven by dynamics, are in 25-06).

Many cells and the average over positions. For each of mutually separated cells c, take insertion involutions $\tau_{c,p}$ at positions $p \in P$ in the cell (P a non-empty finite set), build $t_{c,p}$ by (5), and set $\bar{t}_c = \frac{1}{|P|}\sum_p t_{c,p}$.

Position-Averaged Insertion Estimate (a bounded estimate averaged over positions)

Under the finite configuration-space condition and the positive-weight condition, assume that each $\tau_{c,p}$ is an involution, H is odd under every $\tau_{c,p}$, and the factors of other cells do not move ($t_{d,q}\circ\tau_{c,p} = t_{d,q}$ if $c \neq d$). Then

$$\mathbb{E}[H] = \mathbb{E}\Big[H\prod_c \bar{t}_c\Big], \qquad |H| \leq 1 \text{ implies } |\mathbb{E}[H]| \leq \mathbb{E}\Big[\prod_c |\bar{t}_c|\Big]. \tag{7}$$

Proved here Proof. First, for $\prod_c t_{c,p_c}$ with one position chosen from each cell, an induction adding cells one at a time gives $\mathbb{E}[H\prod_c t_{c,p_c}] = \mathbb{E}[H]$: at the step adding cell d, the product so far $G = \prod_{c\neq d} t_{c,p_c}$ does not move under τ_{d,p_d}, so HG is odd and the Bounded Insertion Identity applies. Expanding $\prod_c \bar{t}_c$ gives the average over the $|P|^{\#\text{cells}}$ choices, hence the first formula. Taking absolute values with $|H| \leq 1$ gives the second. ∎ Each factor lies in $[-1,1]$, and no moment assumptions are needed.

Lean BoundedInsertion (`Bounded.averaged`, `Bounded.averaged_bound`) (hypotheses: X, the set of cells and the set of positions P are finite, P is non-empty, every $\tau_{c,p}$ is an involution, $w > 0$, H is odd under every $\tau_{c,p}$, $t_{d,q}\circ\tau_{c,p} = t_{d,q}$ if $c \neq d$, and for the latter also $|H| \leq 1$. Conclusion: (7)). One step of the induction is BoundedInsertion (`Bounded.multi`).

05

Second-moment forms

The tanh form is bounded, but it does not capture the exponential average (the free energy) as it is. Another bounded form is $\sqrt{\rho}$.

Square-Root Insertion Identity (the Bhattacharyya form)

Under the finite configuration-space condition and the positive-weight condition, if τ is an involution, $H \circ \tau = -H$ and $\rho = w \circ \tau / w$, then $\mathbb{E}[H\sqrt{\rho}] = 0$.

Proved here Proof. $w\sqrt{\rho} = \sqrt{w \cdot w \circ \tau}$ is symmetric under $x \leftrightarrow \tau x$ and H is antisymmetric, so $\sum_x w(x)H(x)\sqrt{\rho(x)}$ changes sign when relabelled by τ, and is 0. ∎

Lean SecondMoment (`SecondMoment.sqrt_identity`) (hypotheses: X is finite, τ is a bijection with $\tau \circ \tau = \mathrm{id}$, $w > 0$, $H \circ \tau = -H$. Conclusion: $\sum wH\sqrt{w \circ \tau / w} = 0$).

Paley–Zygmund-Type Bias Estimate

Under the finite configuration-space condition and the positive-weight condition, take an involution τ_p for each p of a non-empty finite set P, and let H be odd under every τ_p with $|H| \le 1$. Setting $Y := \frac{1}{|P|}\sum_p \sqrt{\rho_p}$ (a weighted count of insertions that flip the sector),

$$\mathbb{E}[H]^2\,\mathbb{E}[Y^2] \;\le\; \mathbb{E}[Y^2] - \mathbb{E}[Y]^2, \qquad \text{that is} \quad \mathbb{E}[H]^2 \le 1 - \frac{\mathbb{E}[Y]^2}{\mathbb{E}[Y^2]}, \qquad \text{and} \;\; \mathbb{E}[Y^2] \le 1.$$

Proved here Proof. Averaging the Square-Root Insertion Identity gives $\mathbb{E}[HY] = 0$. Setting $C = \mathbb{E}[Y^2]$, $B = \mathbb{E}[Y]$ gives $\mathbb{E}[H(C - BY)] = C\,\mathbb{E}[H]$. From Cauchy–Schwarz and $H^2 \le 1$, $C^2\mathbb{E}[H]^2 \le \mathbb{E}[(C - BY)^2] = C^2 - 2CB^2 + B^2C = C(C - B^2)$. Dividing by $C > 0$ ($\rho_p > 0$) gives the first formula. The last inequality follows from $Y^2 \le \frac{1}{|P|}\sum_p \rho_p$ (Cauchy–Schwarz) and, for involutions, $\mathbb{E}[\rho_p] = \sum_x w(\tau_p x)/\sum w = 1$. ∎

Lean SecondMoment (`SecondMoment.pz_bound`) (hypotheses: X and P are finite, P is non-empty, every τ_p is an involution, $w > 0$, H is odd under every τ_p, $|H| \le 1$. Conclusion: $(\sum wH)^2(\sum wY^2) \le (\sum w)\big((\sum wY^2)(\sum w) - (\sum wY)^2\big)$, which normalised is the first formula of (8)). $\mathbb{E}[Y^2] \le 1$ has no Lean check.

(8) says that the more concentrated Y is in the sense of second moments (the closer $\mathbb{E}[Y^2]$ is to $\mathbb{E}[Y]^2$), the smaller the bias. The second moment is always at most 1, so there is no worry about heavy tails as in the variance form of ρ.

Many cells. We also place a form that handles products of insertion ratios over cells.

Many-Cell Second-Moment Estimate

Under the finite configuration-space condition and the positive-weight condition, with $|H| \le 1$, take a function $r_c : X \to \mathbb{R}$ (an averaged insertion ratio) for each c of

a finite set of cells C, and assume **oddness under insertion** — for every $S \subseteq C$, $\mathbb{E}[H \prod_{c \in S} r_c] = (-1)^{|S|}\mathbb{E}[H]$. Then

$$\mathbb{E}\Big[H \prod_{c \in C}(1 - r_c)\Big] = 2^{|C|}\,\mathbb{E}[H], \qquad 4^{|C|}\,\mathbb{E}[H]^2 \;\le\; \mathbb{E}\Big[\prod_{c \in C}(1 - r_c)^2\Big]. \tag{9}$$

Proved here Proof. Expand $\prod_c(1 - r_c) = \sum_{S \subseteq C}(-1)^{|S|}\prod_{c \in S} r_c$; using the assumption each term becomes $\mathbb{E}[H]$, and there are $2^{|C|}$ terms. Cauchy–Schwarz and $H^2 \le 1$ give the second formula. ∎

Lean MultiCell (`MultiCell.bound`) (hypotheses: X and C are finite, $w > 0$, $|H| \le 1$, the equalities of "oddness under insertion" above for every $S \subseteq C$. Conclusion: the second formula of (9)) and MultiCell (`MultiCell.expand_identity`) (the first formula). Within Lean, "oddness under insertion" is **an equality placed as a hypothesis**. For a single bijection it follows from the pushforward equality $\sum w\,(w \circ T/w)\,F = \sum w\,F \circ T^{-1}$ (MultiCell (`MultiCell.pushforward`), hypotheses: finite, $w > 0$), but that the equality holds when insertions in separate cells are composed depends on the commutativity of the maps and the invariance of the ratios, and is outside Lean.

06

Rederiving slab confinement

Take the thin insertion of 25-04. Let the cells be the plaquettes $p \in S(C)$ of the plane inside the loop, with one position ($|P| = 1$), and let τ_p be the flip of z on the vortex sheet S_p. τ_p is itself an involution, and since it does not change the links it preserves the Haar $\times$ counting measure. $H = W(C)$ is odd under each τ_p, and $t_p = (1 - r_p)/(1 + r_p)$ is determined only by the values on S_p, so it does not move under τ_q for $q \neq p$. From the Position-Averaged Insertion Estimate,

$$|\mathbb{E}[W(C)]| \;\le\; \mathbb{E}\Big[\prod_{p \in S(C)} |t_p|\Big] \;\le\; \tanh(\beta N^2)^A \tag{10}$$

($|t_p| \le \tanh(\beta N^2)$ is the one-sheet inequality of 25-04) Proved here . This is exactly (4) of move 4 of 25-04, and the projection to a fixed sector is the same as move 5 of 25-04. The Slab Confinement Theorem is the special case "thin insertion, one position" of the Position-Averaged Insertion Estimate. (10) on the continuous configuration space is the same argument with sums replaced by integrals, and is outside Lean.

07

What it means, and what is missing

The average over positions is entropy. The $\mathrm{Var}(\bar{\rho})$ of the Bias Estimate by Insertion-Ratio Fluctuation is expected to be of order $\approx VK/n$, where K is the number of

positions, among the n, within reach of each other's correlations and V is the variance of one insertion ratio Reading (physics). Increasing the number of positions n plays the role of entropy, and the decay of correlations between positions (the mass gap) plays the role of keeping K small. For insertions of the smooth fractional-instanton type, $V \sim e^{S_{\text{inst}}}$ is expected, the area of the region needed is proportional to V, and the string tension is of order $\sigma \sim e^{-S_{\text{inst}}}/L^2$, the same form as the formula for a dilute vortex gas Reading (physics, a semiclassical estimate; not a proof).

Table 1 Comparison of the three forms (factor per cell)

Form	Factor	Strengths	Weaknesses
Variance of the insertion ratio (Bias Estimate by Insertion-Ratio Fluctuation)	$1-\bar{\rho}$	Captures the free energy (the exponential average) as it is	Heavy tails. The second moment can be large
tanh (Position-Averaged Insertion Estimate)	$\bar{t}_c$	Bounded. Meaning as an acceptance rate (Insertion Fluctuation Relation Lemma)	Not an exponential average, so it captures the entropy of vortex shapes poorly
$\sqrt{\rho}$ (Paley–Zygmund-Type Bias Estimate)	Y	Second moment always at most 1, and it can capture the entropy of shapes too	Needs bounds on the overlaps between positions and shapes

Collapsing onto a chessboard. If there is reflection positivity in the directions of the plane, chessboard estimates collapse things into a single translation-invariant quantity, $\mathbb{E}[\prod_{c\in\text{inside}} |\bar{t}_c|] \le \Lambda^n$ with $\Lambda := (\mathbb{E}[\prod_{\text{all cells}} |\bar{t}_c|])^{1/\#\text{cells}}$, and if $\Lambda < 1$ an area law follows; this is a strand Assumption (where the evidence is: the treatment of reflection positivity is in 26-04 and 26-05. The same structure is in (13)(21) of Kovács–Tomboulis Cited).

What is missing. (i) The large-deviation stage: $\Lambda < 1$ needs the event that insertions are often refused in all cells at once to be exponentially rare in the number of cells. (ii) The ultraviolet stage: the lower bound on the acceptance rate must be uniform in the lattice spacing, and with thin flips (25-04) it gets worse with N. With smooth insertions, it is known that fixed maps pay an extra cost in the ultraviolet (25-07). (iii) An upper bound on second moments: an upper bound on the V of the Bias Estimate by Insertion-Ratio Fluctuation. This book shows none of the three.

A negative control. Four-dimensional $U(1)$ does not confine at weak coupling, so if the bounds of this chapter are applied there, they must stop working somewhere (for example, V becoming large). This check has not yet been done in this book.

> 08

What cannot be said

(a) Confinement. None of the identities and inequalities of this chapter implies confinement. (i)–(iii) of § 07 remain.

(b) The continuous configuration space. All the theorems are finite versions under the finite configuration-space condition, the positive-weight condition and the sector-flipping bijection condition; the version on the Haar measure of $SU(2)$ is the same argument with the measure preservation of the maps (or the Jacobian) added, but it is outside Lean.

(c) The hypothesis of the Many-Cell Second-Moment Estimate. The equalities of "oddness under insertion" are not shown in this chapter for maps composed of insertions in separate cells (in Lean too they are hypotheses).

(d) Novelty. All are elementary combinations of relabelling of variables and Cauchy–Schwarz, and comparison with earlier literature is not done. The summed ensemble and the container identity lie within the framework of Mack–Petkova, the form bounding Wilson loops by twisted partition functions within Tomboulis–Yaffe, and products on a chessboard within Kovács–Tomboulis. The combination of insertion averaged over positions with Cauchy–Schwarz is recorded as a standard technique in proofs of Mermin–Wagner type, and is said to appear in gauge theory in Chatterjee (arXiv:2602.00436) as well (not yet checked). No novelty is claimed for the results of this chapter.

(e) The estimates of § 07. $\mathrm{Var}(\bar{\rho}) \approx VK/n$, $V \sim e^{S_{\mathrm{inst}}}$ and $\sigma \sim e^{-S_{\mathrm{inst}}}/L^2$ are physical estimates, and none is a proof.

(f) The negative control. That the bounds stop working for four-dimensional $U(1)$ has not yet been checked.

09

The table of five grades and the dependency graph

Table 2 The rows of this chapter (conditions: the finite configuration-space condition, the positive-weight condition, the sector-flipping bijection condition)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	Mean Insertion-Ratio Identity: $\mathbb{E}[\rho] = 1$, $\mathbb{E}[\rho\sigma] = -\mathbb{E}[\sigma]$	Proved here	§ 03	The finite configuration-space, positive-weight and sector-flipping bijection conditions. Uses neither involution nor commutativity	InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`) (hypotheses: the finite configuration-space, positive-weight and sector-flipping bijection conditions, $\sigma = \pm 1$)
2	Bias Estimate by Insertion-Ratio Fluctuation: $4\mathbb{E}[\sigma]^2 \le \mathrm{Var}(\bar{\rho})$	Proved here	§ 03	The finite configuration-space and positive-weight conditions, each T_k satisfies the sector-flipping bijection condition, the family is finite and non-empty	InsertionIdentity (`Insertion.sector_bias_le`) (hypotheses: as left)
3	The τ of (4) is an involution	Proved here	§ 04	Sector-flipping bijection condition	—
4	Bounded Insertion Identity: $\mathbb{E}[H] = \mathbb{E}[H \tanh(\Delta S/2)]$	Proved here	§ 04	The finite configuration-space and positive-weight conditions, τ an involution, H odd	BoundedInsertion (`Bounded.single`) (hypotheses: as left)
5	Insertion Fluctuation Relation Lemma: $\mathbb{E}[t] \ge 0$, the acceptance rate (6)	Proved here	§ 04	The finite configuration-space and positive-weight conditions, τ an involution	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
6	Position-Averaged Insertion Estimate: (7)	Proved here	§ 04	The finite configuration-space and positive-weight conditions, involutions, H odd, factors of other cells do not move, $\lvert H\rvert \le 1$	BoundedInsertion (`Bounded.averaged`, `Bounded.averaged_bound`) (hypotheses: as left)
7	Square-Root Insertion Identity: $\mathbb{E}[H\sqrt{\bar\rho}] = 0$	Proved here	§ 05	The finite configuration-space and positive-weight conditions, involution, H odd	SecondMoment (`SecondMoment.sqrt_identity`) (hypotheses: as left)
8	Paley–Zygmund-Type Bias Estimate: the first formula of (8)	Proved here	§ 05	The finite configuration-space and positive-weight conditions, a family of involutions, H odd, $\lvert H\rvert \le 1$	SecondMoment (`SecondMoment.pz_bound`) (hypotheses: as left)
9	$\mathbb{E}[Y^2] \le 1$	Proved here	§ 05	Involutions	—
10	Many-Cell Second-Moment Estimate: (9)	Proved here	§ 05	The finite configuration-space and positive-weight conditions, $\lvert H\rvert \le 1$, "oddness under insertion" assumed	MultiCell (`MultiCell.bound`, `MultiCell.expand_identity`) (hypotheses: as left. "Oddness under insertion" is a hypothesis in Lean too)
11	(10) of slab confinement via thin insertion	Proved here	§ 06	Slab, N fixed, the sector-summed ensemble	— (the core is the bundle of row 6)
12	$\mathrm{Var}(\bar\rho) \approx VK/n$, $\sigma \sim e^{-S_{\mathrm{inst}}/L^2}$	Reading (physics)	§ 07	Dilute region, semiclassical	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
13	The product structure of the chessboard	Cited	(13)(21) of Kovács–Tomboulis	Under reflection positivity	—
14	Collapsing onto Λ by the chessboard; an area law if $\Lambda < 1$	Assumption	Strand: § 07, 26-04, 26-05	Under reflection positivity in the directions of the plane	—

The bundles in the machine-checking column are those confirmed, by re-checking on this book's side, to contain no `sorry` and to use axioms forming a subset of the three standard ones. What is guaranteed is only the algebra of sums over finite sets (23-13). In every chapter Lean names are written in the form "file (namespace.theorem)". The theorems of the namespace Insertion are in the file InsertionIdentity, and those of Bounded in the file BoundedInsertion.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  E[ρ] = 1, E[ρσ] = −E[σ]                        the three conditions
 2  PRV  4E[σ]² ≤ Var(ρ̄) ⟸ 1                            Lean sector_bias_le
 3  PRV  the oriented τ is an involution                bijection condition
 4  PRV  E[H] = E[H t] ⟸ 3                               Lean Bounded.single
 5  PRV  fluctuation relation, acceptance rate ⟸ 3
 6  PRV  average over positions (7) ⟸ 4                  Lean averaged_bound
 7  PRV  E[H√ρ] = 0 ⟸ 3                                  Lean sqrt_identity
 8  PRV  Paley–Zygmund (8) ⟸ 7                           Lean pz_bound
 9  PRV  E[Y²] ≤ 1 ⟸ 1, 3
10  PRV  many cells (9) (oddness under insertion assumed)   Lean MultiCell.bou
11  PRV  slab confinement via thin insertion ⟸ 6     slab, N fixed
12  RDG  VK/n, e^(−S_inst)/L² (physics) ⟸ 2
13  CIT  product structure of the chessboard
14  ASM  collapse to Λ; area law if Λ < 1 ⟸ 6, 13   under reflection positivi
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The only assumption row is the chessboard row (collapse to Λ; area law if $\Lambda < 1$), and it enters as a premise of no row. The "oddness under insertion" of the many-cell estimate is a hypothesis within the statement of the theorem, and does not come from another row. This chapter has neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Proved here Under the finite configuration-space, positive-weight and sector-flipping bijection conditions, $\mathbb{E}[\rho] = 1$, $\mathbb{E}[\rho\sigma] = -\mathbb{E}[\sigma]$, $4\mathbb{E}[\sigma]^2 \leq \mathrm{Var}(\bar{\rho})$	An upper bound on $\mathrm{Var}(\bar{\rho})$ (a bound on the second moment V)
Proved here For involutions and odd H: the tanh form, the fluctuation relation, the average over positions, the $\sqrt{\rho}$ form, the Paley–Zygmund form	The large-deviation stage (rarity of the event of simultaneous refusal in all cells). The ultraviolet stage (a lower bound on the acceptance rate uniform in the lattice spacing)
Proved here The slab-confinement theorem is a special case of thin insertion	Confinement when the T^2 is widened. The negative control of four-dimensional $U(1)$
Assumption The strand of collapsing onto a single quantity Λ by the chessboard	A proof of $\Lambda < 1$

Sources and reproduction

Item	Kind	Source / tools
The Mean Insertion-Ratio Identity, the Bias Estimate by Insertion-Ratio Fluctuation, the Bounded Insertion Identity, the Insertion Fluctuation Relation Lemma, the Position-Averaged Insertion Estimate, the Square-Root Insertion Identity, the Paley–Zygmund-Type Bias Estimate, the Many-Cell Second-Moment Estimate, (10)	Proved here	Proofs in the text, § 03– § 06. The same content as the theorems of the draft paper "Exact centre-vortex insertion identities in the twist-summed ensemble, and bounded, averaged and driven confinement bounds" (the paper on vortex-insertion identities in A-07)
The summed ensemble, the container identity	Cited (indirect)	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117 (as in 25-03; text not checked)
The product structure of the chessboard (13)(21)	Cited	T. G. Kovács, E. T. Tomboulis, arXiv:hep-lat/0108017 (Phys. Rev. D 65 (2002) 074501)
The combination of the average over positions with Cauchy–Schwarz (an example in gauge theory)	Comparison not done	S. Chatterjee, arXiv:2602.00436 (bibliographic data from a checked record; the correspondence of theorems not compared)

Item	Kind	Source / tools
Lean	Lean	InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`, `Insertion.sector_bias_le`), BoundedInsertion (`Bounded.single`, `Bounded.multi`, `Bounded.averaged`, `Bounded.averaged_bound`), SecondMoment (`SecondMoment.sqrt_identity`, `SecondMoment.pz_bound`), MultiCell (`MultiCell.pushforward`, `MultiCell.expand_identity`, `MultiCell.bound`). What the bundles guarantee is only the algebra of finite sums

Words used in this chapter — insertion map, the insertion ratio ρ, the involution oriented by the sector τ, the tanh factor t, acceptance rate, the average over positions, the $\sqrt{\rho}$ form, the Paley–Zygmund form, oddness under insertion. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-06 Driven insertion and the flow lemma — if, instead of applying the map at once, the vortex is inserted slowly by heat-bath dynamics, the Bounded Insertion Identity takes the form of the work W.

Driven insertion and the flow lemma

This chapter is this book's own investigation. The established knowledge it builds on is in 25-03 and 2-08.

What this chapter shows — In 25-05 we obtained $\mathbb{E}[H] = \mathbb{E}[H \tanh(\Delta S/2)]$ from a map τ that inserts a vortex at once. For thin vortices the cost of insertion ΔS is large, proportional to the area, and the bound is almost useless. In this chapter, instead of inserting at once, we consider "driven insertion", which updates only the inside of the container by heat bath while flipping the couplings of the vortex sheet little by little, and show (1) that on the space of paths the combination of time reversal and sector flip is an involution, with weight ratio e^{-W} (W is the work along the path) (the Crooks relation); (2) that from this follow $\mathbb{E}[H] = \mathbb{E}[H \tanh(W/2)]$ (the driven identity) and the Jarzynski equality; (3) for the probability a that one step of a stationary kernel flips the sector, $\theta \le 1 - a$ (the flow lemma), and flip probability $\le 1 - \theta^2$; and (4) that moving containers simultaneously gives $\mathbb{E}\prod \theta \le \mathbb{P}(\text{none flips})$. Finally we record that for a one-step kernel this guarantee reaches only $\theta_0 \sim \frac{1}{2}$, and cannot be used to test the crossover band (26-01).

Prerequisite chapters — 25-03 (containers, frame, the bias of the sector θ, the Container Identity), 25-05 (the Bounded Insertion Identity, the Insertion Fluctuation Relation Lemma, the Position-Averaged Insertion Estimate), 26-01 (the correct form of the band $\ln(1/\theta(L,L)) \ge cL^2$). From Part 2, 2-08 (Gibbs measures, free energy), and from Part 5, 5-07 (heat bath, detailed balance).

Said first — All theorems are statements under "a finite configuration space, positive weights, and kernels that move only the inside of the container", and each row states those conditions. Passing to the continuous configuration space of $SU(2)$ is the same as in 25-05 § 02 (replace the sums by Haar integrals) and is outside Lean. The Crooks relation and the Jarzynski equality are known results of non-equilibrium statistical mechanics; here we write out their proofs in order to apply them to the sector-summed ensemble of the lattice. Confinement follows from none of the inequalities of this chapter.

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The order of this chapter
```

1. Summary — inserting a vortex slowly
2. The setting — the driving procedure and the space of paths
3. The Crooks involution and the driven identity — the Jarzynski Equality, the acceptance rate, many cells
4. Two limits — inserting at once, inserting quasi-statically
5. The Cut-Measure Formula
6. The Flip-Probability Lemma and the Flow Lemma
7. The Simultaneous-Flip Estimate, and the limits of this guarantee (Table 1)
8. What cannot be said
9. The table of five grades and the dependency graph — Table 2

01

Summary — inserting a vortex slowly

1. The insertion of 25-05 was an operation that flips at once the signs of the couplings on a vortex sheet. Its cost ΔS is, for a thin vortex sheet, proportional to the number of plaquettes on the sheet (the area N_T^2 of the T^2), and the factor of the bound $\tanh(\Delta S/2)$ is almost 1.
2. So we split the coupling into m steps and flip it little by little, updating only the inside of the container by heat bath at each step. The surrounding fluctuations relax to fit the new coupling, so the work paid W goes down.
3. On the space of paths, the map "follow time backwards and flip the sector" is an involution, and the weight ratio is e^{-W} (the Crooks Relation) Cited . Applying the Bounded Insertion Identity of 25-05 to this involution gives $\mathbb{E}[H] = \mathbb{E}[H\,\tanh(W/2)]$ Proved here .
4. From the same relation, independently of the number of steps m of the procedure, $\mathbb{E}[e^{-W}\,|\,\text{frame}] = Z_-/Z_+$ (the Jarzynski equality) Cited . With 0 steps, $W = \Delta S$ and we are back to 25-05.
5. For any stationary kernel that flips the sector, the probability a that the sector flips in one step is at most $1-\theta$. In other words $\theta \le 1-a$ (the flow lemma) Proved here . There is also a weaker form, $a \le 1-\theta^2$ Proved here .
6. With the frame fixed the containers are independent, so running kernels in all containers simultaneously gives $\mathbb{E}\prod_t \theta_t \le \mathbb{P}(\text{no container flips})$ Proved here . The right-hand side is a quantity measurable by Monte Carlo, and it estimates an upper bound on the Wilson loop in the correct direction.

7. However, the form of the band needs $a \geq 1 - e^{-cL^{\not 2}}$, that is, a kernel that brings the overlap of the forward and reverse paths to 1 exponentially in the area; the a of the one-step kernels measured is about 0.6–0.73 (acceptance $\min(1, e^{-W})$), and what this guarantee can say goes only up to $\theta \leq \theta_0$ (a constant less than 1) Reading . It cannot be used to test the form of the band, $\theta \leq e^{-cL^2}$. Its proper use is a lower bound on the slab tension with the T^2 fixed.

02

The setting — the driving procedure and the space of paths

We use the sector-summed ensemble of 25-03 and 25-04. The weight on the $SU(2)$ link variables U and the ± 1 plaquette variables z (coclosed) is $w(U, z) = \exp\left(\frac{\beta}{2} \sum_p z_p \operatorname{tr} U_p\right)$. For a plaquette p of the $(3,4)$ plane of the slab $T^2_{N_T} \times (\mathbb{Z}/M)^2$, flipping all the z of the vortex sheet $S_p = \{p\} \times T^2$ flips the sign of the sector seen by a Wilson loop enclosing p (25-04). We take a container $D_p \times T^2$ around p (D_p is a disc in the $(3,4)$ plane), and call the variables outside the container the **frame** Φ.

Throughout this chapter we place the following three.

- **Finite configuration-space condition** X. The continuous case is as noted in 25-05 § 02.
- **Positive-weight condition** $w = e^{-S} > 0$.
- **Kernel moving only inside the container**: the kernels K below update only the variables inside the container and do not change the frame.

Interpolating the coupling. For $\lambda \in [0,1]$, let S_λ be the action in which only the couplings of the plaquettes on S_p are multiplied by $1 - 2\lambda$. $S_0 = S$, and $S_1(x) = S(Rx)$ (R is the flip of z on S_p). S_λ is gauge invariant.

Kernels. Let K_λ be a heat-bath transition kernel that moves only the links inside the container and is reversible (detailed balance) with respect to e^{-S_λ}:

$$e^{-S_\lambda(x)} K_\lambda(x, y) = e^{-S_\lambda(y)} K_\lambda(y, x). \quad (1)$$

The forward procedure. $\lambda_k = k/m$ ($k = 0, \dots, m$). Starting from a configuration x_0 in sector $+$, at each step $k = 1, \dots, m$: "change the coupling $\lambda_{k-1} \to \lambda_k$, add the resulting change of action to the work, then update $x_{k-1} \to x_k$ by K_{λ_k}". The work is

$$W = \sum_{k=1}^{m} \left[S_{\lambda_k}(x_{k-1}) - S_{\lambda_{k-1}}(x_{k-1}) \right]. \quad (2)$$

Rx_m, obtained by finally applying R, is a configuration in sector $-$. From configurations in sector $-$, the same is done by the reverse procedure mapped by R (λ going $1 \to 0$). If $m = 0$ there is no update, and $W = S_1(x_0) - S_0(x_0) = \Delta S$, the single insertion of 25-05.

The space of paths. Let $\Omega = \{(s, x_0, \dots, x_m)\}$ ($s = \pm$ is the sector of the starting point), and set the weight of a path starting in sector $+$ to be

$$\hat{W}(+, x_0, \dots, x_m) = e^{-S_0(x_0)} \prod_{k=1}^{m} K_{\lambda_k}(x_{k-1}, x_k), \tag{3}$$

The weight of a path starting in sector $-$ is defined by the product of the same form for the reverse procedure. The rows of the kernels sum to 1, so projecting the measure on Ω to the starting point gives back the original measure. Hence the expectation of a quantity determined by the starting point alone does not change when computed on Ω Proved here .

The involution. $\iota(+, x_0, \dots, x_m) = (-, Rx_m, \dots, Rx_0)$: follow the path backwards and flip the sector (and conversely for paths starting at $-$). $\iota \circ \iota = \mathrm{id}$. By (1), paths of positive weight are mapped to paths of positive weight.

03

The Crooks involution and the driven identity

Crooks Relation

Under the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container and (1), for every path ω of positive weight,

$$\frac{\hat{W}(\iota\omega)}{\hat{W}(\omega)} = e^{-W(\omega)}, \qquad W \circ \iota = -W. \tag{4}$$

Cited A relation of this form is due to Crooks (source table); here we write out the standard computation adapted to the lattice procedure. Proof. Let $\omega = (+, x_0, \dots, x_m)$. The weight of the reversed path, brought back to the coordinates of $+$ using that $S\lambda$ maps to $S_{1-\lambda}$ under R, is $e^{-S_1(x_m)} \prod_{k=1}^{m} K_{\lambda_k}(x_k, x_{k-1})$. Replacing each factor by $K_{\lambda_k}(x_{k-1}, x_k)\, e^{S_{\lambda_k}(x_k) - S_{\lambda_k}(x_{k-1})}$ using (1), the exponent of the ratio is

$$S_0(x_0) - S_1(x_m) + \sum_{k=1}^{m} \left[S_{\lambda_k}(x_k) - S_{\lambda_k}(x_{k-1}) \right] = \sum_{k=1}^{m} S_{\lambda_{k-1}}(x_{k-1}) - \sum_{k=1}^{m} S_{\lambda_k}(x_{k-1}) =$$

(shift the summation index by one, and let $S_0(x_0)$ cancel against the $k = 1$ term and $-S_1(x_m)$ against the $k = m$ term). Computing (2) along the reversed path, the same terms appear with opposite signs, so $W \circ \iota = -W$. ∎

The Crooks Relation says that "the involution with weight ratio $e^{-\Delta S}$" of (5) of 25-05 has been replaced, on the space of paths, by "an involution with weight ratio e^{-W}". So the Bounded Insertion Identity of 25-05 can be used as it is.

Driven Insertion Identity (the driven identity)

Under the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container and (1), if $H = h(\Phi)\,\sigma$ (h is a function of the frame alone, σ is the sign of the sector of the container), then

$$\mathbb{E}[H] = \mathbb{E}_\Omega\big[H \tanh(W/2)\big]. \quad (5)$$

Proved here Proof. Regard H as a function on Ω evaluated at the starting point. The kernels and R do not change the frame, so h is invariant under ι and σ flips, giving $H \circ \iota = -H$. For the weight $\hat{W}$ and the involution ι on Ω, the factor of (5) of 25-05 is $(\hat{W} - \hat{W} \circ \iota)/(\hat{W} + \hat{W} \circ \iota) = (1 - e^{-W})/(1 + e^{-W}) = \tanh(W/2)$ by (4). From the Bounded Insertion Identity of 25-05, $\mathbb{E}_\Omega[H] = \mathbb{E}_\Omega[H \tanh(W/2)]$. The left-hand side is determined by the starting point alone, so it equals $\mathbb{E}[H]$ (§ 02). ∎

Lean: the core is BoundedInsertion (`Bounded.single`) (hypotheses: a finite set, an involution, $w > 0$, H odd. Conclusion: $\sum wHt = \sum wH$) applied to the finite space of paths Ω (restricted to paths of positive weight). The construction of the space of paths and the Crooks Relation are outside Lean.

Jarzynski Equality

Under the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container and (1), independently of the number of steps m,

$$\mathbb{E}\big[e^{-W} \,\big|\, \Phi,\ \sigma = +\big] = \frac{Z_-(\Phi)}{Z_+(\Phi)}, \quad (6)$$

Here $Z_\pm(\Phi)$ is the partition function of sector $\pm$ with the frame Φ fixed.

Cited Due to Jarzynski (source table). Proof. From (4), $\sum_\omega \hat{W}(\omega) e^{-W(\omega)} = \sum_\omega \hat{W}(\iota\omega)$. The right-hand side is the total weight of the paths starting in sector $-$, which is $Z_-(\Phi)$ since the rows of the kernels sum to 1. Dividing the left-hand side by the total weight $Z_+(\Phi)$ of paths in sector $+$ gives (6). ∎

(6) is a quantity that does not depend on the speed of the procedure (the number of steps m), and can be used as a numerical check. On the other hand, the average of e^{-W} is dominated by rare small W, so with few samples the estimate fluctuates strongly (that is why some rows of the table in 25-08 have negative $-\ln\langle e^{-W}\rangle$).

Driven Insertion Acceptance-Rate Corollary (the acceptance rate)

In the setting of the Driven Insertion Identity, $1-\mathbb{E}_\Omega[\tanh(W/2)] = 2\,\mathbb{E}_\Omega\big[1/(1+e^{W})\big]$. The right-hand side is twice the mean acceptance rate of a move that accepts the endpoint of a driven path by the Barker-type rule.

Proved here One just takes the expectation of $1-\tanh(W/2) = 2/(1+e^{W})$. It is the Insertion Fluctuation Relation Lemma of 25-05 with ΔS replaced by W.

Many cells. For each of mutually separated cells c (containers $D_c \times T^2$), choose a position P_c uniformly and independently from a set of positions P (a non-empty finite set), and perform driven insertion at the chosen position. The path of each cell reads only its own container and its surroundings in the starting configuration.

Many-Cell Driven Estimate

Under the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container and (1), if the containers are separated from each other (no plaquette straddles the insides of two containers), $H = h(\Phi)\prod_c \sigma_c$ and $|H|\le 1$, then

$$|\mathbb{E}[H]| \;\le\; \mathbb{E}\Big[\prod_c \frac{1}{|P|}\sum_{p\in P}\mathbb{E}\big[\,|\tanh(W_{c,p}/2)|\;\big|\;x_0\big]\Big]. \tag{7}$$

Proved here Proof. Build the enlarged space consisting of the starting point x_0, the choice of positions (P_c) and the path of each cell. The involution ι_d of cell d follows the path of cell d backwards and replaces the inside of container d of the starting point by the flip of the endpoint of the path. Since the containers are separated, the weight with the frame fixed splits into a product over containers, and ι_d does not change the path weights or works of other cells. So the weight ratio of ι_d is e^{-W_d}, and the factors $\tanh(W_c/2)$ of the other cells ($c \neq d$) do not move under ι_d. The proof of the Position-Averaged Insertion Estimate of 25-05 (the induction adding cells one at a time) goes through as it is, giving $\mathbb{E}[H] = \mathbb{E}[H\prod_c \tanh(W_{c,P_c}/2)]$. Taking absolute values, and using that with the starting point fixed the paths of the cells and the choices of positions are independent, gives (7). ∎

Unlike the Position-Averaged Insertion Estimate of 25-05, (7) has the absolute value inside the average over positions. It is weaker by the amount that it does not use cancellation of signs between positions.

04

Two limits — inserting at once, inserting quasi-statically

Zero steps. At $m = 0$, $W = \Delta S$, and the Driven Insertion Identity is the Bounded Insertion Identity of 25-05 Proved here . For thin vortices ΔS is proportional to the area of the T^2, and $\tanh(W/2)$ is almost 1.

The quasi-static limit. As $m \to \infty$ (driving slowly), W is expected to approach the conditional free energy of the container with the frame fixed, $\Delta F_D(\Phi) = \ln\big(Z_+(\Phi)/Z_-(\Phi)\big)$ Reading (physics; the standard estimate of thermodynamic integration, which requires the kernel to mix well enough at each λ. Not proved in this book). The factor of the Driven Insertion Identity then becomes $\tanh(\Delta F_D/2) = \theta_D(\Phi)$, and we return to the Container Identity of 25-03 (the Mack–Petkova identity). The reading is that the length m of the procedure connects the crude bound of inserting at once with the exact conditional free energy.

The numerical picture. For $SU(2)$, $\beta = 2.3$, the slab $2 \times 2 \times 8 \times 8$, twist $n_{12} = 1$, disc radius 2, 6 configurations $\times$ 4 positions, the mean work at $m = 0, 4, 16$ steps was 10.7, 6.5, 2.9, and the mean of $\tanh(W/2)$ was 0.990, 0.968, 0.688 Reading . Increasing the number of steps lowers the work and moves the factor away from 1. The records for other β, N_T and container radii are collected in 25-08. If the container is smaller than the vortex, the work does not come down fully even as the number of steps increases (same table).

05

The formula of the cut measure

Inside the container $D \times T^2$, let μ_{cut} be the measure (with the frame Φ fixed) in which only the plaquette terms on the flip sheet $S = \{p_0\} \times T^2$ are removed from the action, and set $X := \sum_{p \in S} z_p \operatorname{tr} U_p$ (z is the representative of sector $+$). The weight on the sheet is $e^{(\beta/2) z_p \operatorname{tr} U_p}$, and in sector $-$ the z on S are flipped, so

Cut-Measure Formula

$$\frac{Z_{\pm}(\Phi)}{Z_{\mathrm{cut}}(\Phi)} = \mathbb{E}_{\mathrm{cut}}\left[e^{\pm\beta X/2}\right], \qquad \theta(\Phi) = \frac{\left|\mathbb{E}_{\mathrm{cut}}[\sinh(\beta X/2)]\right|}{\mathbb{E}_{\mathrm{cut}}[\cosh(\beta X/2)]}. \tag{8}$$

Proved here The first formula is from the definition; the second just puts the first into $\theta = |Z_+ - Z_-|/(Z_+ + Z_-)$ and uses $e^a \mp e^{-a} = 2\sinh a,\ 2\cosh a$.

Lean FlipLemma (`FlipLemma.cut_formula`) (hypotheses: real-valued weights v on a finite set (non-negativity not used either), real-valued X, a real b. Conclusion: $\sum v e^{bX/2} - \sum v e^{-bX/2} = 2\sum v \sinh(bX/2)$, and the equality in sum form for cosh). What is guaranteed is only the algebra of sums.

μ_{cut} has no sectors. $\theta(\Phi)$ can be read as "the strength with which the information of the frame reaches the sheet, crossing the cut measure over the size of the container". Driven insertion can also be set up in the symmetric form $+ \to \text{cut measure} \leftarrow -$ (compatible with the involution of the Crooks Relation).

06

The flip-probability lemma and the flow lemma

Let μ_Φ be the conditional measure with the frame Φ fixed, $\sigma \in \{\pm 1\}$ the sign of the sector of the container, $m_\Phi = \mathbb{E}[\sigma \mid \Phi]$, $\theta(\Phi) = |m_\Phi|$. Let K be a Markov kernel that moves only the inside of the container, and $(X, X') \sim \mu_\Phi \otimes K$ (X drawn from μ_Φ and advanced one step by K gives X').

Flip-Probability Lemma

If K is **stationary** with respect to μ_Φ ($\sum_x \mu_\Phi(x)K(x,y) = \mu_\Phi(y)$), then

$$\mathbb{P}\big(\sigma(X') \neq \sigma(X)\big) \ \le\ 1 - m_\Phi^2 = 1 - \theta(\Phi)^2. \tag{9}$$

Proved here Proof. Since $\sigma = \pm 1$, $\mathbb{P}(\sigma(X') \neq \sigma(X)) = \sum_{x,y} \mu_\Phi(x)K(x,y)(\sigma(x) - \sigma(y))^2/4$. Bound it above by $(a-b)^2 \le 2(a-m)^2 + 2(b-m)^2$, and use that the rows sum to 1 and stationarity to turn both sums into $\sum_x \mu_\Phi(x)(\sigma(x) - m)^2 = 1 - m^2$, giving $\frac{1}{2}(1-m^2) + \frac{1}{2}(1-m^2)$. ∎ Reversibility is not used.

Lean FlipLemma (`FlipLemma.flip_le`) (hypotheses: a finite set, $w \ge 0$, $\sum w = 1$, $K \ge 0$, rows sum to 1, stationarity $\sum_x w_x K_{xy} = w_y$, $\sigma^2 = 1$. Conclusion: $\sum_{x,y} w_x K_{xy}(\sigma_x - \sigma_y)^2/4 \le 1 - (\sum w\sigma)^2$).

Flip-Probability Corollary for Positive Kernels

If K is reversible with respect to μ_Φ and positive as an operator on $L^2(\mu_\Phi)$ ($\langle f, Kf\rangle \geq 0$), the right-hand side of (9) drops to $(1-\theta^2)/2$.

Proved here Proof. $\mathbb{P}(\text{flip}) = (1 - \langle\sigma, K\sigma\rangle)/2$ (inner product in μ_Φ). Setting $\tilde\sigma = \sigma - m$, since the rows of K sum to 1 and it is reversible, $\langle\sigma, K\sigma\rangle = \langle\tilde\sigma, K\tilde\sigma\rangle + m^2 \geq m^2$. ∎ This corollary has no Lean check.

The next form is sharper, and is the one mainly used in this chapter.

Flow Lemma

In the setting of the Flip-Probability Lemma (stationary K), letting $a(\Phi)$ be the probability that the sector flips in one step,

$$a(\Phi) \;\leq\; 1 - \theta(\Phi), \qquad \text{that is} \qquad \theta(\Phi) \;\leq\; 1 - a(\Phi). \tag{10}$$

Proved here Proof. Take the flow $A = \mathbb{P}(\sigma(X) = +, \sigma(X') = -)$ from $+ \to -$ and the flow B from $- \to +$. By stationarity the distribution of the sector of X' is the same as that of X, so $A = B$. Also $A \leq \mathbb{P}(\sigma(X) = +) = (1+m)/2$, $B \leq (1-m)/2$. Hence $a = A + B = 2A \leq 2\min\big((1+m)/2, (1-m)/2\big) = 1 - |m|$. ∎ Since $\theta \leq 1$, $1 - \theta \leq 1 - \theta^2$, and (10) is a sharper form than (9).

Lean FluxLemma (`FluxLemma.flip_le_one_sub_abs`) (hypotheses: a finite set, $w \geq 0$, $\sum w = 1$, $K \geq 0$, rows sum to 1, stationarity, the values of σ are 1 or -1. Conclusion: $\sum_{x,y} w_x K_{xy} (\sigma_x - \sigma_y)^2/4 \leq 1 - |\sum w\sigma|$).

How to use it. If one can build a stationary kernel that flips the sector with flip probability at least c, then $\theta \leq 1 - c$. A kernel that proposes a driven insertion and accepts the endpoint with probability $\min(1, e^{-W})$ is reversible (hence stationary) with respect to μ_Φ by the Crooks Relation, and its flip probability is the acceptance rate. The acceptance rate is at least $\mathbb{P}(W \leq w_0)\, e^{-w_0}$, so there is no need to control the average of e^{-W}; it is enough that "cheap work occurs with positive probability" Proved here .

07

Simultaneous flips, and the limits of this guarantee

Take a finite set S of mutually separated containers, and let $\theta_t(\Phi)$ be the bias of the sector of each container $t \in S$ (the frame Φ is the variables outside all containers).

Simultaneous-Flip Estimate

Under the finite configuration-space condition and the positive-weight condition, if the containers are separated from each other, and for each container t one takes a stationary kernel K_t that moves only the inside of that container and moves all containers one step simultaneously, then

$$\mathbb{E}\Big[\prod_{t\in S}\theta_t(\Phi)\Big] \;\le\; \mathbb{P}\big(S:\text{no container flips}\big). \tag{11}$$

Proved here Proof. With the frame fixed, the insides of the containers are conditionally independent (the conditional independence of 25-03), and the kernels also move independently per container, so $\mathbb{P}(\text{none flips}\,|\,\Phi)=\prod_t(1-a_t(\Phi))$. By the Flow Lemma each factor is at least $\theta_t(\Phi)$. Take expectations. ∎

The container identity of 25-03 had the form $|\langle W(C)\rangle|\le\mathbb{E}\prod_t\theta_t$ (for the containers inside C). Together with (11), an upper bound on the Wilson loop is bounded **in the correct direction** by a quantity measurable by Monte Carlo, $\mathbb{P}(\text{none flips})$ Proved here .

Limits (an important caution). What (10) gives is $\theta\le 1-a$. The flip probability a of a one-step kernel (a kernel that proposes one driven insertion and decides whether to accept it) is about 0.6–0.73 in the range measured with kernels of acceptance $\min(1,e^{-W})$ Computed (values recomputed from the raw W of Tables 1 and 2 of 25-08. The acceptance-rate columns of the tables are means of the Barker-type $1/(1+e^{W})$, and their ceiling $1/(1+e^{\Delta F})\le\frac12$ belongs to the acceptance rule, not to the kernel Paper Lean). So what can be said for containers of size L is $\theta\le\theta_0$ (with θ_0 a constant less than 1) at best. Even if this held uniformly in L, all that follows is

$$\sigma \;\ge\; \frac{\ln(1/\theta_0)}{L^2} \;\longrightarrow\; 0 \qquad (L\to\infty) \tag{12}$$

and this is exactly the "weak form" of 26-01 (a form that $U(1)$ also satisfies) Proved here . To show the correct form of the band $\theta(L,L)\le e^{-cL^2}$ by (10), one needs a kernel with $a\ge 1-e^{-cL^2}$, that is, a kernel that flips the sector almost surely Proved here (a rewriting of (10)). The acceptance rate is sandwiched by the overlap O of the measures of the forward and reverse paths ($e^{-\Delta F}O\le a_F\le\min(e^{-\Delta F},O)$, and for the mixed kernel $(1-\theta)O\le a\le 1-\theta$) Paper Lean (LatticeGaugeAcceptanceSandwich (overlap_sandwich, mixture_sandwich); the assumption is the Crooks-type relation $P_F e^{-W}=e^{-\Delta F}P_R$), and $a\ge 1-e^{-cL^2}$ demands $O\ge 1-e^{-cL^2}$, in terms of dissipation of the order $D\lesssim e^{-2cL^2}$ Reading (Pinsker's inequality read in reverse). There is no prospect of building that with a one-step kernel.

Therefore, **the guarantee from simultaneous flips cannot be used as a numerical test of the crossover band.** The record measured with this tool (25-08 § 06) cannot be read as a numerical picture of the band either. Its proper uses are a lower bound on the slab tension with the size L of the T^2 fixed, and an estimate from above of the chessboard density κ_{cb} (26-05).

Table 1 What the bounds of this chapter can and cannot say (conditions: the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container)

Bound	Form	What can be said	What cannot be said
Driven identity (the Driven Insertion Identity, the Many-Cell Driven Estimate)	$\mathbb{E}[H] = \mathbb{E}[H \tanh(W/2)]$	Increasing the number of steps can move the factor away from 1 (numerics)	That the upper bound on the factor is uniform in L
Jarzynski Equality	$\mathbb{E}[e^{-W}] = Z_-/Z_+$	A numerical check	Precision with few samples (dominated by rare events)
Flip probability (the Flip-Probability Lemma)	$a \le 1 - \theta^2$	$\theta \le \sqrt{1-a}$	— (weaker than the Flow Lemma)
Flow Lemma	$\theta \le 1 - a$	A lower bound on the slab tension with the T^2 fixed	For a one-step kernel only up to $\theta_0 \sim \frac{1}{2}$. The form of the band e^{-cL^2} does not follow
Simultaneous-Flip Estimate	$\mathbb{E}\prod\theta \le \mathbb{P}(\text{no flip})$	An estimator in the correct direction for an upper bound on the Wilson loop	A numerical test of the band

08

What cannot be said

(a) Confinement. None of the identities and inequalities of this chapter implies confinement. One would need the factors of the Driven Insertion Identity and the Many-Cell Driven Estimate to stay away from 1 uniformly in L and the lattice spacing, or a kernel with a close to 1, and this book shows neither.

(b) The quasi-static limit. That $W \to \Delta F_D$ as $m \to \infty$ is the estimate of § 04; this book has no proof of it, including bounds on how the kernels mix.

(c) The continuous configuration space. All theorems are finite versions under the finite configuration-space condition. The version on the Haar measure of $SU(2)$ needs the reversibility of the heat-bath kernels and the measure on the space of paths to be rewritten for the continuous case, and is outside Lean.

(d) The limit of one-step kernels. The a of the kernels measured (0.6–0.73 with acceptance $\min(1, e^{-W})$) is a reading of measurements, not a theorem. The conclusion that the form of the band is not reached rests not on this reading but on the rewriting

of (10) that $a \geq 1 - e^{-cL^{\angle}}$ is needed (the row "only the weak form follows" of Table 2). That the acceptance rates of the tables of 25-08 stay below $\frac{1}{2}$ is the ceiling of the Barker-type rule. Conversely, if there were a kernel with a close to 1, (10) could be used as it is, but no construction of one is known.

(e) Novelty. The Crooks Relation and the Jarzynski Equality are known, and the application to the sector-summed ensemble, the Flip-Probability Lemma, the Flow Lemma and the Simultaneous-Flip Estimate are elementary. Techniques that raise the acceptance rate with non-equilibrium proposals that flip a sector are known in the Monte Carlo literature as tempered transitions and non-equilibrium candidate methods (not yet checked). No novelty is claimed for the results of this chapter.

09

The table of five grades and the dependency graph

Table 2 The rows of this chapter (conditions: the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The marginal of the starting point on the space of paths is the original measure	Proved here	§ 02	The finite configuration-space and positive-weight conditions, the kernel moving only inside the container, rows of the kernels sum to 1	—
2	Crooks Relation: $\hat{W} \circ \iota / \hat{W} = e^{-W}$, $W \circ \iota = -W$	Cited	§ 03, Crooks	The finite configuration-space and positive-weight conditions, the kernel moving only inside the container, detailed balance (1)	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
3	Driven Insertion Identity: $\mathbb{E}[H] = \mathbb{E}_\Omega[H \tanh(W/2)]$	Proved here	§ 03	The finite configuration-space and positive-weight conditions, the kernel moving only inside the container, (1), $H = h(\Phi)\sigma$	BoundedInsertion (`Bounded.single`) applied to Ω (hypotheses: finite, involution, $w > 0$, H odd. The construction of Ω is outside Lean)
4	Jarzynski Equality (independent of the number of steps)	Cited	§ 03, Jarzynski	The finite configuration-space and positive-weight conditions, the kernel moving only inside the container, (1), frame fixed	—
5	Driven Insertion Acceptance-Rate Corollary: $1 - \mathbb{E}[\tanh(W/2)] = 2\times$ acceptance rate	Proved here	§ 03	The setting of row 3	—
6	Many-Cell Driven Estimate: (7)	Proved here	§ 03	Containers separated, $\lvert H\rvert \le 1$, positions chosen independently and uniformly	— (the type of induction is BoundedInsertion (`Bounded.averaged`) of 25-05)
7	$W \to \Delta F_D(\Phi)$ as $m \to \infty$	Reading (physics)	§ 04	The kernels mix well enough at each λ	—
8	The work goes down with the number of steps ($\beta = 2.3$, $2 \times 2 \times 8 \times 8$)	Reading	§ 04, 25-08	$SU(2)$, 6 configurations $\times$ 4 positions	—
9	Cut-Measure Formula (8)	Proved here	§ 05	Frame fixed, terms of the flip sheet removed	FlipLemma (`FlipLemma.cut_formula`) (hypotheses: a finite set, real-valued v, X, b)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
10	Flip-Probability Lemma: flip probability $\leq \frac{1-\theta}{2}$	Proved here	§ 06	Stationary kernels (reversibility not needed)	FlipLemma (`FlipLemma.flip_le`) (hypotheses: $w \geq 0$, $\sum w = 1$, $K \geq 0$, rows sum to 1, stationary, $\sigma^2 = 1$)
11	Flip-Probability Corollary for Positive Kernels: $(1 - \theta^2)/2$ for positive kernels	Proved here	§ 06	Reversible and a positive operator	—
12	Flow Lemma: $\theta \leq 1 - a$	Proved here	§ 06	Stationary kernels (reversibility not needed)	FluxLemma (`FluxLemma.flip_le_one_sub_abs`) (hypotheses: $w \geq 0$, $\sum w = 1$, $K \geq 0$, rows sum to 1, stationary, $\sigma = \pm 1$)
13	The kernel with acceptance $\min(1, e^{-W})$ is stationary, acceptance rate $\geq \mathbb{P}(W \leq w_0) e^{-w_0}$	Proved here	§ 06	The setting of row 2	—
14	Simultaneous-Flip Estimate: $\mathbb{E}\prod \theta_t \leq \mathbb{P}(\text{no flip})$	Proved here	§ 07	Containers separated, a stationary kernel per container	— (the core is row 12)
15	The a of a one-step kernel is 0.6–0.73 in the range measured (acceptance $\min(1, e^{-W})$; the Barker-type columns of the tables are below the ceiling $\frac{1}{2}$ of the rule)	Reading	§ 07, 25-08	The range of β, N_T and containers measured	—
16	What follows from (10) is $\sigma \geq \ln(1/\theta_0)/L^2$ (the weak form). The form of the band needs $a \geq 1 - e^{-cL^2}$	Proved here	§ 07	When only $\theta \leq 1 - a$ is used	—

The bundles in the machine-checking column are those confirmed, by re-checking on this book's side, to contain no `sorry` and to use axioms forming a subset of the three standard ones. What is guaranteed is only the algebra of sums over finite sets (23-13).

`flip_le` and `cut_formula` are in the file FlipLemma, and `flip_le_one_sub_abs` is in the file FluxLemma.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  marginal of start on paths = original measure   rows sum to 1
 2  CIT  weight ratio e^(−W), W∘ι = −W                    detailed balance (1)
 3  PRV  E[H] = E[H tanh(W/2)] ⟸ 1, 2                      Lean Bounded.single
 4  CIT  E[e^(−W) | frame] = Z₋/Z₊ ⟸ 2
 5  PRV  1 − E[tanh(W/2)] = 2 × acceptance ⟸ 3
 6  PRV  many cells (7) ⟸ 2                                containers separated
 7  RDG  m → ∞ gives W → ΔF_D (physics) ⟸ 4
 8  RDG  work goes down with the number of steps          β = 2.3, 2×2×8×8
 9  PRV  Cut-Measure Formula                              Lean cut_formula
10  PRV  flip probability ≤ 1 − θ²                        Lean flip_le
11  PRV  (1 − θ²)/2 for positive kernels ⟸ 10
12  PRV  θ ≤ 1 − a                                        Lean flip_le_one_sub_
13  PRV  min(1, e^(−W)) kernel stationary, lower bound on acceptance ⟸ 2
14  PRV  E∏θ ≤ P(no flip) ⟸ 12                             containers separated
15  RDG  a of one-step kernels is 0.6–0.73 (Metropolis type)   range measured
16  PRV  only the weak form; band needs a ≥ 1 − e^(−cL²) ⟸ 12
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. This chapter has no assumption (Assumption) rows. The reading of the acceptance of one-step kernels is not made a premise of the row "only the weak form follows" (that row is a rewriting that follows from $\theta \le 1 - a$ alone, and the reading of the acceptance is a judgement of how far it can be used). There is neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Proved here Under the finite configuration-space condition, the positive-weight condition and the kernel moving only inside the container, the driven identity $\mathbb{E}[H] = \mathbb{E}[H \tanh(W/2)]$ and its form for many cells	An upper bound on the factor $\tanh(W/2)$ uniform in L and the lattice spacing
Cited The Crooks relation and the Jarzynski equality (written out for the lattice procedure)	A proof of the quasi-static limit $W \to \Delta F_D$
Proved here For stationary kernels, $a \le 1 - \theta^2$, $\theta \le 1 - a$, and simultaneous flips $\mathbb{E}\prod\theta \le \mathbb{P}(\text{no flip})$	A construction of a kernel with a close to 1. The form of the band $\theta \le e^{-cL^2}$

What is said	What is not said
Reading The one-step kernels measured are far from $a \geq 1 - e^{-cL^2}$ (cannot be used to test the band)	The same theorems on the continuous configuration space (outside Lean)

Sources and reproduction

Item	Kind	Source / tools
The Crooks relation (path weight ratio e^{-W})	Cited	G. E. Crooks, "The entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences", Phys. Rev. E 60 (1999) 2721, arXiv:cond-mat/9901352 (bibliographic data checked on the arXiv page. The original source of the Markov-chain form not checked)
The Jarzynski equality	Cited	C. Jarzynski, "A nonequilibrium equality for free energy differences", Phys. Rev. Lett. 78 (1997) 2690, arXiv:cond-mat/9610209 (bibliographic data checked on the arXiv page)
The Driven Insertion Identity, the Driven Insertion Acceptance-Rate Corollary, the Many-Cell Driven Estimate, the Cut-Measure Formula, the Flip-Probability Lemma, the Flip-Probability Corollary for Positive Kernels, the Flow Lemma, the Simultaneous-Flip Estimate	Proved here	Proofs in the text, § 03– § 07. The Driven Insertion Identity has the same content as the driven identity of the draft paper "Exact centre-vortex insertion identities in the twist-summed ensemble, and bounded, averaged and driven confinement bounds" (the paper on vortex-insertion identities in A-07). The Flow Lemma has the same content as the flux bound of the paper on the reduction of the band in A-07
The container identity	Cited (indirect)	G. Mack, V. B. Petkova (as in 25-03)
Tempered transitions, non-equilibrium candidate methods	Comparison not done	The Monte Carlo literature (bibliographic data not checked)
Numerics (§ 04)	Reading	The same measurement as the first row of Table 1 of 25-08
Lean	Lean	FlipLemma (`FlipLemma.flip_le` , `FlipLemma.cut_formula`), FluxLemma (`FluxLemma.flip_le_one_sub_abs`), BoundedInsertion (`Bounded.single`). What the bundles guarantee is only the algebra of finite sums

Words used in this chapter — driven insertion, the number of steps m, the work W, the space of paths Ω, the Crooks involution ι, quasi-static, the cut measure, the flip probability, the flow lemma, simultaneous flips. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-07 Why fixed maps fail — why driven insertion is cheap and a fixed map applied at once is expensive, explained at the one-loop level as the difference of "whether the shape of the fluctuations is changed".

Why fixed maps fail

This chapter is this book's own investigation. The established knowledge it builds on is in 4-06 and 4-04.

What this chapter shows — As seen in 25-05 and 25-06, a fixed map that inserts a vortex at once is expensive, and driven insertion that inserts it slowly by heat bath is cheap. In this chapter we explain the difference in the form of a principle. (1) Bounds from below on the sector ratio Z_-/Z_+ (from above on the sector free energy ΔF) are obtained by the Gibbs variational principle with a map, and the gap is exactly a relative entropy (the principle of direction). (2) In a Gaussian model, the loss of a map that does not change the shape of the fluctuations (an isometric map) does not decrease as the number of degrees of freedom grows, while reshaping the fluctuations with a map of determinant 1 brings the loss down to $\Delta F^2/n$. (3) In the one-loop count $\frac{11}{3} = -\frac{1}{3} + 4$ for gluons in a background magnetic field, the $+4$ of antiscreening comes from a spin term that is traceless in the polarisations, so **at the one-loop level** a comparison by isometric maps can give only the direction opposite to asymptotic freedom, $\beta' \geq \beta + 0.023$, while a map that reshapes the fluctuations reproduces $\beta - \beta' = 0.258$. (4) We set side by side that fixed pointwise maps pay an extra cost in the ultraviolet (numerics in a two-dimensional model), and that if the residual relative entropy grows with the area, no area law follows.

Prerequisite chapters — 25-05 (insertion identities), 25-06 (driven insertion), 26-01 (the correct form of the band). From Part 4, 4-06 (asymptotic freedom, the one-loop coefficient) and 4-04 (renormalisation); from Part 2, 2-08 (free energy, relative entropy).

Said first — All claims of § 03– § 05 are statements **at the one-loop (Gaussian) level**. This chapter makes no non-perturbative claims. Bounding the error of maps that reshape fluctuations at intermediate coupling and large fields is an open problem, and that is where the core of the crossover band (Part 26) lies. The extra ultraviolet cost of § 06 is a numerical reading in a two-dimensional model.

The order of this chapter

1. Summary — whether the shape of the fluctuations is changed

2. The principle of direction — upper bounds come from variation (the Transport Identity)
3. The Gaussian model — the loss of isometric maps, and shaping the fluctuations (Table 1)
4. The one-loop table 11/3 = −1/3 + 4 — the spin term does not appear in first-order traces (Table 2)
5. Consequences for the renormalisation comparison — isometric versus shaping comparison (Table 3)
6. The extra ultraviolet cost of fixed maps — Table 4
7. Deterministic squeezing and the area law — the relative-entropy obstruction of the residual, and additivity
8. What cannot be said
9. The table of five grades and the dependency graph — Table 5

01

Summary — whether the shape of the fluctuations is changed

1. Confinement needs the sector with a vortex not to be too expensive compared with the sector without, that is, a bound **from above** on the sector free energy $\Delta F = \ln(Z_+/Z_-)$ (to make $\theta = \tanh(\Delta F/2)$ small).
2. What gives a bound in that direction is the Gibbs variational principle with a map T that carries the measure of sector $+$ to sector $-$: $\Delta F = \mathbb{E}_+[S \circ T - S - \ln J_T] - \mathrm{KL}$. Discarding the KL (relative entropy) gives an upper bound Proved here .
3. The discarded KL is the loss of the bound. Computing in a Gaussian model, the loss of maps that do not change the shape of the fluctuations (the identity map, group translations, gauge transformations, decimation) does not decrease as degrees of freedom are added. A map that stretches and shrinks the fluctuations while keeping volume brings the loss down to $\Delta F^2/n$ (n is the number of degrees of freedom) Proved here .
4. The coefficient of asymptotic freedom $\frac{11}{3}$ splits at the one-loop level into an orbital part $-\frac{1}{3}$ and a spin term $+4$. The spin term is traceless in the polarisations, so it does not appear in bounds from maps that do not change the shape of the fluctuations (first-order traces) Proved here (at the one-loop level).
5. As a result, at the one-loop level, a renormalisation comparison built with isometric maps can give only $\beta' \geq \beta + 0.023$ per doubling (the same direction as $U(1)$), while a

map that reshapes the fluctuations reproduces the $\beta - \beta' = 0.258$ of asymptotic freedom Proved here (at the one-loop level).

6. In a two-dimensional model, fixed pointwise maps paid an extra cost growing faster than linearly in the lattice size N_T in the twisted direction. In trials that redraw the fluctuations around the new background, this growth disappears Reading .
7. If the residual KL of a deterministic map grows with the area of the sheet, no area law follows from the variational bound Proved here (under that assumption). The role of the transport road lies in a qualitative target weaker than the form of the band itself.

02

The principle of direction — upper bounds come from variation

Let X_+, X_- be the sets of configurations of sectors $+$ and $-$, $\mu_\pm = e^{-S}/Z_\pm$ the Gibbs measures on each, and set the sector free energy to be $\Delta F = \ln(Z_+/Z_-)$ (the same orientation as ΔF_D of 25-06). The bias of the sector is $\theta = \tanh(|\Delta F|/2)$, so what is needed in the direction of confinement is an **upper bound** on ΔF, in other words a **lower bound** on Z_-/Z_+.

Transport Identity (the transport identity, the Gibbs variational principle)

If $T : X_+ \to X_-$ is a measurable bijection with Jacobian J_T relative to the reference measure (Haar $\times$ counting; Lebesgue in the Gaussian model), and $T_\# \mu_+$ is the pushforward of μ_+ by T, then

$$\Delta F = \mathbb{E}_+\left[S \circ T - S - \ln J_T\right] - \mathrm{KL}\left(T_\# \mu_+ \,\|\, \mu_-\right). \quad (1)$$

In particular $\Delta F \le \mathbb{E}_+[S \circ T - S - \ln J_T]$, with equality only when $T_\# \mu_+ = \mu_-$ (T carries μ_+ exactly to μ_-).

Proved here A standard computation. Proof. The density of $\nu = T_\# \mu_+$ is $\nu(y) = \mu_+(x)/J_T(x)$ at $y = Tx$. Hence

$$\mathrm{KL}(\nu \| \mu_-) = \mathbb{E}_+\left[\ln \mu_+(x) - \ln J_T(x) - \ln \mu_-(Tx)\right] = \mathbb{E}_+\left[S(Tx) - S(x) - \ln J_T(x)\right] - \ln Z$$

Rearranging gives (1). The inequality follows from $\mathrm{KL} \ge 0$. ∎ On a finite configuration space (the finite configuration-space condition of 25-05), if T is a bijection then $J_T = 1$, and the first term of the right-hand side of (1) is the $\mathbb{E}_+[\Delta S]$ of 25-05.

The other direction. The structural tools used in this book do not bound ΔF from above. Reflection positivity gives $Z_- \le Z_+$ ($\theta \ge 0$, $\Delta F \ge 0$), but that is a **lower bound** on ΔF, the direction opposite to confinement (26-04) Proved here . Chessboard estimates replace $\mathbb{E}\prod\theta$ by a single density κ_{cb}, but saying $\kappa_{cb} < 1$ still needs an upper bound on ΔF (26-05). The Tomboulis–Yaffe inequality carries a bound on θ to the Wilson loop, and does not bound θ itself (26-01). Within the range of this book's tools, upper bounds on ΔF have come only from variation of the type (1) (trial maps, trial measures) and from explicit constructions Reading (a judgement arranging the list of tools, not a theorem). We call this the **principle of direction**. A candidate tool for crossing the band must first of all give a bound in this direction.

03

The Gaussian model — isometric maps and reshaping maps

We look at how large the KL of (1) is in a Gaussian model where everything can be computed. Take two Gaussian measures on $\mathbb{R}^n$, $\mu_0 = N(0, \Delta_0^{-1})$ and $\mu_B = N(0, \Delta_B^{-1})$ (actions $S_0 = \frac{1}{2}x^{\mathsf{T}}\Delta_0 x$, $S_B = \frac{1}{2}x^{\mathsf{T}}\Delta_B x$; Δ_0, Δ_B positive definite). μ_0 plays the role of "the fluctuations of sector $+$ (no background)", and μ_B that of "the fluctuations of sector $-$ (around the vortex background B)". The exact value is

$$\Delta F = -\ln\frac{Z_B}{Z_0} = \tfrac{1}{2}\ln\det(\Delta_B\Delta_0^{-1}) = \tfrac{1}{2}\sum_k \ln\lambda_k, \quad (2)$$

λ_k are the eigenvalues of $\Delta_0^{-1/2}\Delta_B\Delta_0^{-1/2}$. Putting a linear map T into (1), the bound, using $M := \Delta_0^{-1/2}T^{\mathsf{T}}\Delta_B T\Delta_0^{-1/2}$, is

$$\mathbb{E}_0\big[S_B\circ T - S_0\big] - \ln|\det T| = \tfrac{1}{2}(\operatorname{tr} M - n) - \ln|\det T| \quad (3)$$

(from $\mathbb{E}_0[x^{\mathsf{T}}Ax] = \operatorname{tr}(A\Delta_0^{-1})$).

A **map isometric for the fluctuations** is a linear map with $T^{\mathsf{T}}\Delta_0 T = \Delta_0$ (it does not change the shape of the fluctuations of μ_0, i.e. it preserves μ_0). The identity map, linearisations of group translations and gauge transformations, and the linearisation of decimation (the operation of discarding some of the fine degrees of freedom and taking the rest as they are as degrees of freedom of the coarse lattice) belong to this class (for the treatment of decimation, see § 08 (b)).

Isometric-Map Loss Proposition (the loss of isometric maps is not diluted by degrees of freedom)

If T is isometric for the fluctuations, the bound (3) is $\frac{1}{2}\sum_k(\lambda_k - 1)$ independently of T, and its difference from the exact value (2) is

$$\tfrac{1}{2}\sum_k \big(\lambda_k - 1 - \ln\lambda_k\big) \;\geq\; 0. \tag{4}$$

Proved here Proof. $O := \Delta_0^{1/2} T \Delta_0^{-1/2}$ is an orthogonal matrix satisfying $O^{\mathsf{T}} O = 1$, and $M = O^{\mathsf{T}} (\Delta_0^{-1/2} \Delta_B \Delta_0^{-1/2}) O$. Hence $\operatorname{tr} M = \sum_k \lambda_k$ and $|\det T| = 1$. The difference is the sum of $t - 1 - \ln t \geq 0$. ∎ (4) is exactly the relative entropy $\mathbf{KL}(\mu_0 \| \mu_B)$ of the two Gaussian measures. If the background changes the fluctuations in only a few directions (finitely many λ_k different from 1), (4) does not decrease as n is made large.

Fluctuation-Shaping Proposition (shaping with a map of determinant 1 gives loss $\Delta F^2/n$)

Taking the infimum of the bound (3) over all linear maps with $\det T = 1$, with $r := \det(\Delta_B \Delta_0^{-1}) = e^{2\Delta F}$,

$$\inf_{\det T = 1} \Big(\tfrac{1}{2}(\operatorname{tr} M - n) \Big) = \tfrac{n}{2}\big(r^{1/n} - 1\big), \qquad 0 \;\leq\; \tfrac{n}{2}\big(e^{2\Delta F/n} - 1\big) - \Delta F \;\leq\; \frac{\Delta F^2}{n}\, e^{2|\Delta}$$

Proved here Proof. If $\det T = 1$ then $\det M = r$, and conversely any positive definite M with $\det M = r$ is realised by $T = \Delta_B^{-1/2} M^{1/2} \Delta_0^{1/2}$ (times an orthogonal matrix; $\det T = r^{-1/2} r^{1/2} = 1$). By the arithmetic–geometric mean inequality $\operatorname{tr} M \geq n (\det M)^{1/n} = n r^{1/n}$, with equality at $M = r^{1/n} \cdot 1$. The bound on the difference follows, with $u = 2\Delta F/n$, from $e^u - 1 - u \leq \frac{u^2}{2} e^{|u|}$. ∎

The map of the Fluctuation-Shaping Proposition preserves volume ($\ln J_T = 0$), stretches and shrinks the fluctuations in the few directions the background changes, and spreads that change of volume thinly over the many remaining directions, $\Delta F/n$ each. **What is needed is not having a Jacobian (entropy), but changing the shape of the fluctuations.**

Table 1 An example in the Gaussian model: eigenvalues 0.2, 0.3, 3, 4 (all the rest 1)

Bound	$n = 10$	$n = 100$	$n = 10^4$
Exact value (2)	-0.16425	-0.16425	-0.16425
Isometric map (Isometric-Map Loss Proposition)	$+1.75$	$+1.75$	$+1.75$
Optimal map of determinant 1 (Fluctuation-Shaping Proposition)	-0.1616	-0.1640	-0.16425

Proved here (values of the closed formulae (2), (4), (5)). When small and large eigenvalues are mixed, the first-order trace $\frac{1}{2}\sum(\lambda_k - 1)$ and $\frac{1}{2}\sum \ln \lambda_k$ can have opposite signs. In this example the change of sector is exactly **cheap** ($\Delta F < 0$), yet the bound of the isometric map says it is expensive ($+1.75$). The spin term of the next section has exactly this form.

04

The one-loop table 11/3 = −1/3 + 4

Place a constant abelian magnetic field B in the (1,2) plane as background, and consider the fluctuations of the $SU(2)$ gluons around it in Feynman gauge. The operator of the quadratic action of the fluctuations is

$$M_B = -D^2\,\delta_{\mu\nu} - 2iF_{\mu\nu} \quad (6)$$

where the first term is the orbital part (the square of the covariant derivative) and the second term is the spin term. The ghost operator is $-D^2$. The heat-kernel traces (per volume, omitting the common factor $1/(16\pi^2 s^2)$ and the group factor $C_A = 2$ = the number of charged components), as functions of the proper time s, are

$$\text{vector (4 polarisations)}: \ \frac{Bs}{\sinh Bs}\big(2 + 2\cosh 2Bs\big), \qquad \text{ghosts}: \ -2\,\frac{Bs}{\sinh Bs} \quad (7)$$

With $x = Bs$, using $x/\sinh x = 1 - x^2/6 + O(x^4)$ and $\cosh 2x = 1 + 2x^2 + O(x^4)$ and picking out the coefficient of x^2 gives the following table.

Table 2 Breakdown of the coefficient of $B^2 s^2$ (at the one-loop level)

Part	Form of the heat-kernel trace	Coefficient of $B^2 s^2$	Meaning
Orbital (4 orbital degrees of freedom of the vector − 2 of the ghosts)	$\frac{Bs}{\sinh Bs}\,(4-2)$	$-\frac{1}{3}$	Diamagnetic (screening). The same type as a charged particle of $U(1)$
Spin term	$\frac{Bs}{\sinh Bs}\,2(\cosh 2Bs - 1)$	$+4$	Paramagnetic (antiscreening). Peculiar to the non-abelian case
Total	Sum of (7)	$+\frac{11}{3}$	The one-loop coefficient of asymptotic freedom (the $\frac{11}{3}$ of 4-06)

Proved here (at the one-loop level. The expansion is a two-line computation: orbital $2(1 - x^2/6) = 2 - x^2/3$, spin $(1 - x^2/6)\cdot 4x^2 = 4x^2 + O(x^4)$.) In the conventions of this count, the coefficient $+\frac{11}{3}$ is the direction of asymptotic freedom and $-\frac{1}{3}$ the opposite direction.

First-Order Vanishing of the Spin Term (the spin term does not appear in first-order traces — at the one-loop level)

At the Gaussian level, when comparing the operator without background $M_0 = -\partial^2\delta_{\mu\nu}$ with the M_B of (6), the bound (1) from a map isometric for the fluctuations is the first-order trace $\frac{1}{2}\mathrm{tr}\big((M_B - M_0)M_0^{-1}\big)$, and the contribution of the spin term is

exactly 0. This bound is at least the exact value for the orbital part alone, $\frac{1}{2}\ln \det(M_{\mathrm{orb}} M_0^{-1})$.

Proved here (at the one-loop level) Proof. M_0^{-1} is proportional to $\delta_{\mu\nu}$ in the polarisation indices, and $F_{\mu\nu}$ is antisymmetric, so $\mathrm{tr}_{\mu\nu}(F_{\mu\nu}) = 0$. Hence the first-order trace of the spin term is 0, the bound equals $\frac{1}{2}\mathrm{tr}((M_{\mathrm{orb}} - M_0)M_0^{-1})$, and by the inequality $t - 1 \geq \ln t$ of the Isometric-Map Loss Proposition it is at least $\frac{1}{2}\ln\det(M_{\mathrm{orb}} M_0^{-1})$. ∎ The spin term takes effect at second order ($\cosh 2Bs - 1 = O(B^2)$ in Table 2) and gives the $+4$ of antiscreening. A bound that looks only at first-order traces loses it completely. (Traces and determinants of infinite-dimensional operators are statements under the regularisation usually done in one-loop computations.)

05

Consequences for the renormalisation comparison

The renormalisation comparison of Part 26 (26-06) aims to bound from above the vortex free energy of $(2N, \beta)$, with the whole lattice doubled, by that of the original size (N, β'), and to say $\beta' < \beta$ (the direction in which the coupling becomes stronger, asymptotic freedom). At the one-loop level, the coefficient c around the vortex background fixes the shift of the coupling per doubling, and with $\beta = 4/g^2$ and $b_0 = c\,C_A/(16\pi^2)$, $\beta - \beta' = 8b_0 \ln 2$.

Isometric versus Shaping Comparison Proposition (comparison of the two maps — at the one-loop level)

(a) If the bound (1) is built with maps isometric for the fluctuations, the coefficient that can be captured at the one-loop level is at most $c = -\frac{1}{3}$, and only $\beta' \geq \beta + 0.023$ per doubling can be given (the same as $U(1)$, the direction opposite to asymptotic freedom).
(b) A map that shapes the fluctuations while preserving volume (the Fluctuation-Shaping Proposition) reproduces the exact $\frac{1}{2}\ln\det$ with error $O(\Delta F^2/n)$, and gives $c = \frac{11}{3}$, $b_0 = 22/(48\pi^2)$, and $\beta - \beta' = 0.258$ per doubling.

Proved here (at the one-loop level) Proof. (a) follows from the First-Order Vanishing of the Spin Term and the fact that the bound is an upper bound (the Transport Identity). The number is $8 \cdot \frac{-1/3 \cdot 2}{16\pi^2}\ln 2 = -\frac{0.2575}{11} \approx -0.0234$. (b) applies the Fluctuation-Shaping Proposition to the fluctuations shell by shell, and the number is $8 \cdot \frac{22}{48\pi^2}\ln 2 \approx 0.2575$.

∎

Table 3 Shift of the coupling per doubling $\beta - \beta'$ ($SU(2)$, $\beta = 4/g^2$)

From what	$\beta - \beta'$	Grade	Conditions written in the same row
The exact one-loop coefficient $\frac{11}{3}$	0.258	Proved here	One-loop level
Bound from a map shaping the fluctuations (Isometric versus Shaping Comparison Proposition (b))	0.258 (error $O(\Delta F^2/n)$)	Proved here	One-loop level, Gaussian bound
Bound from an isometric map (Isometric versus Shaping Comparison Proposition (a))	≤ -0.023	Proved here	One-loop level. The same direction as $U(1)$
Test doubling the whole lattice ($4^4 \to 8^4$, 1–3 seeds per β)	$+0.16$ to $+0.25$	Reading	$\beta = 2.5$ to 3.0. The range at each β is as in 26-06 Table 4 (for example $+0.13$ to $+0.19$ at $\beta = 2.7$, $+0.19$ to $+0.5$ at $\beta = 2.5$). At $\beta = 2.3$, $\gtrsim 0.1$. Small lattices
The same test for $U(1)$ ($(N, M) = (4, 8) \to (8, 16)$, $\beta = 1.5$)	< -0.2	Reading	Opposite direction ($\beta' > 1.7$, off the curve). 48 replicas, minimum exchange acceptance 0.37. One seed (26-06 Table 2)

The attempt to build a renormalisation comparison with Migdal–Kadanoff-type decimation (Tomboulis 2007) was criticised on the point that the same argument also flows to strong coupling for $U(1)$ Cited (Ito–Seiler, Kanazawa; source table).

Decimation belongs on the side that does not change the shape of the fluctuations, so the Isometric versus Shaping Comparison Proposition (a) can be read as giving a one-loop-level explanation of this criticism Reading (physics; an interpretation in which a one-loop computation pins down the gap in a non-perturbative argument, not a proof of the criticism itself).

A screening criterion. A candidate method for crossing the band must have **a linearisation of the trial map that is non-isometric for the fluctuations** — this is a necessary condition at the one-loop level Proved here (a restatement of the Isometric versus Shaping Comparison Proposition (a); at the one-loop level). The view that the same condition is needed at the non-perturbative level is Reading (physics). The driven insertion of 25-06 is cheap because the heat bath relaxes the shape of the fluctuations to fit the new coupling, which is consistent with this criterion Reading (physics).

> 06

The extra ultraviolet cost of fixed maps

The Isometric-Map Loss Proposition said that "the loss of isometric maps is not diluted by degrees of freedom". Numerics show that with fixed pointwise maps on the lattice, the loss moreover **grows** as the lattice spacing is made finer.

A two-dimensional model. Place the two-dimensional $O(3)$ model (spins $s_x \in S^2$, action $-\beta \sum s_x \cdot s_y$) on the strip $\mathbb{Z} \times (\mathbb{Z}/N_T)$, with a twist of a π rotation about the z axis in the periodic direction. The classical vacua are the north and south poles, and the action of the kink joining them (the two-dimensional version of the fractional instanton) approaches $2\pi\beta$ from below. It is a light test bed for the small twisted T^2 in four dimensions.

Table 4 The cost of kink insertion on the twisted two-dimensional strip (action in units with β removed; the extra cost is $\beta \times (\mathbb{E}[\Delta S] - \text{classical value})$)

Trial	β	$N_T = 2$	$N_T = 4$	$N_T = 8$	Conditions
Extra cost of the fixed pointwise rotation (the minimal rotation from vacuum to kink at each point)	2	-2.1	$+4.1$	$+14.7$	Standard deviations at $N_T = 8$ are 4.6 ($\beta = 2$) and 2.6 ($\beta = 4$). The number of configurations is not in the record
Same as above	4	-2.0	$+4.6$	$+16.2$	

Reading The extra cost hardly depends on β and grows faster than linearly in N_T. This is the opposite of the expected "decrease by $\ln N_T$ due to the running coupling".

On the other hand, in trials that do not carry the fluctuations over on the old background but **redraw them from the Gaussian measure around the new background (the kink)**, the difference of the bound becomes the ratio of one-loop determinants $\frac{1}{2} \ln[\det' H_{\text{kink}} / \det H_{\text{vac}}]$. Its value (with the two collective coordinates removed) changes only logarithmically, $-3.68, -2.94, -2.43, -2.11, -1.88$ at $N_T = 3, 4, 6, 8, 10$, and the quantity including the measure of the rotational and translational collective coordinates was almost constant, -4.91 to -4.85 for $N_T = 4$ to 12 Reading (numerical computation of eigenvalues; no error estimate). The faster-than-linear growth has disappeared.

A four-dimensional slab. For $SU(2)$, $\beta = 2.3$, the slab $2 \times 2 \times 8 \times 8$, an insertion that spreads the flux smoothly in a fixed colour direction (σ_3) raised the cost instead of lowering it: the mean of ΔS was 21.9 at spreading radius 1 and 36.0 at radius 2 Reading . This can be read as being because, in configurations without gauge fixing, the field in a fixed direction does not line up with the colour direction at each point Reading (physics). On the same slab, driven insertion (25-06) brought the mean work down to 2.9 with 16 steps.

How to read it. Fixed pointwise maps move the fluctuations "on a fixed background", so they fail to capture the difference of entropy of the fluctuations on the new background. The picture is that the loss of the Isometric-Map Loss Proposition piles up for each fine degree of freedom of the lattice Reading (physics; whether it is proportional to N_T is a reading from three numerical points, and there is no general bound). The average of insertions (25-05), driven insertion (25-06) and redrawing the fluctuations are all tools for avoiding this cost.

07

Deterministic squeezing and the area law

The correct form of the band was $\theta(L,L) \le e^{-cL^2}$ (26-01). Since $\theta = \tanh(\Delta F/2) \le \Delta F/2$, showing this variationally requires the upper bound on $\Delta F(L,L)$ to be as small as order $2e^{-cL^2}$.

Relative-Entropy Obstruction to the Area Law (if the residual KL grows with the area, no area law follows)

If, for a map T, the residual $\mathrm{KL}(T_{\#}\mu_+ \| \mu_-)$ of (1) is $\ge c'A$ ($c' > 0$) in the area A of the vortex sheet, then the bound $\mathbb{E}_+[S \circ T - S - \ln J_T]$ from (1) is at least $c'A$, and the upper bound on θ derived from it is at least $\tanh(c'A/2)$, approaching 1 as $A \to \infty$. In particular no bound of the form $\theta \le e^{-\sigma A}$ follows.

Proved here Proof. From (1), $\mathbb{E}_+[\cdots] = \Delta F + \mathrm{KL} \ge \mathrm{KL} \ge c'A$ (taking the orientation of the sectors so that $\Delta F \ge 0$). $\tanh$ is monotone. ∎ This is a statement under an assumption, and whether the assumption holds must be checked map by map.

Additivity of Relative Entropy over Containers (for independent containers the residual KL adds up)

If, with the frame fixed, the insides of containers $D_1, \dots, D_k$ are conditionally independent (25-03) and the map is a product over containers $T = T_1 \times \cdots \times T_k$, then $\mathrm{KL}(T_{\#}\mu_+ \| \mu_- \mid \Phi) = \sum_j \mathrm{KL}(T_{j\#}\mu_{+,j} \| \mu_{-,j} \mid \Phi)$. If each term is at least $c_0 > 0$, the residual is proportional to the number of containers, that is, to the area.

Proved here The standard fact that the relative entropy of a product measure is the sum of the relative entropies of the factors. ∎

That for fixed maps (not adapted to the fluctuations) the residual of each container is bounded below by $c_0 > 0$, so that the assumption of the Relative-Entropy Obstruction to the Area Law holds, is the shared judgement of the study from two viewpoints,

constructive field theory and probability Reading (physics, an estimate; the same goes for the judgement that the ceiling of triangular maps is of perimeter type). Under this judgement the transport road is not a tool for showing the form of the band itself, but a tool for a weaker qualitative target — $\Delta F(L, L) \to 0$ (at any speed). $\Delta F(L, L) \to 0$ is enough to rule out the Coulomb phase of $U(1)$ (where $\Delta F(L, L)$ approaches a positive constant) and the deconfined phase of $\mathbb{Z}_2$, but it yields neither confinement nor a mass gap Reading (physics; numerics in 26-06).

08

What cannot be said

(a) The non-perturbative level. The First-Order Vanishing of the Spin Term, the Isometric versus Shaping Comparison Proposition and the top three rows of Table 3 are all statements at the one-loop (Gaussian) level. Bounding the error of maps that shape the fluctuations at intermediate coupling and large fields (local relative entropy near the vortex sheet, and higher loops) is an open problem, and this book does not show it.

(b) The classification of isometric maps. The identity map and the linearisations of group translations and gauge transformations are isometric by definition. Putting decimation on the isometric side is a classification in the sense that it carries the fluctuations of the kept degrees of freedom over as they are; decimation is not a bijection, so this is not a literal application of the Isometric-Map Loss Proposition.

(c) The principle of direction. "Upper bounds on ΔF come only from variation and explicit constructions" is a judgement about the list of this book's tools, not a theorem.

(d) The extra ultraviolet cost. Table 4 is three numerical points in a two-dimensional model, and does not show that four-dimensional gauge theory grows in the same way. In four dimensions cancellations through Ward identities are said to be possible, and whether they occur depends on gauge fixing and the covariance of the map.

(e) The assumption of the Relative-Entropy Obstruction to the Area Law. That the residual KL of fixed maps is proportional to the area can be said only in the product case of the Additivity of Relative Entropy over Containers, assuming that the residual of each container is bounded below; for general maps it is an estimate.

(f) Novelty. The Gibbs variational principle, the relative entropy of Gaussian measures and the one-loop computation of the background field are all standard. No novelty is claimed for the view that lines them up as a screening criterion of "whether the shape of the fluctuations is changed". For scalar fields, Polchinski-type transport maps are known (Bauerschmidt–Bodineau–Dagallier); no example for gauge fields was found in the range searched (not yet checked).

09

The table of five grades and the dependency graph

Table 5 The rows of this chapter (§ 03– § 05 are all at the one-loop (Gaussian) level)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	Transport Identity: $\Delta F = \mathbb{E}_+[S \circ T - S - \ln J_T] - \mathrm{KL}$	Proved here	§ 02 (standard)	Measurable bijection, Jacobian defined	—
2	Reflection positivity gives only $\Delta F \geq 0$ (the direction opposite to confinement)	Proved here	§ 02, 26-04	Reflection directions with the twist fixed	—
3	The principle of direction: upper bounds on ΔF only from variation and explicit constructions	Reading	§ 02	A judgement about the list of this book's tools	—
4	Isometric-Map Loss Proposition: $\frac{1}{2}\sum(\lambda_k - 1 - \ln \lambda_k)$ (not diluted in n)	Proved here	§ 03	Gaussian model, $T^{\mathsf{T}}\Delta_0 T = \Delta_0$	—
5	Fluctuation-Shaping Proposition: the optimum over maps of determinant 1, and loss $\leq (\Delta F^2/n)e^{2\lvert\Delta F\rvert/n}$	Proved here	§ 03	Gaussian model, linear maps	—
6	Values of the example of Table 1	Proved here	§ 03	Eigenvalues 0.2, 0.3, 3, 4	—
7	Table 2: $\frac{11}{3} = -\frac{1}{3} + 4$ ($+4$ is the spin term)	Proved here	§ 04	One-loop level, constant abelian background, Feynman gauge	—
8	First-Order Vanishing of the Spin Term: the spin term does not appear in first-order traces	Proved here	§ 04	One-loop level	—
9	Isometric versus Shaping Comparison Proposition (a): isometric comparison gives $\beta' \geq \beta + 0.023$	Proved here	§ 05	One-loop level, per doubling	—
10	Isometric versus Shaping Comparison Proposition (b): shaping comparison gives $\beta - \beta' = 0.258$	Proved here	§ 05	One-loop level, per doubling	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
11	$\beta - \beta'$ of the doubling test (positive for $SU(2)$, negative for $U(1)$)	Reading	§ 05, 26-06	Small lattices, a few values of β, 1–3 seeds	—
12	Migdal–Kadanoff interpolation does not distinguish $U(1)$ (criticism)	Cited	Ito–Seiler, Kanazawa	On the argument of Tomboulis 2007	—
13	A one-loop-level explanation of row 12	Reading (physics)	§ 05	One-loop level	—
14	Screening criterion: the linearisation must be non-isometric for the fluctuations	Proved here	§ 05	A necessary condition at the one-loop level (Reading non-perturbatively)	—
15	The extra cost of fixed pointwise maps is at least linear in N_T	Reading	§ 06	Two-dimensional $O(3)$, twisted strip, $N_T = 2, 4, 8$	—
16	Redrawing the fluctuations makes the growth disappear	Reading	§ 06	Same, $N_T = 3$ to 12, no error estimate	—
17	In four dimensions, insertion spread in a fixed direction raises the cost	Reading	§ 06	$SU(2)$, $\beta = 2.3$, $2 \times 2 \times 8 \times 8$	—
18	Relative-Entropy Obstruction to the Area Law: if the residual KL is $\geq c' A$, no area law follows	Proved here	§ 07	Assuming a lower bound on the residual KL	—
19	Additivity of Relative Entropy over Containers: for independent containers the KL adds up	Proved here	§ 07	With the frame fixed the containers are independent, and the map is a product	—
20	For fixed maps the residual of each container is bounded below	Reading (physics)	§ 07	Estimate	—
21	$\Delta F(L, L) \to 0$ rules out the Coulomb phase and the deconfined $\mathbb{Z}_2$ phase	Reading (physics)	§ 07, 26-06	Yields neither confinement nor a gap	—

This chapter has no machine-checked bundles.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  ΔF = E₊[S∘T − S − ln J] − KL                          bijection, Jacobia
 2  PRV  reflection positivity gives only ΔF ≥ 0                26-04
 3  RDG  principle of direction (judgement on the tools) ⟸ 1, 2
 4  PRV  loss of isometric maps not diluted in n ⟸ 1             Gaussian
 5  PRV  shaping with det 1, loss ~ ΔF²/n ⟸ 1                    Gaussian
 6  PRV  example of Table 1 ⟸ 4, 5
 7  PRV  11/3 = −1/3 + 4                                       one-loop level
 8  PRV  spin term absent from first-order trace ⟸ 4, 7          one-loop level
 9  PRV  isometric comparison β' ≥ β + 0.023 ⟸ 1, 8              one-loop level
10  PRV  shaping comparison β − β' = 0.258 ⟸ 5, 7                one-loop level
11  RDG  sign of doubling test (SU(2) +, U(1) −)               small lattices
12  CIT  criticism of Migdal–Kadanoff interpolation
13  RDG  one-loop explanation of row 12 (physics) ⟸ 9, 12
14  PRV  screening criterion: non-isometric ⟸ 9                  one-loop level
15  RDG  extra cost of fixed maps (two dimensions)
16  RDG  redrawing removes the growth (two dimensions)
17  RDG  fixed-direction insertion is expensive (four-dimensional slab)
18  PRV  residual KL ≥ c'A gives no area law ⟸ 1                  with assumption
19  PRV  KL adds up over independent containers
20  RDG  residual of fixed maps bounded below per container (estimate) ⟸ 19
21  RDG  role of ΔF(L,L) → 0 (physics) ⟸ 18, 20
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The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. There are no assumption (Assumption) rows. The conclusion of the Relative-Entropy Obstruction to the Area Law applies to fixed maps only when the estimate "for fixed maps the residual of each container is bounded below" is correct, and that distinction is expressed by the grade (Reading) of the last row (the role of $\Delta F(L,L) \to 0$). This chapter has no node of confinement, no node of the mass gap, and no non-perturbative node.

What this chapter can say

What is said	What is not said
Proved here The transport identity (the Gibbs variational principle), and that the loss of its bound is a relative entropy	Bounds on the error of maps reshaping the fluctuations at the non-perturbative level
Proved here In the Gaussian model, the loss of isometric maps is not diluted, and the loss of reshaping maps is $\Delta F^2/n$	Literal extension to operations that are not bijections, such as decimation

What is said	What is not said
Proved here At the one-loop level $\frac{11}{3} = -\frac{1}{3} + 4$, the isometric comparison gives $\beta' \geq \beta + 0.023$, the reshaping comparison $\beta - \beta' = 0.258$	The renormalisation comparison at intermediate coupling (the core of the band in Part 26)
Reading Fixed pointwise maps pay an extra cost in the ultraviolet (three points in a two-dimensional model)	The same growth in four-dimensional gauge theory
Proved here If the residual KL is proportional to the area, no area law follows (with an assumption)	That the assumption holds for fixed maps in general

Sources and reproduction

Item	Kind	Source / tools
The seven propositions of § 02– § 07, Table 1, Table 2	Proved here	Proofs and computations in the text, § 02– § 07. The First-Order Vanishing of the Spin Term, the Isometric versus Shaping Comparison Proposition and Table 2 have the same content as the section on the renormalisation comparison in the draft paper (the paper on the reduction of the band in A-07)
The one-loop coefficient of asymptotic freedom	Cited	As in 4-06
The argument by Migdal–Kadanoff interpolation	Cited	E. T. Tomboulis, arXiv:0707.2179
Its criticism	Cited	K. R. Ito, E. Seiler, arXiv:0711.4930. T. Kanazawa, arXiv:0805.2742 (checking as in 26-06)
Polchinski-type transport maps (scalar fields)	Comparison not done	Bauerschmidt–Bodineau–Dagallier (bibliographic data not checked)
The numerical rows of Table 3	Reading	26-06 (the test doubling the whole lattice)
Table 4 and the two-dimensional numerics of § 06	Reading	Computations on the twisted strip of the two-dimensional $O(3)$ model done in this book's work (cooled kinks, the pointwise rotation map, eigenvalues of the Hessian). The error estimates are rough
The four-dimensional slab numerics of § 06	Reading	The same measurement as the numerical section of the paper on vortex-insertion identities in A-07

Words used in this chapter — the transport identity, the Gibbs variational principle, the principle of direction, maps isometric for the fluctuations, reshaping the fluctuations, the spin term, antiscreening, the residual relative entropy, the screening criterion. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-08 The measured record — the numerics of Part 25 (the work of driven insertion, the vortex free energy when the T^2 is widened, the string tension of slabs) gathered in one place, with conditions.

The measured record

> This chapter is this book's own investigation. The established knowledge it builds on is in 25-03 and 5-07.

What this chapter shows — We collect the numerical record of Part 25 in one place. There are three series: (1) the work and acceptance rate of driven insertion (25-06), (2) the vortex free energy of the global flux when the twisted T^2 is widened (with a $U(1)$ control), and (3) the string tension of slabs (Creutz ratios). Finally we annotate the measurement of simultaneous flips, and why it cannot be used as a test of the band.

Prerequisite chapters — 25-06 (driven insertion, the flow lemma), 25-03 (containers, the bias of the sector θ), 26-02 (the lateral-widening inequality, monotonicity of the string tension). From Part 5, 5-07 (Monte Carlo) and 5-08 (string tension, Creutz ratios).

Said first — The numbers of this chapter are **all numerical readings** (label "Reading"), and none is a proof. Each row states the group, β, the size of the T^2 N_T, the size of the plane M, the number of measurements and the error. Where the error is not in the record, we write "not in the record". The lattices are small, and β takes a few values. The rows with $N_T = 2$ are a region where the T^2 is only 2 in lattice units and lattice artefacts are large.

The order of this chapter

1. Summary — three series
2. Common setting and how to read it
3. The work of driven insertion — Tables 1 and 2
4. The vortex free energy when the T^2 is widened — Tables 3, 4 and 5
5. The string tension of slabs — Tables 6 and 7
6. The measurement of simultaneous flips, and why it cannot be used as a test of the band — Tables 8 and 9
7. What cannot be said
8. The table of five grades and the dependency graph — Table 10

01

Summary — three series

1. **Driven insertion.** On $SU(2)$ slabs, the cost of inserting a vortex in one go (0 steps) is large and proportional to the area of T^2, at 11.8, 24.2 and 44.9 ($N_T = 2, 3, 4$, $\beta = 2.3$), but driving it over 256 steps brought it down to 0.50–0.63 Reading (rows A, B and C of Table 1. In rows G and H, under different conditions, the work over 256 steps stays high, at 3.64 and 6.67). When the container (a disc) is smaller than the size of the vortex, increasing the number of steps does not bring it all the way down.
2. Fixing the container radius at 4 and widening to $N_T = 4, 6, 8$ ($\beta = 2.5$, 1024 steps), the work was 1.1–1.6, with no clear increase in N_T Reading (10 configurations $\times$ 3 positions).
3. **The vortex free energy.** The free energy ΔF of the global flux sector increased as T^2 was widened for $U(1)$ ($\beta = 1.5$), and did not increase, or decreased, for $SU(2)$ ($\beta = 2.4, 2.5$) Reading ($M = 8$ and 12; the errors are 0.04–0.78 for $U(1)$ and 0.1–0.36 for $SU(2)$).
4. **String tension of slabs.** For $SU(2)$, $\beta = 2.5$, $M = 16$, the Creutz ratio $\chi(4)$ increased for $N_T = 4 \to 8$ and was nearly saturated at $N_T = 10$. No significant overshoot is seen Reading (2000 measurements).
5. **Simultaneous flips.** The measurement driving containers simultaneously cannot be used as a numerical picture of the crossover band (25-06 § 07). It remains only as a tool for lower bounds on the slab tension with the T^2 fixed.

02

Common setting and how to read it

- **Lattice.** The slab $(\mathbb{Z}/N_T)^2 \times (\mathbb{Z}/M)^2$. Directions 1 and 2 form the twisted T^2 (twist $n_{12} = 1$), and directions 3 and 4 the plane. $SU(2)$ uses the Wilson action, and $U(1)$, as a control, an action of the same form.
- **Driven insertion.** The work W (equation (2) of 25-06) when only the inside of a container, a disc of the plane (radius r) $\times$ T^2, is updated by heat bath and the couplings of the vortex sheet are changed $1 \to -1$ in m steps. The acceptance rate is the mean of $1/(1 + e^W)$ (Barker-type; the acceptance rate of the Driven Insertion Acceptance-Rate Corollary of 25-06, equal to $\frac{1}{2}(1 - \mathbb{E}\tanh(W/2))$). Only the simultaneous flips of Tables 8 and 9 use the mean of $\min(1, e^{-W})$ (Metropolis-type).
- **Vortex free energy.** The $\Delta F = \ln(Z(1,0)/Z(1,1))$ of the two sectors of the flux $n_{34} \in \{0, 1\}$ through the whole $(3, 4)$ torus of the plane, estimated by replica exchange interpolating the coupling and Bennett's acceptance ratio (BAR). The bias is $\theta = \tanh(\Delta F/2)$. The errors are the record's estimates.
- **String tension.** The Creutz ratio $\chi(R)$ built from $R \times R$ Wilson loops in the plane. At small R it contains the Coulomb term and overestimates the tension.
- **Physical size.** In the record, $N_T = 8$ at $\beta = 2.5$ is estimated as $L\sqrt{\sigma} \approx 1.5$, $N_T = 8$ at $\beta = 2.4$ as $L\sqrt{\sigma} \approx 2.2$, and $N_T = 4$ at $\beta = 2.7$ as $L\sqrt{\sigma} \approx 0.4$ Reading (converted from known values of $a\sqrt{\sigma}$). Only the last is in the semiclassical region of a small T^2.

- $N_T = 2$. The rows where the T^2 is 2 in lattice units are a region of large lattice artefacts. They are excluded when reading trends.

03

The work of driven insertion

All are $SU(2)$, twist $n_{12} = 1$. The number of samples is "number of configurations × number of insertion positions per configuration". The figures in brackets for W are the standard deviations of the samples (from the records of the run outputs), not the errors of the means. Samples at different positions in the same configuration are not independent, so the error of the mean should be taken as larger than the standard deviation divided by the square root of the number of samples.

Table 1 Driven insertion (comparison of 0 steps with the slowest driving. Left and right of the slash are the two values of the number of steps or of the radius)

Run	β	N_T	M	r	Steps m	Samples	Mean of W (s.d.)	Mean of $\tanh(W/2)$	Acceptance rate
P	2.3	2	8	2	0 / 4 / 16	6 × 4	10.7 / 6.5 / 2.9 (not in the record)	0.990 / 0.968 / 0.688	0.005 / 0.016 / 0.156
A	2.3	2	8	2	0 / 256	12 × 4	11.8 (2.5) / 0.50 (0.71)	1.00 / 0.22	0.000 / 0.39
B	2.3	3	8	2	0 / 256	12 × 4	24.2 (4.6) / 0.52 (0.84)	1.00 / 0.21	0.000 / 0.39
C	2.3	4	8	2	0 / 256	8 × 4	44.9 (6.8) / 0.63 (1.19)	1.00 / 0.24	0.000 / 0.38
D	2.3	2	10	3	256	12 × 4	0.19 (0.74)	0.09	0.46
E	2.5	4	10	2 / 4	1024	8 × 3	2.40 (2.11) / 0.44 (1.21)	0.67 / 0.17	0.17 / 0.41
F	2.7	4	12	2 / 4	1024	6 × 3	6.00 (1.34) / 3.77 (1.18)	0.99 / 0.92	0.006 / 0.04
G	2.7	4	18	8	256 / 1024	4–6 × 3	3.64 (3.98) / 2.72 (2.53)	0.54 / 0.59	0.23 / 0.20
H	2.5	8	12	4	256 / 1024	4–6 × 3	6.67 (2.63) / 0.51 (1.64)	0.98 / 0.17	0.012 / 0.42

Row P is the measurement cited in 25-06 § 04. Rows P, A and D are $N_T = 2$ (the region of lattice artefacts).

Reading Reading : (i) The cost at 0 steps is roughly proportional to N_T^2 (11.8, 24.2, 44.9 for A, B, C). (ii) If the container is large enough, the work of driven insertion comes down to near 0. (iii) The container radius needed appears to roughly double each time the lattice spacing halves (2 at $\beta = 2.3$, 4 at 2.5; at 2.7, 4 is not enough and even 8 does not reach near 0). This is consistent with the thickness of the vortex being fixed by a physical length. (iv) The work 0.51 at 1024 steps for H ($N_T = 8$) is almost the same as 0.44 for E ($N_T = 4$), but at 256 steps it is high, 6.67, depending strongly on the speed of driving.

Table 2 Driven insertion with more statistics ($SU(2)$, twist $n_{12} = 1$)

Run	β	N_T	M	r	Steps m	Samples	Mean of W (s.d.)	$-\ln\langle e^{-W}\rangle$	Mean of $\tanh(W/2)$	Acceptance rate

Run	β	N_T	M	r	Steps m	Samples	Mean of W (s.d.)	$-\ln\langle e^{-W}\rangle$	Mean of $\tanh(W/2)$	Acceptance rate
I4	2.5	4	12	4	1024	10×3	1.32 (1.12)	0.72	0.48	0.26
I6	2.5	6	12	4	1024	10×3	1.14 (1.03)	0.59	0.44	0.28
I8	2.5	8	12	4	1024	10×3	1.64 (1.59)	0.22	0.52	0.24
J	2.7	4	18	8	4096	Not in the record	2.84 (0.84)	2.44	0.85	0.07

Reading Reading : (i) I: fixing the container radius at 4 and widening to $N_T = 4, 6, 8$, the work is 1.1–1.6, with no clear increase in N_T (the error of the mean is the standard deviation $/\sqrt{30}$, about 0.2–0.3). The 0.44 and 0.51 of E and H in Table 1 were fluctuations from little statistics. (ii) J: at $\beta = 2.7$, $r = 8$, driving four times more slowly than G still gives $W \approx 2.8$ (2.7 with 1024 steps for G), and the Jarzynski estimate is 2.4. Most of the work can be read not as dissipation due to the speed of driving but as a genuine free energy. $N_T = 4$ at $\beta = 2.7$ is a physically small T^2 ($L\sqrt{\sigma} \approx 0.4$), on the side of the dilute region, so the reading is that the vortex really is expensive (physics).

On the Jarzynski estimate. $-\ln\langle e^{-W}\rangle$ is an estimate of the sector free energy by the Jarzynski Equality of 25-06, but it is dominated by rare small W, so with few samples it fluctuates strongly. It is a column not shown in Table 1, but negative values have occurred (for example -0.30 for E at radius 4 with 1024 steps, -0.085 for A with 64 steps). This column is a guide for checking, and the values themselves are not reliable.

The acceptance rule. The acceptance-rate columns of Tables 1 and 2 are means of the Barker-type $1/(1+e^{W})$. Recomputing the mean of the Metropolis-type $\min(1, e^{-W})$ from the raw W of the same record gives, in the slowest-driving rows, A 0.641, B 0.647, C 0.595, D 0.729 and E ($r = 4$, 1024 steps) 0.607, all above $\frac{1}{2}$ Computed . Pointwise $\frac{1}{2}\min(1, e^{-x}) \le 1/(1+e^{x}) \le \min(1, e^{-x})$, and under a Crooks-type relation the Barker-type mean is at most $1/(1+e^{\Delta F})$ (at most $\frac{1}{2}$ if $\Delta F \ge 0$) Paper Lean (LatticeGaugeAcceptanceSandwich (half_metro_le_fermi, fermi_le_metro, barker_le_of_crooks)). That the acceptance rates of the tables stay below $\frac{1}{2}$ is the ceiling of this rule, not a physical limit of the kernel.

The limits of this series. What confinement on R^4 needs is that the bias becomes small exponentially in the area as L grows (26-01), and that is not visible in this measurement. β takes a few values between 2.3 and 2.7, and the error estimates are still rough.

04

The vortex free energy when the T² is widened

The container is the whole plane ($M \times M$), and the $T^2 = N_T \times N_T$ is widened. It is a test of the sign of the lateral-widening inequality (26-02). All by replica exchange + BAR, twist $n_{12} = 1$, 24 replicas (unless otherwise stated). The number of iterations is not in the record.

Table 3 ΔF, plane $M = 8$

Theory	β	$N_T = 2$	$N_T = 4$	$N_T = 6$	$N_T = 8$

Theory	β	$N_T = 2$	$N_T = 4$	$N_T = 6$	$N_T = 8$
$U(1)$ (control)	1.5	-0.03 ± 0.04	0.83 ± 0.11	2.10 ± 0.47	5.14 ± 0.78
$SU(2)$	2.5	-0.01 ± 0.12	0.63 ± 0.18	0.33 ± 0.22	1.51 ± 0.29

Table 4 ΔF, plane $M = 12$ (remeasurement with a larger container)

Theory	β	$N_T = 4$	$N_T = 6$	$N_T = 8$
$U(1)$ (control)	1.5	0.20 ± 0.21	1.35 ± 0.30	1.72 ± 0.47 (40 replicas; with 24, 0.54 ± 0.53, with poor mixing)
$SU(2)$	2.5	0.57 ± 0.22	0.42 ± 0.24	-0.15 ± 0.36

Table 5 ΔF, $SU(2)$, $\beta = 2.4$, plane $M = 8$

N_T	**4**	**8**
ΔF	0.31 ± 0.10	0.11 ± 0.25

Reading Reading : (i) For $U(1)$, ΔF increases as the T^2 is widened (at $M = 8$, θ goes $0 \to 0.39 \to 0.78 \to 0.99$; at $M = 12$, $0.20 \to 1.35 \to 1.72$). (ii) For $SU(2)$, in three settings ($M = 8$ and 12 at $\beta = 2.5$, $M = 8$ at $\beta = 2.4$), it does not increase or decreases when widened. The exception is 1.51 at $N_T = 8 = M$ (the symmetric torus T^4_8) for $\beta = 2.5$, $M = 8$. (iii) For $N_T = 4 \to 8$, $\ln(1/\theta)$ goes 0.93 $\to$ 0.012 for $U(1)$ (about 1/80) and 1.18 $\to$ 0.45 for $SU(2)$ (about 1/2.6) ($M = 8$). This can be read as a numerical sign of the sign of the lateral-widening inequality. (iv) That $N_T = 8$ at $\beta = 2.4$ is smaller than at $\beta = 2.5$ is consistent with the same lattice being physically larger ($L\sqrt{\sigma} \approx 2.2$ against 1.5).

Limits. This is a regime where θ is not close to 1 (it is $O(1)$), so the $U(1)$ ratio is also held down because $\theta \leq 1$, and in this geometry the lateral-widening inequality has little power to discriminate. The errors are large, 0.04–0.78 for $U(1)$ and 0.1–0.36 for $SU(2)$, and β is one or two points. $\beta = 1.5$ for $U(1)$ and $\beta = 2.5$ for $SU(2)$ are not matched to "the same physics". The $N_T = 2$ column is in the region of lattice artefacts.

05

The string tension of slabs

$SU(2)$, $\beta = 2.5$, plane $M = 16$, twist $n_{12} = 1$. We measured the Creutz ratio $\chi(R)$ of Wilson loops in the plane for each size N_T of the T^2. It is a test of "monotonicity of the reduced two-dimensional string tension $\sigma_{\text{eff}}(N_T)$" contained in the lateral-widening inequality (strong form) of 26-02, and of an "overshoot" in which the tension once becomes large at $L \sim 1/\Lambda$. In the record, the four-dimensional value at $\beta = 2.5$ is given as $\sigma a^2 \approx 0.036$ Reading .

Table 6 Creutz ratios, 400 measurements

N_T	$\chi(3)$	$\chi(4)$	$\chi(5)$
2	0.098 ± 0.006	0.034 ± 0.029	-0.02 ± 0.14
4	0.113 ± 0.007	0.070 ± 0.018	0.07 ± 0.06
6	0.108 ± 0.003	0.089 ± 0.013	0.11 ± 0.05
8	0.113 ± 0.003	0.051 ± 0.010	0.07 ± 0.04

Table 7 Creutz ratios, 2000 measurements (5 times the statistics) and $N_T = 10$

N_T	$\chi(3)$	$\chi(4)$	$\chi(5)$
4	0.103 ± 0.002	0.059 ± 0.009	0.03 ± 0.04
6	0.115 ± 0.001	0.061 ± 0.004	0.084 ± 0.019
8	0.116 ± 0.002	0.076 ± 0.002	0.070 ± 0.022
10	0.112 ± 0.002	0.071 ± 0.002	0.076 ± 0.009

Reading Reading : (i) With 400 measurements $\chi(4)$ was largest at $N_T = 6$ (0.089), and its difference from $N_T = 8$ was about 2.3 standard deviations. It looked like a sign of overshoot, but with 2000 measurements $N_T = 6$ gives 0.061 ± 0.004, so it was **a statistical fluctuation**. (ii) With 2000 measurements $\chi(4)$ increases for $N_T = 4 \to 8$, and at $N_T = 10$ it is 0.071 (1.7 standard deviations from $N_T = 8$), almost saturated. No significant overshoot is seen. This is consistent with monotonicity of the tension (a necessary condition of 26-02). (iii) $\chi(4)$ contains the Coulomb term, so it comes out larger than the four-dimensional $\sigma a^2 \approx 0.036$. We read only the trend in N_T, not the absolute value. (iv) The $N_T = 2$ row (Table 6) is in the region of lattice artefacts, with only 400 measurements.

Limits. β is the single value 2.5, and the plane the single size $M = 16$. From $\chi(5)$ on the errors are large, and the value of the tension itself is not determined. $N_T = 10$ is about 2 in $L\sqrt{\sigma}$, and the trend as $L \to \infty$ is an extrapolation.

06

The measurement of simultaneous flips, and why it cannot be used as a test of the band

A measurement that uses the Simultaneous-Flip Estimate of 25-06 ($\mathbb{E}\prod\theta \le \mathbb{P}(\text{no container flips})$) numerically. $SU(2)$, $\beta = 2.5$, containers placed with spacing B, and driven insertions (512 steps) with acceptance $\min(1, e^{-W})$ performed simultaneously in 4 containers per configuration. We write $\bar{a}$ for the mean acceptance rate.

Table 8 Container radius fixed at $r = 4$ ($B = 10$, $M = 20$, 512 steps)

N_T	Configurations	$\bar{a}$	Mean of W	W/N_T^2
4	30	0.36 ± 0.03	1.6	0.10
6	20	0.25 ± 0.04	2.6	0.07
8	10	0.12 ± 0.03	4.5	0.07
10	6	0.02 ± 0.01	8.2	0.08

Table 9 Remeasurement with container radius proportional to N_T ($r = N_T$, $B = 2r + 2$, $M = 2B$, 512 steps)

N_T	r	Configurations	$\bar{a}$	Mean of W
4	4	30	0.35 ± 0.03	1.6
6	6	11	0.34 ± 0.05	2.0
8	8	3	0.16 ± 0.08	4.5

Reading Reading : (i) Table 8: widening the T^2 with the radius fixed, the work increases roughly in proportion to N_T^2, and the acceptance rate falls. The size of the vortex is fixed by the size of the T^2, so the reading is that a container of radius $r < L$ cannot hold the vortex, which then pays a cost proportional to the area as a sheet inside a thin tube. The lateral-widening inequality is a claim about containers of size at least L, so this is not a counterexample. The guaranteed tension ($-\ln \mathbb{P}(\text{no flip})$ divided by the number of containers and B^2) was about 0.005 at $N_T = 4$. (ii) Table 9: if the containers are enlarged together with L, the acceptance rate is almost constant for $N_T = 4 \to 6$ ($N_T = 8$ cannot be judged from 3 configurations).

Why it cannot be used as a test of the band. This measurement **cannot be read** as "a numerical picture of the uniform slab tension (the crossover band)". The reason is as in 25-06 § 07: the guarantee of the Flow Lemma is $\theta \le 1 - a$, and obtaining the form of the band $\theta \le e^{-cL^2}$ needs a kernel with $a \ge 1 - e^{-cL^2}$, which requires bringing the overlap O of the measures of the forward and reverse paths to 1 exponentially in the area (in terms of dissipation, $D \lesssim e^{-2cL^2}$) (25-06 § 07). The kernels measured are far from that, so all that can be said is "for containers of size L, $\theta \le \theta_0$ (with θ_0 a constant less than 1)", and the resulting $\sigma \ge \ln(1/\theta_0)/L^2$ is the weak form that $U(1)$ also satisfies (26-01). **This measurement can be used only as a tool for measuring lower bounds on the slab tension with the T^2 fixed.** The continuation of Table 9 at $N_T = 6, 8$ has been stopped.

07

What cannot be said

(a) Proof. None of the numbers of this chapter is a proof, and the hypotheses of no inequality have been checked.

(b) The band. The trends of Table 2, Tables 3–5 and Table 7 do not show the crossover band ($\theta \le e^{-cL^2}$ uniform in L) numerically. The T^2 measured go only up to about $L\sqrt{\sigma} \approx 2$, and $L \to \infty$ is an extrapolation.

(c) Errors. The error estimates of driven insertion are still rough (sample standard deviations only), the errors of replica exchange are the record's estimates, and the treatment of autocorrelation is not in the record.

(d) Lattice artefacts. The rows with $N_T = 2$ are the region of lattice artefacts. Even at $N_T = 4$, small-lattice corrections remain.

(e) Balance of the control. $\beta = 1.5$ for $U(1)$ and $\beta = 2.5$ for $SU(2)$ are not a comparison with matched physical sizes.

(f) Simultaneous flips. Tables 8 and 9 cannot be read as a numerical picture of the band.

08

The table of five grades and the dependency graph

Table 10 The rows of this chapter (all numerical readings. Conditions in the scope column of each row)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The cost of insertion at 0 steps is roughly proportional to N_T^2	Reading	Table 1 (A, B, C)	$SU(2)$, $\beta = 2.3$, $M = 8$, $r = 2$, 8–12 configurations $\times$ 4	—
2	If the container is large enough, the driven work comes down to near 0	Reading	Table 1	$\beta = 2.3$ to 2.5, 256–1024 steps, includes $N_T = 2$ rows	—
3	The container radius needed roughly doubles each time the lattice spacing halves	Reading	Table 1 (E, F, G)	$\beta = 2.3, 2.5, 2.7$, little statistics	—
4	With radius 4 fixed, the work does not clearly increase for $N_T = 4 \to 8$	Reading	Table 2 (I)	$\beta = 2.5$, $M = 12$, 1024 steps, 10 configurations $\times$ 3, error 0.2–0.3	—
5	The work at $\beta = 2.7$, $r = 8$ is free energy, not dissipation	Reading (physics)	Table 1 (G), Table 2 (J)	$N_T = 4$, $M = 18$, the Jarzynski estimate fluctuates strongly	—
6	Widening the T^2 increases the ΔF of $U(1)$	Reading	Tables 3 and 4	$\beta = 1.5$, $M = 8$, 12, errors 0.04–0.78	—
7	Widening the T^2 does not increase, or decreases, the ΔF of $SU(2)$ (except $N_T = M = 8$)	Reading	Tables 3–5	$\beta = 2.4, 2.5$, $M = 8, 12$, errors 0.1–0.36	—
8	Rows 6 and 7 are a sign of the sign of the lateral-widening inequality	Reading (physics)	§ 04 ⟸ 6, 7	Weak discriminating power in the region where θ is $O(1)$	—
9	The overshoot at $N_T = 6$ is a statistical fluctuation	Reading	Tables 6 and 7	$\beta = 2.5$, $M = 16$, 400 and 2000 measurements	—
10	$\chi(4)$ increases for $N_T = 4 \to 8$ and nearly saturates at $N_T = 10$	Reading	Table 7	$\beta = 2.5$, $M = 16$, 2000 measurements, includes the Coulomb term	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
11	With the radius fixed, the work of simultaneous flips grows in proportion to N_T^2	Reading	Table 8	$\beta = 2.5$, $r = 4$, 6–30 configurations × 4 containers	—
12	The acceptance rate of one-step kernels is, in the range measured, at most 0.46 Barker-type and at most 0.73 Metropolis-type (Tables 8 and 9 at most 0.36 Metropolis-type)	Reading	Tables 1, 2, 8 and 9	The range of β, N_T and containers measured	—
13	The measurement of simultaneous flips cannot be used to test the band	Proved here	§ 06, 25-06 § 07 ⟸ 12	When only the Flow Lemma $\theta \leq 1 - a$ is used	— (the core is FluxLemma (`FluxLemma.flip_le_one_sub_abs`) of 25-06)

Only the row "the measurement of simultaneous flips cannot be used to test the band" is a row of logic rather than numerics: it is an application of the row "only the weak form follows" of 25-06, that only the weak form follows from $\theta \leq 1 - a$; the reading of the acceptance-rate row shows how far that takes effect.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  RDG  cost at 0 steps ∝ N_T²                          β = 2.3, M = 8
 2  RDG  driven work → near 0 in large containers        β = 2.3–2.5
 3  RDG  radius needed doubles per halving of a ⟸ 2
 4  RDG  radius 4, widening N_T: no increase             β = 2.5, 10 confs × 3
 5  RDG  work at β = 2.7 is free energy (physics)        Jarzynski fluctuates
 6  RDG  ΔF of U(1) grows with T²                        β = 1.5
 7  RDG  ΔF of SU(2) does not grow                       β = 2.4, 2.5
 8  RDG  sign of the lateral-widening inequality (physics) ⟸ 6, 7
 9  RDG  overshoot at N_T = 6 is a fluctuation           2000 measurements
10  RDG  χ(4) grows and saturates ⟸ 9                     includes Coulomb term
11  RDG  fixed-radius simultaneous flips ∝ area
12  RDG  one-step acceptance: Barker ≤ 0.46, Metropolis ≤ 0.73 (range measured)
13  PRV  simultaneous flips cannot test the band ⟸ 12   Flow Lemma of 25-06
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. There are no assumption rows. There is neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Reading Driven insertion brings the work down to near 0 if the container is larger than the vortex ($SU(2)$, $\beta = 2.3$ to 2.5)	That the upper bound on the work is uniform in L and the lattice spacing
Reading Widening the T^2 makes the vortex of $U(1)$ more expensive and that of $SU(2)$ not ($\beta = 2.4, 2.5$, $M = 8, 12$)	The lateral-widening inequality. Discrimination in the region where θ is small
Reading The string tension of slabs (Creutz ratios) grows with NT and saturates, with no significant overshoot ($\beta = 2.5$, 2000 measurements)	The value of the tension itself. The behaviour as $L \to \infty$
Proved here The guarantee of simultaneous flips cannot be used to test the band	A numerical test of the band with a kernel with $a \to 1$

Sources and reproduction

Item	Kind	Source / tools
All the numbers of Tables 1–9	Reading	The Monte Carlo record of this book's work (heat bath, driven insertion, replica exchange with BAR, Creutz ratios). Compared against the output records of each run; the standard deviations of Table 1 were copied from the output records
Row P of Table 1	Reading	The same measurement as the numerical section of the paper on vortex-insertion identities in A-07 (A-07)
Tables 3–5, Table 7	Reading	The same measurement as the numerical section of the paper on the reduction of the band in A-07
$L\sqrt{\sigma}$ and the four-dimensional $\sigma a^2 \approx 0.036$	Reading	Conversion in the record (by known values of $a\sqrt{\sigma}$; the values of the original source not compared)
The Jarzynski Equality	Cited	The source table of 25-06
The logic of why simultaneous flips cannot be used to test the band	Proved here	25-06 § 07

Words used in this chapter — driven insertion, number of steps, work, acceptance rate, the free energy of the global flux, replica exchange, BAR, Creutz ratio, overshoot, simultaneous flips, guaranteed tension. Their meanings are in A-02 Glossary (the words of Part 25 are to be added). The numerics of Part 26 (the test doubling the whole lattice, step functions, column biases) are in 26-09.

Next chapter to read: 25-09 Where this road stops — confirms that widening the T^2 is exactly the crossover band, and summarises what the road from the direction of space has reached.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space the band, lateral widening, the end of the road · Kiichi, Shiori and Rin

Where this road stops — widening the T² is the band

> This chapter is a map of open questions. It marks the boundary between the parts that close and the parts that do not.

What this chapter shows — At which stage the road from the direction of space, starting from a small twisted T^2, stops. It stops at the stage of widening the side L of the T^2 beyond the correlation-length scale $1/\Lambda$, and this is the crossover band (the band for short). At the two ends of the band — the region of dilute fractional instantons where L is small, and the confinement region where L is large — there is a prospect that the needed inequality holds, and all that remains is to pass non-perturbatively through the middle, $L \sim 1/\Lambda$.

Prerequisite chapters — 25-00 (the entrance of this part), 25-02 (vortices and fractional instantons), 25-04 (slab confinement), 16-09 (the band stage in the table of the seven stages a proof must contain).

Said first — This is a short guiding chapter, and it places no new proofs. Every statement crossing the band is an assumption, and the prospects at the two ends are physical estimates resting on one-loop and dilute-gas counting, or on the picture of confinement.

The order of this chapter

1. The stage where it stops — what is in hand on a small T^2, and the stage of widening
2. The two ends and the middle of the band — the dilute region, the confinement region, and between them (Table 1)
3. The statement to show in the band — the correct form of the band, and intermediate goals
4. To Part 26

01

The stage where it stops — what is in hand on a small T^2, and the stage of widening

The road of this part went by way of a small twisted T^2 in order to split the four-dimensional problem in two (25-00). Since the twist removes the zero modes, the theory stays weakly coupled at scales up to the size L of the T^2, and the vortex sheets winding the T^2 close locally as fractional instantons. In this way the problem splits into an intermediate goal where only the ultraviolet wall remains (the gap and confinement on a small T^2; among the seven stages a

proof must contain, the small-T^2 gap stage and the small-T^2 confinement stage) and a stage where all other walls gather (widening the T^2, the band stage).

What is in hand on the side of the small T^2 is all either conditional or stated with the lattice spacing fixed. With the lattice spacing and the T^2 fixed, the area law holds at every β, but the tension is fixed by the cost of thin vortices and becomes 0 in the continuum limit Proved here (25-04). The insertion identities and averaged bounds reduce confinement to a single local quantity Proved here (25-05). Confinement and the gap on a small T^2 with the lattice spacing sent to 0 are a strand resting on assumptions such as ultraviolet control Assumption . Ultraviolet control in this part is "that the same construction as the one published on 2026-11-10 stands under twisted boundary conditions"; the published construction is only for a periodic cubic box, so this is an open condition (the arrow of the small-T^2 gap and the arrow of the small-T^2 area law in 27-01).

What the problem statement asks for is the theory on $\mathbb{R}^4$, so $L \to \infty$ is needed. One must widen L from well below the correlation length $1/\Lambda$ to well above it, keeping a bound with the same constant at every L. This stage is where this road stops, and in this book we call it **the crossover band** (the band for short) Assumption . It is a different word from the "windows" of Parts 23 and 24 (the site window, the window in $\hbar$, the time window) (note in 26-00).

02

The two ends and the middle of the band

The statement to show in the band, in the language of the bias of the sector of a container $\theta(R; T^2_L)$ (container side R, side L of the transverse T^2, 25-03), has the form that the bias does not grow when the transverse T^2 is widened (the lateral-widening inequality, 26-02). The situation differs at the two ends and in the middle.

Table 1 The two ends and the middle of the band (Λ is the dynamical scale, L the side of the transverse T^2)

Region	Picture	When the transverse T^2 is widened	Label	Conditions written in the same row
$L \ll 1/\Lambda$ (dilute region)	Vortices are self-dual lumps of charge $\frac{1}{2}$ (fractional instantons), and the action $4\pi^2/g^2$ does not depend on the shape of the T^2. In the dilute gas $\ln(1/\theta) \propto A_D\, L^{5/3}$ (A_D is the area of the container)	$\ln(1/\theta)$ grows and the bias becomes smaller. The exponent $5/3 = 11/3 - 2$ is the margin by which the coefficient of asymptotic freedom beats the entropy of the two transverse dimensions	Reading (physics: one-loop and dilute-gas counting). The formula of the tension $\Lambda^2 (\Lambda L)^{5/3}$ is Cited	Dilute region, one loop, SU(2). For U(1) the same counting gives exponent -2

Region	Picture	When the transverse T^2 is widened	Label	Conditions written in the same row
$L \gg 1/\Lambda$ (confinement region)	The free energy of the vortex sheet in a container is tension $\times$ area, and the dependence on the area of the T^2 is only the entropy of the position of the vortex sheet (logarithmic)	The ratio of biases stays of order s^2	Reading (physics: on the picture of confinement)	A picture that assumes confinement. Not shown
$L \sim 1/\Lambda$ (middle)	Neither the weak-coupling picture nor the strong-coupling picture can be used	The statement to be shown itself	Assumption	Needs non-perturbative input. Structural inputs (reflection positivity, duality, the transfer matrix) alone do not give it (26-04)

Putting U(1) beside it as a control: the flux of $U(1)$ can only spread uniformly, and $U(1)$ on a slab becomes at long distances a two-dimensional $U(1)$ with coupling e^2/A_T (A_T is the area of the T^2), so $\ln(1/\theta) \propto A_D/A_T$, and the ratio of biases when the T^2 is widened (after over before) becomes arbitrarily large Reading (physics: the estimate counting of the paper on the reduction of the band). Weakly coupled four-dimensional $\mathbb{Z}_2$ breaks it in the same way. What separates SU(2) from U(1) is that non-abelian flux can localise into lumps of finite action independent of scale (26-03). The measured record (the vortex free energy when the T^2 is widened goes up for U(1) and not for SU(2); the slab tension grows and saturates) is in 25-08 and 26-09 Reading .

03

The statement to show in the band — the correct form of the band, and intermediate goals

The goal of the band is not "the bias stays uniformly away from 1" ($\sup_L \theta(L;T_L^2) \le \theta_0 < 1$). This form is satisfied by four-dimensional $U(1)$ too, and the tension it gives is $\ln(1/\theta_0)/L^2 \to 0$ (26-01). The correct form of the band is that the vortex free energy of the container becomes small exponentially in the area,

$$\ln \frac{1}{\theta(L;T_L^2)} \;\ge\; c\,L^2 \qquad \text{(in physical units, uniform in the lattice spacing } a\text{)} \quad (1)$$

a local version, conditioned on the frame, of 't Hooft's criterion for confinement Assumption . (1) follows by iterating doubling, given the base point, frame decorrelation and the lateral-widening inequality (the Doubling-Iteration Theorem of 27-01 § 03 and its premises). In the language of the chessboard, the content of (1) becomes two uniformities: the uniform slab tension and the uniform slab gap (26-05).

As intermediate goals before reaching (1), two weaker statements have been named Assumption . (a) That the vortex free energy of the container approaches 0 as $L \to \infty$ (a restatement of there being no deconfined phase). (b) $\ln(1/\theta) \geq cL^2/\ln L$ (an almost-area law). Neither has been shown.

04

To Part 26

This road came as far as just before the band and stopped. What is needed in the band can be narrowed in form with the tools of this part: the correct form of the band (1), the reduction to one inequality, where non-abelianness enters, and that structure alone cannot cross, in that order. The road from the direction of time (Parts 23 and 24) also arrives at a uniformity of the same type when its four-dimensional goal is written in the form "a constant uniform in $\beta \to \infty$" (23-15 § 08). That the two roads meet at this band is treated in the entrance of Part 26.

What this chapter can say

What is said	What is not said
That the stage at which the road from the direction of space stops is the stage of widening the transverse T^2 beyond the correlation-length scale (the band). The correct form of the band (1), and the naming of the exclusion of a deconfined phase (a)	Each of (1), (a) and (b) is Assumption
The prospects at the two ends: $5/3 = 11/3 - 2$ in the dilute region, logarithmic entropy in the confinement region Reading (physics)	Passing non-perturbatively through the middle $L \sim 1/\Lambda$. Proofs of the prospects at the two ends

Sources and reproduction

Item	Kind	Source / tools
The string tension $\Lambda^2(\Lambda L)^{5/3}$ on a small T^2	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166, equations (2.32) and (2.37) (the source table of 25-02)
$5/3 = 11/3 - 2$ in the dilute region, -2 for $U(1)$	Reading (physics)	The proposition "the inequality separates theories" in the paper on the reduction of the band (draft, A-07) (the paper itself states that it is an estimate)
The correct form of the band (1) and the counterexample $U(1)$	Assumption, Reading	26-01. The Tomboulis–Yaffe inequality is Cited

Item	Kind	Source / tools
Exclusion of a deconfined phase (a)	Assumption	The row "positivity of the 't Hooft tension" in the ledger of 27-02

Words used in this chapter — the crossover band (the band), the lateral-widening inequality, the bias of the sector $\theta(R;T_L^2)$, the dilute region, the confinement region, intermediate goals. Their meanings are in A-02 Glossary.

Next chapter to read: 26-00 Entrance — confluence — that the two roads arrive at a statement of one type, "the same constant at all scales".

Entrance — confluence

> This chapter is a guide. § 03 and § 05 are a map of open questions.

This part is a record of the study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — That the road entering from the direction of time (Parts 23 and 24) and the road entering from the direction of space (Part 25) both arrive at a statement of one type: "it holds at every scale with the same constant". In this part we call the interval of scales that this statement has to cross **the crossover band** (**the band** for short). In the figure of the ladder of scales (Figure 2) we see how far the weak-coupling tools and the strong-coupling tools each reach, and where the gap is. We put the breakdown of the band (the goal, the reduction, the negative lemmas, the electric flux, the place of the non-abelian, β-monotonicity, the renormalisation comparison, the chessboard) into one table, and give a map of the thirteen chapters of Part 26.

Prerequisite chapters — 23-15 § 07, 23-15 § 08 (the one line on uniformity), 24-07 § 01 (the dimensionless coupling), 4-07 (dimensional transmutation), 25-00 (the entrance to the road from the direction of space), 16-09 (the seven stages a proof must contain).

Said first — This page is a map and sets out no new theorem. Figure 2 is a figure of physical estimates, and each row carries a label. "The two roads meet" is a statement at **the level of the goal**. It is not a statement that the **walls at which the roads now stop** are the same (§ 05). No chapter of this part shows the mass gap, or confinement in the continuum limit.

The order of this chapter

1. One type that the two roads reach — "the same constant at every scale"
2. A note on words — the "band" of this part and the "window" of Parts 23 and 24 are different things
3. The ladder of scales — which tool works at which scale (Figure 2)
4. The breakdown of the band — eight components and their labels (Table 1)
5. Confluence at the level of the goal, difference at the level of the wall (Table 2)
6. The map of Part 26 — thirteen chapters, and the one line of 24-07 (Table 3)
7. The table of the five grades and the dependency graph

01

One type that the two roads reach — "the same constant at every scale"

From Part 23 onwards, this book has approached the problem of the mass gap from two directions.

The first is the road from the direction of time (Parts 23 and 24). Around **the diffusive-relaxation-time assumption** — that the time over which fluctuations are smoothed out grows as the square of the correlation length (the upper side is the relaxation-time upper-bound assumption, the lower side the relaxation-time lower-bound assumption; 23-01 § 05) — we compared the relaxation time on the lattice with the correlation length. The statement aimed at on the four-dimensional lattice is the physical-unit target level of Table 5 in 23-15 § 07: the mass in physical units of the lattice, $m_{\mathrm{lat}}(\beta)/a(\beta)$, tends to a positive finite value. To get there by combining the one-sided bridge (23-01 § 06) with the upper-bound assumption, the constant K of the upper-bound assumption must be uniform as $\beta \to \infty$.

The second is the road from the direction of space (Part 25). We wrap two directions of space on a small torus T^2_L with the 't Hooft twist $n_{12} = 1$, and look at confinement on the remaining plane $\mathbb{R}^2$. If T^2_L is small enough, the twist removes the zero modes, the theory stays weakly coupled up to the scale L, and the vortices can be treated as fractional instantons (25-00, 25-02). The goal of this road is that the string tension $\sigma_{\mathrm{eff}}(L)$ of the two-dimensional theory obtained by shrinking the transverse directions to T^2_L is **positive uniformly in** L, that is,

$$\inf_{L \geq L_0} \sigma_{\mathrm{eff}}(L) \;>\; 0 \qquad \text{(physical units, uniform in the lattice spacing } a\text{)} \tag{1}$$

(the uniform slab tension; the definition is in 26-02).

The two goals have the same form. Neither is "it holds at some scale"; both are statements that "**it holds at every scale, with the same constant**". They ask that the constant does not break down at any intermediate scale, from the lattice spacing a across the physical length $1/\Lambda$ up to large scales. The one line of 23-15 § 08 — what is missing is not the number of steps but uniformity, holding at every scale with the same constant — was a statement of this type written from the side of the first road. We also saw at the end of 23-15 § 07 that it appears in changed form: uniform as $\hbar \to 0$ at the first step (0+1 dimensions), uniform as $\beta \to \infty$ on the four-dimensional lattice.

In four dimensions, between the weak-coupling scales and the strong-coupling scales, there is a scale $L \sim 1/\Lambda$ at which the size of the coupling is of order 1. Λ is a scale born quantum mechanically through dimensional transmutation (4-07), and measured in lattice units it diverges as $a \to 0$. Neither of the two goal statements holds unless it passes through this scale with a uniform constant. In this part we call the interval of scales around $L \sim 1/\Lambda$ **the crossover band**, and we call the statement the two roads reach "a statement that crosses the band".

This confluence also agrees with an arrangement made outside our study. In the division of the continuum tasks (the task of a uniform gap in 16-07), the uniform gap $\inf_L \Delta_L > 0$ is set as one task, and the two roads attack it in two languages. In the table of the seven stages a proof must contain in 16-09, the band corresponds to the "band stage".

02

A note on words — the "band" of this part and the "window" of Parts 23 and 24 are different things

Parts 23 and 24 used the word "window" in at least three senses. **The "band" of this part is different from all of them.** If the words were the same, the two roads would read as if they stopped at the same place, so we separate them here.

Table 0 The "window" and the "band" compared, and notes on other words and symbols

Word	Meaning	Chapters	Relation to the band
The window of sites (the window of A_W, the window mark σ_W, the marginal log-Sobolev input for the window)	An interval of finitely many sites on the path in 0+1 dimensions	23-09	Of finite length in lattice units. Not the band
The window in $\hbar$	The range of $\hbar$ that can be checked with computer assistance	23-07, 23-08, 23-15	An interval of a parameter. Not an interval of scales
The time window (the window factor φ, the In-Window Supremum Proposition)	Test functions restricted to an interval of time	24-04, 24-05	The name of a tool. Not the band
The crossover band (the band)	The interval around the scale $L \sim 1/\Lambda$ at which weak coupling gives way to strong coupling	Part 26	—
The "band" of Parts 23 and 24	The "band statement" of 23-01 § 05 and 23-15 (5) (the statement that a quantity lies in a band of width a constant multiple of the square of a length), ℓ_{band} of 23-09 (a length concerning the summit band), and the "two-dimensional strip" of 24-02 § 03, § 05 and 24-09 (a narrow strip-shaped lattice). The "twisted strip" of the two-dimensional model in 25-07 § 06 also means a strip-shaped lattice	23-01, 23-09, 23-15, 24-02, 24-09, 25-07	A word with another meaning. Not the band of this part (an interval of scales)
Three meanings of θ	The threshold θ of 23-09 (θ_B, θ_σ and others, values that divide configurations into the valley side and the summit side), the sector bias $\theta(R; T_L^2)$ of this part ((4) of 25-01), and the vacuum angle θ of Tanizaki–Ünsal (set to $\theta = 0$ in 25-02)	23-09, 25-01, 25-02	Only the sector bias is a quantity of this part. The other two are different quantities

Word	Meaning	Chapters	Relation to the band
L and L_0	In 26-11, L is the blocking factor and L_0 is the side of the box (as in 7-05)	7-05, 26-11	Different from the L of Parts 25 and 26 (the size of T^2)

There are other clashes of symbols, so in this part we write by the following conventions.

- We call by name the property that the vortex free energy increases monotonically in β — "**β-monotonicity**" — and the inequality comparing the vortex free energy of the doubled lattice with that of the coarse lattice — "**the renormalisation comparison**".
- We call the assumptions only by name: **the base point at small T^2**, **frame decorrelation**, **the lateral-widening inequality** (strong form, weak form), **the uniform slab tension**, **the uniform slab gap**.
- We write the chessboard density as κ_{cb} (to distinguish it from κ of Part 23).

03

The ladder of scales — which tool works at which scale

Figure 2 sets out, in the language of the road from the direction of space, the scale L (the size of the small torus T^2_L, or the size of the box being compared) vertically. The bottom is short distance (the lattice spacing a), the top is long distance. Each row carries a label at its right end. Figure 1 (the flow of the logic) is in 16-09. The number and title of Figure 2 are the same as those of Figure 2 in 25-00.

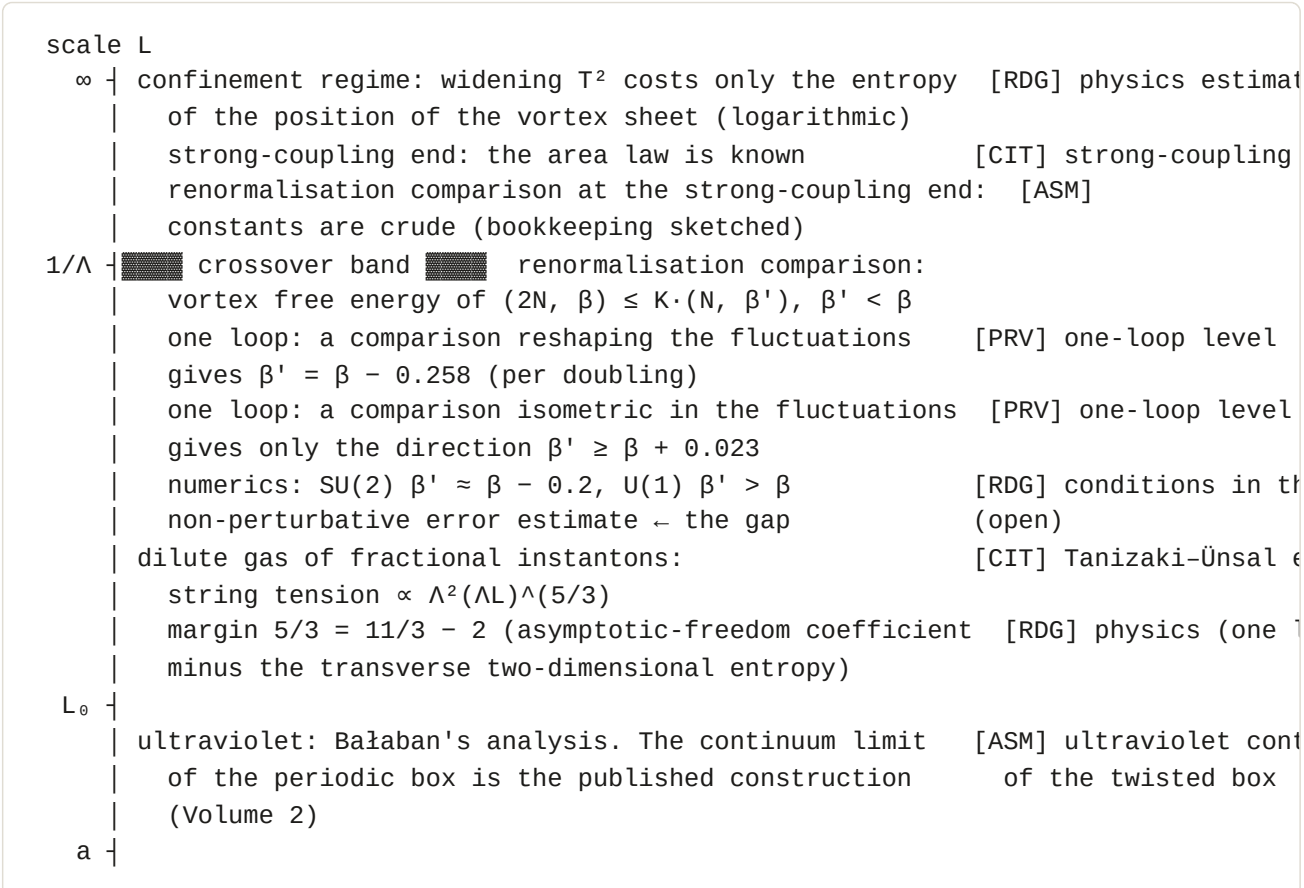

Figure 2 The ladder of scales. The weak-coupling tools (bottom) and the strong-coupling tools (top) each work at their own end; only the non-perturbative estimate in the band (L ~ 1/Λ) is missing. Labels are per row.

Just three points on how to read the figure.

(i) The shift of β per doubling. "β′ = β − 0.258" is the value coming from the coefficient $11/3 = -1/3 + 4$ of the one-loop heat kernel of fluctuations in a background magnetic field; in the convention $\beta = 4/g^2$ it is $\beta - \beta' = 8b_0 \ln 2$, with $b_0 = 22/(48\pi^2)$. A comparison that reshapes the fluctuations reproduces this value. In a comparison isometric in the fluctuations (the identity map, translation, gauge transformation, Migdal–Kadanoff-type decimation), the spin term $+4$ does not contribute to the first-order estimate, and only $\beta' \geq \beta + 0.023$ on the $-1/3$ side (the same direction as U(1)) comes out. Both are **statements at the one-loop level**, not non-perturbative comparisons. The breakdown into tables and the computation are in 25-07; the comparison itself is treated in 26-06.

(ii) The conditions of the numerical "β′ ≈ β − 0.2". This is a value read from measurements of the vortex free energy ΔF for $SU(2)$ with twist $n_{12} = 1$, by replica exchange and Bennett's acceptance-ratio method: the whole lattice was doubled ($N_T \times M = 4 \times 8 \to 8 \times 16$ and $4^4 \to 8^4$), and the value on the large lattice was fitted to the β' curve of the small lattice. For $\beta = 2.5$–3.0, with 1–3 seeds, the $\beta - \beta'$ read off is $+0.13$–$+0.25$ (the width of the reading is of order ± 0.05–0.1; judging from the differences between seeds, the errors are rather underestimated). For $U(1)$ at $\beta = 1.5$, $4 \times 8 \to 8 \times 16$, $\beta - \beta' < -0.2$ (the opposite direction). The lattices are small and the statistics few, so this is a reading of the direction, not a determination of the value [Reading]. The table is in 26-09.

(iii) The gap. The strong-coupling end and the one-loop end are each held by a different tool. In the band in the middle, there is no tool that estimates the error of the comparison non-perturbatively (at intermediate coupling, large fields included). The "▒" row of the figure is the subject of this whole part.

04

The breakdown of the band — eight components and their labels

Table 1 lists, in the language of the road from the direction of space, the components involved in the statement that crosses the band. Labels are per row, and conditions are written in the same row. The labels mean what they mean in the five grades of Part 23: [Proved here] a proof set out on paper; [Cited] known, checked in the original source; [Computer-assisted]; [Reading] a numerical reading (physical estimates are noted "physics"); [Assumption]. The Lean column is outside the labels and lists names and assumptions.

Table 1 The breakdown of the band (labels per row; "—" marks a row with no machine checking)

Component	Content in one line	Label	Conditions in the same row	Lean	Chapter

Component	Content in one line	Label	Conditions in the same row	Lean	Chapter
Goal	"The sector bias θ stays away from 1 uniformly in the scale" is not enough. What is needed is $\ln(1/\theta(L,L)) \geq cL^2$ (an area exponent). To the area law by the Tomboulis–Yaffe inequality	Proved here (the logic of insufficiency), Cited (Tomboulis–Yaffe), Reading (the U(1) count, physics)	The U(1) row: Coulomb phase, two-dimensional U(1) at long distance, an estimate	—	26-01
Reduction	Under the three of the base point, frame decorrelation and the lateral-widening inequality, the goal follows from the doubling iteration $\theta(2R) \leq K\theta(R)^4$	Proved here (conditional)	Under the three assumptions. $L \geq L_{\min}(a)$, with no lattice artefacts. On a dyadic sequence of scales	`Doubling.iterate`, `decay` (only the algebra of the iteration)	26-02
Negative lemma	Reflection positivity, the centre and duality identities, and the positivity of the transfer matrix hold in the same way for $\mathbb{Z}_2$ and U(1), so the lateral-widening inequality does not follow from them alone	Proved here	In the form "from this set of inputs alone"	—	26-04
Electric flux	$\theta = Y(1)/Y(0) \geq 0$ (the ratio of the partition functions with electric flux 1 and 0; a lattice version of 't Hooft duality)	Proved here	For $\theta \leq 1$, the positivity of the twisted partition function is used as an assumption	`FluxDuality` (the algebra of the spectral form)	26-04
The place of the non-abelian	In the dilute regime $\ln(1/\theta) \propto A_D L^{5/3}$, with the exponent margin $5/3 = 11/3 - 2$. For U(1), -2	Reading (physics), Cited (Tanizaki–Ünsal)	Dilute regime, one loop. The place where the non-abelian works "within this reduction"	—	26-03

Component	Content in one line	Label	Conditions in the same row	Lean	Chapter
β-monotonicity	The vortex free energy increases in β. A problem of the same weight as the β-monotonicity of the string tension in lattice units	Assumption (unproved for SU(2))	Proofs exist only in limited cases (small two-dimensional lattices; near $\beta \to 0$ and $\beta \to \infty$ on a fixed lattice), and the book does not set them out	—	26-06
Renormalisation comparison	The vortex free energy of $(2N, \beta) \leq K \cdot (N, \beta')$, $\beta' < \beta$. The only place asymptotic freedom enters	Proved here (one-loop level), Reading (the doubling test), Assumption (the non-perturbative comparison)	On the strong-coupling side, crude bookkeeping of constants; on the weak-coupling side, the one-loop level. U(1) goes the other way	—	26-06
Chessboard	Under reflection positivity in z, frame decorrelation is replaced by a single density κ_{cb}. The band is concentrated into the uniform slab tension and the uniform slab gap	Proved here (the chessboard at fixed L), Assumption (the uniform slab tension, the uniform slab gap)	If L is fixed. Making it uniform in L is open	`ReflPos` (the algebra of the kernel)	26-05

What the Lean column guarantees is only the algebra of finite sums and real numbers. That θ is the sector bias, the Haar measure and the positivity of the transfer matrix lie outside Lean. Names and assumptions are written in the classification table of each chapter.

Of the rows of Table 1, **non-abelian dynamics enters only the rows concerned with "the lateral-widening inequality" and "the uniform slab tension"** (the third assumption of the reduction, the place of the non-abelian, the renormalisation comparison). These have a form that fails for U(1) and $\mathbb{Z}_2$. The other rows are structural facts that hold for any group, or arrangements of logic.

05

Confluence at the level of the goal, difference at the level of the wall

The phrase "the two roads reach the same wall" is **correct at the level of the goal, and not accurate at the level of the walls at which they now stop**.

The correct part is as in § 01. The four-dimensional goal of Part 23 (the physical-unit target level of 23-15: the upper-bound assumption and the one-sided bridge with the same constant

as $\beta \to \infty$) and the goal (1) of the road from the direction of space are both statements that must cross the band.

The inaccurate part is the **present wall** of Part 23. The "summit-curvature shortfall wall" of 23-09 § 06 (that the diagonal $15/4$ of one site at the summit does not balance the sum of couplings 4) was the wall of the first step of the staircase, on the double well in 0+1 dimensions. The length at which things cross over there is a **classical** length fixed by the first conjugate point of the Jacobi equation at the summit; it does not depend on $\hbar$ and is finite in lattice units. The length $1/\Lambda$ at which things cross over in the band is born quantum mechanically and diverges in lattice units. Table 2 sets out the differences.

Table 2 The wall of the first step and the band (details in 26-07)

	The wall of the first step (Part 23)	**The crossover band (Part 26)**
Place	0+1 dimensions, double well, the lattice with $\varepsilon = \frac{1}{2}$	Four-dimensional lattice, $SU(2)$
Crossover length	The first conjugate point at the top (classical, independent of $\hbar$)	$1/\Lambda$ (born quantum mechanically through dimensional transmutation)
Size in lattice units	Finite	Diverges as $a \to 0$
Tools at the two ends	Convexity for short chains, mixing for long chains	One loop and ultraviolet analysis at weak coupling, expansion at strong coupling
Type	The same: there are different tools at the two ends, the estimate of the worst configuration stops in the middle, and there is a prospect that uniformity comes out on moving to averaging tools	

So the wall of the first step is **a miniature** of the band, but it is not the band itself. At the first step one can think of a road that crosses with a finite block longer than the crossover length and strong mixing (labelled [Assumption]), but in four dimensions there is no "finite block longer than the crossover length". This difference, and what the two have in common in type, are written from the side of time in 26-07. The reason for the difference — that the four-dimensional coupling is dimensionless and has no classical length — is in 24-07 § 01.

06

The map of Part 26 — thirteen chapters, and the one line of 24-07

Table 3 The chapters of Part 26 (labels are those of the main rows of each chapter; conditions in the same row)

Chapter	Title	One line	Main labels and conditions
26-00	Entrance — confluence	This page	Map (physics figure + the label of each row)
26-01	The correct form of the band	θ moving away from 1 is not enough; $\ln(1/\theta(L,L)) \geq cL^2$ is needed	[Proved here] the logic of the shortfall; [Cited] Tomboulis–Yaffe; [Reading] the U(1) count (physics)

Chapter	Title	One line	Main labels and conditions
26-02	Reduction to a single inequality	The lateral-widening inequality (strong form, weak form) and iterated doubling. The Conditional Reduction Theorem	[Proved here] under the three assumptions of the base point, frame decorrelation and lateral widening
26-03	The only place where non-abelian enters	$5/3 = 11/3 - 2$; the contrast with U(1) and $\mathbb{Z}_2$	[Reading] physics (dilute regime, one loop; "within this reduction")
26-04	Structure alone cannot cross	Reflection positivity, duality and the transfer matrix are the same for $\mathbb{Z}_2$ and U(1). $\theta = Y(1)/Y(0) \geq 0$	[Proved here] "with this set of inputs alone"
26-05	The chessboard and simultaneous flips	Frame decorrelation turned into the density κ_{cb}. The band is concentrated into the uniform slab tension and the uniform slab gap	[Proved here] for fixed L; [Assumption] uniformity
26-06	Renormalisation comparison — a comparison that distinguishes groups	Band = β-monotonicity + renormalisation comparison. Asymptotic freedom enters only through $\beta' < \beta$	[Proved here] one-loop level; [Reading] doubling test; [Assumption] non-perturbative
26-07	The band seen from the side of time	The form of the statement that corresponds to the band on the road of Part 23. The wall of the first step is a miniature of the band but not the band	[Proved here] the computation of the first conjugate point; [Assumption] the statement that crosses the band itself
24-07	Precisely because it is four-dimensional	Kept in Part 24. The dimensionless coupling and topology — why the length of the band is born quantum mechanically	The label of each row as in 24-07
26-08	A map of the roads tried and stopped	The roads tried and why they stopped. Tomboulis's (2007) claim, and the criticism that it does not distinguish U(1)	[Cited] only literature that has been checked; unchecked items are marked
26-09	The measured record	Widening T^2 makes the U(1) vortex costlier, but not the SU(2) one. The slab tension increases and saturates. The test of doubling the lattice	[Reading] lattice size, β, number of seeds and error on each row
26-10	Candidate new viewpoints	Four faces of the middle band. Including viewpoints that the study found negative	[Reading] physics estimate

Chapter	Title	One line	Main labels and conditions
26-11	Ultraviolet control — what it means to close at infinitely many steps	The foundation under both roads. What closing at one step means, three difficulties, the five steps of the series and where it stopped, a proposed statement	[Proved here] Background–Fluctuation Decomposition Lemma, Coupling-Smallness Induction Lemma (the one-step remainder = the net one-step bracket of the coupling is unproved); [Cited] the five steps of Bałaban's series; [Reading] the class of one-link heat kernels (one link, registered range); [Assumption] the ultraviolet statement for a small twisted four-dimensional torus (proposal, unproved)
26-12	A sieve for arguments about the band — seven test models	Six models besides U(1), listing for each type of argument which models it fails. A table sorting this book's theorems and the candidate statements for the band into true, false or undefined for each model	[Cited], [Reading], [Proved here], [Assumption] (only what was checked in the literature is Cited; computations on small lattices are conditional)

We recommend reading in numerical order. In 26-01 and 26-02 we narrow the band down to a single inequality, in 26-03 to 26-05 we see what enters where in that inequality, and in 26-06 and 26-07 we look at the band itself from two directions. 26-08 to 26-10 are records and candidates. 26-11 is the foundation under the two roads (ultraviolet control), the content of the "ultraviolet control" row of the ledger of conditions. 26-12 is a list of models against which to test arguments about the band. The conditional statement as a whole is gathered in Part 27 (27-01).

07

The table of the five grades and the dependency graph

The labels of the rows of Tables 1 and 3 are fixed in the classification table of each chapter. The rows this page sets out itself are the following twelve.

Table 4 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	Λ is born through dimensional transmutation and, in lattice units, diverges as $a \to 0$	Cited	4-07	One- and two-loop flow	—
2	A comparison that reshapes the fluctuations gives $\beta - \beta' = 8b_0 \ln 2 = 0.258$	Proved here	25-07	One-loop level, per doubling	—
3	A comparison isometric in the fluctuations gives only the direction $\beta' \geq \beta + 0.023$	Proved here	25-07	One-loop level	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
4	The doubling test: $\beta - \beta' \approx +0.13$–$+0.25$ for SU(2), the opposite direction for U(1)	Reading	§ 03 (ii), 26-09	$4 \times 8 \to 8 \times 16$, $4^4 \to 8^4$, $\beta = 2.5$–3.0, 1–3 seeds	—
5	The string tension on a small T^2, $\propto \Lambda^2 (\Lambda L)^{5/3}$	Cited	Tanizaki–Ünsal eq. (2.37)	Dilute regime, one loop	—
6	The margin $5/3 = 11/3 - 2$ (the coefficient of asymptotic freedom beats the transverse entropy)	Reading (physics)	26-03	Dilute regime, one loop	—
7	The area law at strong coupling	Cited	5-02	Small β	—
8	The renormalisation comparison at the strong-coupling end	Assumption	26-06 (bookkeeping sketched)	Very small β	—
9	Ultraviolet control (the twisted small-T^2 box; the continuum limit in the periodic cubic box was shown by the published construction, Volume 2)	Assumption	7-07, 27-02	From a to L_0	—
10	The goals of the two roads are both statements of "the same constant at every scale", and cross the band	Reading (physics)	§ 01	The level of the goal	—
11	The crossover length at the wall of the first step is classical and finite in lattice units	Proved here	26-07 § 02	0+1 dimensions, $\varepsilon = \frac{1}{2}$	—
12	The wall of the first step is a miniature of the band, not the band	Reading (physics)	§ 05	A comparison of types	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  dimensional transmutation: Λ is quantum, diverges in lattice units  4-07
 2  PRV  reshaping comparison gives β − β' = 0.258                          one-loop leve
 3  PRV  isometric comparison: direction β' ≥ β + 0.023                     one-loop leve
 4  RDG  doubling test: SU(2) β' < β, U(1) the opposite                     small lattice
 5  CIT  string tension ∝ Λ²(ΛL)^(5/3)                                      dilute, one
 6  RDG  margin 5/3 = 11/3 − 2 ⟸ 5                                           physics
 7  CIT  strong-coupling area law                                           small β
 8  ASM  renormalisation comparison at the strong-coupling end              bookkeeping s
 9  ASM  ultraviolet control                                                a to L₀
10  RDG  the goals of the two roads are of one type and cross the band ⟸ 1   level
11  PRV  the crossover length of the first step is classical, finite        26-07 §02
12  RDG  the first-step wall is a miniature of the band, not the band ⟸ 1, 10, 11
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The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The rows of Figure 2 are a rearrangement of the rows of Table 4 from the reshaping comparison to ultraviolet control, and the figure itself creates no new node. The conclusion-side nodes (that the goals of the two roads are of the same type, and that the wall of the first step is a miniature of the band) are "Reading (physics)", and this page has no node for the statement that crosses the band itself (the uniform slab tension, the uniformity of the upper-bound assumption). That is placed as an assumption in the ledger of conditions of Part 27.

What this chapter can say

What is said	What is not said
Reading (physics) That the goals of the two roads are of the same type (the same constant at every scale), and that both cross the band	Crossing the band. The goal of either road
Proved here The direction of the comparison in the band at the one-loop level (the difference between the reshaping comparison and the isometric comparison)	The non-perturbative comparison. Estimating the error at the intermediate couplings of the band
Reading That in the doubling test on small lattices the directions for SU(2) and U(1) separate	Determining the value. Behaviour on large lattices
Reading (physics) That the wall of the first step is a miniature of the band, not the band	Getting past the wall of the first step. Whether the tools of the first step cross the band

Sources and reproduction

Item	Kind	Source / tools
The string tension on a small twisted T^2	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166 (Prog. Theor. Exp. Phys. 2022), eqs. (2.32) and (2.37). Chapter 25-02
Upper bound on the Wilson loop (from the vortex free energy)	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. T. Kanazawa, arXiv:0808.3442 (generalisation to SU(N)). Chapter 26-01
The strong-coupling area law, dimensional transmutation, the multiscale ultraviolet analysis	Cited	Sources of chapters 5-02, 4-07, 7-07
The coefficient of the one-loop heat kernel and the shift per doubling	Proved here	Chapter 25-07 (one-loop level)
Numbers from the doubling test	Reading	Replica exchange and Bennett's acceptance-ratio method. The table in chapter 26-09 (lattice size, β, seeds, errors)
The goal of Part 23 and the wall of the first step	This book	23-15 § 07, § 08, 23-09 § 06, 24-07 § 01

Terms used in this chapter — the crossover band (the band), the ladder of scales, the uniform slab tension, the lateral-widening inequality, frame decorrelation, the base point at small T^2, β-monotonicity, the renormalisation comparison, the chessboard density κ_{cb}, the diffusive-relaxation-time assumption. Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-01 The correct form of the band — why "θ moves away from 1" is not enough.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band the sector bias, the area exponent, Tomboulis–Yaffe · Kiichi, Shiori and Rin

The correct form of the band

> This chapter is this book's own investigation. § 06 is a map of open questions.

What this chapter shows — How the statement that crosses the band should be written in the language of the sector bias θ. The naive form "θ stays away from 1 uniformly in the scale" is not enough: the lower bound on the string tension that the container estimate gives from that form is $\ln(1/\theta_0)/L^2$, which goes to 0 as $L \to \infty$, and weakly coupled four-dimensional U(1) can be estimated to satisfy that form. What is needed is the form in which the bias becomes small with an area exponent, $\ln(1/\theta(L, L)) \geq cL^2$; this is the local form of 't Hooft's criterion for confinement, conditioned on the frame. We also see how the Tomboulis–Yaffe inequality connects decay of the vortex free energy with an area exponent to the area law.

Prerequisite chapters — 26-00 (the band), 25-01 (the twist and 't Hooft flux), 25-03 (containers and the sector bias), 25-04 (slab confinement), 5-06 (U(1) and the non-abelian case).

Said first — The proofs in this chapter are only manipulations of elementary inequalities. The count for the Coulomb phase of U(1) is an estimate by an effective theory ([Reading], physics), not a theorem. Of the conclusion "the naive form is not enough", the part that can be said on paper is "the lower bound that the container estimate gives from that form goes to 0"; the part "from that form, confinement does not follow by any road" stands on the U(1) estimate. This chapter shows neither confinement nor the correct form (7).

The order of this chapter

1. The naive form — θ stays away from 1 uniformly in the scale
2. What follows from the naive form — the lower bound on the tension goes to 0 as L→∞
3. The count for the Coulomb phase of U(1) — it satisfies the naive form and does not confine
4. The correct form — an area exponent and the Tomboulis–Yaffe inequality
5. The touchstone — the behaviour for U(1), $\mathbb{Z}_2$ and SU(2) (Table 1)

6. What this form imposes on the band
7. The table of the five grades and the dependency graph

01

The naive form of the band — θ stays away from 1 uniformly in the scale

We take the notation from 25-03. We write the Wilson action of $SU(2)$ on the lattice $(\mathbb{Z}/L)^2 \times (\mathbb{Z}/M)^2$ as the ensemble summed over Mack–Petkova twists (links U and closed ± 1 variables z_p, with weight $\exp(\frac{\beta}{2}\sum_p z_p \operatorname{tr} U_p)$), and fix the background twist $n_{12} = 1$ in the two transverse directions 1 and 2. For a disc or square D in the plane (directions 3 and 4), we call $D \times T^2_L$ the **container**, the $\mathbb{Z}_2$ flux $\sigma_D = \prod_{p \in D \times \{y_0\}} z_p$ through D its **sector**, and define the **sector bias**, conditioned on all the variables outside the container (the **frame**) Φ, as

$$\theta_D(\Phi) \;:=\; \big|\, \mathbb{E}[\sigma_D \mid \Phi] \,\big| \;\in [0,1] \quad (1)$$

We write $\theta(R; T^2_L)$ for its value at a typical frame for a container of size R with transverse T^2_L, and in particular $\theta(L,L) := \theta(L; T^2_L)$. Given the frame, the insides of the containers are conditionally independent, and for disjoint containers D_j inside a loop C in the plane

$$|\langle W(C)\rangle| \;\le\; \mathbb{E} \prod_j \theta_{D_j}(\Phi) \quad (2)$$

holds (the Mack–Petkova container estimate Cited , 25-03).

The line of the road from the direction of space was this: confinement is visible on a small T^2_L (25-04), and carrying it to $L \to \infty$ gives confinement on $\mathbb{R}^4$. So the first statement of the band that comes to mind is the following **naive form of the band** (the naive form, below).

(naive form) $\qquad \sup_{L \ge L_0} \theta(L,L) \;\le\; \theta_0 \;<\; 1 \qquad$ (at a typical frame, uniform in the lattice sp

It is the statement that "even if the container and the transverse torus are enlarged together, the sector bias does not stick to 1" — disorder remains at every scale.

02

What follows from the naive form — the lower bound on the tension goes to 0 as L → ∞

Tension Lower Bound from the Naive Form Proved here Suppose that, for some L, when the plane is tiled with square containers of side L, the product on the right-hand side of (2) splits into the product of the individual factors up to a constant $C_{\mathrm{D}} \geq 1$ (frame decorrelation Assumption), and that each factor is at most θ_0. Then for a square loop C of area A (with side a multiple of L)

$$|\langle W(C)\rangle| \;\leq\; C_{\mathrm{D}}\,\theta_0^{A/L^2} \;=\; C_{\mathrm{D}}\,e^{-\sigma_L A}, \qquad \sigma_L := \frac{\ln(1/\theta_0)}{L^2}. \tag{4}$$

Proof Place A/L^2 square containers of side L inside C and use (2) and the assumption: $|\langle W(C)\rangle| \leq C_{\mathrm{D}} \prod_j \theta_0 = C_{\mathrm{D}}\theta_0^{A/L^2}$. Rewriting the exponent gives (4). ∎

The tension σ_L in (4) is positive for each L. But the naive form (3) gives only *one* constant θ_0, so the lower bound on the tension obtained by this road is $\ln(1/\theta_0)/L^2$, which goes to 0 in the four-dimensional limit ($L \to \infty$, where the slab $T^2_L \times \mathbb{R}^2$ approaches $\mathbb{R}^4$). In physical units, whatever multiple of $1/\Lambda$ we take L to be, the lower bound is $\Lambda^2 \ln(1/\theta_0)/(L\Lambda)^2$, and it gives no uniform tension.

What is said here is that "the lower bound that follows from (3) **by the container estimate** goes to 0". That a theory satisfying (3) does not confine does not follow from the Tension Lower Bound from the Naive Form. The U(1) estimate in the next section fills that step.

03

The count for the Coulomb phase of U(1) — it satisfies the naive form and does not confine

We apply the same language of containers to compact $U(1)$ in four dimensions (the twist enters in the same way, as a centre flux with values ± 1). What follows is an estimate by an effective theory Reading (physics).

Weakly coupled $U(1)$ is in the Coulomb phase, and on the slab $T^2_L \times \mathbb{R}^2$, after integrating over the transverse directions at long distance, it becomes two-dimensional $U(1)$ with coupling e^2/A_T ($A_T = L^2$ is the area of T^2). In two-dimensional $U(1)$, the partition function for flux Φ through a region of area A_D can be written as a Poisson sum over the wrapping k:

$$Z(\Phi) \;\propto\; \sum_{k\in\mathbb{Z}} e^{ik\Phi}\, e^{-e^2 A_D k^2/(2A_T)} \tag{5}$$

The sectors $\sigma_D = \pm 1$ correspond to $\Phi = 0$ and $\Phi = \pi$, so when $q := e^{-e^2 A_D/(2A_T)}$ is small

$$\theta(R; T_L^2) \;=\; \frac{Z(0) - Z(\pi)}{Z(0) + Z(\pi)} \;\approx\; 2q \;=\; 2\exp\Big(-\frac{e^2 A_D}{2A_T}\Big). \quad (6)$$

Taking the container size $R = L$, A_D/A_T is a number independent of L (1 for a square), so $\theta(L, L)$ is a constant independent of L and satisfies the naive form (3). On the other hand, that weakly coupled four-dimensional $U(1)$ does not confine is a theorem (Guth 1980, Fröhlich–Spencer 1982, Villain-type action Cited , 5-06). Together, the two give the conclusion that **(3) cannot lead to confinement**. But (6) is an estimate, so this conclusion carries the same label, [Reading] (physics).

(6) is consistent with the string tension $\sigma \sim e^2/(2L^2)$ of U(1) reduced to two dimensions. U(1) on the slab confines, but its tension goes to 0 as $1/L^2$. The $\ln(1/\theta_0)/L^2$ of the Tension Lower Bound from the Naive Form has exactly the same form as this behaviour.

Note The classical energy count we tried first (the energy $(\beta/2)(\pi/L^2)^2 L^4 = \beta\pi^2/2$ of the flux π spread uniformly over a container of radius L does not depend on L) gives the same conclusion, but it drops the sum over the wrapping k, so we put the forms (5) and (6) in the text.

04

The correct form of the band — an area exponent

To get a uniform tension, the bias must become small with an exponent in the area of the container. In physical units,

$$\text{(correct form)} \qquad \ln\frac{1}{\theta(L,L)} \;\geq\; c\,L^2 \qquad (L \geq L_0, \text{ at a typical frame, uniform in } a) \quad (7)$$

This is the correct form of the band. Seen from the side of the container estimate, (7) turns directly into a tension independent of L.

Area Law from the Area Exponent Proved here Under the same frame-decorrelation assumption as the Tension Lower Bound from the Naive Form, if $\theta \leq e^{-cL^2}$ for containers of side L, then for square loops with side a multiple of L, $|\langle W(C)\rangle| \leq C_{\mathrm{D}}\, e^{-cA}$.

Proof Replacing θ_0 by e^{-cL^2} in the proof of the Tension Lower Bound from the Naive Form gives $C_{\mathrm{D}}(e^{-cL^2})^{A/L^2} = C_{\mathrm{D}} e^{-cA}$. ∎

(7) is 't Hooft's criterion for confinement — that the free energy of a vortex (a centre flux) becomes small with an exponent in the area of the cross-section — put in a local

form conditioned on the frame. The global form can be written with partition functions with twisted boundary conditions. On the torus, define the vortex free energy by $e^{-F_v} := Z_-/Z_+$ as the ratio of the partition function Z_- with twist flux in the plane $\mu\nu$ of the loop to the one Z_+ without; then the global version of the sector bias is

$$\theta \;=\; \frac{Z_+ - Z_-}{Z_+ + Z_-} \;=\; \tanh\frac{F_v}{2} \tag{8}$$

(from the definition, Proved here). Since $\tanh x \le x$, and $\operatorname{artanh}\theta \le \theta/(1-\theta^2) \le 2\theta$ for $0 \le \theta \le 1/\sqrt{2}$, give $\theta \le F_v/2 \le 2\theta$, the conditions $\ln(1/\theta) \ge cL^2$ and $F_v \lesssim e^{-cL^2}$ are the same up to constants.

It is the Tomboulis–Yaffe inequality that connects the global form to the area law Cited . In $SU(2)$ lattice gauge theory, for a rectangle C (area A_C) in the plane of the directions with periods L_μ, L_ν,

$$\langle W(C)\rangle \;\le\; 2\Big\{\tfrac{1}{2}\big(1-e^{-F_v}\big)\Big\}^{A_C/(L_\mu L_\nu)} \tag{9}$$

holds. Here F_v is the free energy of the twist flux in the same plane as C; the vortex sheet wraps the remaining two directions and winds once around C (this is a different twist from the background twist n_{12} of the slab). T. Kanazawa extended this inequality to $SU(N)$ and to general classical spin systems, using reflection positivity Cited .

Tension Lower Bound from Tomboulis–Yaffe Proved here Under (9), if $F_v \le e^{-\rho L_\mu L_\nu}$ (with $\rho > 0$ independent of the size of the torus), then $\langle W(C)\rangle \le 2e^{-\rho A_C}$; that is, the string tension is at least ρ.

Proof From $1-e^{-F} \le F$, $\tfrac{1}{2}(1-e^{-F_v}) \le \tfrac{1}{2}e^{-\rho L_\mu L_\nu}$. Raising to the power $A_C/(L_\mu L_\nu)$ gives $2^{-A_C/(L_\mu L_\nu)}e^{-\rho A_C} \le e^{-\rho A_C}$. ∎

ρ is called the 't Hooft string tension. The assumption of the Tension Lower Bound from Tomboulis–Yaffe and (7) are conditions of the same type — the former a ratio of global partition functions, the latter the bias of a container conditioned on the frame. The global form does not need frame decorrelation; in exchange, the base point (the estimate on a small T^2) must also be shown for the global partition functions (the note in 26-02).

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05
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The touchstone — the behaviour for U(1), $\mathbb{Z}_2$ and SU(2)

An argument that crosses the band must fail for a theory that does not confine. The naive form (3) does not pass this touchstone; the correct form (7) is expected to pass. All the behaviour in Table 1 is estimated Reading (physics).

Table 1 The behaviour of $\theta(L, L)$ (estimates; four dimensions, weak coupling)

Theory	$\theta(L, L)$ as L grows	Naive form (3)	Correct form (7)	Confinement
$U(1)$ (Coulomb phase)	Constant ((6))	Satisfied	Not satisfied	No Cited
$\mathbb{Z}_2$ (weak coupling)	Tends to 1 (the flux makes a wall of the area of T^2)	Not satisfied	Not satisfied	No
$SU(2)$	Tends to 0 (expected)	Satisfied (expected)	Satisfied (expected)	Yes (physical expectation)

Satisfying the naive form is not a touchstone for confinement, because U(1) satisfies it. The right way to impose "failing for U(1)" is as "θ does not become small with an area exponent as L grows".

06

What this form imposes on the band

(7) is a statement that joins the two sides of the band. At small L (the regime of a dilute gas of fractional instantons), $\ln(1/\theta)$ is expected to grow as the product of the string tension $\propto \Lambda^2(\Lambda L)^{5/3}$ and the area of the container (26-03); at large L (the confinement regime) it is expected to take the form σL^2 minus the logarithm of the entropy of the position of the vortex sheet Reading (physics). (7) asks that the region in between — $L \sim 1/\Lambda$ — be passed with a single constant c.

Two intermediate goals weaker than (7) can also be named.

- (a) $F_v(L, L) \to 0$: excluding a non-confining phase. Stronger than the naive form, but gives no uniform lower bound on the tension.
- (b) $\ln(1/\theta(L, L)) \geq cL^2/\ln L$: almost area law. In the form of the Tension Lower Bound from the Naive Form, the lower bound on the tension is $c/\ln L$, still not uniform.

Both are unsolved.

The conclusion of this chapter also changes the standing of the tools of Part 25. The flow lemma $\theta \leq 1 - a$ (25-06) gives only an estimate of the form $\theta_0 \sim \frac{1}{2}$ — the naive

form (3) — because the flip probability a of a one-step kernel can be at most of order $\frac{1}{2}$. So that tool cannot be used to test the band; its use is limited to supporting a lower bound on the slab tension at fixed L Proved here .

In the next chapter, 26-02, we reduce (7) to a single inequality that fixes the container and widens only the transverse torus.

07

The table of the five grades and the dependency graph

Table 2 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The container estimate (2)	Cited	Mack–Petkova, 25-03	The ensemble summed over twists	—
2	Frame decorrelation (the product in (2) splits up to a constant C_{D})	Assumption	A-06	—	—
3	Tension Lower Bound from the Naive Form: the lower bound on the tension that the container estimate gives from the naive form is $\ln(1/\theta_0)/L^2 \to 0$	Proved here	Proof in § 02	Under frame decorrelation, loops with side a multiple of L	—
4	Weakly coupled four-dimensional U(1) does not confine	Cited	Guth, Fröhlich–Spencer (5-06)	Villain-type action	—
5	In the Coulomb phase of U(1), $\theta \approx 2\exp(-e^2 A D/(2AT))$, and $\theta(L, L)$ is constant	Reading (physics)	§ 03	Coulomb phase, two-dimensional U(1) at long distance, an estimate	—
6	The naive form cannot lead to confinement	Reading (physics)	§ 03	On the estimate for the Coulomb phase of U(1)	—
7	$\theta = \tanh(F_v/2)$, $\theta \le F_v/2 \le 2\theta$ ($\theta \le 1/\sqrt{2}$)	Proved here	§ 04	Global form	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
8	The Tomboulis–Yaffe inequality (9)	Cited	Tomboulis–Yaffe (the form of the formula checked in Kanazawa's restatement)	$SU(2)$, lattice, torus	—
9	Extension of (9) to $SU(N)$ and classical spin systems	Cited	Kanazawa (abstract)	Under reflection positivity	—
10	Tension Lower Bound from Tomboulis–Yaffe: if $F_v \le e^{-\rho L_\mu L_\nu}$, the tension is $\ge \rho$	Proved here	Proof in § 04	Under the Tomboulis–Yaffe inequality	—
11	Area Law from the Area Exponent: if $\theta \le e^{-cL^2}$, then $\lvert\langle W\rangle\rvert \le C_{\mathrm{D}} e^{-cA}$	Proved here	Proof in § 04	Under frame decorrelation	—
12	The behaviour in Table 1 (U(1) constant, $\mathbb{Z}_2$ to 1, SU(2) to 0)	Reading (physics)	§ 05	Four dimensions, weak coupling, an estimate	—
13	The Flow Lemma gives only estimates of the naive form	Proved here	§ 06, 25-06	One-step kernel	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  container estimate |W| ≤ E∏θ                    ensemble summed over
 2  ASM  frame decorrelation (constant C_D)
 3  PRV  lower bound of the naive form ln(1/θ₀)/L² → 0 ⟸ 1, 2
 4  CIT  weakly coupled 4D U(1) does not confine        Villain type
 5  RDG  θ in the Coulomb phase of U(1) is independent of L  estimate
 6  RDG  the naive form cannot lead to confinement ⟸ 3, 4, 5
 7  PRV  θ = tanh(F_v/2), θ ≤ F_v/2 ≤ 2θ                global form
 8  CIT  the Tomboulis–Yaffe inequality                 SU(2)
 9  CIT  extension to SU(N)                             abstract
10  PRV  if F_v ≤ e^(−ρLL), tension ≥ ρ ⟸ 8
11  PRV  if θ ≤ e^(−cL²), |W| ≤ C_D e^(−cA) ⟸ 1, 2
12  RDG  the behaviour in Table 1 ⟸ 5                    estimate
13  PRV  the Flow Lemma gives only naive-form estimates ⟸ 3
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The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. Of the conclusion nodes of this chapter, the ones that stand on paper are four: the Tension Lower Bound from the Naive Form, the Tension Lower Bound from Tomboulis–Yaffe, the Area Law from the Area Exponent,

and the Flow Lemma's estimate; all are elementary manipulations on an assumption row (frame decorrelation) or a literature row (the Tomboulis–Yaffe inequality). The whole of "the naive form is not enough" has an estimate row (the count for the Coulomb phase of U(1)) as a premise, so it is "Reading (physics)". There is no node for the correct form (7) itself. That is the statement of the band, and this chapter does not show it.

What this chapter can say

What is said	What is not said
Proved here The lower bound on the tension that the container estimate gives from the naive form is $\ln(1/\theta_0)/L^2$ and goes to 0 (under frame decorrelation). With the area-exponent form the tension is uniform (under the same assumption)	Frame decorrelation itself. The correct form (7)
Cited + Proved here From the Tomboulis–Yaffe inequality, a lower bound on the tension from decay of the vortex free energy with an area exponent	The decay of the global vortex free energy itself
Reading (physics) The Coulomb phase of U(1) satisfies the naive form. So the naive form cannot lead to confinement by any road	A proof of (6) as a theorem. An estimate of the errors of the effective theory of U(1)
Reading (physics) Table 1, the touchstone	That $\theta(L, L) \to 0$ for SU(2)

Sources and reproduction

Item	Kind	Source / tools
The Tomboulis–Yaffe inequality (9)	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. The form of the formula and the setting checked in eq. (1) of T. Kanazawa, arXiv:0805.2742 and in the restatement in arXiv:0808.3442 (the text of the original paper not obtained)
Extension to $SU(N)$	Cited	T. Kanazawa, Generalizing the Tomboulis–Yaffe inequality to SU(N) lattice gauge theories and general classical spin systems, arXiv:0808.3442 (checked in the abstract)
The container estimate, the ensemble summed over twists	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442, 125 (1980) 117. Chapter 25-03
't Hooft flux and the criterion for confinement	Cited	G. 't Hooft, Nucl. Phys. B153 (1979) 141. Chapter 25-01

Item	Kind	Source / tools
Non-confinement of four-dimensional U(1)	Cited	A. H. Guth, Phys. Rev. D 21 (1980) 2291. J. Fröhlich, T. Spencer, Commun. Math. Phys. 83 (1982) 411. Chapter 5-06
The count for the Coulomb phase of U(1), (5) and (6)	Reading	Text § 03 (an estimate by an effective theory; Poisson sum)
Tension Lower Bound from the Naive Form, Area Law from the Area Exponent, Tension Lower Bound from Tomboulis–Yaffe, the estimate (8)	Proved here	Proofs in § 02 and § 04 of the text

Terms used in this chapter — container, sector, frame, the sector bias θ, the naive form, the correct form (an area exponent), the vortex free energy F_v, the 't Hooft string tension, frame decorrelation, touchstone. Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-02 Reduction to a single inequality — the correct form, as a single inequality that widens the transverse directions.

Reduction to a single inequality

> This chapter is this book's own investigation. § 08 is a map of open questions.

What this chapter shows — The conditional reduction theorem, which derives the correct form of the band $\ln(1/\theta(L,L)) \geq cL^2$ (26-01) under three assumptions — the small-T² base point, frame decorrelation and the lateral-widening inequality — and its proof. Alternating between nesting, which widens the container in the directions of the plane, and an inequality that fixes the container and widens the transverse T^2, we obtain decay with an area exponent from the fact that the sequence of biases t_k satisfies $t_{k+1} \leq Kt_k^4$ (the doubling iteration). We also look at the strong and weak forms of the lateral-widening inequality, at the fact that the strong form is stronger than confinement (it includes monotonicity of the slab tension), and at why we state things restricted to scales $L \geq L_{\min}(a)$ free of lattice artefacts.

Prerequisite chapters — 26-01 (notation, the container estimate, the Area Law from the Area Exponent), 25-03 (nesting of containers), 25-04 (slab confinement), A-04 (Lean).

Said first — The Conditional Reduction Theorem and the Reduction Theorem in the Weak Form are conditional, and none of the three assumptions has been shown. In particular the lateral-widening inequality is the only place in this reduction where non-abelian dynamics enters, and the difficulty of the band has only moved there. The conclusion is a statement on the dyadic sequence of scales $R_k = 2^k R_0$, and whether the constants are uniform in the lattice spacing a is decided by whether the constants of the assumptions are uniform. What was checked in Lean is only the real-number algebra of the iteration.

The order of this chapter

1. Two moves — nesting in the directions of the plane, and widening the transverse T^2
2. Three assumptions — the base point, frame decorrelation, the lateral-widening inequality (strong form, weak form)
3. The Conditional Reduction Theorem and its proof
4. The Reduction Theorem in the Weak Form, and what replaces the jump step

5. The strong form of the lateral-widening inequality is stronger than confinement — monotonicity of the slab tension
6. Lattice artefacts and $L_{\min}(a)$
7. Reading the Lean statements — Doubling (`Doubling.iterate`, `Doubling.decay`)
8. What remains
9. The table of the five grades and the dependency graph

01

Two moves — nesting in the directions of the plane, and widening the transverse T^2

The notation is as in 26-01 § 01. $\theta(R;T^2_L)$ is the sector bias at a typical frame of the container $D_R \times T^2_L$ of size R in the directions of the plane, and we write $\theta(R,R) := \theta(R;T^2_R)$. The goal is, for some $c>0$ uniform in physical units,

$$\ln \frac{1}{\theta(R,R)} \;\geq\; c\,R^2 \qquad (1)$$

There are two moves that change $\theta(R;T^2_L)$.

Nesting in the directions of the plane. Keeping T^2_L fixed, split a large container R by an interior frame Φ_{int} into small containers $R_1,\dots,R_k$; conditioned on the frame, the small containers are independent, so

$$\big|\mathbb{E}[\sigma_R \mid \Phi_R]\big| \;\leq\; \mathbb{E}\Big[\prod_i \theta_{R_i} \;\Big|\; \Phi_R\Big] \qquad (2)$$

holds (the nested form of the Mack–Petkova container identity Cited , 25-03). If the container is enlarged s times, the right-hand side is a product of s^2 factors. If the factors are close to their typical value and nearly independent of each other, $\theta(sR;T^2_L) \approx \theta(R;T^2_L)^{s^2}$, and the disorder propagates to large scales. But this is a matter inside a slab with T^2 fixed, and it has the same content as slab confinement (25-04).

Widening the transverse T^2. (1) asks that the container and the transverse torus be enlarged *together*. Nesting does not change T^2, so it alone does not reach (1). Replacing T^2_L by $T^2_{L'}$ changes the very surface that the vortex sheet wraps, and a relation connecting containers at the two scales does not come out of structure (26-04). Setting this relation as an assumption is the lateral-widening inequality of the next section.

02

Three assumptions

We call the assumptions by name. All lengths are in physical units and at least $L_{\min}(a)$, where lattice artefacts can be neglected (§ 06).

The base point at small T^2 Assumption For some physical $L_0 \geq L_{\min}(a)$ and $\varepsilon_0 \in (0,1)$,

$$\theta(L_0; T^2_{L_0}) \;\leq\; 1-\varepsilon_0. \tag{3}$$

In the estimate of a dilute gas of fractional instantons, $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$ (25-02). A line of showing it under ultraviolet control and uniform acceptance of insertions has been considered, but it has not been shown.

Frame decorrelation Assumption There is a constant $C_{\mathrm{D}} \geq 1$ such that for all R, L and integers $s \geq 1$,

$$\theta(sR; T^2_L) \;\leq\; C_{\mathrm{D}}\, \theta(R; T^2_L)^{s^2}. \tag{4}$$

The right-hand side of (2) is the average of a product of factors that share a frame, and Hölder's inequality alone loses the power (only the form $\mathbb{E}\theta^4_{2R} \leq \mathbb{E}\theta^4_R$ comes out). (4) is the assumption that this product splits, up to a constant, into the product of typical values. If L is fixed, it can be replaced, under reflection positivity of z, by the chessboard density κ_{cb} (26-05). Read in terms of κ_{cb}, the power in the nesting holds with constant 1 for all L (the Nested Supermultiplicativity Lemma of 26-05, and the Reduction Read in the Chessboard Density in § 03). The form uniform in L with θ as it is remains open.

The lateral-widening inequality (strong form) Assumption There is a constant C such that for $L_{\min}(a) \leq L \leq L' \leq R$,

$$\theta(R; T^2_{L'}) \;\leq\; C\Big(\frac{L'}{L}\Big)^2\, \theta(R; T^2_L). \tag{5}$$

It is the statement that, even if the container is fixed and the transverse T^2 is widened by a factor $s = L'/L$, the bias grows by at most a factor Cs^2. The factor s^2 is the entropy of the transverse position of the vortex sheet. Setting $L' = L$, we need $C \geq 1$ (if $\theta > 0$).

The lateral-widening inequality (weak form) Assumption For the dyadic scales $R_k = 2^k R_0$, there are $0 \leq \varepsilon_k < 1$ with $\sum_k \varepsilon_k < \infty$ and a constant $C' \geq 0$ such that

$$\ln \frac{1}{\theta(R_{k+1}; T^2_{R_{k+1}})} \;\geq\; (1-\varepsilon_k)\, \ln \frac{1}{\theta(R_{k+1}; T^2_{R_k})} \;-\; C'. \tag{6}$$

It is the statement that at each step of doubling the transverse directions one may lose a fraction ε_k of $\ln(1/\theta)$ and a constant C', provided the sum of the fractions lost is finite. The strong form (5) at $s = 2$ loses only the constant $\ln(4C)$ from $\ln(1/\theta)$, so it is the case $\varepsilon_k = 0$ of (6).

03

The conditional reduction theorem

Conditional Reduction Theorem Proved here Assume the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form). Set $K := 4C\,C_{\mathrm{D}}$ (≥ 1), and choose an integer $s \geq 1$ so that

$$t_0 \;:=\; C\,C_{\mathrm{D}}\,s^2(1-\varepsilon_0)^{s^2} \;<\; K^{-1/3} \quad (7)$$

(this is possible because $s^2(1-\varepsilon_0)^{s^2} \to 0$). Setting $R_0 := sL_0$ and $R_k := 2^k R_0$, for all $k \geq 0$

$$\theta(R_k, R_k) \;\leq\; K^{-1/3}\big(K^{1/3}t_0\big)^{4^k}, \qquad \text{in particular} \qquad \ln\frac{1}{\theta(R_k,R_k)} \;\geq\; \gamma\,R_k^2, \quad \gamma := \frac{\ln(1}$$

Moreover, under frame decorrelation, by the Area Law from the Area Exponent in 26-01, for square loops on the slab $T^2_{R_k} \times \mathbb{R}^2$ with side a multiple of R_k, $|\langle W(C)\rangle| \leq C_{\mathrm{D}} e^{-\gamma A}$. The lower bound γ on the tension does not depend on k.

Proof In three steps.

Step 1 (jump). Using frame decorrelation with $R = L = L_0$ and s, and putting in the base point at small T^2, gives $\theta(sL_0; T^2_{L_0}) \leq C_{\mathrm{D}}(1-\varepsilon_0)^{s^2}$. Next, using the lateral-widening inequality (strong form) with $L = L_0$ and $L' = R = sL_0$,

$$t_0' \;:=\; \theta(R_0,R_0) \;=\; \theta(sL_0; T^2_{sL_0}) \;\leq\; C\,s^2 \cdot C_{\mathrm{D}}(1-\varepsilon_0)^{s^2} \;=\; t_0.$$

Step 2 (one doubling). Set $t_k := \theta(R_k, R_k)$. Using frame decorrelation with $R = L = R_k$ and $s = 2$ gives $\theta(2R_k; T^2_{R_k}) \leq C_{\mathrm{D}} t_k^4$. Using the lateral-widening inequality (strong form) with $L = R_k$ and $L' = R = 2R_k$,

$$t_{k+1} \;=\; \theta(2R_k; T^2_{2R_k}) \;\leq\; 4C \cdot C_{\mathrm{D}}\,t_k^4 \;=\; K\,t_k^4. \quad (9)$$

Step 3 (iteration). Setting $c := K^{1/3} \geq 1$, (9) becomes $t_{k+1} \leq c^3 t_k^4$. Multiplying both sides by c gives $ct_{k+1} \leq (ct_k)^4$, so by induction $ct_k \leq (ct_0')^{4^k} \leq (ct_0)^{4^k}$ (using $0 \leq ct_0' \leq ct_0$). Hence $t_k \leq c^{-1}(ct_0)^{4^k}$. From $4^k = R_k^2/R_0^2$, $ct_0 < 1$ and $c \geq 1$,

$$\ln\frac{1}{t_k} \;\geq\; 4^k \ln\frac{1}{ct_0} + \ln c \;\geq\; \frac{\ln(1/(ct_0))}{R_0^2}\, R_k^2 \;=\; \gamma R_k^2.$$

For the last claim, take $L = R_k$ in the Area Law from the Area Exponent and replace e^{-cL^2} by $e^{-\gamma R_k^2}$. ∎

The algebra of the iteration in step 3 — from $t_{k+1} \leq c^3 t_k^4$ to $ct_k \leq (ct_0)^{4^k}$ and $t_k \leq (ct_0)^{4^k}/c$ — is checked in Lean (§ 07).

What the Conditional Reduction Theorem does not say.

- The conclusion is a statement on the dyadic sequence R_k. Extending it to the scales in between, $R_k < R < R_{k+1}$, needs an additional property such as monotonicity of $\theta(R, R)$ in R, which we do not assume here.
- The constant γ is uniform in the lattice spacing a only when the constants $\varepsilon_0, C_{\mathrm{D}}, C$ of the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form), and L_0, are uniform in a.
- Frame decorrelation is used in two places — the power in the nesting (steps 1 and 2) and the bridge from the bias to the Wilson loop (the Area Law from the Area Exponent).
- The meaning of "typical frame" is put inside frame decorrelation. We have not used a form that treats the tail of the distribution of frames (the rarity of bad frames) separately.

Note (the global version). If the same chain is written with the ratio of twisted partition functions on the torus $T_L^2 \times T_M^2$ ((8) of 26-01) instead of θ conditioned on the frame, the bridge to the area law is carried by the Tomboulis–Yaffe inequality (the Tension Lower Bound from Tomboulis–Yaffe in 26-01), and frame decorrelation is not needed. In exchange, the base point at small T^2 must also be shown for the global partition functions — as a lower bound saying that vortices are *cheap*. This is also open.

Reduction Read in the Chessboard Density (the reduction read in terms of κ_{cb}) Proved here For the $\kappa_{\mathrm{cb}}(R; L)$ of 26-05 (the density with the container taken to be the whole interior of the block; M fixed to a multiple of all the $2R_k$ of the steps considered), assume

$$\begin{aligned}
&(\mathrm{A_{cb}}) \quad \kappa_{\mathrm{cb}}(L_0; L_0) \leq 1 - \varepsilon_0, \\
&(\mathrm{C_{cb}}) \quad \kappa_{\mathrm{cb}}(R; L') \leq C\,(L'/L)^2\, \kappa_{\mathrm{cb}}(R; L) \quad (L_{\min}(a) \leq L \leq L' \leq R)
\end{aligned}$$

The formula label $(\mathrm{A_{cb}})$ is the chessboard-density base point, $(\mathrm{C_{cb}})$ is the lateral comparison of the chessboard density, and $(\mathrm{B_0})$ below is the base-point translation

inequality (names as in A-06). If the Nested Supermultiplicativity Lemma of 26-05 ($\kappa_{\mathrm{cb}}(sR;L) \le \kappa_{\mathrm{cb}}(R;L)^{s^2}$, constant 1, all L) is used in place of frame decorrelation, the proof of the Conditional Reduction Theorem goes through as it is with $C_{\mathrm{D}} = 1$, $K = 4C$, $t_0 = C\,s^2(1-\varepsilon_0)^{s^2} < K^{-1/3}$, and $t_k := \kappa_{\mathrm{cb}}(R_k;R_k) \le K^{-1/3}(K^{1/3}t_0)^{4^k}$. The step down to loops is made directly by (5) of 26-05, so the C_{D} of the Area Law from the Area Exponent is not needed either. Step 3 is the same algebra as Doubling.iterate, and the doubling form is Lean: LatticeGaugeNestedBlocks (doubling_iterate, doubling_bound). In this form frame decorrelation does not remain as an independent assumption, and the content uniform in L goes into $(\mathrm{C}_{\mathrm{cb}})$. If the base point at small T^2 is kept with θ as it is, adding the single-scale line

$$(\mathrm{B}_0)\quad 1-\kappa_{\mathrm{cb}}(L_0;L_0) \;\ge\; \eta\,\big(1-\theta(L_0;L_0)\big) \quad (\eta>0 \text{ uniform in } a\ )$$

gives $(\mathrm{A}_{\mathrm{cb}})$ with $\varepsilon_0 \to \eta\varepsilon_0$ Proved here . None of $(\mathrm{A}_{\mathrm{cb}})$, $(\mathrm{C}_{\mathrm{cb}})$, (B_0) has been shown Assumption .

04

Reduction with the weak form

Even with the weak form (6), the iteration keeps the area exponent. But the jump step cannot be made with the weak form, so the base point has to be set sufficiently small at one scale.

Reduction Theorem in the Weak Form Proved here Assume frame decorrelation, and the lateral-widening inequality (weak form) on the dyadic sequence $R_k = 2^k R_0$, and set $P := \prod_k(1-\varepsilon_k) > 0$, $A := \ln C_{\mathrm{D}} + C'$, $u_k := \ln(1/\theta(R_k,R_k))$. If $u_0 > A/(3P)$, then for all k

$$u_k \;\ge\; \Big(P\,u_0 - \frac{A}{3}\Big)4^k \;=\; \gamma'\,R_k^2, \qquad \gamma' := \frac{Pu_0 - A/3}{R_0^2} > 0. \tag{10}$$

Proof Using frame decorrelation with $s=2$ gives $\ln(1/\theta(R_{k+1};T_{R_k}^2)) \ge 4u_k - \ln C_{\mathrm{D}}$. Putting this into (6) and using $0 \le 1-\varepsilon_k \le 1$ and $\ln C_{\mathrm{D}} \ge 0$ gives $u_{k+1} \ge (1-\varepsilon_k)(4u_k - \ln C_{\mathrm{D}}) - C' \ge 4(1-\varepsilon_k)u_k - A$. Setting $v_k := u_k/4^k$, $v_{k+1} \ge (1-\varepsilon_k)v_k - A\,4^{-(k+1)}$. For $P_k := \prod_{i<k}(1-\varepsilon_i)$ and $S_k := \sum_{j=1}^{k} 4^{-j}$ we show $v_k \ge P_k v_0 - A S_k$ by induction: for $k=0$ it is an equality. If it holds for k, then from $AS_k \ge 0$ and $1-\varepsilon_k \le 1$, $v_{k+1} \ge (1-\varepsilon_k)P_k v_0 - AS_k - A4^{-(k+1)} = P_{k+1}v_0 - AS_{k+1}$. From $v_0 = u_0 > 0$, $P_k \ge P$ and $S_k < 1/3$, $v_k \ge Pu_0 - A/3$. ∎

The strong form is the case $\varepsilon_k = 0$ ($P=1$), $C' = \ln(4C)$, and the condition there, $u_0 > (\ln C_{\mathrm{D}} + \ln 4C)/3 = \ln K/3$, is the same condition as $t_0 < K^{-1/3}$ in the Conditional

Reduction Theorem. The condition on w_0 in the Reduction Theorem in the Weak Form moves the role that the jump step played in the Conditional Reduction Theorem onto the assumption of the base point.

05

The strong form of the lateral-widening inequality is stronger than confinement — monotonicity of the slab tension

We define the string tension of the two-dimensional theory obtained by shrinking the transverse directions to T_L^2 (the slab tension) as

$$\sigma_{\mathrm{eff}}(L) \;:=\; \liminf_{M\to\infty} \frac{-\ln\theta(M;T_L^2)}{M^2} \quad (11)$$

The uniform slab tension ((1) of 26-00) is $\inf_L \sigma_{\mathrm{eff}}(L) > 0$.

Slab-Tension Monotonicity Proposition Proved here If the strong form (5) holds for all $R \geq L'$, then $\sigma_{\mathrm{eff}}(L') \geq \sigma_{\mathrm{eff}}(L)$ for $L_{\min}(a) \leq L \leq L'$.

Proof For $M \geq L'$, taking the logarithm of (5) gives $-\ln\theta(M;T_{L'}^2) \geq -\ln\theta(M;T_L^2) - \ln\bigl(C(L'/L)^2\bigr)$. Dividing by M^2 and taking the lower limit as $M\to\infty$, the constant term vanishes and $\sigma_{\mathrm{eff}}(L') \geq \sigma_{\mathrm{eff}}(L)$. ∎

So the strong form includes that the slab tension does **not decrease** in L. Confinement (the area law) needs only $\inf_L \sigma_{\mathrm{eff}}(L) > 0$, so the strong form is a stronger assumption than confinement. At small L the tension is expected to grow as $\sigma_{\mathrm{eff}} \propto \Lambda^{11/3}L^{5/3}$, and at large L to approach the tension σ_∞ of $\mathbb{R}^4$ Reading (physics); but if at $L \sim 1/\Lambda$, σ_{eff} goes once above σ_∞ and then comes back (overshoots), the strong form with constant C is false, while confinement can still hold. The weak form allows this kind of overshoot, as long as the sum of the fractions lost is finite.

On the numerical side, on the slab with $SU(2)$, $\beta = 2.5$, $M = 16$, the Creutz ratio of the 4×4 loop is $0.059,\ 0.061,\ 0.076,\ 0.071$ for $N_T = 4, 6, 8, 10$ (errors 0.009–0.002, 2000 measurements): it increases and saturates, and no significant overshoot is seen Reading . The Creutz ratio contains the short-distance Coulomb term, so we read only the trend in N_T, not the values. The lattice is at a single β and the range of N_T is narrow, so this is not a reading about the band as a whole. The table of measurements is in 26-09.

The form that widens the transverse directions in one direction only (making T^2 a rectangle $L_1 \times L_2$ and enlarging only L_1) is expected not to be treatable in the same way as the square form, because the small direction of the two behaves like the time of

finite temperature Reading (physics). The chain of this chapter uses only the square form $L_1 = L_2$.

06

Lattice artefacts and $L_{\min}(a)$

We state (5) and (6) restricted to $L \geq L_{\min}(a)$, where lattice artefacts can be neglected. $L_{\min}(a)$ is kept constant in physical units, and in lattice units $L_{\min}/a \to \infty$. The goal of the band (1) is a statement about the continuum limit uniform in a, so this restriction is natural.

That the restriction is needed also shows in the numbers. For $SU(2)$, $\beta = 2.5$, $M = 8$, going $N_T = 2 \to 4$ changes the vortex free energy as $-0.01 \pm 0.12 \to 0.63 \pm 0.18$ and the bias as $\theta \lesssim 0.06 \to 0.31$, a ratio of about 5 Reading . This sits at the boundary, within errors, of the $Cs^2 = 4C$ that the strong form allows at $s = 2$. It can be read as follows: on the twisted $2 \times 2\ T^2$ with $N_T = 2$, the vortex enters cheaply as a thin vortex on the lattice rather than as a smooth fractional instanton, and the slab tension comes out large. The strong form cannot be tested with values from small lattices.

07

Reading the Lean statements — `Doubling.iterate`, `decay`

The algebra of step 3 of the Conditional Reduction Theorem is two theorems in the Lean 4 file `Doubling.lean` (namespace `Doubling`). We read the statements in the words of the text.

`Doubling.iterate`: for a real number c and a real sequence $t : \mathbb{N} \to \mathbb{R}$, under the assumptions $0 \leq c$, $0 \leq t_k$ for all k, and $t_{k+1} \leq c^3 t_k^4$ for all k, we have $c\,t_k \leq (c\,t_0)^{4^k}$ for all k.

`Doubling.decay`: under the assumptions $0 < c$, $0 \leq t_k$, $t_{k+1} \leq c^3 t_k^4$, we have $t_k \leq (c\,t_0)^{4^k}/c$ for all k.

The K of the text is c^3 in Lean. What machine checking guarantees is only the algebra of these real inequalities. That t_k is the sector bias, and that the recursion (9) holds — that is, the base point at small T^2, frame decorrelation and the lateral-widening inequality — lie outside Lean. The axioms of the two theorems are, in the record, the three standard ones (`propext`, `Classical.choice`, `Quot.sound`), and they have passed this book's re-check (A-04).

08

What remains

The Conditional Reduction Theorem reduced the correct form of the band to three assumptions. The three are different in character.

- **The base point at small T^2** is a statement on the weak-coupling side, and a line of showing it under ultraviolet control and uniform acceptance of insertion has been considered (Part 25, 27-02).
- **Frame decorrelation**, read in terms of κ_{cb}, holds with constant 1 for all L by the Nested Supermultiplicativity Lemma of 26-05, and does not remain as an independent assumption (the Reduction Read in the Chessboard Density in § 03). The content uniform in L moves to $(\mathrm{C}_{\mathrm{cb}})$, which is the lateral-widening inequality (strong form) read in terms of κ_{cb}. In exchange, the base point at small T^2 and the lateral-widening inequality (strong form) must be read in terms of κ_{cb} (or the single line (B_0) added).
- **The lateral-widening inequality** is the only place in this reduction where non-abelian dynamics enters. It fails for U(1) and $\mathbb{Z}_2$, and is expected to hold for SU(2) in the dilute regime with the margin $5/3 = 11/3 - 2$ (26-03, [Reading] physics). It does not follow from structural inputs alone, such as reflection positivity or duality (26-04).

What is open is to show the strong form (or the weak form) of the lateral-widening inequality non-perturbatively through the region between the dilute regime and the confinement regime — $L \sim 1/\Lambda$. The difficulty of the band has gathered into this single inequality.

09

The table of the five grades and the dependency graph

Table 1 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	Nesting in the directions of the plane (2)	Cited	Mack–Petkova, 25-03	T^2 fixed	—
2	The base point at small T^2 (3)	Assumption	25-02, 27-02	$L_0 \geq L_{\min}(a)$	—
3	Frame decorrelation (4)	Assumption	26-05, A-06	Constant $C_{\mathrm{D}} \geq 1$	—
4	The lateral-widening inequality (strong form) (5)	Assumption	A-06	$L_{\min}(a) \leq L \leq L' \leq R$, square T^2	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
5	The lateral-widening inequality (weak form) (6)	Assumption	A-06	Dyadic sequence, $\sum \varepsilon_k < \infty$	—
6	The strong form is the case $\varepsilon_k = 0$, $C' = \ln 4C$ of the weak form	Proved here	§ 02	—	—
7	Iteration: if $t_{k+1} \le c^3 t_k^4$, then $ct_k \le (ct_0)^{4^k}$ and $t_k \le (ct_0)^{4^k}/c$	Proved here	§ 03 step 3	Real-number algebra	Doubling (`Doubling.iterate`) (assumptions: $0 \le c$, $0 \le t_k$, $t_{k+1} \le c^3 t_k^4$), Doubling (`Doubling.decay`) (assumptions: $0 < c$ and the same two)
8	Conditional Reduction Theorem: under the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form), $\ln(1/\theta(R_k, R_k)) \ge \gamma R_k^2$, and the area law on the slab	Proved here	Proof in § 03, the Area Law from the Area Exponent in 26-01	Under the three assumptions, dyadic sequence, $L \ge L_{\min}(a)$	— (only the iteration part; the iteration row of the table)
9	Reduction Theorem in the Weak Form: under frame decorrelation, the lateral-widening inequality (weak form) and $u_0 > A/(3P)$, $u_k \ge \gamma' R_k^2$	Proved here	Proof in § 04	Under the two assumptions and the condition on the size of the base point	—
10	Slab-Tension Monotonicity Proposition: under the strong form, σ_{eff} does not decrease in L	Proved here	Proof in § 05	$L \ge L_{\min}(a)$	—
11	σ_{eff} grows as $L^{5/3}$ at small L, and goes to σ_∞ at large L	Reading (physics)	§ 05, 26-03	Dilute regime, one loop / confinement regime	—
12	The Creutz ratio increases and saturates for $N_T = 4 \to 10$, no significant overshoot	Reading	§ 05, 26-09	SU(2), $\beta = 2.5$, $M = 16$, 2000 measurements, trend only	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
13	For $N_T = 2 \to 4$ the ratio of biases is about 5	Reading	§ 06	SU(2), $\beta = 2.5$, $M = 8$, at the boundary within errors	—
14	The lateral-widening inequality must be stated restricted to $L \geq L_{\min}(a)$	Reading (physics)	§ 06	Lattice artefacts on a small T^2	—
15	The lateral-widening inequality (strong form) fails for U(1) and $\mathbb{Z}_2$, with margin $5/3$ for dilute SU(2)	Reading (physics)	26-03	Dilute regime, one loop. "Within this reduction"	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  nesting in the directions of the plane (2)            T² fixed
 2  ASM  base point at small T² θ(L₀;T²_L₀) ≤ 1 − ε₀
 3  ASM  frame decorrelation θ(sR;T²_L) ≤ C_D θ(R;T²_L)^(s²)
 4  ASM  lateral widening, strong form θ(R;T²_L') ≤ C(L'/L)² θ(R;T²_L)
 5  ASM  lateral widening, weak form (finite total loss)
 6  PRV  the strong form is a special case of the weak form ⟸ 4
 7  PRV  t_(k+1) ≤ c³t_k⁴ ⇒ ct_k ≤ (ct₀)^(4^k)                 Lean Doubling.i
 8  PRV  Conditional Reduction Theorem: ln(1/θ(R_k,R_k)) ≥ γR_k² ⟸ 1, 2, 3, 4
 9  PRV  Reduction Theorem in the Weak Form ⟸ 3, 5
10  PRV  Slab-Tension Monotonicity: strong form ⇒ σ_eff does not decrease ⟸ 4
11  RDG  shape of σ_eff (grows at small L, saturates at large L)  physics
12  RDG  the Creutz ratio increases and saturates              β = 2.5, M = 16
13  RDG  N_T = 2 → 4 gives a ratio of about 5                  β = 2.5, M = 8
14  RDG  restriction to L ≥ L_min(a) is needed ⟸ 13
15  RDG  lateral widening fails for U(1), Z₂; margin 5/3 for dilute SU(2)   26
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The premises of the nodes of the two reduction theorems always include assumption rows (the base point at small T^2, frame decorrelation, the two forms of the lateral-widening inequality). The assumption rows do not contain the quantity of the conclusion (the area exponent of $\theta(R_k, R_k)$): the base point at small T^2 is a statement at one small scale, frame decorrelation a statement of nesting with T^2 fixed, and the lateral-widening inequality (strong form) a statement of comparison with the container fixed. The numerical rows (the Creutz ratio on the slab, the ratio of biases) and the estimate rows (the shape of the slab tension, the failure depending on the group) are premises of no theorem.

What this chapter can say

What is said	What is not said
Proved here Under the three of base point, frame decorrelation and the lateral-widening inequality (strong form), $\ln(1/\theta(R_k, R_k)) \geq \gamma R_k^2$ on the dyadic sequence of scales, and a tension on the slab independent of k (the Conditional Reduction Theorem)	Any of the three assumptions. The scales in between. Uniformity of the constants in a (it depends on the constants of the assumptions)
Proved here With the weak form too, the same conclusion if the base point is small enough at one scale (the Reduction Theorem in the Weak Form)	The weak form itself. A jump from the weak form alone
Proved here The strong form includes monotonicity of the slab tension and is stronger than confinement (the Slab-Tension Monotonicity Proposition)	That the slab tension does not overshoot
Reading The Creutz ratio on the slab increases and saturates. On a small T^2, lattice artefacts touch the boundary of the strong form	The behaviour of the slab tension across the whole band. Values on large lattices
Lean: the real-number algebra of the iteration (Doubling (`Doubling.iterate`, `Doubling.decay`))	That the bias satisfies the recursion (outside Lean)

Sources and reproduction

Item	Kind	Source / tools
The container identity and nesting	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442, 125 (1980) 117. Chapter 25-03
Conditional Reduction Theorem, Reduction Theorem in the Weak Form, Slab-Tension Monotonicity Proposition	Proved here	Proofs in the text, § 03– § 05
The form of the string tension on a small T^2 (the base point and the small-L side of σ_{eff})	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166, eqs. (2.32) and (2.37). Chapter 25-02
The Creutz ratio on the slab, the bias for $NT = 2 \to 4$	Reading	Monte Carlo for SU(2) (Creutz ratio), and replica exchange and Bennett's acceptance-ratio method (vortex free energy). The table in chapter 26-09
Lean	Lean	Doubling (`Doubling.iterate`, `Doubling.decay`). Guarantees only the real-number algebra. A-04

Terms used in this chapter — nesting in the directions of the plane, the lateral-widening inequality (strong form, weak form), the base point at small T^2, frame decorrelation, the doubling iteration, jump, the slab tension σ_{eff}, overshoot, the scale of lattice artefacts $L_{\min}(a)$. Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-03 The only place where non-abelian enters — the $5/3 = 11/3 - 2$ that decides the sign of the lateral-widening inequality.

The only place where non-abelian enters

> This chapter is this book's own investigation. The established knowledge it builds on is in 25-02 and 3-05.

What this chapter shows — Where non-abelian dynamics enters within the reduction of 26-02. Of the three assumptions (the small-T^2 base point, frame decorrelation, the lateral-widening inequality) and the nesting inequality, only the lateral-widening inequality fails for U(1) and $\mathbb{Z}_2$, and we see that what fixes its sign is the exponent margin in the dilute regime $5/3 = 11/3 - 2$ — the margin by which the coefficient of asymptotic freedom $11/3$ beats the 2 of the number of positions of a lump in the plane (the transverse two-dimensional entropy). For contrast we set side by side that abelian flux can only spread out uniformly, so that the vortex free energy of U(1) increases when T^2 is widened, and that a vortex of $SU(2)$ can localise in a scale-invariant self-dual lump.

Prerequisite chapters — 26-01 (the sector bias θ, the U(1) count), 26-02 (the three assumptions and the Conditional Reduction Theorem), 25-02 (vortices and fractional instantons), 3-05 (instantons and the Bogomolny bound), 4-06 (asymptotic freedom), 5-06 (U(1) and the non-abelian).

Said first — The "only place" of the title is a way of speaking **within this reduction**. In another reduction (the renormalisation comparison of 26-06), the non-abelian input enters a place of a different form. The counts of the behaviour of U(1), $\mathbb{Z}_2$ and $SU(2)$ are counts that the original paper itself states to be estimates, and in this chapter too they carry the label [Reading] (physics). We say nothing outside the dilute regime and one loop. The propositions set out on paper are only an elementary lemma that flux spreads out uniformly, and arrangements of logic.

The order of this chapter

1. The question — what fixes the sign of the lateral-widening inequality
2. Abelian flux can only spread out uniformly — the Flux-Equalisation Lemma and the counts for U(1) and $\mathbb{Z}_2$
3. A non-abelian vortex can localise in a scale-invariant lump — the self-dual lump with $Q = \frac{1}{2}$
4. The margin $5/3 = 11/3 - 2$ — the count for a dilute gas and the Tanizaki–Ünsal formula
5. The only place "within this reduction" — the Contrapositive of the Reduction, and its limits
6. Numerical signs — the vortex free energy when T^2 is widened (Table 2)
7. The table of the five grades and the dependency graph

01

The question — what fixes the sign of the lateral-widening inequality

The notation is as in 26-01 § 01 and 26-02 § 01. $\theta(R;T^2_L)$ is the sector bias, in a typical frame, of the container $D\times T^2_L$ of size R in the planar directions (area A_D), with the twist $n_{12}=1$ fixed on the transverse torus T^2_L (area $A_T=L^2$). The Conditional Reduction Theorem of 26-02 derived the correct form of the band, $\ln(1/\theta(R,R))\geq cR^2$, from three assumptions. The third of them was the lateral-widening inequality (strong form)

$$\theta(R;T^2_{L'})\;\leq\;C\Big(\frac{L'}{L}\Big)^2\,\theta(R;T^2_L)\qquad(L_{\min}(a)\leq L\leq L'\leq R)\tag{1}$$

If we widen the transverse T^2 while keeping the container fixed, which way does the sector bias θ move? The smaller θ is, the greater the disorder of the sector (the more vortices), so the question is whether widening the transverse directions makes vortices **costlier** (θ moves towards 1) or **not costlier**. Even if they become costlier, (1) allows it only up to the factor s^2 of the number of positions ($s=L'/L$).

We write the answer of this chapter first. In an abelian theory the flux of a vortex can only spread out uniformly over the whole transverse cross-section, so the wider the transverse directions, the costlier the vortex, and (1) fails. In $SU(2)$ a vortex can localise in a self-dual lump of size about L, and its action does not depend on the size, so classically the cost of widening the transverse directions is 0. The remaining quantum effects lean, in the dilute regime, to the side of asymptotic freedom, and (1) is expected to hold with margin $5/3$.

02

Abelian flux can only spread out uniformly

First we set out one elementary fact about the classical action.

Flux-Equalisation Lemma Proved here Let Ω be a planar region of area $|\Omega|$, and f a square-integrable real function on Ω with $\int_\Omega f=\Phi$. Then

$$\int_\Omega f^2\;\geq\;\frac{\Phi^2}{|\Omega|},\tag{2}$$

with equality only when f is almost everywhere equal to the constant $\Phi/|\Omega|$ on Ω.

Proof (2) follows from the Cauchy–Schwarz inequality $\big(\int_\Omega f\cdot 1\big)^2\leq\int_\Omega f^2\cdot\int_\Omega 1$. Equality holds only when f and 1 are proportional, that is, only when f is constant. ∎

The action of an abelian gauge field, $\frac{1}{2e^2}\int F_{\mu\nu}F_{\mu\nu}$, is the square of the field strength: a linear field theory. Consider a configuration in which the same flux Φ passes through each cross-section $D\times\{y\}$ ($y\in T^2_L$) of the container $D\times T^2_L$. Using the Flux-Equalisation Lemma on each cross-section and integrating over the transverse T^2_L, we get

$$\frac{1}{2e^2}\int_{D\times T^2_L}F_{34}^2\;\geq\;\frac{A_T}{2e^2A_D}\,\Phi^2\tag{3}$$

Proved here (under the condition that the flux through each cross-section is the same Φ; a lower bound on the classical action). Equality holds for the configuration in which the flux spreads out uniformly over the whole container, and **localising the flux only increases the action**. The right-hand side is proportional to the transverse area A_T, so if we fix the container and widen the transverse directions by a factor s, the minimal action of the flux π (corresponding to the sector $\sigma_D = -1$) becomes s^2 times larger. An abelian vortex becomes costlier when the transverse directions are widened: that is the answer at the classical level.

The count at the quantum level is as in 26-01 § 03 Reading (physics). Seen at long distance on the slab $T^2_L \times \mathbb{R}^2$, U(1) in the Coulomb phase is a two-dimensional U(1) (coupling e^2/A_T), and the partition function with flux Φ is the Poisson sum over wrappings k, $Z(\Phi) \propto \sum_k e^{ik\Phi} e^{-e^2 A_D k^2/(2A_T)}$. This is the sum over k of the classical action $\frac{A_T}{2e^2 A_D}(\Phi + 2\pi k)^2$ of (3), rewritten with the Poisson summation formula, and when $q := e^{-e^2 A_D/(2A_T)}$ is small

$$\theta(R; T^2_L) \;\approx\; 2q \;=\; 2\exp\Big(-\frac{e^2 A_D}{2A_T}\Big), \qquad \ln\frac{1}{\theta} \;\approx\; \frac{e^2 A_D}{2A_T} \;\propto\; A_D\, L^{-2}. \tag{4}$$

If the transverse directions are widened by a factor s, A_T becomes s^2 times larger and q becomes q^{1/s^2}, so $\theta_{\text{new}} \approx 2(\theta_{\text{old}}/2)^{1/s^2}$. The ratio is

$$\frac{\theta(R; T^2_{sL})}{\theta(R; T^2_L)} \;\approx\; 2^{\,1-1/s^2}\, \theta(R; T^2_L)^{-(1-1/s^2)} \;\longrightarrow\; \infty \qquad (\theta(R; T^2_L) \to 0,\ s > 1) \tag{5}$$

and it cannot be bounded by any constant Cs^2. **For U(1), the lateral-widening inequality (1) fails** Reading (physics; one line of algebra once (4) is granted). The weak form (the lateral-widening inequality (weak form) of 26-02) also fails, because the slab tension goes to 0 like e^2/L^2 Reading (physics).

The four-dimensional $\mathbb{Z}_2$ theory is, at weak coupling (large β), in a non-confining phase (5-10). There the flux through the container creates a wall of area A_T spanning the whole transverse T^2, and its tension does not depend on A_D. Widening the transverse directions increases the cost of the wall in proportion to A_T, and (1) fails Reading (physics). What the two abelian theories have in common is that **the cost of the flux increases with the transverse area**: in U(1) because of linearity (the Flux-Equalisation Lemma), in $\mathbb{Z}_2$ because of the area of the wall.

03

A non-abelian vortex can localise in a scale-invariant lump

In $SU(2)$ things change. On $T^2_L \times \mathbb{R}^2$ with the twist $n_{12} = 1$, the topological charge Q can take half-integer values ('t Hooft's twist, 25-01 Cited). And the action has the Bogomolny bound

$$S \;\geq\; \frac{8\pi^2}{g^2}\,|Q| \tag{6}$$

(3-05 Cited), with equality for self-dual (or anti-self-dual) configurations. A configuration with $Q = \frac{1}{2}$ — a fractional instanton — has action at least $4\pi^2/g^2$, and plays the role of a vortex that flips the planar sector (25-02).

The right-hand side of (6) depends neither on the size L of T^2 nor on the size R of the container. This is because the classical action of four-dimensional Yang–Mills does not change under a change of scale (the coupling g is dimensionless, 24-07 § 01). The estimate of the original paper is that the minimal-action configuration with $Q = \frac{1}{2}$ is a self-dual lump localised in a region of size about L, whose action equals the bound $4\pi^2/g^2$ and does not depend on the ratio of the transverse and longitudinal sizes Reading (physics). On the lattice, it has been measured that the action of cooled fractional instantons approaches $\pi^2\beta$ from below (with $\beta = 4/g^2$, $4\pi^2/g^2 = \pi^2\beta$; the values obtained by cooling are upper bounds on the minimal action on the lattice) (the table in 25-02).

Here the difference from the Flux-Equalisation Lemma of § 02 becomes clear.

- **Abelian**: the action is a quadratic form, so the minimal action for a given flux is attained by the configuration that spreads the flux out uniformly, and its value is proportional to the transverse area A_T ((3)). Localising does not make it cheaper; one pays the cost over the whole region it is spread over.
- **Non-abelian**: the self-duality equations are non-linear, and a solution of finite action $4\pi^2/g^2$ can be localised to a size of about L. As long as the container is large enough to hold the lump ($R \geq L'$), widening the transverse directions from L to L' only requires re-spreading the lump, and the classical cost is 0.

This is not a difference of local kinematics, but a difference in the existence of solutions of the self-duality equations — in non-linearity. Since at the classical level the cost of lateral widening is 0, the sign of (1) is fixed by quantum effects: the number of positions of the vortices, and the running of the coupling. That is the count of the next section.

04

The margin $5/3 = 11/3 - 2$

The count for a dilute gas. For $L\Lambda \ll 1$ (Λ is the scale of dimensional transmutation, 4-07), the coupling $g(L)$ is small, and the vortices (lumps with $Q = \pm\frac{1}{2}$) can be treated as a dilute gas, far apart from one another. For $SU(N)$, Tanizaki–Ünsal derived the ground-state energy and the string tension from the facts that on a small twisted T^2 the centre vortices behave like points on the plane $\mathbb{R}^2$ and that the weight of each is $Ke^{-SI/N}$ ($S_I = 8\pi^2/g^2$, with K the prefactor per unit area):

$$E_k(\theta) \;=\; -2Ke^{-SI/N}\cos\frac{\theta - 2\pi k}{N} \;\sim\; -\Lambda^2(\Lambda L)^{5/3}\cos\frac{\theta - 2\pi k}{N}, \qquad T_R \;\sim\; \Lambda^2(\Lambda L)^{5/3}\sin^2\frac{\pi|R|}{N} \quad (\theta =$$

(Tanizaki–Ünsal, eqs. (2.32) and (2.37) Cited , dilute regime, one loop). The θ here is the vacuum angle, a symbol different from this book's sector bias θ. The exponent $5/3$ comes out as follows: from the one-loop running of the coupling $8\pi^2/g^2(L) = \frac{11N}{3}\ln\frac{1}{L\Lambda}$ (4-06 Cited) we get $e^{-SI/N} = (L\Lambda)^{11/3}$, and from the size L of a lump the number of positions per unit area is $1/L^2$, so $(L\Lambda)^{11/3}/L^2 = \Lambda^2(\Lambda L)^{5/3}$.

We move on to the sector bias. In $SU(2)$, let ζ be the density per unit area of vortices and antivortices together, and approximate the positions of the vortices by independent Poisson points Reading (physics; the dilute-gas approximation). The sector σ_D of the container is $(-1)^n$ for the number n of vortices inside D. For n in a Poisson distribution with mean m,

$$\mathbb{E}\big[(-1)^n\big] \;=\; \sum_{n\ge 0} e^{-m}\frac{m^n}{n!}(-1)^n \;=\; e^{-2m} \qquad (8)$$

so Proved here (an elementary identity), putting $m = \zeta A D$,

$$\theta(R;T_L^2) \;\approx\; e^{-2\zeta A D}, \qquad \ln\frac{1}{\theta} \;\approx\; 2\zeta A D \;\propto\; A D\,\Lambda^{11/3}L^{5/3} \qquad (\Lambda \text{ fixed}) \qquad (9)$$

Reading (physics; dilute regime, one loop). Set beside (4), the power in L of $\ln(1/\theta)$ with the container fixed is as follows.

Table 1 The power in L of $\ln(1/\theta)$ with the container fixed (estimates; four dimensions, weak-coupling side)

Theory	Form of the vortex cost	Power of $\ln(1/\theta)$	On widening the transverse directions	Lateral-widening inequality (1)
$U(1)$ (Coulomb phase)	The flux spreads out uniformly over the transverse cross-section (the Flux-Equalisation Lemma)	L^{-2}	θ moves towards 1	Fails ((5))
$\mathbb{Z}_2$ (weak coupling)	A wall of area A_T	— (with the wall cost $\propto L^2$, $\theta \to 1$)	θ moves towards 1	Fails
$SU(2)$ (dilute regime)	A self-dual lump of size L, weight $(L\Lambda)^{11/3}$, density of positions $1/L^2$	$L^{+5/3} = L^{11/3-2}$	θ decreases	Holds (the ratio is at most 1)

The power $5/3$ for $SU(2)$ is the difference between the coefficient of asymptotic freedom $11/3$ and the 2 of the number of positions of a lump in the plane (what this part has been calling "the transverse two-dimensional entropy", Figure 2 of 26-00). The power -2 for U(1) can be read as the form in which there is no asymptotic-freedom term and only the term of the number of positions remains. **The margin $5/3 > 0$ by which asymptotic freedom beats the transverse two-dimensional entropy is the sign of the lateral-widening inequality itself** Reading (physics).

From (9), the ratio when the transverse directions are widened by a factor s in $SU(2)$ in the dilute regime is

$$\frac{\theta(R;T_{sL}^2)}{\theta(R;T_L^2)} \;\approx\; \exp\Big(-2A D\big(\zeta(sL)-\zeta(L)\big)\Big) \;\le\; 1 \qquad (10)$$

($\zeta \propto L^{5/3}$ is increasing), and (1) holds with $C = 1$, without using the factor s^2 Reading (physics). But (9) and (10) can be used only in the dilute regime $L\Lambda \ll 1$. In the confining regime ($L\Lambda \gg 1$) one expects $\ln(1/\theta) \approx \sigma A D - \ln(A_T/\xi^2)$ (ξ is the correlation length); the dependence on A_T is then only the logarithm of the entropy of the position of the vortex sheet, and (1) is expected to hold within the factor s^2 Reading (physics). **There is no estimate that passes non-perturbatively between the two regimes, $L \sim 1/\Lambda$ — the band.**

05

The only place "within this reduction"

We set out the components of the chain of the Conditional Reduction Theorem of 26-02 according to how they behave when the group is changed.

- **The nesting in the planar directions** ((2) of 26-02) is a statement that comes from the Markov property — that conditioning on the frame makes the insides of the containers independent — and from the container identity of Mack–Petkova, and it does not use the non-commutativity of the group Cited .
- **The base point at small T²** is an estimate that holds for U(1) too. Putting $R = L$ in (4), $\theta(L, L) \approx 2e^{-e^2/2}$ is a constant less than 1 independent of L (26-01 § 03) Reading (physics).
- **Frame decorrelation**, if L is fixed, is replaced under reflection positivity of z by the chessboard density κ_{cb} (26-05), and its structure is the same for $\mathbb{Z}_2$ and U(1) too (26-04). There is no estimate that U(1) fails in the form uniform in L (the uniform slab gap); U(1) is expected to fail only on the side of the uniform slab tension Reading (physics, 26-05 § 05).
- **The lateral-widening inequality** fails for U(1) and $\mathbb{Z}_2$ (§ 02), and is expected to hold with margin $5/3$ for $SU(2)$ in the dilute regime (§ 04).

Contrapositive of the Reduction Proved here If, in some theory, the conclusion of the Conditional Reduction Theorem of 26-02 ($\ln(1/\theta(R_k, R_k)) \geq \gamma R_k^2$ on a dyadic sequence of scales, and a tension on the slab independent of k) is false, then at least one of the base point at small T², frame decorrelation and the lateral-widening inequality (strong form) is false in that theory.

Proof The contrapositive of the Conditional Reduction Theorem. ∎

For U(1) at weak coupling, the tension on the slab is estimated to go to 0 like e^2/L^2 (26-01 § 03), and the conclusion of the Conditional Reduction Theorem is false. By the Contrapositive of the Reduction one of the three assumptions is false, and in the list above the base point at small T² holds and the lateral-widening inequality (strong form) fails. **Within this reduction, the only assumption that is necessarily used to exclude U(1) and $\mathbb{Z}_2$ is the lateral-widening inequality**, and that is where non-abelian dynamics enters Reading (physics; resting on estimates of the behaviour of U(1), $\mathbb{Z}_2$ and $SU(2)$).

Two qualifications go with this way of speaking.

- **It depends on the reduction.** "The only place" is a way of speaking within the chain of 26-02. In another reduction that splits the band into "β-monotonicity" and "the renormalisation comparison" (26-06), asymptotic freedom enters through the shift of the coupling in the comparison, $\beta' < \beta$. It is natural to read this as the same physics entering a differently shaped inequality in each reduction, but the precise correspondence between the two reductions has not been shown.
- **The estimates of the sign are only at the two ends.** The margin $5/3$ is a count in the dilute regime ($L\Lambda \ll 1$) at one loop, and the "only a logarithm" of the confining regime is also an estimate. There is no tool that shows the lateral-widening inequality in the band $L \sim 1/\Lambda$. The difficulty of the band is concentrated in this one interval of scales, within this one inequality.

06

Numerical signs — the vortex free energy when T^2 is widened

These are values of the vortex free energy $\Delta F = F_v$ (the ratio to the partition function with the twist flux n_{34} inserted, $\theta = \tanh(\Delta F/2)$) when the transverse T^2 is widened to $N_T \times N_T$, in the global form in which the container is the whole of the planar directions (T^2_M with period M) ((8) of 26-01), measured by replica exchange and Bennett's acceptance-ratio method [Reading] . The lattice is $N_T^2 \times M^2$, with twist $n_{12} = 1$ and the Wilson action. The errors are the estimates given by the source, and the source does not state the number of seeds.

Table 2 ΔF when the transverse T^2 is widened (global form; [Reading])

Theory, β, M	$N_T = 2$	4	6	8
$U(1)$, $\beta = 1.5$, $M = 8$	-0.03 ± 0.04	0.83 ± 0.11	2.10 ± 0.47	5.14 ± 0.78
$SU(2)$, $\beta = 2.5$, $M = 8$	-0.01 ± 0.12	0.63 ± 0.18	0.33 ± 0.22	1.51 ± 0.29
$U(1)$, $\beta = 1.5$, $M = 12$	—	0.20 ± 0.21	1.35 ± 0.30	1.72 ± 0.47
$SU(2)$, $\beta = 2.5$, $M = 12$	—	0.57 ± 0.22	0.42 ± 0.24	-0.15 ± 0.36
$SU(2)$, $\beta = 2.4$, $M = 8$	—	0.31 ± 0.10	—	0.11 ± 0.25

The reading is as follows [Reading] .

- For $U(1)$, at both values of M, ΔF increases when the transverse directions are widened (the vortex becomes costlier). At $M = 8$, θ is $0.39,\ 0.78,\ 0.99$ at $N_T = 4, 6, 8$, and $\ln(1/\theta)$ goes $0.93 \to 0.012$ for $N_T = 4 \to 8$ (about a factor of 80 smaller). This is the direction predicted in § 02.
- For $SU(2)$ the increase is much gentler, or there is none. At $M = 8$, $\beta = 2.5$, $\ln(1/\theta)$ goes $1.19 \to 0.45$ for $N_T = 4 \to 8$ (about a factor of 2.6 smaller), and the ratio of θ, $0.64/0.31 \approx 2$, is within the $4C$ that the lateral-widening inequality (strong form) allows at $s = 2$. At $M = 12$ and at $\beta = 2.4$, ΔF decreases for $N_T = 4 \to 8$. The exception is $N_T = 8$ at $\beta = 2.5$, $M = 8$ (the symmetric torus $N_T = M$), where ΔF rises from 0.33 to 1.51.

We write the limits with the same weight. Since this is a regime where θ is of order 1, the ratio of θ for U(1) is also held to about 2.5 because $\theta \leq 1$, and the discriminating power in this geometry is weak. β takes one or two points for each theory, and $\beta = 1.5$ for U(1) and $\beta = 2.5$ for $SU(2)$ are not matched to the same physics. The errors are 0.04–0.8. The values at $N_T = 2$ are in the region of lattice artefacts (26-02 § 06). **These are signs of a direction, not the conclusion of a test of the lateral-widening inequality.** The detailed tables of the measurements are in 25-08 and 26-09.

07

The table of the five grades and the dependency graph

Table 3 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The Bogomolny bound $S \geq 8\pi^2 \lvert Q\rvert /g^2$	Cited	3-05	Continuum classical action	—
2	Under the twist $n_{12} = 1$, Q can take half-integer values	Cited	't Hooft, 25-01	$T^2_L \times \mathbb{R}^2$, $SU(2)$	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
3	One-loop running $8\pi^2/g^2(L) = \frac{11N}{3}\ln\frac{1}{L\Lambda}$	Cited	4-06	One loop, $L\Lambda \ll 1$	—
4	The string tension on a small twisted T^2, $T_R \sim \Lambda^2(\Lambda L)^{5/3}\sin^2(\pi\lvert R\rvert/N)$	Cited	Tanizaki–Ünsal eqs. (2.32), (2.37)	Dilute regime, one loop, $\theta = 0$	—
5	Flux-Equalisation Lemma: $\int f^2 \geq \Phi^2/\lvert\Omega\rvert$, with equality when constant	Proved here	Proof in § 02	Square-integrable	—
6	Lower bound (3) on the abelian action: $\geq \frac{A_T}{2e^2 A_D}\Phi^2$, equality for the uniform configuration	Proved here	The Flux-Equalisation Lemma on each cross-section	The same flux Φ through each cross-section, classical action	—
7	In the Coulomb phase of U(1), $\theta \approx 2e^{-e^2 A_D/(2A_T)}$	Reading (physics)	26-01 § 03	Coulomb phase, two-dimensional U(1) at long distance, an estimate	—
8	For U(1) the lateral-widening inequality fails (the ratio (5) is unbounded)	Reading (physics)	§ 02	On top of the estimate for the Coulomb phase of U(1)	—
9	For $\mathbb{Z}_2$ at weak coupling the lateral-widening inequality fails (a wall of area A_T)	Reading (physics)	§ 02, 5-10	Weak coupling, an estimate	—
10	The minimal action with $Q = \frac{1}{2}$ in $SU(2)$ is a self-dual lump of size about L, and the classical cost of lateral widening is 0	Reading (physics)	§ 03, 25-02	Stated as an estimate by the original paper; the container large enough to hold the lump	—
11	The Poisson identity $\mathbb{E}[(-1)^n] = e^{-2m}$	Proved here	(8) of § 04	Poisson distribution with mean m	—
12	In a dilute gas $\theta \approx e^{-2\zeta A_D}$, $\ln(1/\theta) \propto A_D L^{5/3}$	Reading (physics)	(9) of § 04	Dilute regime, one loop, approximation by independent points	—
13	The margin $5/3 = 11/3 - 2$ fixes the sign of the lateral-widening inequality	Reading (physics)	§ 04	Dilute regime, one loop	—
14	For dilute $SU(2)$ the ratio on lateral widening is at most 1 ((10))	Reading (physics)	§ 04	$L\Lambda \ll 1$	—
15	In the confining regime the dependence on A_T is only logarithmic	Reading (physics)	§ 04	$L\Lambda \gg 1$, an estimate	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
16	Contrapositive of the Reduction: if the conclusion of the Conditional Reduction Theorem is false, one of the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form) is false	Proved here	§ 05 (contrapositive)	—	—
17	Within this reduction, only the lateral-widening inequality excludes U(1) and $\mathbb{Z}_2$	Reading (physics)	§ 05	Within the chain of 26-02	—
18	On lateral widening, ΔF increases for U(1); for $SU(2)$ it increases gently or not at all	Reading	§ 06 Table 2	$M = 8, 12$, U(1) $\beta = 1.5$, SU(2) $\beta = 2.4$, 2.5, $N_T = 2$–8, errors 0.04–0.8, the exception at $N_T = M = 8$	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  Bogomolny bound S ≥ 8π²|Q|/g²                              continuum, classical
 2  CIT  Q half-integer under the twist                             T²_L × R²
 3  CIT  8π²/g²(L) = (11N/3) ln(1/LΛ)                               one loop
 4  CIT  T_R ~ Λ²(ΛL)^(5/3) sin²(π|R|/N)                            dilute, one loop
 5  PRV  Flux-Equalisation Lemma: ∫f² ≥ Φ²/|Ω|
 6  PRV  abelian action ≥ (A_T/2e²A_D)Φ² ⟸ 5                         same flux through each cr
 7  RDG  U(1): θ ≈ 2exp(−e²A_D/2A_T)                                estimate
 8  RDG  lateral widening fails for U(1) ⟸ 6, 7
 9  RDG  lateral widening fails for Z₂                              weak coupling, wall
10  RDG  SU(2) vortex is a self-dual lump of size L ⟸ 1, 2          classical cost 0
11  PRV  E[(−1)^n] = e^(−2m)
12  RDG  θ ≈ e^(−2ζA_D), ln(1/θ) ∝ A_D L^(5/3) ⟸ 3, 4, 11           dilute, one loop
13  RDG  margin 5/3 = 11/3 − 2 is the sign ⟸ 8, 10, 12
14  RDG  ratio ≤ 1 for dilute SU(2) ⟸ 12
15  RDG  only a logarithm in the confining regime                   estimate
16  PRV  Contrapositive of the Reduction                            26-02
17  RDG  in this reduction only lateral widening separates groups ⟸ 8, 9, 13, 16
18  RDG  direction of Table 2                                       small lattices, one or two
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The nodes that stand on paper are four: the Flux-Equalisation Lemma, the lower bound on the abelian action, the Poisson identity and the Contrapositive of the Reduction, all of them elementary. The claims of the chapter (that the margin 5/3 = 11/3 − 2 fixes the sign, and that within this reduction only the lateral-widening inequality separates the groups) are "Reading (physics)", and their premises include estimate rows (the Coulomb phase of U(1), the self-dual lump of SU(2), the bias of the dilute gas). The numerical row (the direction of Table 2) is not a premise of any conclusion. There is no node for the lateral-widening inequality itself. It is placed as an assumption in the ledger of conditions of 27-02.

What this chapter can say

What is said	What is not said
Proved here For the abelian action, the minimal action for a given flux is attained by the uniformly spread configuration and is proportional to the transverse area (under the condition of the same flux through each cross-section; classical action)	A proof, as a theorem, of the quantum U(1) count (4)
Reading (physics) For U(1) and $\mathbb{Z}_2$ the lateral-widening inequality fails; for dilute $SU(2)$ it holds with the margin $5/3 = 11/3 - 2$	The lateral-widening inequality in the band $L \sim 1/\Lambda$. The sign outside the dilute regime and one loop
Proved here + Reading (physics) Within this reduction, only the lateral-widening inequality excludes U(1) and $\mathbb{Z}_2$	An "only place" independent of the reduction. The precise correspondence with the reduction of 26-06
Reading On small lattices, the direction in which widening the transverse directions makes the U(1) vortex costlier and the $SU(2)$ one not (with one exception)	Behaviour on large lattices and at many values of β. The values of the constants of the inequality

Sources and reproduction

Item	Kind	Source / tools
The gas of centre vortices on a small twisted T^2 and the string tension (7)	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166 (Prog. Theor. Exp. Phys. 2022), eqs. (2.32) and (2.37). Chapter 25-02
The Bogomolny bound, half-integer topological charge	Cited	Sources of chapters 3-05 and 25-01 (G. 't Hooft, Nucl. Phys. B153 (1979) 141 and others)
One-loop running of the coupling	Cited	Sources of chapter 4-06
The count for the Coulomb phase of U(1); the estimate that the lateral-widening inequality separates the groups	Reading (physics)	The proposition "the inequality separates the theories" of the paper on the reduction of the band (a paper that came out of this book, A-07) (the paper itself states that it is heuristic). Chapter 26-01
Flux-Equalisation Lemma, (3), (8), Contrapositive of the Reduction	Proved here	The proofs in § 02, § 04, § 05 of the text
The vortex free energies of Table 2	Reading	Replica exchange and Bennett's acceptance-ratio method. The tables of chapters 25-08 and 26-09

Terms used in this chapter — the lateral-widening inequality, the sector bias θ, the self-dual lump (fractional instanton), the dilute gas, the margin $5/3$, the transverse two-dimensional entropy, the Coulomb phase, the crossover band (the band). Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-04 Structure alone cannot cross — why reflection positivity and duality alone do not give the lateral-widening inequality.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band the negative lemma, the electric flux, reflection positivity of z · Kiichi, Shiori and Rin

Structure alone cannot cross

> This chapter is this book's own investigation. § 01 is established knowledge; § 05 is a map of open questions.

What this chapter shows — We set out the structural tools of lattice gauge theory — the positivity of the transfer matrix, the centre and duality identities, reflection positivity, the symmetries of the lattice, the finite lemmas — in the language of the twisted slab, and check that every one of them holds as it stands for $\mathbb{Z}_2$ and U(1). From this follows the negative lemma: **from this set of inputs alone**, the lateral-widening inequality cannot be derived. To cross the band one needs a dynamical input that excludes the non-confining phase. On the way, we set out with proofs two things that can certainly be said from the side of structure: the electric-flux form of the sector bias, $\theta = Y(1)/Y(0) \geq 0$, and the positivity of reflections through lattice points for the z ensemble. The latter is an input to the chessboard estimate of the next chapter.

Prerequisite chapters — 26-01 (the sector bias and the global form (8)), 26-02 (the lateral-widening inequality), 26-03 (that it fails for U(1) and $\mathbb{Z}_2$), 5-03 (reflection positivity and the transfer matrix), 25-01 (the twist and 't Hooft flux), A-04 (Lean).

Said first — The negative lemma is a statement of the form "**from this set of inputs alone**": an arrangement of logic with the list of inputs made explicit. It does not say that none of the inputs is of any use (reflection positivity is used in 26-05). "The lateral-widening inequality fails for U(1) and $\mathbb{Z}_2$" is an estimate of 26-03 ([Reading], physics), so the last step of the conclusion of the negative lemma inherits that label. What was checked in Lean is only the spectral algebra of the electric-flux form and the algebra of the final sum in reflection positivity.

The order of this chapter

1. The list of structural inputs — what we call "structure"
2. The electric-flux form — $\theta = Y(1)/Y(0) \geq 0$ and the tanh comparison
3. Reflection positivity of z — reflections through lattice points, and how transverse reflections break down under the twist
4. Lemma that Structural Inputs Alone Do Not Suffice — from this set of inputs alone

5. The inputs needed — what excludes the non-confining phase
6. Reading the Lean statements — `FluxDuality`, `ReflPos`
7. The table of the five grades and the dependency graph

01

The list of structural inputs — what we call "structure"

Of the five inputs in this section, three (positivity of the transfer matrix, the centre and duality identities, the lattice symmetries) summarise existing knowledge. The introduction is in 5-03 § 05 (the transfer matrix) and 5-03 § 04 (reflection positivity).

In lattice gauge theory with the Wilson action on the lattice $\Lambda = (\mathbb{Z}/L_1) \times (\mathbb{Z}/L_2) \times (\mathbb{Z}/M)^2$, we place the background twist $n_{12} = 1$ in the two transverse directions 1 and 2 (25-01). Directions 3 and 4 are the plane. What this chapter calls the **structural inputs** is the following set $\mathcal{I}$ of five.

1. **Positivity of the transfer matrix**: the transfer matrix T with a planar direction (3 or 4) as time is a positive operator (5-03, Osterwalder–Seiler, Lüscher Cited).
2. **The centre and duality identities**: inserting twist flux is inserting a centre transformation, and it can be decomposed by the projections onto electric flux ('t Hooft Cited). The Electric-Flux Decomposition Lemma of § 02 is one of this form.
3. **Reflection positivity in the planar directions**: the positivity of reflections through lattice points in directions 3 and 4 for the z ensemble with the twist fixed (the reflection positivity of z in § 03).
4. **Symmetries of the lattice**: translations, rotations, reflections, and the fact that they preserve the class of the twist.
5. **The finite lemmas**: the container identity and nesting (25-03), the slab estimate $\sup_\Phi \theta_D \le \tanh(\beta L^2)^{|D|}$ (25-04), the flow lemma $\theta \le 1 - a$ (25-06). All of them use only properties of measures and finite sums.

The question is whether the lateral-widening inequality ((5) of 26-02) follows from $\mathcal{I}$. The answer, in § 04, is "it does not". Before getting there, we set out two things that can certainly be said about the sector bias within $\mathcal{I}$.

02

The electric-flux form — $\theta = Y(1)/Y(0) \geq 0$

Take the lattice to be $T^2_{L1\times L2} \times T^2_{M\times M}$, and write $Z(1, n34)$ for the partition function with, in addition to $n12 = 1$, the twist flux $n34 \in \{0, 1\}$ through the plane (3,4) inserted. The global bias of (8) of 26-01 was

$$\theta \;:=\; \frac{Z(1,0) - Z(1,1)}{Z(1,0) + Z(1,1)} \;=\; \tanh\frac{F_v}{2}, \qquad e^{-Fv} := \frac{Z(1,1)}{Z(1,0)} \quad (1)$$

We choose direction 3 as time. The space at one time is $T^2_{L1\,L2} \times T^1_M$ (directions 1, 2, 4), and in it there is the spatial magnetic flux $m_4 = n_{12} = 1$. The transfer matrix $T_3 \geq 0$ commutes with the centre transformation $\hat{C}_4$ of direction 4 (the operation that multiplies the links going once round direction 4 by the centre element -1), and $\hat{C}_4^2 = 1$. When space is split by the eigenvalues $(-1)^{e4}$ of $\hat{C}_4$, $e_4 \in \{0, 1\}$ is the **electric flux** in direction 4 ('t Hooft Cited). Let the projections be P_e and put

$$Y(e) \;:=\; \mathrm{Tr}\big(P_e\, T_3{}^M\big) \qquad (e = 0, 1) \quad (2)$$

This defines Y(e).

Electric-Flux Decomposition Lemma (the electric-flux form) Proved here In the setting above, with $Y(0), Y(1) \geq 0$,

$$Z(1,0) = Y(0) + Y(1), \qquad Z(1,1) = Y(0) - Y(1). \quad (3)$$

Hence when $Z(1,0) + Z(1,1) > 0$, $\theta = Y(1)/Y(0) \geq 0$. If moreover the positivity assumption for the twisted partition function holds, then $\theta \leq 1$; that is, with $F_v \geq 0$, $0 \leq \theta \leq 1$.

Proof The partition function is the trace of the transfer matrix, $Z(1,0) = \mathrm{Tr}\, T_3{}^M$, and the twist $n34 = 1$ through the plane (3,4) is the same as inserting $\hat{C}_4$ somewhere around the time circle, so $Z(1,1) = \mathrm{Tr}(\hat{C}_4 T_3{}^M)$ ('t Hooft's construction Cited). Substituting $\hat{C}_4 = P_0 - P_1$ and $1 = P_0 + P_1$ gives (3). P_e is an orthogonal projection commuting with T_3, so $P_e T_3{}^M P_e = P_e T_3{}^M$ is a positive operator and its trace is at least 0. When $Z(1,0) + Z(1,1) = 2Y(0) > 0$, putting (3) into (1) gives $\theta = 2Y(1)/2Y(0)$. From $Z(1,1) = Y(0) - Y(1) \geq 0$, $Y(1) \leq Y(0)$. ∎

Two remarks. First, $\theta \geq 0$ and $|Z(1,1)| \leq Z(1,0)$ are **spectral** facts that follow from (3) and $Y(e) \geq 0$ alone, but the positivity assumption for the twisted partition function needed for $\theta \leq 1$ is not. $Z(1,1)$ is at least 0 because it is the integral of a positive function containing the twisted plaquette weight $e^{-\frac{\beta}{2}\mathrm{tr}U_p}$, but this is the positivity of the Boltzmann weight and does not follow from the spectrum of the transfer matrix.

Second, the Electric-Flux Decomposition Lemma is a lattice version of 't Hooft's electric–magnetic duality: "the vortex free energy (magnetic flux) is small $\Longleftrightarrow$ the state with electric flux $e_4 = 1$ is heavy compared with $e_4 = 0$". In the confining regime $Y(1)/Y(0)$ is expected to be small because of the electric string tension Reading (physics, 26-05 § 04).

We add one elementary lemma, used when writing lateral widening in the global form.

tanh Comparison Lemma Proved here Let $\theta = \tanh(F/2)$, $\theta' = \tanh(F'/2)$, $F, F' \geq 0$. If $F' \leq KF$ $(K \geq 1)$, then $\theta' \leq K\theta$.

Proof $\tanh$ is monotonically increasing and concave on $[0, \infty)$ with $\tanh 0 = 0$, so for $K \geq 1$, $\tanh(Kx) \leq K \tanh x$ (for a concave function g with $g(0) = 0$, $g(x) = g(\frac{1}{K} \cdot Kx + (1 - \frac{1}{K}) \cdot 0) \geq \frac{1}{K} g(Kx)$). Hence $\theta' = \tanh(F'/2) \leq \tanh(KF/2) \leq K \tanh(F/2) = K\theta$. ∎

That is, lateral widening in the form of the vortex free energy, $F_v(L') \leq C(L'/L)^2 F_v(L)$, implies lateral widening in the form of the bias.

03

Reflection positivity in z — reflections through lattice points, and transverse reflections

Setting. Take the torus $\Lambda = (\mathbb{Z}/L)^2 \times (\mathbb{Z}/M)^2$ with M even. The variables are the links $U_\ell \in SU(2)$ and the plaquette variables $z_p \in \{\pm 1\}$, and we impose the following constraints.

- For every three-dimensional cube c, $\prod_{p \in \partial c} z_p = 1$ (z is a closed $\mathbb{Z}_2$ 2-cochain).
- Fixing the global class, $n_{12}(z) := \prod_{p \in \Pi} z_p = -1$. Π is a (1,2) plane with (x_3, x_4) fixed, and since z is closed this does not depend on the position of Π.

The measure is

$$d\mu \;=\; Z^{-1} \prod_{\ell} dU_\ell \sum_{z} \mathbf{1}[\delta z = 1]\, \mathbf{1}[n_{12}(z) = -1]\, \exp\Big(\frac{\beta}{2} \sum_{p} z_p \operatorname{tr} U_p\Big) \tag{4}$$

(dU is the Haar measure). This is the ensemble of 26-01 § 01 with the twist summed over, with the transverse twist fixed.

Reflection. Let Θ be the reflection through lattice points in direction 3, $x_3 \mapsto -x_3 \pmod M$. The fixed planes are $P_0 = \{x_3 = 0\}$ and $P_1 = \{x_3 = M/2\}$. It acts on links by $U_{(x,\mu)} \mapsto U_{(\Theta x, \mu)}$ for directions $\mu \neq 3$ and by $U_{(x,3)} \mapsto U^{-1}_{(\Theta x - e_3, 3)}$ for direction 3, on plaquette variables by $z_p \mapsto z_{\Theta p}$, and on functions by $(\Theta G)(U, z) := G(\Theta U, \Theta z)$.

Reflection positivity of z Proved here For any bounded function G depending only on the variables of the right half $\Lambda_+ = \{0 \le x_3 \le M/2\}$ (including the variables on the planes $P_0 \cup P_1$), $\langle G \cdot \Theta G \rangle_\mu \ge 0$. The same holds for direction 4.

Proof We show it in five steps.

Step 1 (splitting the variables). Split the variables into three. φ: the variables on the planes $P_0 \cup P_1$ (the links in directions 1, 2, 4 within the planes, and the z of the plaquettes of types (1,2), (1,4), (2,4) within the planes). ψ_+: the right variables (the direction-3 links leaving $0 \le x_3 < M/2$, the links in other directions with $0 < x_3 < M/2$, the plaquettes of type $(3,\nu)$ with $[x_3, x_3+1] \subset [0, M/2]$, and the z of in-plane-type plaquettes with $0 < x_3 < M/2$). $\psi_- := \Theta\psi_+$: the left variables. Θ does not move φ.

Step 2 (decomposing the weight). The plaquette weight $e^{\frac{\beta}{2} z_p \operatorname{tr} U_p}$ belongs to exactly one of right, left and the planes, since p does, and goes into that one. The weights of plaquettes on the planes go into $A(\varphi)$. The cube constraints split in the same way: a cube containing direction 3 spans $[x_3, x_3+1]$ and so depends only on the variables of one side and of the planes; a cube not containing direction 3 depends only on the right variables if $0 < x_3 < M/2$, and only on the plane variables if it lies on a plane. The global constraint n_{12}, measured on $\Pi \subset P_0$, is a function of φ. So the weight has the form $W = A(\varphi)\, F(\psi_+, \varphi)\, F'(\psi_-, \varphi)$, with $A \ge 0$, $F \ge 0$.

Step 3 (the relation between the left and right factors). The action and the constraints are invariant under Θ. Even when the orientation of a plaquette is reversed, in $SU(2)$ we have $\operatorname{tr} U_{\Theta p} = \operatorname{tr} U_p^{-1} = \operatorname{tr} U_p$. Cubes go to cubes, and the plane on which n_{12} is measured is fixed. Hence $F'(\psi_-, \varphi) = F(\Theta\psi_-, \varphi)$.

Step 4 (invariance of the measure). The Haar measure is invariant under taking inverses, and the counting measure of z is invariant under permutations.

Step 5 (positivity). From the above,

$$Z\,\langle G \cdot \Theta G \rangle \;=\; \int d\varphi\, A(\varphi) \Big[\int d\psi_+ F(\psi_+, \varphi) G(\psi_+, \varphi) \Big] \Big[\int d\psi_- F(\Theta\psi_-, \varphi) G(\Theta\psi_-, \varphi) \Big] \;=$$

∎

The algebra from the last equality to the inequality (in the version with finite sums) is checked by ReflPos in Lean (`ReflPos.refl_pos`) (§ 06). The algebra, after splitting into factors, of the Cauchy–Schwarz form that follows from positivity, $\langle G \cdot \Theta H \rangle^2 \le \langle G \cdot \Theta G \rangle \, \langle H \cdot \Theta H \rangle$ (for real G, H), is ReflPos (`ReflPos.refl_cs`). This is an input to the Chessboard Estimate (26-05).

The reflection positivity of z is the Osterwalder–Seiler argument for reflections through lattice points Cited (5-03) with the $\mathbb{Z}_2$ plaquette variables and the closedness constraint added. The key points are that each cube constraint fits on one side plus the planes, and that the global class n_{12} lies in a plane orthogonal to the direction of reflection (3) and can be measured on the plane P_0.

Transverse reflections. If we try the same argument with reflections in directions 1 and 2, the twist gets in the way.

Lemma on the Non-Positivity of Lateral Reflections Proved here In $SU(2)$, write $e^{\frac{\beta}{2}\mathrm{tr}U} = \sum_j c_j \chi_j(U)$ (χ_j is the character of spin j, $c_j = \frac{2(2j+1)}{\beta} I_{2j+1}(\beta) > 0$). Then the twisted plaquette weight is

$$e^{-\frac{\beta}{2}\mathrm{tr}U} = \sum_j (-1)^{2j} c_j \chi_j(U) \quad (5)$$

and the coefficients for half-integer j are negative. Hence, with the twist fixed, the weights of plaquettes straddling the plane of a reflection in the transverse directions 1 and 2 are not of positive type, and the standard argument for reflection positivity does not hold there.

Proof -1 is a centre element of $SU(2)$, and $-\mathrm{tr}\,U = \mathrm{tr}(-U)$, $\chi_j(-U) = (-1)^{2j}\chi_j(U)$. Hence (5) follows from $e^{-\frac{\beta}{2}\mathrm{tr}U} = e^{\frac{\beta}{2}\mathrm{tr}(-U)} = \sum_j c_j \chi_j(-U)$. The formula for c_j and its positivity are the standard expansion via the modified Bessel functions $I_n(\beta) > 0$ ($\beta > 0$) Cited . Being a weight of positive type (coefficients of the character expansion at least 0) is the property the standard argument needs for the plaquettes straddling the plane of reflection, and (5) does not satisfy it. ∎

The consequence of the Lemma on the Non-Positivity of Lateral Reflections is that **reflection positivity gives no convexity or monotonicity in the transverse size** L. The lateral-widening inequality is a comparison in L, so the road to obtaining it from reflections is closed from the start. What the Lemma on the Non-Positivity of Lateral Reflections says is that the standard argument cannot be used; it does not show that positivity of transverse reflections fails for all functions.

04

The negative lemma — from this set of inputs alone

Lemma that Structural Inputs Alone Do Not Suffice (the negative lemma) Proved here The set of structural inputs $\mathcal{I}$ of § 01 (positivity of the transfer matrix, the centre and duality identities (including the Electric-Flux Decomposition Lemma), reflection positivity in the planar directions (the reflection positivity of z), the symmetries of the

lattice, the finite lemmas) holds, with the same proofs, in the four-dimensional Wilson theory in which the group is changed from $SU(2)$ to $\mathbb{Z}_2$ or U(1) and the twist variables are kept as ± 1. Hence, if there were an argument deriving the lateral-widening inequality using $\mathcal{I}$ alone, the same argument would derive the lateral-widening inequality for $\mathbb{Z}_2$ and U(1) too. Contrapositively, **if the lateral-widening inequality is false for either $\mathbb{Z}_2$ or U(1), it cannot be derived from this set of inputs alone**.

Proof For each input, we check the properties of the group it uses.

- Positivity of the transfer matrix: the Osterwalder–Seiler argument holds for the Wilson action of a compact group, for actions whose character-expansion coefficients are positive. $e^{\beta \operatorname{Re} U_p}$ for $\mathbb{Z}_2$ and U(1) satisfies this Cited .
- The centre and duality identities: all that is used is that -1 is in the centre of the group (the centre of $\mathbb{Z}_2$ and of U(1) is the group itself and contains -1), and that twist flux is the insertion of a centre transformation. The proof of the Electric-Flux Decomposition Lemma goes through as it stands.
- Reflection positivity: the only properties of the group used in the proof of the reflection positivity of z are $\operatorname{tr} U_{\Theta p} = \operatorname{tr} U_p^{-1} = \operatorname{tr} U_p$ in step 3 and the invariance of the Haar measure under inverses in step 4. For U(1), writing the action with $\operatorname{Re} U_p$, $\operatorname{Re} U^{-1} = \operatorname{Re} U = \operatorname{Re} U$; for $\mathbb{Z}_2$, $U^{-1} = U$; both hold.
- Symmetries of the lattice: the definitions of the action and of the class of the twist have a form independent of the group, so they are the same.
- The finite lemmas: the container identity uses independence conditioned on the frame (the Markov property), the slab estimate uses conditioning on the orbits of flips of the plaquette variables, and the flow lemma uses only stationarity; none uses the non-commutativity of the group.

The first half is shown, so the second half is logic: if an argument uses only the properties of $\mathcal{I}$, it gives the same conclusion in every theory that satisfies $\mathcal{I}$. ∎

The premise of the Lemma that Structural Inputs Alone Do Not Suffice, "the lateral-widening inequality is false for $\mathbb{Z}_2$ or U(1)", is the estimate of 26-03 § 02 Reading (physics; the count for the Coulomb phase of U(1) and the count for the wall of $\mathbb{Z}_2$ at weak coupling). Granting it, the conclusion is that **the lateral-widening inequality does not follow from the structural inputs alone**. This last step inherits the label of its premise, [Reading] (physics).

The same argument applies as it stands to the weak form (the lateral-widening inequality (weak form) of 26-02) and to the uniform slab tension. For U(1) and $\mathbb{Z}_2$ the slab tension is estimated to go to 0 as L grows, so neither follows from the same set of inputs alone Reading (physics).

We also write what the Lemma that Structural Inputs Alone Do Not Suffice does not say. First, it does not say that $\mathcal{I}$ is useless for the band. If L is fixed, the reflection positivity of z replaces frame decorrelation by the chessboard density κ_{cb} (26-05). Second, what comes out when one dynamical input is added to $\mathcal{I}$ is outside the lemma. Third, at strong coupling (small β) U(1) also confines, so what the lemma separates is not the groups themselves but **whether there is a non-confining phase**.

The thin-vortex floor — tools of structure alone give only the floor. Even when correlation inequalities and covering comparisons are added to the structural inputs, what follows is a lower bound (a floor) that does not distinguish between groups. For the $SU(2)$ Wilson measure on the slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, for every real β, every $N, M \geq 1$ and background $n_{12} \in \{0,1\}$,

$$\theta = \frac{Z_{n34=0} - Z_{n34=1}}{Z_{n34=0} + Z_{n34=1}} \;\leq\; \tanh\!\left(N^2|\beta|\right)^{M^2} \qquad (6)$$

holds Proved here (the thin-vortex floor. The $SU(2)$ system is bounded above by a $\mathbb{Z}_2$ sign system, and the four-dimensional $\mathbb{Z}_2$ system is moved to a two-dimensional system of plaquettes and evaluated in closed form. Checked in Lean as a theorem about the concrete $SU(2)$ Wilson measure). The type of the inequality lies on the line of Mack–Petkova and Fröhlich Cited (not yet checked against the text of the originals). When the coupling is positive, in terms of the tension, $\sigma_N(\beta) \geq -\ln\tanh(N^2\beta) \approx 2e^{-2N^2\beta}$, which goes rapidly to 0 with N and is not uniform in N (inside the strong-coupling window there is a form in which the rate per unit area does not depend on the transverse size; the prefactor remains. Outside the window there is no uniform form. 20-06). The right-hand side is the value of a thin vortex made by stacking N^2 plaquettes along the plane, and the same comparison holds in the same form for the $\mathbb{Z}_2$ system. A comparison of this type becomes an equality for the Gaussian model, which sits at the limit of the class of models it applies to, so the comparison alone does not give that the floor keeps being exceeded at a uniform rate (the tension-doubling inequality of 26-06 § 08) Reading . Where the Lemma that Structural Inputs Alone Do Not Suffice says "does not follow from this set of inputs alone", the same point said from the side of what does follow is "what follows is only the floor". Above the floor is where the non-abelian dynamical input of § 05 is needed. The statement and proof of this floor are in 21-02.

05

The inputs needed — what excludes the non-confining phase

By the Lemma that Structural Inputs Alone Do Not Suffice, an argument that crosses the band must use at least one input outside $\mathcal{I}$. That input must be false for U(1) and $\mathbb{Z}$

2 at weak coupling — the non-confining phase. The candidates that the study of this book can name are all properties of non-abelian dynamics.

Table 1 Candidate inputs needed for the band (none is shown; the labels are estimates for the candidates)

Input	For U(1) and $\mathbb{Z}_2$	Where it enters	Chapter
Asymptotic freedom (the sign of the running of the coupling)	U(1) does not run or runs the other way; $\mathbb{Z}_2$ has no continuous running	The margin $5/3$ in the dilute regime; $\beta' < \beta$ in the renormalisation comparison	26-03 § 04, 26-06
Localisation in a self-dual lump (non-linearity)	Abelian flux can only spread out uniformly	The classical cost of lateral widening is 0	26-03 § 03
Ultraviolet control (multiscale analysis)	For U(1), a different form (conformal at long distance)	The base point at small T^2	the base point at small T^2 in 26-02, 27-02

Each row of Table 1 is an estimate, [Reading] (physics), and none of the inputs is in a form usable in the band $L \sim 1/\Lambda$. On the strong-coupling side, there is a line of argument by a polyhedral expansion in which the lateral-widening inequality holds with the constant $C \approx 1$ (there is a report that the factor $(L'/L)^2$ is consistent with the literature on the leading strong-coupling term (Münster 1981), but checking is not yet done). However, an estimate uniform in the planar size M has not been confirmed, and this book places it under [Assumption]. At strong coupling U(1) also confines, so this row does not contradict the Lemma that Structural Inputs Alone Do Not Suffice.

The Lemma that Structural Inputs Alone Do Not Suffice makes explicit a result that was expected; there is no surprise in it. Its role is to give one **touchstone** for reading arguments about the band: if an argument proceeds using only the properties of $\mathcal{I}$, it gives the same conclusion for U(1), so there is an error somewhere. The record in 26-08 of a road that drew the criticism that it does not distinguish U(1) is a real example of this touchstone.

06

Reading the Lean statements — `FluxDuality`, `ReflPos`

The spectral part of the Electric-Flux Decomposition Lemma and the algebra of step 5 of the reflection positivity of z are five theorems in the Lean 4 files `FluxDuality.lean` and `ReflPos.lean`. We read the statements in the words of the text. All of them are statements about sums of real numbers over a finite set ι (or Φ, Ψ).

`FluxDuality.split`: for families of reals $t, c : \iota \to \mathbb{R}$, under the assumption "for every i, $c_i = 1$ or $c_i = -1$", $\sum_i t_i - \sum_i c_i t_i = 2\sum_{c_i=-1} t_i$ and $\sum_i t_i + \sum_i c_i t_i = 2\sum_{c_i=1} t_i$. In

the text t_i are the eigenvalues of T_3^M and c_i the eigenvalues of $\hat{C}_4$, and the two formulas are $Z(1,0) \mp Z(1,1) = 2Y(1),\ 2Y(0)$ of (3).

`FluxDuality.twisted_le`: under the assumptions "for every i, $0 \le t_i$" and "$c_i = \pm 1$", $\left|\sum_i c_i t_i\right| \le \sum_i t_i$. This is $|Z(1,1)| \le Z(1,0)$ of the text.

`FluxDuality.bias_mem`: under the assumptions "$0 \le t_i$", "$c_i = \pm 1$", "$0 \le \sum_i c_i t_i$", "$0 < \sum_i t_i + \sum_i c_i t_i$", $0 \le \theta \le 1$ for $\theta := (\sum t - \sum ct)/(\sum t + \sum ct)$. **The third assumption $Z_{11} := \sum ct \ge 0$ is used for $\theta \le 1$**; it is not a spectral fact, and corresponds to the positivity of the Boltzmann weight in the text.

`ReflPos.refl_pos`: for finite sets Φ, Ψ and real functions $A : \Phi \to \mathbb{R}$, $F, G : \Psi \to \Phi \to \mathbb{R}$, under the assumption "for every φ, $0 \le A(\varphi)$", $\sum_\varphi \sum_{\psi_1} \sum_{\psi_2} A(\varphi) F(\psi_1, \varphi) F(\psi_2, \varphi) G(\psi_1, \varphi) G(\psi_2, \varphi) \ge 0$. It is the finite-sum version of $\int A \left| \int FG \right|^2 \ge 0$ in step 5 of the reflection positivity of z, and the sign of F does not matter.

`ReflPos.refl_cs`: under the same assumption $0 \le A$, for $g(\varphi) := \sum_\psi F(\psi, \varphi) G(\psi, \varphi)$ and $h(\varphi) := \sum_\psi F(\psi, \varphi) H(\psi, \varphi)$, $\left(\sum_\varphi A\, g\, h\right)^2 \le \left(\sum_\varphi A\, g^2\right)\left(\sum_\varphi A\, h^2\right)$. It writes the Cauchy–Schwarz form for reflections, $\langle G \cdot \Theta H \rangle^2 \le \langle G \cdot \Theta G \rangle \langle H \cdot \Theta H \rangle$, as algebra after the weight has been split into factors.

What machine checking guarantees is only this algebra of finite sums and real numbers. That the transfer matrix is positive, that it commutes with $\hat{C}_4$, that the twist is the insertion of $\hat{C}_4$, that the weight splits into the form $A \cdot F \cdot F$ (steps 1–4 of the reflection positivity of z), and the Haar measure lie outside Lean; the proofs in the text and the literature carry them. The axioms of the five theorems are, in the record, the three standard ones (`propext`, `Classical.choice`, `Quot.sound`), and they pass this book's re-check (A-04).

07

The table of the five grades and the dependency graph

Table 2 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	Positivity of the transfer matrix $T \ge 0$	Cited	Osterwalder–Seiler, Lüscher, 5-03	Wilson action of a compact group	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
2	The twist $n34$ is the insertion of the centre transformation $\hat{C}_4$; e_4 is an eigenvalue of $\hat{C}_4$	Cited	't Hooft, 25-01	Groups whose centre contains -1	—
3	Electric-Flux Decomposition Lemma: $Z(1,0) = Y(0) + Y(1)$, $Z(1,1) = Y(0) - Y(1)$, $\theta = Y(1)/Y(0) \geq 0$	Proved here	Proof in § 02 $\Longleftarrow$ 1, 2	$T^2_{L_1 L_2} \times T^2_{M \times M}$, $n_{12} = 1$, $Z(1,0) + Z(1,1) > 0$	FluxDuality (`FluxDuality.split`) (assumption: $c_i = \pm 1$), `twisted_le` (assumptions: $0 \leq t_i$, $c_i = \pm 1$)
4	$\theta \leq 1$ ($F_v \geq 0$)	Proved here	§ 02 $\Longleftarrow$ 3	Uses the positivity assumption for the twisted partition function (positivity of the Boltzmann weight)	FluxDuality (`FluxDuality.bias_mem`) (assumptions: $0 \leq t_i$, $c_i = \pm 1$, $Z_{11} \geq 0$, $Z_{10} + Z_{11} > 0$)
5	tanh Comparison Lemma: if $F' \leq KF$ ($K \geq 1$), then $\tanh(F'/2) \leq K \tanh(F/2)$	Proved here	Proof in § 02 (concavity of $\tanh$)	$F, F' \geq 0$	—
6	Reflection positivity of z: positivity of reflections through lattice points in directions 3 and 4 for the z ensemble	Proved here	Proof in § 03 (the Osterwalder–Seiler argument with z and the constraints added)	$SU(2)$, M even, twist $n_{12} = 1$ fixed	ReflPos (`ReflPos.refl_pos`) (only the algebra of step 5; assumption: $0 \leq A$), `refl_cs` (assumption: $0 \leq A$)
7	Character-expansion coefficients $c_j = \frac{2(2j+1)}{\beta} I_{2j+1}(\beta) > 0$	Cited	Standard expansion (cross-checked numerically)	$SU(2)$, $\beta > 0$	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
8	Lemma on the Non-Positivity of Lateral Reflections: the twisted plaquette weight has negative coefficients for half-integer j; transverse reflections are not of positive type	Proved here	Proof in § 03 ⟸ 7	Twist fixed, reflections in directions 1 and 2	—
9	Lemma that Structural Inputs Alone Do Not Suffice: $\mathcal{I}$ holds for $\mathbb{Z}_2$ and U(1) too, and an argument from $\mathcal{I}$ alone gives the same conclusion there	Proved here	Proof in § 04 ⟸ 1, 3, 6	The form "from this set of inputs alone"; twist ± 1	—
10	For U(1) and $\mathbb{Z}_2$ the lateral-widening inequality is false	Reading (physics)	26-03 § 02	Weak coupling, an estimate	—
11	The lateral-widening inequality (strong and weak forms) and the uniform slab tension do not follow from the structural inputs alone	Reading (physics)	§ 04 ⟸ 9, 10	On top of the estimate that the lateral-widening inequality is false for U(1) and Z_2	—
12	Reflection positivity gives no convexity or monotonicity in the transverse size L	Proved here	§ 03 ⟸ 8	In the sense that the standard argument cannot be used	—
13	At strong coupling the lateral-widening inequality holds with $C \approx 1$	Assumption	Line of argument: polyhedral expansion (an estimate uniform in the planar size is unconfirmed)	Small β	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
14	In the confining regime $Y(1)/Y(0)$ is small because of the electric string tension	Reading (physics)	26-05 § 04	$L\Lambda \gg 1$, an estimate	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  positivity of the transfer matrix                    compact group, 
 2  CIT  twist is a centre insertion, e₄ an eigenvalue        't Hooft
 3  PRV  Electric-Flux Decomposition Lemma: θ = Y(1)/Y(0) ≥ 0 ⟸ 1, 2   Lean s
 4  PRV  θ ≤ 1 ⟸ 3                                            uses Z(1,1) ≥
 5  PRV  tanh Comparison Lemma: F' ≤ KF ⇒ θ' ≤ Kθ             K ≥ 1
 6  PRV  reflection positivity of z (directions 3, 4)         Lean refl_pos,
 7  CIT  character-expansion coefficients c_j > 0             SU(2)
 8  PRV  Lemma on the Non-Positivity of Lateral Reflections ⟸ 7   twist fixed
 9  PRV  Structural Inputs Alone Do Not Suffice ⟸ 1, 3, 6      from this set
10  RDG  lateral widening false for U(1), Z₂                  26-03, estimate
11  RDG  does not follow from structural inputs alone ⟸ 9, 10
12  PRV  no convexity or monotonicity in L from reflections ⟸ 8
13  ASM  C ≈ 1 at strong coupling                             line of argumen
14  RDG  Y(1)/Y(0) small in the confining regime              estimate
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The nodes that stand on paper are the Electric-Flux Decomposition Lemma, $\theta \leq 1$, the tanh Comparison Lemma, the reflection positivity of z, the Lemma on the Non-Positivity of Lateral Reflections, the Lemma that Structural Inputs Alone Do Not Suffice, and the absence of convexity and monotonicity; they rest on the literature rows (positivity of the transfer matrix, the insertion of the centre transformation, the character-expansion coefficients). Only the claim of the chapter title (that the lateral-widening inequality and the uniform slab tension do not follow from the structural inputs alone) has an estimate row (that the lateral-widening inequality is false for U(1) and Z_2) as a premise, and it is "Reading (physics)". The assumption row (that the lateral-widening inequality holds at strong coupling) is not a premise of any conclusion.

The machine-checked form of reflection positivity is in 20-01, and the sign of θ is in 21-06.

What this chapter can say

What is said	What is not said
Proved here The global bias of the sector is the ratio of electric fluxes $\theta = Y(1)/Y(0) \in [0,1]$ (the upper bound uses the positivity of the weight)	An estimate of the size of $Y(1)/Y(0)$. The electric string tension in the band
Proved here The z ensemble with the twist fixed is positive under reflections through lattice points in directions 3 and 4. Transverse reflections are not of positive type because of the twist	That positivity of transverse reflections fails for all functions. Convexity in L
Proved here The set of structural inputs $\mathcal{I}$ holds for $\mathbb{Z}_2$ and U(1) too, and an argument from $\mathcal{I}$ alone gives the same conclusion there (from this set of inputs alone)	What comes out when a dynamical input is added to $\mathcal{I}$
Reading (physics) The lateral-widening inequality and the uniform slab tension do not follow from the structural inputs alone	A proof, as a theorem, that the lateral-widening inequality is false for U(1) and $\mathbb{Z}_2$
Lean: the spectral algebra (FluxDuality (`FluxDuality.split`, `FluxDuality.twisted_le`, `FluxDuality.bias_mem`)) and the algebra of the final sum of the reflection (ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`))	The transfer matrix, the centre transformation, the decomposition of the weight, the Haar measure (outside Lean)

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity and positivity of the transfer matrix	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. M. Lüscher, Commun. Math. Phys. 54 (1977) 283. Chapter 5-03
Electric and magnetic flux, the twist and centre transformations	Cited	G. 't Hooft, Nucl. Phys. B153 (1979) 141. Chapter 25-01
The ensemble with the twist summed over	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442. Chapter 25-03
Electric-Flux Decomposition Lemma, tanh Comparison Lemma, reflection positivity of z, Lemma on the Non-Positivity of Lateral Reflections, Lemma that Structural Inputs Alone Do Not Suffice	Proved here	The proofs in § 02– § 04 of the text. The same content as the propositions "site-reflection positivity" and "no-go from structure" of the paper on the reduction of the band (a paper that came out of this book, A-07)
The formula for the character-expansion coefficients	Cited	Standard expansion. Checked to agree to 8 decimal places with numerical integration over the Haar measure for $\beta = 0.5, 2.5$, $j \le 3/2$
The line of argument by the strong-coupling polyhedral expansion	Assumption	Line of argument only. The literature for the comparison of the leading term (Münster 1981) is not yet checked

Item	Kind	Source / tools
Lean	Lean	FluxDuality (`FluxDuality.split`, `FluxDuality.twisted_le`, `FluxDuality.bias_mem`), ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`). They guarantee only the algebra of finite sums and real numbers. A-04

Terms used in this chapter — structural inputs, the transfer matrix, the centre transformation, the electric flux e_4, the sector bias θ, the z ensemble, reflection through lattice points, positive type, the negative lemma, the lateral-widening inequality, touchstone. Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-05 The chessboard and simultaneous flips — using the reflection positivity of z to replace frame decorrelation by a single density.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band
the chessboard density κ_cb, simultaneous flips, two uniformities · Kiichi, Shiori and Rin

The chessboard and simultaneous flips

> This chapter is this book's own investigation. § 04 and § 05 are a map of open questions; § 06 is guide material.

What this chapter shows — The second assumption of the reduction in 26-02, "frame decorrelation", was the assumption that the product of the biases of containers sharing a frame splits into a product of typical values. If the size L of the transverse torus is fixed, the reflection positivity of z (the Reflection-Positivity Theorem for z in 26-04) and the Chessboard Estimate replace this assumption by **a single density** κ_{cb}. On a slab with L fixed, the Wilson loop is bounded by $\kappa_{\mathrm{cb}}^{n(C)}$, and the only work left is the estimate $\kappa_{\mathrm{cb}} < 1$. As tools for that estimate we set side by side the slab estimate and simultaneous flips, and we also write why simultaneous flips cannot be used to test the band. Finally we see that the band is concentrated into two uniformities, "the uniform slab tension" and "the uniform slab gap".

Prerequisite chapters — 26-02 § 02 (frame decorrelation), 26-04 § 03 (z reflection positivity), 25-03 (containers), 25-04 (slab confinement), 25-06 (the flow lemma and simultaneous flips), 5-03 (reflection positivity).

Said first — The paper results of this chapter (the Chessboard Estimate with the Lateral Size Fixed and the Simultaneous-Flip Estimate) are statements of the form **if L is fixed**. There is not a single estimate uniform in L. The two uniformities of the band remain [Assumption]. The Chessboard Estimate itself is a standard tool (Fröhlich–Israel–Lieb–Simon); what is new is only the construction that applies it to the biases of containers. The subscript of the symbol κ_{cb} is there to distinguish it from the other κ of this book (§ 06).

The order of this chapter

1. What is replaced — frame decorrelation, and the power lost to Hölder
2. The Chessboard Estimate — the chessboard with the lateral size fixed, and the density κ_cb
3. Upper bounds on κ_{cb} — the slab estimate and the Simultaneous-Flip Estimate, and their limits
4. Where the band appears again — the dilute regime, the confining regime, and in between

5. The band concentrated — the uniform slab tension and the uniform slab gap (Table 1)
6. The uses of the symbol κ compared (Table 2)
7. The table of the five grades and the dependency graph

01

What is replaced — frame decorrelation, and the power lost to Hölder

The notation is as in 26-01 § 01. The sector bias $\theta_D(\Phi) = |\mathbb{E}[\sigma_D \mid \Phi]|$ of a container $D \times T_L^2$ is a function of the frame Φ, and if mutually disjoint containers D_j in the plane lie inside the loop C, then

$$|\langle W(C)\rangle| \;\le\; \mathbb{E}\prod_j \theta_{D_j}(\Phi) \qquad (1)$$

(Mack–Petkova Cited). The right-hand side is the mean of a product of factors **sharing the same frame**. If the factors were independent it would split into $\prod_j \mathbb{E}\theta_{D_j}$, but the frame field is connected between containers, so it does not. If we use the Hölder inequality, $\mathbb{E}\prod_{j=1}^{n}\theta_j \le \prod_j (\mathbb{E}\theta_j^n)^{1/n}$, each factor on the right is the n-th root of the mean of the n-th power of θ; this is not a product of typical values θ, and a power proportional to the area is lost (in the nested form only $\mathbb{E}\theta_{2R}^4 \le \mathbb{E}\theta_R^4$ comes out).

So 26-02 § 02 set frame decorrelation,

$$\theta(sR; T_L^2) \;\le\; C_{\mathrm{D}}\,\theta(R; T_L^2)^{s^{\measuredangle}} \qquad (2)$$

as an assumption Assumption . (2) played two roles: the power of the nesting (the jump step and the doubling step of the Conditional Reduction Theorem of 26-02), and the bridge from the bias to the Wilson loop (the Area Law from the Area Exponent in 26-01). This chapter moves the role of splitting the product, **on a slab with L fixed**, to a single translation-invariant quantity.

02

The chessboard estimate — the density κ_{cb}

Setting. Consider the z ensemble with the twist $n_{12} = 1$ fixed ((4) of 26-04) on the lattice $(\mathbb{Z}/L)^2 \times (\mathbb{Z}/M)^2$. Divide the plane (3,4) into square blocks Q_t of side B ($t \in \mathcal{T}$, including the boundary lattice points), and let M be a multiple of $2B$. Inside each block put one container $D_t \times T_L^2$, at positions carried into one another by reflections.

Chessboard Estimate (Fröhlich–Israel–Lieb–Simon) Cited Suppose the measure is reflection positive for reflections in planar directions 3 and 4 through all the lattice planes on the boundaries of the blocks. For each t let f_t be a bounded real function depending only on the variables of the block Q_t (boundary included), and let $f_t^{(s)}$ be the function obtained by carrying f_t to the block Q_s by a composition of reflections. Then

$$\Big|\Big\langle \prod_{t\in\mathcal{T}} f_t \Big\rangle\Big| \;\le\; \prod_{t\in\mathcal{T}} \Big\langle \prod_{s\in\mathcal{T}} f_t^{(s)} \Big\rangle^{1/|\mathcal{T}|}. \tag{3}$$

This is the standard argument of Fröhlich–Israel–Lieb–Simon, obtained by using the reflection Cauchy–Schwarz form repeatedly (26-04 § 03; Lean's ReflPos (`ReflPos.refl_cs`) is the algebra after splitting into those factors). (The original paper is Fröhlich–Israel–Lieb–Simon (1978). In this book we checked from the abstract of M. Biskup's lecture notes, arXiv:math-ph/0610025, that this is the standard form of obtaining the Chessboard Estimate from reflection positivity.)

In the z ensemble, reflection positivity is given by the reflection positivity of z in 26-04. The reflection positivity of z speaks of reflections through $x_3 = 0$ and $x_3 = M/2$, but the measure is invariant under translations in direction 3, so the same holds for reflections through any lattice plane $x_3 = c$. Direction 4 is the same.

Next we check that the bias is a function of the block. Given the frame, the interiors of the containers $D_t \times T_L^2$ are conditionally independent, and the conditional distribution depends only on the variables of the shell just outside the container (the Markov property). Since the container lies inside the block, the shell lies in the block, and $\theta_t := \theta_{D_t}$ is a function of the variables of Q_t. Also, a reflection preserves the measure and carries the sector $\sigma_D = \prod_{p\in D\times\{y_0\}} z_p$ to the sector of the reflected container, so the function obtained by carrying θ_t to the block Q_s by reflection is θ_s.

Chessboard Estimate with the Lateral Size Fixed (the chessboard with L fixed) Proved here Fix L and M. In the setting above, put

$$\kappa_{\mathrm{cb}} \;:=\; \Big(\mathbb{E}\prod_{t\in\mathcal{T}} \theta_t\Big)^{1/|\mathcal{T}|} \tag{4}$$

Then for every subset $S \subset \mathcal{T}$, $\mathbb{E}\prod_{t\in S}\theta_t \le \kappa_{\mathrm{cb}}^{|S|}$. In particular, if $n(C)$ is the number of blocks whose container lies wholly inside the loop C,

$$|\langle W(C)\rangle| \;\le\; \kappa_{\mathrm{cb}}^{\,n(C)}. \tag{5}$$

Proof In (3) put $f_t = \theta_t$ if $t \in S$ and $f_t = 1$ if $t \notin S$. Since $\theta_t \ge 0$, the left-hand side is $\mathbb{E}\prod_{t\in S}\theta_t$. On the right, the factors with $t \in S$ are $\big(\mathbb{E}\prod_{s\in\mathcal{T}}\theta_s\big)^{1/|\mathcal{T}|} = \kappa_{\mathrm{cb}}$, because $\theta_t^{(s)} = \theta_s$

as seen above, and the factors with $t \notin S$ are 1. Hence $\mathbb{E}\prod_{t\in S}\theta_t \le \kappa_{\mathrm{cb}}^{|S|}$. For (5), take D_j in (1) to be the containers inside C and use this estimate. ∎

Since a block of side B occupies the area B^2, (5) gives the lower bound $\ln(1/\kappa_{\mathrm{cb}})/B^2$ on the tension (for square loops whose side is a multiple of B). **With L fixed, z reflection positivity has closed, so the gap to the area law on the slab has become only the estimate $\kappa_{\mathrm{cb}} < 1$** Proved here (a statement with L and M fixed). No assumption about the distribution of the frame, such as frame decorrelation (2), is needed.

κ_{cb} is the geometric mean per block of the product of the biases when containers are placed in all blocks at once. It is not "the bias of one typical container" but "a quantity that looks at all containers at once", so the correlation of the frames is absorbed into κ_{cb}. In exchange, the smallness of κ_{cb} can no longer be read as a property of one container.

Nested Supermultiplicativity Lemma Proved here Take the container to be the whole interior of the block (let D_t be all the (3,4) plaquettes in the interior of Q_t, so that the shell $\partial Q_t \times T_L^2$ lies in the closed block Q_t). Write $\kappa_{\mathrm{cb}}(R;L)$ for (4) for blocks of side R (with M fixed). For an integer $s \ge 1$, if $2sR$ divides M, then for all L

$$\kappa_{\mathrm{cb}}(sR;L) \;\le\; \kappa_{\mathrm{cb}}(R;L)^{s^2}. \quad (5a)$$

Proof A large block Q'_u of side sR is the union of s^2 small blocks of side R, and since the (3,4) plaquettes lie in the interiors of the blocks, the sector is $\sigma'_u = \prod_{t\subset u}\sigma_t$. The union Φ' of the shells of the small blocks contains the union Φ of the shells of the large blocks, and by the Markov property the small containers are conditionally independent given Φ', so $\theta'_u = |\mathbb{E}[\sigma'_u \mid \Phi]| \le \mathbb{E}[\prod_{t\subset u}\theta_t \mid \Phi]$ ((2) of 26-02, the nested form of Mack–Petkova). $X_u := \prod_{t\subset u}\theta_t$ is a function of the variables of the closed large block u, and given Φ the interiors of the large blocks are conditionally independent, so $\prod_u \mathbb{E}[X_u \mid \Phi] = \mathbb{E}[\prod_u X_u \mid \Phi]$. Averaging, $\mathbb{E}\prod_u \theta'_u \le \mathbb{E}\prod_t \theta_t$. The numbers of blocks satisfy $|\mathcal{T}| = s^2|\mathcal{T}'|$, so take the $1/|\mathcal{T}'|$-th power of both sides. ∎

The Nested Supermultiplicativity Lemma does not use reflection positivity. It uses only the Markov property and the nested form, and the constant is 1, independent of L, R and s. The Chessboard Estimate is needed only for the step down from κ_{cb} to a single container ($\mathbb{E}\theta_t \le \kappa_{\mathrm{cb}}$) and to the Wilson loop ((5)). The abstract form on a finite probability space, with the large block as a fibre of s^2 small blocks and the inequality of means placed as an assumption, is Lean: LatticeGaugeNestedBlocks (nested_supermultiplicative). The Markov property is outside Lean.

03

Upper bounds on κ_{cb} — the slab estimate and simultaneous flips, and their limits

(i) The slab estimate. For each face $p \in D$ of the container, the operations that flip the z on the face $\{p\} \times T_L^2$ commute with one another, each flips the sector, and none changes the frame. Conditioning on their orbit gives $\mathbb{E}[\sigma_D \mid \text{orbit}] = \pm \prod_{p \in D} \tanh X_p$, and since $|X_p| \le \beta L^2$, $\sup_\Phi \theta_D(\Phi) \le \tanh(\beta L^2)^{|D|}$ (25-04 Proved here). From $\theta_t \le \sup \theta$,

$$\kappa_{\mathrm{cb}} \;\le\; \tanh(\beta L^2)^{|D|} \;<\; 1. \quad (6)$$

So the area law on a slab with L fixed already follows. But the tension is $-\ln \tanh(\beta L^2) \approx 2e^{-2\beta L^2}$ (per unit area), which for $\beta = 2.5$, $L = 4$ is $2e^{-80}$. **An estimate that goes through the supremum (sup) over frames gets worse superexponentially in L,** so an estimate uniform in L can only come from a mean form — a quantity like κ_{cb}.

(ii) Simultaneous flips. The flow lemma (25-06) says that for any Markov kernel K that preserves the conditional distribution $\mu(\cdot \mid \Phi)$ of the interior of the container, if $a(\Phi)$ is the probability that the sector changes in one step, then $\theta_D(\Phi) \le 1 - a(\Phi)$ Proved here (it uses only stationarity; reversibility is not needed).

Simultaneous-Flip Estimate Proved here With the frame held fixed, run a stationary kernel independently for one step inside each of the mutually disjoint containers D_t ($t \in S$), starting from the conditional distribution. Then

$$\mathbb{E} \prod_{t \in S} \theta_t \;\le\; \mathbb{P}\big(S\text{: no container sector changes}\big). \quad (7)$$

Proof Fix the frame Φ. The interiors of the containers are conditionally independent and the kernels also run independently, so the conditional probability that no container changes is $\prod_{t \in S}(1 - a_t(\Phi))$. From the flow lemma, $\theta_t(\Phi) \le 1 - a_t(\Phi)$, so $\prod_t \theta_t(\Phi) \le \mathbb{P}(\text{none changes} \mid \Phi)$. Averaging over Φ gives (7). ∎

The right-hand side of (7) is a quantity that can be measured by Monte Carlo, so with $S = \mathcal{T}$, $\kappa_{\mathrm{cb}} \le \mathbb{P}(\text{none changes})^{1/|\mathcal{T}|}$ becomes a tool that estimates the upper bound on the Wilson loop **in the correct direction**. The natural kernel is the dynamically driven insertion (25-06, acceptance probability $\min(1, e^{-W})$).

(iii) The limits. The flip probability a of a one-step kernel is, in the range measured, away from 1 (0.6–0.73 for the driven-insertion kernel with acceptance $\min(1, e^{-W})$, 25-08 Computed). So what (7) can guarantee is of the form $\theta_t \le \theta_0$ (with θ_0 a constant less than 1), and κ_{cb} cannot be made small exponentially in the area. Even if this is uniform in L for blocks of size $B \sim L$, the lower bound on the tension that comes out is $\ln(1/\theta_0)/L^2 \to 0$, which is only the naive form of the band treated by the Tension Lower Bound from the

Naive Form in 26-01 — a form that U(1) also satisfies Proved here (logic). **Simultaneous flips cannot be used to test the band.** To check the correct form of the band, $\theta \le e^{-cL^2}$, one needs a kernel with $a \to 1$, and no construction of one is known (25-06 § 07). Treating the measurements of simultaneous flips as "the numerical face of the band" has been withdrawn (25-08); the correct use of this tool is the lower bound on the tension of a slab with L fixed — the estimation of κ_{cb}.

04

Where the band appears again — the dilute regime, the confining regime, and in between

The Chessboard Estimate with the Lateral Size Fixed was a statement with L fixed. When L moves, the block size B needed to show $\kappa_{\mathrm{cb}} < 1$, and the quality of the estimate, change from scale to scale. Everything below is an estimate Reading (physics).

- **The dilute regime ($\Lambda L \ll 1$).** For a container of side ℓ, $\mathbb{E}[-\ln \theta_D] \approx \sigma_{\mathrm{TU}} \ell^2$, with $\sigma_{\mathrm{TU}} \propto \Lambda^2 (\Lambda L)^{5/3}$ (26-03 § 04). The influence of the frame decays over distances of about L/π. The criterion holds, but the blocks needed have $B \gtrsim \sigma_{\mathrm{TU}}^{-1/2}$, and their volume grows exponentially in $1/g^2(L)$.
- **The confining regime ($\Lambda L \gg 1$).** In the form of the electric flux (the Electric-Flux Decomposition Lemma of 26-04), $\theta \approx Y(1)/Y(0) \approx c\,\sigma L^2 e^{-\sigma \ell^2}$ (the factor L^2 is the entropy of the transverse position of the electric string). The block size needed grows only as $\sqrt{\ln L}$.
- **In between ($\Lambda L \sim 1$) — the band.** Two uniformities are needed (§ 05).

As $a \to 0$, both B and L become infinite in lattice units, so checking finitely many boxes is not enough. An inequality that connects the scales is needed, and that is the problem of the band.

Why the usual Dobrushin-type criteria cannot be used. With the frame fixed, the sectors do not interact with one another. All the dependence is in the frame field, and since a continuous field follows the boundary, the influence coefficient measured in total variation is nearly 1. Conditions of the Dobrushin, Dobrushin–Shlosman and van den Berg–Maes type are not satisfied as they stand, and we expect another input to be needed, such as contraction on good blocks (of the type of Bałaban's multiscale analysis) Reading (physics). The original papers of these criteria have not been checked in this book.

05

The band concentrated — the uniform slab tension and the uniform slab gap

In the language of the chessboard, the band is concentrated into the following two statements of uniformity. Neither has been shown, and we set them as assumptions.

The uniform slab tension Assumption The string tension of the two-dimensional theory obtained by shrinking the transverse directions to T^2_L (the slab tension, (11) of 26-02) is positive uniformly in L:

$$\inf_{L \geq L_0} \sigma_{\mathrm{eff}}(L) \;>\; 0 \qquad (\text{physical units, uniform in the lattice spacing } a). \tag{8}$$

In the language of containers it has the form $\inf_L \mathbb{E}[-\ln \theta_D]/(\ell\Lambda)^2 > 0$ and corresponds to the content of the lateral-widening inequality (weak form) of 26-02. The chessboard criterion only carries it to the area law. For U(1) the tension is $\sim e^2/L^2$ for every L, and it is expected to fail **here only** Reading (physics).

The uniform slab gap Assumption That κ_{cb} can be compared across scales: that the estimates of the correlations between biases, of the probability of bad blocks and of the fluctuations of the shells are uniform in L (this corresponds to the mass gap of the slab being uniform in L). This is what frame decorrelation (2) of 26-02 really is when made uniform in L. By the Nested Supermultiplicativity Lemma (§ 02), within a fixed L the density κ_{cb} can be carried from scale R to sR with constant 1, so the content of this statement that is uniform in L moves to the comparison across L

$$(\mathrm{Ccb}) \quad \kappa_{\mathrm{cb}}(R; L') \leq C\,(L'/L)^2\,\kappa_{\mathrm{cb}}(R; L) \qquad (L_{\min}(a) \leq L \leq L' \leq R)$$

— the lateral-widening inequality of 26-02 read in terms of κ_{cb} Paper (the Reduction Read in the Chessboard Density in 26-02 § 03). There is no nesting in the lateral directions (tiling T^2_{2L} by four copies of T^2_L makes the twist $4 \equiv 0 \pmod 2$), so the ensemble with $n_{12} = 1$ cannot be formed).

Table 1 The band concentrated (labels per row; conditions in the same row)

Statement	Label	Conditions in the same row	Correspondence with 26-02	For U(1)
The chessboard with L fixed: $\lvert\langle W\rangle\rvert \leq \kappa_{\mathrm{cb}}^{n(C)}$	Proved here	L and M fixed; under the reflection positivity of z (26-04 § 03)	Takes over, at fixed L, the role of frame decorrelation of "splitting the product"	Holds as well
$\kappa_{\mathrm{cb}} < 1$ (fixed L)	Proved here	From the slab estimate (6). The tension is extremely small, $\approx 2e^{-2\beta L^2}$	—	Holds as well

Statement	Label	Conditions in the same row	Correspondence with 26-02	For U(1)
The uniform slab tension (8)	Assumption	Physical units, uniform in a	The content of the weak form of the lateral-widening inequality	Expected to fail [Reading]
The uniform slab gap	Assumption	Estimates of correlations, bad blocks and shell fluctuations uniform in L. Read in terms of κ_{cb}, within a fixed L constant 1 by the Nested Supermultiplicativity Lemma, and the rest is $(\mathrm{C}_{\mathrm{cb}})$	Frame decorrelation made uniform in L	No estimate that it fails [Reading]

In this way the non-abelian content of the band is placed on the side of the uniform slab tension, and the probabilistic content of crossing scales on the side of the uniform slab gap. Read in terms of κ_{cb}, the content of the uniform slab gap that is uniform in L moves to $(\mathrm{C}_{\mathrm{cb}})$ and becomes the same statement as the lateral-widening inequality Paper (the Nested Supermultiplicativity Lemma). The logical relation to the uniform slab tension (8) (whether one follows from the other) has not been examined.

06

The uses of the symbol κ compared

This book has used κ in several senses. The chessboard density of this chapter is written κ_{cb}, and the subscript is never dropped. Table 2 lists the other quantities written with the same letter. The quantities are not related to one another.

Table 2 Quantities written with the letter κ

Symbol	Meaning	Chapter
κ_{cb}	The chessboard density: the geometric mean per block of the product of the biases of the containers of all blocks (4)	This chapter, 26-00 Table 1
κ (the one-sided bridge)	The comparison constant of the One-Sided Bridge Proposition (of the form $C_P \geq \kappa \xi^2$)	23-01 § 06
κ_{BE}	The lower bound on the Bakry–Émery curvature	24-09
$\hbar\kappa$, κ_f	An upper bound on the strength of the coupling to neighbouring sites; the strength of the Brascamp–Lieb cross term	23-06
$\kappa = 3/1000$	The lower bound outside the defects in Barta's row	23-09
κ (Part 24)	The comparison constant between the Langevin metric and the metric of the kinetic term	24-09
$\kappa_{\mathrm{C7}}(k)$	The convexity constant of a chain of length k (the same quantity as $\kappa(k)$ of 23-11)	26-07
κ_w	Adapted stiffness: the variance of the force along the generator of the twist (width w; one of the candidate quantities for the band)	26-10

The whole correspondence of symbols is in A-01.

07

The table of the five grades and the dependency graph

Table 3 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The container estimate (1)	Cited	Mack–Petkova, 25-03	The ensemble summed over twists	—
2	Frame decorrelation (2)	Assumption	26-02, A-06	Constant C_{D}	—
3	Reflection positivity of z (directions 3 and 4, any lattice plane)	Proved here	26-04 § 03, translation invariance	$SU(2)$, twist fixed, M even	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`) (algebra only; assumption: $0 \le A$)
4	The Chessboard Estimate (3)	Cited	Fröhlich–Israel–Lieb–Simon (checked from the abstract of lecture notes)	Under reflection positivity, $2B \mid M$	—
5	θ_t is a function of the variables of the block and is carried to θ_s by reflection	Proved here	§ 02 (Markov property, reflection invariance of the measure)	The container is placed inside the block	—
6	Chessboard Estimate with the Lateral Size Fixed: $\mathbb{E}\prod_S \theta_t \le \kappa_{\mathrm{cb}}^{\lvert S\rvert}$, $\lvert\langle W(C)\rangle\rvert \le \kappa_{\mathrm{cb}}^{n(C)}$	Proved here	Proof in § 02 $\Longleftarrow$ 1, 3, 4, 5	If L is fixed (M also fixed)	—
7	The slab estimate $\sup_\Phi \theta_D \le \tanh(\beta L^2)^{\lvert D\rvert}$, hence $\kappa_{\mathrm{cb}} < 1$	Proved here	25-04, § 03 (6)	L fixed, tension $\approx 2e^{-2\beta L^2}$	—
8	The Flow Lemma $\theta \le 1 - a$	Proved here	25-06	Stationary kernel (reversibility not needed)	FluxLemma (`FluxLemma.flip_le_one_sub_abs`) (assumptions: $w \ge 0$, $K \ge 0$, row sums 1, w stationary, $\sum w = 1$, $\sigma = \pm 1$)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
9	Simultaneous-Flip Estimate: $\mathbb{E}\prod_S \theta_t \leq \mathbb{P}(\text{none changes})$	Proved here	Proof in § 03 $\Longleftarrow$ 8	Frame fixed, one independent step, starting from the conditional distribution	—
10	The flip probability of a one-step kernel levels off at about $\frac{1}{2}$	Reading	Measurements of 25-08	The driven-insertion kernel	—
11	Simultaneous flips give only an estimate of the naive form and cannot be used to test the band	Proved here	§ 03 (iii) $\Longleftarrow$ 9, the Tension Lower Bound from the Naive Form in 26-01	Logic assuming the levelling off of the one-step flip probability	—
12	Criteria and block sizes in the dilute and confining regimes	Reading (physics)	§ 04	$\Lambda L \ll 1$, $\Lambda L \gg 1$, estimate	—
13	Dobrushin-type criteria cannot be used as they stand	Reading (physics)	§ 04	The estimate that the influence coefficient is nearly 1; original papers not yet checked	—
14	The uniform slab tension (8)	Assumption	§ 05, A-06	Physical units, uniform in a	—
15	The uniform slab gap	Assumption	§ 05, A-06	Uniform in L	—
16	U(1) fails only in the uniform slab tension	Reading (physics)	§ 05	Estimate of the tension $\sim e^2/L^2$	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  container estimate |W| ≤ E∏θ                              ensemble summed ov
 2  ASM  frame decorrelation (what this chapter replaces at fixed L)
 3  PRV  reflection positivity of z (26-04)                        Lean refl_pos, ref
 4  CIT  Chessboard Estimate                                       Fröhlich–Israel–Li
 5  PRV  θ_t is a block function, reflected to θ_s                 Markov property
 6  PRV  Chessboard Estimate with the Lateral Size Fixed: |W| ≤ κ_cb^n(C) ⟸ 1,
 7  PRV  κ_cb ≤ tanh(βL²)^|D| < 1                                  L fixed, tension t
 8  PRV  θ ≤ 1 − a                                                 Lean flip_le_one_s
 9  PRV  Simultaneous-Flip Estimate: E∏θ ≤ P(none changes) ⟸ 8
10  RDG  one-step flip probability levels off near ½               measurement
11  PRV  simultaneous flips cannot test the band ⟸ 9, 10            26-01 Tension Lo
12  RDG  criteria and block sizes in the two end regimes           estimate
13  RDG  Dobrushin type not usable as it stands                    estimate
14  ASM  the uniform slab tension
15  ASM  the uniform slab gap
16  RDG  U(1) fails only in the uniform slab tension ⟸ 14, 15      estimate
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The main conclusion of this chapter (the Chessboard Estimate with the Lateral Size Fixed) stands on paper, and its only premises are literature rows (the container estimate, the Chessboard Estimate) and paper rows (the reflection positivity of z, block functions and reflection). The assumption row (frame decorrelation) is not among the premises of this conclusion — that is the point of this chapter. "Simultaneous flips cannot be used to test the band" is a row of logic, but it has a measurement row (the levelling off of the one-step flip probability) as a premise, so it is a statement within the range where that levelling off holds. The two assumptions of the band (the uniform slab tension, the uniform slab gap) do not contain the quantity of the conclusion (κ_{cb} with L fixed), and are placed in the ledger of conditions of Part 27 (27-02).

The machine-checked form of the chessboard estimate is in 22-01.

What this chapter can say

What is said	What is not said
Proved here If L is fixed, z reflection positivity and the chessboard estimate bound the Wilson loop by $\kappa_{\mathrm{cb}}^{n(C)}$, and frame decorrelation is not needed	An estimate of κ_{cb} uniform in L (the uniform slab gap)
Proved here Simultaneous flips are a tool that estimates κ_{cb} in the correct direction, and with a one-step kernel they cannot be used to test the band	A kernel with $a \to 1$. A numerical test of the band

What is said	What is not said
Reading (physics) The band is concentrated into the uniform slab tension and the uniform slab gap, and U(1) fails only in the former	Either of the two uniformities. The logical relation between the two
Lean: the algebra of the final sums of reflection, and a finite version of the flow lemma	The chessboard estimate itself, the Markov property, the Haar measure (outside Lean)

Sources and reproduction

Item	Kind	Source / tools
The chessboard estimate	Cited	The standard argument of J. Fröhlich, R. Israel, E. H. Lieb, B. Simon (1978). Checked from the abstract of M. Biskup, Reflection positivity and phase transitions in lattice spin models, arXiv:math-ph/0610025 (the text of the original paper not obtained). Examples of use in lattice gauge theory (Borgs–Seiler) not yet checked
The container estimate, the ensemble summed over twists	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442. Chapter 25-03
Reflection positivity of z	Proved here	Chapter the reflection positivity of z in 26-04
Chessboard Estimate with the Lateral Size Fixed, Simultaneous-Flip Estimate, (6)	Proved here	Proofs in § 02 and § 03 of the text. The same content as the corollaries "chessboard" and "simultaneous flips" of the paper on the reduction of the band (a paper that came out of this book, A-07)
The levelling off of one-step kernels; the withdrawal of the measurements of simultaneous flips	Reading	The table in chapter 25-08
The criteria of Dobrushin, Dobrushin–Shlosman, van den Berg–Maes	—	Not yet checked (names only)
Lean	Lean	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`), FluxLemma (`FluxLemma.flip_le_one_sub_abs`). They guarantee only the algebra of finite sums and real numbers. A-04

Terms used in this chapter — frame decorrelation, the Chessboard Estimate, the chessboard density κ_{cb}, block, simultaneous flips, the Flow Lemma, the uniform slab tension, the uniform slab gap, the crossover band (the band). Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-06 Renormalisation comparison — a comparison that distinguishes groups — another reduction, which splits the band into β-monotonicity and the renormalisation comparison.

Renormalisation comparison — a comparison that distinguishes groups

This chapter is this book's own investigation. § 04, § 07 and § 08 are a map of open questions.

What this chapter shows — We split the crossover band into two statements. One is that, with the lattice geometry held fixed, the vortex free energy increases in β (β-monotonicity). The other is that the lattice obtained by doubling the whole lattice can be compared, up to a constant factor, with the original lattice at a stronger coupling (the renormalisation comparison). Asymptotic freedom enters only through the "direction of strengthening the coupling" $\beta' < \beta$ of the renormalisation comparison. β-monotonicity looks like an easy component, but it turned out to be a problem of the same weight as the monotonicity in β of the string tension in lattice units. The renormalisation comparison holds at both the strong-coupling end and the one-loop end, but on the strong-coupling side it is a line of proof with extremely crude constants, and on the weak-coupling side it is a computation at the one-loop level. In the band between them, comparing $SU(2)$ and $U(1)$ numerically, the direction of the comparison does not change over the whole band for $SU(2)$, and for $U(1)$ it reverses at the position of the known phase transition. Finally, from the fact that the band is a bounded interval in β, we prove an elementary lemma that reduces the renormalisation comparison inside the band to "the two-sided area law at fixed β (with an error of the order of the perimeter)", and say where in the band the difficulty gathers.

Prerequisite chapters — 26-01 (the correct form of the band, $\ln(1/\theta) \geq cL^2$), 26-02 (reduction to a single inequality), 25-01 (the twisted T^2 and 't Hooft flux), 25-07 (why fixed maps fail, and the one-loop table), 4-06 (asymptotic freedom), 5-02 (the strong-coupling expansion), 5-06 ($U(1)$ and the non-abelian).

Said first — The level is [Res]. This chapter does not show the renormalisation comparison at any β of the band. What it shows is the range and conditions in which the comparison holds at the two ends, numerical readings, and one reduction lemma (§ 06). The assumption of the reduction lemma contains the unsolved problem of

confinement at intermediate coupling on the four-dimensional $SU(2)$ lattice. We write the chessboard density as κ_{cb}.

The order of this chapter

1. Splitting the band in two — β-monotonicity and the renormalisation comparison
2. β-monotonicity is not easy — rewritten with the electric flux, it stands beside the monotonicity of the string tension
3. The comparison that holds at the two ends — the strong-coupling end and the one-loop end (Table 1)
4. The map over β — labels interval by interval (Figure 1)
5. Numerics — doubling the lattice — $SU(2)$ and $U(1)$ (Tables 2–4)
6. The band is a bounded interval — the Lemma Reducing the Comparison inside the Band, and the Corollary Crossing the Band in Finitely Many Steps
7. The four faces of the band in the middle — candidate local quantities (Table 5)
8. Conditions an argument closing the band must satisfy — the criterion "distinguish from the U(1) transition"
9. What cannot be said
10. The table of the five grades and the dependency graph — Table 6

01

Splitting the band in two — β-monotonicity and the renormalisation comparison

The object is the slab of 25-01. On the lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ we put the $SU(2)$ Wilson action and insert the twist $n_{12} = 1$ in the $(1,2)$ plane. We take the flux n_{34} of the $(3,4)$ plane as the sector, and write the vortex free energy as

$$\Delta F(N,M,\beta) \;=\; -\ln \frac{Z_{n_{34}=1}(N,M,\beta)}{Z_{n_{34}=0}(N,M,\beta)}, \qquad \theta = \tanh \frac{\Delta F}{2} \tag{1}$$

θ is the quantity used in 't Hooft's criterion, and the correct form of the band was "$\ln(1/\theta(L,L)) \geq cL^2$ uniformly in L" (26-01). The larger the vortex free energy, the smaller θ, which is the confining side.

We split the statement that crosses the band into the following two Assumption (both set as unproved statements).

$$\text{(Mon)}\quad \frac{\partial}{\partial\beta}\Delta F(N,M,\beta)\ \geq\ 0 \qquad \text{(with the lattice geometry fixed)}\ (2)$$

$$\text{(RG-cmp)}\quad \Delta F(2N,2M,\beta)\ \leq\ K\cdot\Delta F(N,M,\beta'),\qquad \beta'<\beta,\qquad K \text{ independent of } N,M$$

> The formula label (Mon) stands for β-monotonicity, and (RG-cmp) for the renormalisation comparison (names as in A-06). The (Mon) of (5) below is the same.

The left-hand side of (2) can be written as an equality, as an average of the difference of actions: $\partial_\beta\Delta F=\langle\Delta S\rangle$ (the difference of the average actions in the sectors with and without a vortex) Proved here (just differentiate the partition function in β). (3) is the comparison that a vortex on the lattice coarsened by a factor 2 costs no more than a vortex on the lattice of the original size at a stronger coupling.

The scope of (2). (2) is a statement about the slab of this chapter — the group $SU(2)$, the background twist $n_{12}=1$ and the slab geometry $(\mathbb{Z}/N)^2\times(\mathbb{Z}/M)^2$. It is not a general property of non-abelian groups: on small lattices for the finite non-abelian groups Q_8 and D_4 there are points where $\partial_\beta\Delta F<0$ even without a background twist, and, with the background, there are points where it fails in geometries that are long in the transverse directions Reading (floating-point values from transfer matrices of finite groups; two independent implementations agree). On the other hand, for the $\mathbb{Z}_2$ sign system, whose only variables are the signs of the plaquettes, monotonicity of the type of (2) ($\mathbb{Z}_2$ monotonicity) holds for non-negative couplings Proved here (a proof of correlation-inequality type; the statement for abstract sign systems was checked in Lean; the application to the lattice of the four-dimensional torus is on paper).

Together, the two give a line that crosses the band. Iterating (3) as $\beta_0>\beta_1>\beta_2>\cdots$, the vortex free energy on the lattice enlarged 2^k times is bounded by K^k times the value on the original lattice at coupling β_k. Once β_k enters the strong-coupling range, the vortex free energy there is small with an area exponent, and it becomes doubly exponentially small with each step, so it beats the loss K^k. (2) plays the role of an adjustment: nothing is lost if, in the course of the iteration, β' is chosen smaller than necessary.

The rate form of (3). The constant K in (3) is bookkeeping; the content is a comparison of slab tensions. Write the slab tension, and that quantity multiplied by the number of plaquettes, as

$$\sigma_N(\beta) := \lim_{M\to\infty} \frac{-\ln \Delta F(N,M,\beta)}{M^2}, \qquad s(N,\beta) := N^2 \sigma_N(\beta) \quad (3')$$

(assuming the limit exists; otherwise the lower limit). Taking the logarithm of both sides of (3), dividing by M^2 and letting $M \to \infty$, K disappears, and what remains is $s(N,\beta') \le s(2N,\beta)$ Proved here (this form follows from (3); the converse needs control of the difference between ΔF and the area term). In this book, this rate form is read as the main form of the renormalisation comparison. If one looks only at symmetric boxes $N = M$ and allows $K > 1$, $\beta' < \beta$ holds even for $U(1)$ Reading. What forces $\beta' \ge \beta$ is the side $M \to \infty$ with N fixed, and the rate form carries that direction.

Asymptotic freedom enters only through $\beta' < \beta$ in (3). (2) is a statement of a form that contains no mechanism that distinguishes groups. For $\mathbb{Z}_2$ and $U(1)$, it follows from Ginibre's inequality when there is no twisted background, but the slab of this chapter has the twist $n_{12} = 1$, and in that case it does not follow from Ginibre's inequality even for $\mathbb{Z}_2$ Assumption (the application of Ginibre's inequality has not been checked against the text of the original source). The direction of (3), on the other hand, changes with the group. For $U(1)$ one loop gives $\beta' \ge \beta$, and the iteration does not head for strong coupling: it fails, correctly. That $\beta' < \beta$ for the non-abelian $SU(2)$ is a restatement of asymptotic freedom (4-06), that is, of the effective coupling becoming stronger when the lattice is coarsened. In this sense (3) has to be "a comparison that distinguishes groups".

Example. At the strong-coupling end, (3) holds with $K \approx 4$ (§ 03). Doubling the lattice makes the area of the vortex sheet 4 times larger, and it balances at the β' that makes the rate $\sigma(\beta)$ of the area exponent 4 times larger, that is, $\sigma(\beta') = 4\sigma(\beta)$. For $\beta = 1.0$, $\beta' \approx 0.013$ (Table 1). At strong coupling the coupling moves a long way in one step.

Tomboulis (2007) tried to build a comparison of type (3) with Migdal–Kadanoff decimation, and Ito–Seiler and Kanazawa pointed out that the flow heads for strong coupling also for $U(1)$ Cited (Ito–Seiler reduce the remaining step to the inequality $A(\alpha) > A^{+}(\alpha)$ of their eq. (3.5), and write that it has not been shown). The history and the checking are in 26-08. What this chapter needs is just one point: a comparison that does not distinguish groups cannot cross the band.

Looking again at the direction of the comparison in the language of effective actions: at tree level the coarse action of the genuine one step is no greater than the Migdal–Kadanoff action (in the coarse directions it is smaller, down to 1/4, and it agrees at long wavelengths); in the language of couplings this is $\beta/4 \le \beta' \le \beta$, the direction favourable to confinement Proved here (lower side) Reading (the upper side is the numerics of the measured boxes). It still cannot be used for the comparison, because the inequality of partition functions $Z \le Z_{\mathrm{MK}}$ does not lift to the form with the coarse

variables fixed. In the one-loop asymptotics the speed of the Migdal–Kadanoff flow (0.2503 per step) is only slightly slower than the genuine speed 0.2575, and the direction is favourable; but in U(1) neither runs, and the difference between the groups shows up on the Migdal–Kadanoff side only in an exponentially small flow Reading . The one-step sandwich $\langle W\rangle_L < \langle W\rangle_R < \langle W\rangle_{\mathrm{MK}}$ of the same observable holds in all 12 rows for SU(2) and U(1), and does not distinguish the groups Reading (Metropolis, $4^4 \to 2^4$, $8^4 \to 4^4$, $\beta \in \{2, 4, 8\}$).

The inequality $A(\alpha) > A^{+}(\alpha)$ to which Ito–Seiler reduced the problem is the same as the monotonicity "the vortex free energy grows with the coupling", and in the numerics of small boxes it holds even for deconfined U(1) Reading (4^4, one direction of the MK image, $\alpha \in \{0.9, 1.0\}$). What has to distinguish the groups is the extension beyond it, and the device that moves the interpolation parameter away from 1.

Measuring "the Migdal–Kadanoff flow is slower than the genuine flow" with the twist free energy, over one step of the orbit (2.47 → 2.19 in the effective β), in the $4^4/2^4$ box the genuine one step looked faster by 0.15 converted to Wilson's β ($R = 1.34(7)$); but in $6^4/3^4$ the difference is 0.02(2) ($R = 1.17(20)$), which is within the resolution of the measurement (statistics, and the systematic floor of 0.1–0.25 of the ΔF method). The speeds of the two flows look nearly equal inside the band, and the 7% discretisation of the 2^4 box may have produced most of the initial difference Reading .

02

β-monotonicity is not easy — rewritten with the electric flux, it stands beside the monotonicity of the string tension

β-monotonicity looks like "a structural component that can probably be shown". In fact it is not easy. Rewriting it with the electric flux makes this clear.

Splitting the partition function of the sector with flux n_{34}, by the Fourier transform on $\mathbb{Z}_2$, into the partition functions Y_0, Y_1 with electric flux $e \in \{0, 1\}$ gives $Z_{n_{34}=0} = Y_0 + Y_1$, $Z_{n_{34}=1} = Y_0 - Y_1$. Setting $E_e = \frac{d}{d\beta} \ln Y_e$,

$$\frac{\partial \Delta F}{\partial \beta} = \frac{2\, Y_0 Y_1\, (E_1 - E_0)}{Y_0^2 - Y_1^2}, \qquad \theta = \frac{Y_1}{Y_0} \quad (4)$$

Proved here Proof. Differentiate $\Delta F = \ln(Y_0 + Y_1) - \ln(Y_0 - Y_1)$, insert $Y_e' = E_e Y_e$ and bring to a common denominator; the numerator is $(E_0 Y_0 + E_1 Y_1)(Y_0 - Y_1) - (E_0 Y_0 - E_1 Y_1)(Y_0 + Y_1) = 2 Y_0 Y_1 (E_1 - E_0)$. The expression for θ is $\tanh(\Delta F/2) = (e^{\Delta F} - 1)/(e^{\Delta F} + 1)$ with $e^{\Delta F} = (Y_0 + Y_1)/(Y_0 - Y_1)$ inserted. ∎

If $Y_1 > 0$ (the positive case of $\theta = Y(1)/Y(0) \geq 0$ in 26-04), then from (4)

$$(\mathrm{Mon}) \iff E_1 \geq E_0 \iff \theta \text{ increases in } \beta \iff \text{the electric-flux free energy } -\ln(Y_1/Y_0)$$

Proved here . In the confining phase, the free energy of the electric flux is of the order of the product of the string tension and the length the flux passes through. So it can be read that, if β-monotonicity holds on every lattice, it follows that the string tension in lattice units σa^2 decreases in β Assumption (basis: the step connecting the electric-flux free energy with the string tension has not been written out). **The monotonicity in β of the string tension in lattice units has not been proved for $SU(2)$.** Even the monotonicity in β of the expectation value of the Wilson loop has not been shown, as far as we searched. β-monotonicity is a problem at least of the same weight as this Assumption . The initial estimate that "β-monotonicity is structural and can probably be shown" is corrected here.

The partial results to hand are the following Assumption (the proofs of none of them are set out in this book). In two dimensions with 1 or 2 sheets (by the monotonicity of the conditional energy of the $O(4)$ chain); near $\beta \to 0$ on any fixed lattice; near $\beta \to \infty$ in the twisted n_{12} background ($\partial_\beta \Delta F$ approaches a positive constant because $SU(2)$ has no flat connection carrying both n_{12} and n_{34}). No counterexample has been found in any of the closed two-dimensional formulas ($N \leq 100$, $\beta \leq 20$), the exhaustive enumeration of $2 \times 2 \times 2 \times 1$ for $\mathbb{Z}_2$, or the four-dimensional Monte Carlo on 1^4 and 2^4 Reading .

Two reasons why it gets stuck are recorded. In the Mack–Petkova representation, when the $SO(3)$ configuration carries a twist class, the conditional Ginibre inequality goes the wrong way, and $\partial_\beta \Delta F$ becomes exactly "the difference of the two contributions we want to measure". In the character expansion, in three or more dimensions the sign of the spin-network integrals is not fixed Reading (physical estimate).

Consequence. The chain across the band does not need β-monotonicity on every lattice. If (3) is shown directly in the form $\Delta F(2N, 2M, \beta) \leq K \Delta F(N, M, \beta')$, the freedom in choosing β' absorbs the role of β-monotonicity. So below we concentrate on the renormalisation comparison.

03

The comparison that holds at the two ends — the strong-coupling end and the one-loop end

The strong-coupling end

We write the coefficient of the character expansion of the Wilson action in the fundamental representation as $u(\beta) = I_2(\beta)/I_1(\beta)$ ($u \approx \beta/4$ as $\beta \to 0$). Expanding the

vortex free energy of the slab by the strong-coupling expansion (5-02), the leading terms are the N^2 flat sheets wrapping the $(3,4)$ plane (area M^2, weight u^{M^2}). The record contains a proposition and a line of proof of the following form.

Strong-Coupling-End Expansion Proposition. There are absolute constants u_0, K_0, C such that, for $u(\beta) \le u_0$ and all $N, M \ge 2$,

$$\Delta F(N,M,\beta) = 2N^2 e^{-\sigma_N(\beta) M^2}(1+\epsilon), \qquad |\epsilon| \le C M^2 (K_0 u)^{\max(1, M/100)} \le C' u \tag{6}$$

holds. σ_N depends on β, N and n_{12} but not on M, and $\sigma_\infty = -\ln u - 4u^4 + O(u^5)$. Assumption (line: in the polymer expansion the sign of the twist is fixed by the $\mathbb{Z}_2$ homology class; integrating the flat parts of the sheet by gluing two-dimensional characters gives exactly u^{M^2}; a Peierls estimate for contours on the sheet. The bookkeeping of the interaction between volume polymers and sheet contours (step 5 of the proof) is sketched, and the proof is not set out in this book. The form of the expansion is of the Münster (1981) type, but the text of the original source has not been checked)

Comparison at the Strong-Coupling End. Defining β' by $\sigma_N(\beta') = 4\sigma_{2N}(\beta)$, for all N, $M \ge 2$, $\Delta F(2N, 2M, \beta)/\Delta F(N, M, \beta') = 4(1 + O(u))$, $\beta' < \beta$, $u' = u^4(1 + O(u^4))$ Assumption (algebra on top of the Strong-Coupling-End Expansion Proposition).

The constants are extremely crude: $u_0 \approx 10^{-80}$, which in β is about 4×10^{-80}. On paper, the strong-coupling end is closed only in the range of extremely strong coupling. This is a matter of constants, and there is a prospect of large improvement by tightening the counting, but the expansion in flat sheets stops at roughening ($\beta_R \approx 1.9$), so the strong-coupling side alone cannot cover the band Reading (physical estimate). $U(1)$ also confines at strong coupling and satisfies (3). The difference between the groups appears on the weak-coupling side.

Table 1 The root β' of the leading-order strong-coupling equation $u(\beta') = u(\beta)^4$ (the movement of the coupling in one step) Reading

β	$u(\beta)$	β'	$\beta - \beta'$
0.5	0.1237	0.0009	0.499
1.0	0.2402	0.0133	0.987
1.5	0.3441	0.0561	1.444
2.0	0.4331	0.1409	1.859

The β of Table 1 lie outside the range of the Strong-Coupling-End Expansion Proposition ($u \le u_0$). The table is a rough guide to the direction and size of the comparison from the leading terms only, not a claim that the comparison holds.

The one-loop end

On the weak-coupling side we build (3) as a variational inequality. For a measurable bijection T carrying the measure μ_0 of the sector $n_{34}=0$ to the sector 1, the Gibbs variational principle gives

$$\Delta F \;=\; \mathbb{E}_{\mu_0}\big[S\circ T - S - \ln J_T\big] - \mathrm{KL}\big(T_{\#}\mu_0 \,\|\, \mu_1\big) \;\le\; \mathbb{E}_{\mu_0}\big[S\circ T - S - \ln J_T\big] \quad (7)$$

Cited (the standard form of the Gibbs variational principle; J_T is the Jacobian of T). The right-hand side is written only with averages in μ_0, so the estimate needs only one side. We consider the following construction: block the configuration of the fine lattice $(2N,\beta)$ and map it to the coarse lattice (N,β'), use a nearly optimal insertion map on the coarse lattice, and carry the fluctuations when going back to the fine lattice. Depending on how the fluctuations are carried, the one-loop answer splits in two.

The one-loop count is done with the trace of the heat kernel of the $SU(2)$ fluctuations (Feynman gauge) in a constant background magnetic field B. Per volume, apart from $1/(16\pi^2 s^2)$, the vector (4 polarisations) gives $\frac{Bs}{\sinh Bs}(2+2\cosh 2Bs)$ and the ghosts give $-2\frac{Bs}{\sinh Bs}$. Inserting $\frac{x}{\sinh x} = 1-\frac{x^2}{6}+O(x^4)$ and $\cosh 2x = 1+2x^2+O(x^4)$ for $x = Bs$,

$$\underbrace{\frac{x}{\sinh x}(2+2\cosh 2x) - \frac{2x}{\sinh x}}_{\text{total}} = 2+\tfrac{11}{3}x^2+O(x^4), \qquad \underbrace{\frac{4x}{\sinh x} - \frac{2x}{\sinh x}}_{\text{orbital only}} = 2-\tfrac{1}{3}x^2+O(x^4)$$

and the $+4$ in $\frac{11}{3} = -\frac{1}{3}+4$ comes from the spin term $\cosh 2Bs$ Proved here (one-loop level; the expansion is a two-line computation). $\frac{11}{3}$ is the coefficient of asymptotic freedom. The table and the details of the derivation are in 25-07; here we use only the conclusion.

- In **an isometric comparison** (a map isometric in the fluctuations: the identity map, translation in the group, gauge transformation, Migdal–Kadanoff-type decimation), the estimate (7) is the first-order trace $\frac{1}{2}\operatorname{tr}((M_B - M_0)M_0^{-1})$, and the contribution of the spin term is exactly 0. The estimate is worse than the orbital value alone, so what it can give is at best the $-\frac{1}{3}$ side, that is, $\beta' \ge \beta + 0.023$ (per doubling): the direction against asymptotic freedom, the same direction as $U(1)$ Proved here (one-loop level).
- In **a comparison that reshapes the fluctuations** (changing the shape of the fluctuations by an optimal linear map of determinant 1), the exact $\frac{1}{2}\ln\det$ is reproduced with error $\sim \Delta F^2/n$ (n is the number of degrees of freedom), giving $\beta' = \beta - 0.258$ (per doubling) Proved here (one-loop level).

Where the numbers come from: with $\beta = 4/g^2$ and $b_0 = 22/(48\pi^2)$, the one-loop movement of the coupling per doubling is $\beta - \beta' = 4 \cdot 2b_0 \ln 2 = 0.2575$. For the orbital part alone, replacing $\frac{11}{3}$ by $-\frac{1}{3}$ gives $0.2575 \times (-\frac{1}{11}) = -0.0234$ Proved here (arithmetic).

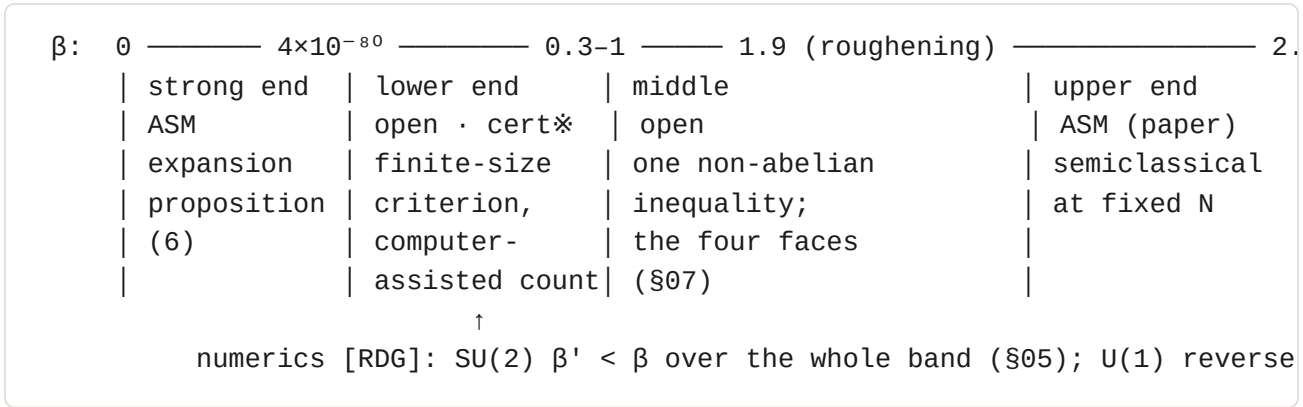
This explains as a number, at the one-loop level, why the method of Tomboulis (decimation = isometric) could not distinguish groups. The mechanism that sees asymptotic freedom lies in the reshaping of the fluctuations. What remains is to control the error in the estimate of the reshaped transport (the local KL near the vortex sheet and the higher loops) at intermediate coupling and in large fields, and this is open.

Always read the statements at the two ends with their conditions. One cannot say "the renormalisation comparison holds at both ends, so only the width of the band remains". The strong-coupling side is a line, part of whose bookkeeping is sketched, in the range $u \le u_0 \approx 10^{-80}$, and the weak-coupling side is a computation at the one-loop level. A non-perturbative weak-coupling comparison has not been shown at any β.

04

The map over β — labels interval by interval

Figure 1 sets out, over the β of the Wilson action, the status of the renormalisation comparison and of confinement interval by interval. The boundaries of the intervals are rough guides, and none of them is a threshold of a theorem.



- **The strong-coupling end** Assumption : the Strong-Coupling-End Expansion Proposition. With the constant $u_0 \approx 10^{-80}$ as it stands, the proof cannot be set out in the book, so it is "Assumption". At the same strong-coupling end there are results of this book checked in Lean (20-06): the area law, and a vortex floor whose rate per unit area does not depend on the transverse size (the prefactor remains). Both are statements at fixed lattice spacing, only inside the strong-coupling window (β below about 10^{-3}), and are not the renormalisation comparison itself. Outside the window there is no form uniform in the transverse size.

- **The lower end** ($\beta \approx 0.3$–1): open. However, the upper side of the two-sided area law at fixed β at the strong-coupling end has been extended (※ in Figure 1). Splitting an element of $SU(2)$ into a centre sign and a section of $SO(3)$ ($U = \sigma V$), the twist free energy is sandwiched by means of ratios for a $\mathbb{Z}_2$ gauge glass with the $SO(3)$ background frozen (couplings $|K_p| \le \beta$, signs random) Paper . Checking the Kotecký–Preiss criterion Cited for the polymer expansion of its closed sheets (the form, uniform in the background, of the mixing of box magnetic flux δ_C (26-10 § 03)) by full enumeration up to 10 plaquettes and a tree bound on the tail, $|\Delta F(N, M, \beta)| \le 8aN^2 e^{-bM^2}$ ($a = 0.0454$, $b = 0.01$; lowering β makes b larger) holds for $N, M \ge 11$ and all $\beta \le 0.1039$ Computer-assisted (interval arithmetic; for all $N, M \ge 2$, $\beta \le 0.0680$. The final arithmetic of the rational version $\beta \le 0.1029$ is Lean: CenterFluxPolymerCertificate (tree_k, kp_bound, kp_lt_one, rate_rational, final_real). $\tanh \beta \le \beta$, the lemma of the induction on trees and the KP theorem are outside Lean). This is of the form of the upper side of the two-sided area law at fixed β; the rate b is much smaller than the true σ, and there is no lower side (an upper bound on the string tension). The same splitting holds for $U(1)$ and for $\mathbb{Z}_2$, so this estimate does not distinguish groups and cannot cross the transition $\beta \approx 0.44$ of pure $\mathbb{Z}_2$ Reading . $\beta \ge 0.12$ cannot be shown at the threshold of the enumeration. It is likely an estimate of a known type, and no novelty is claimed. The candidates above that are a Dobrushin–Shlosman-type finite-size criterion using the influence $\delta(L_0, \beta)$ on the inside when the centre magnetic flux on the boundary of the box is changed, and a road that shows the growth constant of the admissible sheets by computer-assisted enumeration and raises u_0 to 0.04–0.12 (the range of the prospect is a physical estimate). Neither is expected to reach above $\beta \approx 1$.
- **The middle** ($\beta \approx 1$–2.7): open. One inequality that uses non-commutativity is needed, and there are four candidate local quantities to carry it (§ 07).
- **The upper end** (large β at fixed N): for the direction of confinement there is a paper proof in the Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3, which came out of this book's study; $-\ln \theta \ge c\beta^{-a} e^{-\beta(S_* + \varepsilon)} M^2$ at fixed N). However, it cites the general chessboard theorem and a Morse–Bott lemma, and the proof is not set out in this book, so we label it "Assumption" Assumption . The reverse estimate uniform in M, and whether the rate S_* equals the minimal action $S_{\min}$, are open. Increasing N is the same problem as the middle.
- **One loop** Proved here : (8) of § 03 and the two comparisons. This is the perturbative level.

How the width of the band is counted depends on where the lower end is put. The strong-coupling expansion is considered numerically effective up to about $\beta \approx 0.3$–0.5, and the record writes this band as "roughly 0.5 to 2.7 in the β of the Wilson action"

Reading (physical estimate). On paper the band is much wider, $[4 \times 10^{-80},\ 2.7]$, because u_0 is so small.

05

Numerics — doubling the lattice

We test the renormalisation comparison directly with numerics. The procedure is to measure the vortex free energy on the fine lattice and read off, from the ΔF curve of the coarse lattice, the β' of the coarse lattice that gives the same value. Every measurement makes samples that cross sectors by replica exchange and obtains the free-energy difference by Bennett's acceptance ratio method (BAR). The errors are statistical; at some points the difference between two runs with different seeds (initial values of the random numbers) is 1–2 times the displayed error, so the errors may come out on the small side. Every number is a reading, not a proof Reading .

Table 2 Doubling the whole lattice of the slab: $(N, M) = (4, 8) \to (8, 16)$ $(n_{12} = 1)$ Reading

Group	β	$\Delta F(4, 8, \beta)$	$\Delta F(8, 16, \beta)$	Seeds	Matching β'	$\beta - \beta'$
$SU(2)$	2.7	2.97 ± 0.33	0.92 ± 0.30 (weighted mean of 0.65 ± 0.39 and 1.31 ± 0.48)	2	≈ 2.57 (2.45–2.65 including the error of the curve)	$\approx +0.13$ ($+0.05$–$+0.25$)
$SU(2)$	2.6	1.43 ± 0.19	-0.50 ± 0.34 (consistent with 0)	2	$\lesssim 2.45$	$\gtrsim +0.15$
$SU(2)$	2.5	0.63 ± 0.18	0.41 ± 0.34	1	2.2–2.5	0–$+0.3$
$U(1)$	1.5	0.83 ± 0.11	2.32 ± 0.45 (48 replicas, minimum exchange acceptance rate 0.37)	1	> 1.7 (outside the curve)	< -0.2

The curves on the coarse side are, for $SU(2)$, $\Delta F(4, 8, \beta') = 0.05, 0.00, 0.31, 0.63, 1.43, 2.97$ at $\beta' = 2.2, 2.3, 2.4, 2.5, 2.6, 2.7$, and for $U(1)$, $0.42, 0.55, 0.83, 0.92, 1.46$ at $\beta' = 1.3$–1.7. For $SU(2)$, doubling lowers the vortex free energy ($2.97 \to 0.92$ at $\beta = 2.7$), and the matching β' is smaller than β. For $U(1)$ it rises ($0.83 \to 2.32$, about 3σ), and $\beta' > \beta$. The direction of the comparison separated for the two groups. The first $U(1)$ run (32 replicas) had a minimum exchange acceptance rate of 0.03 and mixed badly, and is not used.

Table 3 ΔF on the symmetric torus N^4 $(n_{12} = 1)$, and β' for one step $2^4 \to 4^4$. One seed each Reading

Group	β	$\Delta F(2^4)$	$\Delta F(4^4)$	$\beta - \beta'$
$SU(2)$	1.75, 2.0	0.573, 1.39	0.01, -0.03	> 0.75 ($\beta' < 1.0$ = outside the 2^4 curve)
$SU(2)$	2.25	2.71	0.50	0.53
$SU(2)$	2.5	4.29	2.46	0.30

Group	β	$\Delta F(2^4)$	$\Delta F(4^4)$	$\beta - \beta'$
$SU(2)$	2.75	5.95	5.10	0.13
$SU(2)$	3.0	7.88	7.33	0.07
$U(1)$	0.9	1.15	0.05	> 0.3
$U(1)$	1.0	1.80	1.07	$+0.12$
$U(1)$	1.1	2.46	2.71	-0.04
$U(1)$	1.2	3.10	3.68	-0.11
$U(1)$	1.4	4.18	4.46	-0.05

The errors are 0.01–0.08 on 2^4 and 0.03–0.26 on 4^4. For $SU(2)$, $\beta - \beta' > 0$ at every β that can be read (1.75–3.0), that is, the step function of one step has no fixed point. The closer to strong coupling, the larger the movement in one step, in a form that connects to the strong-coupling values of Table 1. The decrease to 0.07 on the weak-coupling side is likely a lattice artefact, because 2^4 is too small for the one-loop regime. For $U(1)$, **the sign reverses between** $\beta = 1.0$ **and** 1.1. This is at the position of the known phase transition $\beta_c \approx 1.01$: in the confining phase $\beta' < \beta$, in the Coulomb phase $\beta' > \beta$.

Table 4 $4^4 \to 8^4$ for $SU(2)$ ($n_{12} = 1$). ΔF is the simple mean of the per-seed values, and the error is half the difference between seeds Reading

β	$\Delta F(4^4)$	$\Delta F(8^4)$	Seeds (8^4)	$\beta - \beta'$
2.3	—	-0.04 ± 0.10 (≈ 0)	1	$\gtrsim 0.1$
2.5	2.46 (1 seed)	0.48 ± 0.48	1	$\approx +0.25$ ($+0.19$–$+0.5$)
2.7	4.72 ± 0.25	3.11 ± 0.46	2	$+0.16$ ($+0.13$–$+0.19$)
2.9	6.56 ± 0.33	4.68 ± 0.63	3	$+0.20$ ($+0.12$–$+0.29$)
3.0	7.58 ± 0.25	5.78 ± 0.40	2	$+0.18$ ($+0.14$–$+0.25$)

For the 4^4 curve we also used the points $\beta = 2.6$ (3.96 ± 0.13, 1 seed), 2.75 (5.18 ± 0.10) and 2.8 (5.60 ± 0.27). For $4^4 \to 8^4$ too, $\beta - \beta' > 0$ at every β measured, with a mean of about $+0.2$. This is close to the one-loop asymptotic value 0.258 and to the value of about 0.23 expected from the known values of $a\sqrt{\sigma}$ Reading .

How to read this. The test of the sign of the step function distinguishes whether there is a phase transition. $U(1)$ reverses at the position of the transition, and $SU(2)$ does not reverse in the range measured. This is the numerical face of the criterion "distinguish from the U(1) transition" of § 08. However, the lattices are small, up to 8^4, the β points are few, and there are only 1–3 seeds. "For $SU(2)$ the sign does not change over the whole band" is a reading on these lattices and points.

06

The band is a bounded interval — reduction to the area law at fixed β

The band is a bounded closed interval $I = [\beta_a, \beta_b]$ in β. A comparison that crosses infinitely many scales shrinks to finitely many steps on a bounded interval. We write this in the form of the following lemma.

Assumption of two-sided area laws inside the band (the two-sided area law at fixed β on the band I). There are constants $N_1, M_0 \geq 2$, $c_-, c_+ > 0$, $p \geq 0$ and a function $\sigma_N(\beta) > 0$ such that

- Two-sided area law at fixed β: for all $\beta \in I$, $N \geq N_1$, $M \geq M_0$, $c_- N^2 e^{-\sigma_N(\beta) M^2 - pM} \leq \Delta F(N, M, \beta) \leq c_+ N^2 e^{-\sigma_N(\beta) M^2 + pM}$.
- Uniform positivity and continuity of the rate: with $\sigma_* := \inf_{N \geq N_1,\, \beta \in I} \sigma_N(\beta) > 0$, for some $\delta > 0$: if $\beta, \beta' \in I$ and $|\beta - \beta'| \leq \delta$, then for all $N \geq N_1$, $\sigma_N(\beta') \leq \sigma_{2N}(\beta) + \sigma_*$.

The upper side of the two-sided area law at fixed β is the area law (with an error pM of the order of the perimeter), and the lower side is that "vortices are not too cheap", that is, an upper bound on the string tension. The uniform positivity and continuity of the rate says that the rate is uniformly positive on the band and does not change much when β is moved a little.

Lemma Reducing the Comparison inside the Band. Under the assumption of two-sided area laws inside the band, for $\beta \in I$, $\beta' \in [\beta - \delta, \beta) \cap I$, $N \geq N_1$, $M \geq M_0$,

$$\Delta F(2N, 2M, \beta) \leq K \cdot \Delta F(N, M, \beta'), \qquad K = 4 \frac{c_+}{c_-} \exp\Big(\frac{9p^2}{8\sigma_*}\Big) \tag{9}$$

holds. K does not depend on N, M or β.

Proved here Proof. Since $2N \geq N_1$ and $2M \geq M_0$, the upper side of the two-sided area law at fixed β gives $\Delta F(2N, 2M, \beta) \leq c_+ (2N)^2 e^{-\sigma_{2N}(\beta)(2M)^2 + p(2M)} = 4c_+ N^2 e^{-4\sigma_{2N}(\beta) M^2 + 2pM}$. The lower side gives $\Delta F(N, M, \beta') \geq c_- N^2 e^{-\sigma_N(\beta') M^2 - pM}$. Setting $\eta := 4\sigma_{2N}(\beta) - \sigma_N(\beta')$, the uniform positivity and continuity of the rate gives $\eta \geq 3\sigma_{2N}(\beta) - \sigma_* \geq 2\sigma_*$ (using $\sigma_{2N}(\beta) \geq \sigma_*$). The ratio is

$$\frac{\Delta F(2N, 2M, \beta)}{\Delta F(N, M, \beta')} \leq 4 \frac{c_+}{c_-} e^{-\eta M^2 + 3pM} \leq 4 \frac{c_+}{c_-} e^{9p^2/(4\eta)} \leq 4 \frac{c_+}{c_-} e^{9p^2/(8\sigma_*)}$$

The second inequality is the maximum $9p^2/(4\eta)$ of the quadratic $-\eta M^2 + 3pM$, and the third is $\eta \geq 2\sigma_*$. ∎

Corollary Crossing the Band in Finitely Many Steps. The band can be crossed in at most $\lceil(\beta_b - \beta_a)/\delta\rceil$ steps, and the loss over them is K to the power of the number of steps, a constant independent of N and M Proved here . The same proof goes through if β' is lowered further, as long as it satisfies $\sigma_N(\beta') \le 4\sigma_{2N}(\beta) - 2\sigma_*$.

Calibration. The Strong-Coupling-End Expansion Proposition ((6)) gives the two-sided area law at fixed β with $p = 0$ and $c_\pm = 2(1 \pm C'u)$. Then (9) holds with $K = 4(1 + C'u)/(1 - C'u)$, which agrees with $4(1 + O(u))$ of the Comparison at the Strong-Coupling End in § 03 Proved here (the Strong-Coupling-End Expansion Proposition is an assumption, so the agreement is an agreement of form).

What the Lemma Reducing the Comparison inside the Band says. The renormalisation comparison inside the band reduces to "confinement at fixed β (two-sided asymptotics with an error of the order of the perimeter)" at each β of the band. The comparison across scales itself is not needed. The band is entered when the physical box is large, so the two-sided area law at fixed β is needed only in the range where N, M is sufficiently larger than the correlation length $\xi(\beta)$.

What the Lemma Reducing the Comparison inside the Band does not say. The assumption of two-sided area laws inside the band is unproved at every β of the band. The upper side is itself the unsolved problem of confinement at intermediate coupling on the four-dimensional $SU(2)$ lattice. As far as we searched, there is no proof of confinement on the four-dimensional $SU(2)$ lattice at $\beta \approx 1$. The lower side (an upper bound on the string tension to the accuracy of the order of the perimeter) has not been written out either. The lemma does not depend on the group; all the non-commutativity is in the assumption of two-sided area laws inside the band. For $U(1)$ ($\beta \in (1.01, 2.7)$), the assumption of two-sided area laws inside the band is false, so no conclusion follows. This lemma is a reduction, not a tool, and we do not say it has made the band easier.

Where the difficulty gathers Reading (physical estimate). After the Lemma Reducing the Comparison inside the Band, what remains inside the band are problems at fixed β (estimates uniform in the volume, non-convex large fields, the absence of a phase transition). The difficulty of crossing scales does not vanish from inside the band; it gathers at **the upper end of the band** and at **the seam with ultraviolet control**. At the upper end of the band, $\xi(\beta)$ exceeds 10 in lattice units, and the road of checking the two-sided area law at fixed β by computations in finite boxes stops on size. And beyond the upper end of the band ($\beta \to \infty$) is the range taken on by ultraviolet control (7-07), and the two must be joined with the same constant. In terms of the three roots of 24-07 § 05, inside the band it is the source "no global gauge fixing" (the non-convexity of large fields) and excluding a phase transition, and the seam at the upper end is the

source "no expansion linking scales" (there is no expansion that joins the ultraviolet and the infrared).

Example. On the strong-coupling side ($u \le u_0$), a statement of exactly the form of the base point at small T^2 exists as the Strong-Coupling-End Expansion Proposition (at the level of a line of proof). The work of pushing the lower end of the band up to $\beta \approx 0.3$–1 (the lower end of § 04) is, in the words of the Lemma Reducing the Comparison inside the Band, "widening the interval on which the two-sided area law at fixed β holds", and needs no comparison across scales.

07

The four faces of the band in the middle

It has been narrowed down to this: crossing the band in the middle needs one inequality that uses non-commutativity. On **which local quantity to write it in**, there are four candidates. We call them "faces" in the sense that it is enough to show any one of them uniformly in the scale, in a form that fails for $U(1)$. Table 5 lists only the names and the expected behaviour; the detailed examination is in 26-10. Every row is an estimate Reading (physical estimate).

Table 5 The four faces of the band in the middle (expectation rows are physical estimates; numbers are readings)

Face	Local quantity	Expectation for $SU(2)$	Where it fails for $U(1)$
κ_w	Softness in the twist direction: the variance of the force along the generator of the twist (in a strip of width w)	Decreases with w (the soft direction of Nielsen–Olesen; a non-perturbative version of the one-loop $+4$ spin term)	No spin term, $\kappa_w \ge c\beta$
$\delta(L_0)$	The total variation distance on the inner cube when the centre magnetic flux on the boundary of the box is changed	Goes to 0 exponentially in L_0	Remains $O(\beta)$ in the Coulomb phase
$f(\ell)$	The twist free energy of an $\ell \times \ell$ column with its outer frame held fixed	Of the type $e^{-\sigma \ell^2}$	Of the type $\pi^2\beta/\ell^2$ ($\approx 2\beta$ for $\mathbb{Z}_2$)
Pressure margin	The pressure difference $p_\infty - p_{G,L}$ between the system with and without the twist pattern (the exponent of the chessboard density κ_{cb})	A positive margin remains and gives the area law at fixed β	Does not produce a small κ_{cb}

A numerical example. Measuring the column face $f(\ell)$ by the twist bias $\mathbb{E}|\theta_B|$ on $T_4^2 \times \ell \times \ell$ with the outer frame held fixed, $SU(2)$ ($\beta = 2.3$) gives $0.23, 0.07, 0.09$ at $\ell = 4, 5, 6$, and the Coulomb phase of $U(1)$ ($\beta = 1.3$) gives $0.61, 0.49, 0.41$ (0.25 at $\ell = 8$); for $\ell \ge 5$,

$SU(2)$ is 1/5 to 1/7 of $U(1)$ Reading . However, the $SU(2)$ values are close to the statistical floor, and at this size $e^{-\sigma \ell^2}$ cannot be told apart from a power.

For each of the four faces, the comparison across scales (for example $f(2\ell, 2N, \beta) \leq Kf(\ell, N, \beta')$) is a local version with the same difficulty as the global renormalisation comparison, and has not become easier. What changing the face gains is that the object of the proof becomes "a statement about a finite region with its boundary held fixed".

08

Conditions an argument closing the band must satisfy — the criterion "distinguish from the U(1) transition"

The criterion "distinguish from the U(1) transition". $U(1)$ has, inside this band ($\beta \approx 1.01$), a phase transition to a non-confining phase. An argument that closes the band must show that $SU(2)$ has no phase transition, distinguishing it from $U(1)$. This is of the same weight as the unsolved problem that "four-dimensional $SU(2)$ has no transition to a non-confining phase at zero temperature", and proposals that do not satisfy this condition are discarded from the start.

The sign of the step function in § 05 was the numerical face of this criterion. Put another way, **a proof that closes the band is a proof that the direction $\beta' < \beta$ of the comparison does not change over the whole band for $SU(2)$**. Purely structural arguments (reflection positivity, duality, the transfer matrix) hold in the same form for $\mathbb{Z}_2$ and $U(1)$, so they alone do not satisfy the criterion (26-04). Renormalisation that does not distinguish groups (the isometric comparison, the Migdal–Kadanoff flow) does not satisfy it either (§ 03).

Even without closing the whole band at once, there are intermediate goals that satisfy the criterion "distinguish from the U(1) transition".

1. $\Delta F(L, L) \to 0$ (at any speed). In the Coulomb phase of $U(1)$, $\Delta F(L, L)$ tends to $\pi^2 \beta_R /2 - \ln 2$ (the classical half-flux value $\pi^2 \beta_R/2$ minus $\ln 2$ for the two orientations $\pm\pi$ of the half flux; β_R is the renormalised coupling), and in the non-confining phase of $\mathbb{Z}_2$ it increases, so showing this would exclude the known non-confining phases Assumption (the behaviour of the phases is an estimate). It does not give the mass gap. Numerically, for $U(1)$ ($\beta = 1.2$), $2^4 \to 4^4$ is flat at $3.10 \to 3.68$, and for $SU(2)$ ($\beta = 2.75$) it is $5.95 \to 5.10$ Reading .
2. The tension-doubling inequality: there are $\delta_0 > 0$ and N_1 such that $4\sigma_{2N}(\beta) \geq e^{\delta_0} \sigma_N(\beta)$ for all $N \geq N_1$, that is, $s(2N, \beta) \geq e^{\delta_0} s(N, \beta)$ (notation as in the rate form of the renormalisation comparison in § 01). In the Coulomb phase of $U(1)$, s is constant

across scales and sits exactly on the boundary $\delta_0 = 0$, and in the non-confining phase of $\mathbb{Z}_2$, $s \to 0$ and it is on the negative side, so it satisfies the criterion "distinguish from the U(1) transition" Reading (the values for the three phases come from closed-form models and small lattices). What follows is $s(2^k N_1) \geq e^{k\delta_0} s(N_1) \to \infty$, which excludes the Coulomb phase and the non-confining phases of discrete groups Proved here (the arithmetic of iteration). To pass to the first statement above, $\Delta F(L, L) \to 0$, control of the difference between the values on symmetric boxes and the slab tension is needed separately. Neither the mass gap nor the area law follows. Its row in the list of premises is in 28-04.

3. $\ln(1/\theta(L, L)) \geq cL^2/\ln L$ (nearly the area law). Through the Tomboulis–Yaffe inequality Cited, the Wilson loop decreases faster than the perimeter law (of the type $\langle W \rangle \leq 2\exp(-cA/\ln A)$).
4. The band itself: $\ln(1/\theta(L, L)) \geq cL^2$ (26-01).

Reading the band in the language of the gap — the band comparison (in the language of the gap). The vortex free energy is a quantity on the side of electric flux $e \neq 0$ ((4)(5) of § 02), while the mass gap needs the side $e = 0$. A comparison crossing the band on the neutral side can be written as the absence of a fixed point of the doubling map of $x(L) = L\, m_0(L)$, the gap of the box measured in units of the box size; that is, in the form $x(2L) \geq x(L) + \delta$ on the interval of the band (with δ independent of the lattice spacing) Assumption. In the Coulomb phase of $U(1)$, x stays at the photon value 2π (in a periodic box $x(2L)/x(L) = 1/\cos(\pi/2L) \to 1$), so this form satisfies the criterion "distinguish from the U(1) transition" Paper. The form of a ratio of rates, $m_0(2L) \geq (\frac{1}{2} - \varepsilon) m_0(L)$, is satisfied trivially by $U(1)$, with $m_0(2L)/m_0(L) = 1/(2\cos(\pi/2L)) > \frac{1}{2}$, so it cannot be used to distinguish Paper. The content of the distinction lies in the step δ being uniform. Of the four faces (§ 07), κ_W and $\delta(L_0)$ (extended to all boundary conditions) also act on the neutral side, while $f(\ell)$ and the pressure margin are quantities on top of the centre twist and belong only to the side of confinement Reading. The Tomboulis–Yaffe inequality is a structural inequality that holds with the same proof for $U(1)$, and its output is in the charged sector, so it is a tool for stepping down to the side of confinement, not a tool for the band comparison (in the language of the gap) Reading. The criterion on the gap side is narrower than the one on the confinement side, in that it need not exclude the non-confining phase of $\mathbb{Z}_2$ (which also has a gap). None of these rows has been shown.

The test models other than U(1) (Z_2, finite non-abelian groups, the adjoint Higgs limit, mixed actions, SU(N), G_2) are listed in 26-12.

09

What cannot be said

(a) We show neither the renormalisation comparison nor β-monotonicity at any β of the band.

(b) The strong-coupling end is a line, part of whose bookkeeping is sketched, in the range $u \le u_0 \approx 10^{-80}$. The weak-coupling end is at the one-loop level, not a non-perturbative claim. Do not read "it holds at both ends" with these two conditions removed.

(c) The Lemma Reducing the Comparison inside the Band is a reduction, and the assumption of two-sided area laws inside the band contains an unsolved problem. We do not say that the lemma made the band easier.

(d) The numbers of § 05 are readings on small lattices up to 8^4, with few β and 1–3 seeds, and are not evidence that there is no phase transition.

(e) The four faces of Table 5 are candidates, and none of them has an estimate uniform in the scale.

(f) The relation between β-monotonicity and the monotonicity of the string tension includes a step that is a reading (§ 02). Nor do we say that β-monotonicity is false.

10

The table of the five grades and the dependency graph

Table 6 The rows of this chapter ("—" marks a row with no machine checking; the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Basis	Scope
1	$\partial_\beta \Delta F = \langle \Delta S \rangle$	Proved here	§ 01	With the lattice geometry held fixed
2	The Gibbs variational principle (7)	Cited	§ 03	Measurable bijection T
3	(4)(5): β-monotonicity $\Leftrightarrow E_1 \ge E_0$ $\Leftrightarrow \theta$ increases in β	Proved here	§ 02	$Y_1 > 0$
4	β-monotonicity has the same weight as the β-monotonicity of the string tension in lattice units	Assumption	§ 02 (the step connecting the electric flux with the tension has not been written out)	On every lattice
5	Partial results on β-monotonicity (1 and 2 sheets in two dimensions, $\beta \to 0$, $\beta \to \infty$ in the twisted background)	Assumption	§ 02 (the proofs are not set out in the book)	The conditions of each case

Row	Statement (one line)	Grade	Basis	Scope
6	Strong-Coupling-End Expansion Proposition: the form (6) of the vortex free energy at the strong-coupling end	Assumption	Line: the bookkeeping of step 5 of the proof is sketched	$u \le u_0 \approx 10^{-80}$, N, $M \ge 2$
7	The Comparison at the Strong-Coupling End $K = 4(1 + O(u))$, $\beta' < \beta$	Assumption	Algebra on top of the Strong-Coupling-End Expansion Proposition	As above
8	The expansion (8) of the heat-kernel trace: $\frac{11}{3} = -\frac{1}{3} + 4$	Proved here	§ 03	One loop, constant background
9	The isometric comparison gives $\beta' \ge \beta + 0.023$ (against asymptotic freedom)	Proved here	§ 03, 25-07	One-loop level, per doubling
10	The comparison that reshapes the fluctuations gives $\beta' = \beta - 0.258$	Proved here	§ 03, 25-07	One-loop level, per doubling
11	Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3; the direction of confinement at fixed N and large β)	Assumption	A paper proof in the paper (with citations). Not set out in the book	Fixed N
12	Table 2: for $(4, 8) \to (8, 16)$, $SU(2)$ gives $\beta' < \beta$, $U(1)$ gives $\beta' > \beta$	Reading	§ 05	Lattices, β and seeds as in the table
13	Tables 3 and 4: for $SU(2)$ the sign does not change; for $U(1)$ it reverses at $\beta = 1.0$–1.1	Reading	§ 05	Up to 8^4, 1–3 seeds
14	Lemma Reducing the Comparison inside the Band: the comparison (9) inside the band under the assumption of two-sided area laws inside the band	Proved here	§ 06	Two-sided area law at fixed β, uniform positivity and continuity of the rate, $N \ge N_1$, $M \ge M_0$
15	Corollary Crossing the Band in Finitely Many Steps: the band can be crossed in finitely many steps, with a loss independent of N, M	Proved here	§ 06	As above
16	Calibration: $K = 4(1 + C'u)/(1 - C'u)$ in the form of the Strong-Coupling-End Expansion Proposition	Proved here	§ 06	Agreement of form (the Strong-Coupling-End Expansion Proposition is an assumption)

Row	Statement (one line)	Grade	Basis	Scope
17	The difficulty gathers at the upper end of the band and at the seam with ultraviolet control	Reading	§ 06 (physical estimate)	—
18	The four faces of the band in the middle and their expectations	Reading	§ 07 (physical estimate)	—
19	The criterion "distinguish from the U(1) transition": the transition of $U(1)$ inside the band, and the condition on an argument closing the band	Reading	§ 08 (the position of the transition is a known numerical value; physical estimate)	—
20	$\Delta F(L, L) \to 0$ excludes the known non-confining phases	Assumption	§ 08 (the behaviour of the phases is an estimate)	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  ∂_β ΔF = ⟨ΔS⟩
 2  CIT  Gibbs variational principle (7)
 3  PRV  β-monotonicity ⇔ E₁ ≥ E₀ ⇔ θ increases ⟸ 1                Y₁ > 0
 4  ASM  β-monotonicity weighs as much as β-monotonicity of the tension ⟸ 3
 5  ASM  partial results on β-monotonicity
 6  ASM  Strong-Coupling-End Expansion Proposition                u ≤ u₀ ≈ 10⁻
 7  ASM  Comparison at the Strong-Coupling End, K ≈ 4 ⟸ 6
 8  PRV  11/3 = −1/3 + 4                                        one loop
 9  PRV  isometric comparison gives β' ≥ β + 0.023 ⟸ 2, 8        one loop
10  PRV  reshaping comparison gives β' = β − 0.258 ⟸ 2, 8        one loop
11  ASM  Confinement Lower Bound by Fractional Instantons        fixed N
12  RDG  direction of (4,8)→(8,16) separates by group
13  RDG  sign of the step function: SU(2) unchanged, U(1) reverses
14  PRV  Lemma Reducing the Comparison inside the Band
15  PRV  the band takes finitely many steps ⟸ 14
16  PRV  calibration ⟸ 6, 14                                     agreement of
17  RDG  difficulty at the upper end and the UV seam ⟸ 14
18  RDG  the four faces ⟸ 9, 10, 12, 13
19  RDG  criterion "distinguish from the U(1) transition" ⟸ 13
20  ASM  ΔF(L,L) → 0 excludes non-confining phases ⟸ 19
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The two-sided area law at fixed β and the uniform positivity and continuity of the rate are written as conditions in the row of the Lemma Reducing the Comparison inside the Band and are not made nodes, because the assumption rows do not contain the quantity of the conclusion (the comparison that crosses the band). Among the nodes of this chapter there is no node for the comparison at any β of the band, none for confinement, and none for the mass gap.

What this chapter can say

What is said	What is not said
Proved here β-monotonicity and its restatement (4)(5) in terms of the electric flux. That, at the one-loop level, the isometric comparison gives only the direction against asymptotic freedom ($\beta' \geq \beta + 0.023$), and the comparison that reshapes the fluctuations gives $\beta - \beta' = 0.258$	β-monotonicity itself. The non-perturbative weak-coupling comparison
Proved here Lemma Reducing the Comparison inside the Band: that the comparison inside the band reduces to the two-sided area law at fixed β (the assumption of two-sided area laws inside the band)	A proof of the assumption of two-sided area laws inside the band at any β. Having made the band easier
Assumption The Comparison at the Strong-Coupling End ($u \leq u_0 \approx 10^{-80}$, part of the bookkeeping sketched). The upper end at fixed N (the Confinement Lower Bound by Fractional Instantons; Theorem A of Paper 3)	A proof set out in the book. Raising u_0 to a practical range
Reading That on lattices up to 8^4 the direction of the comparison does not change for $SU(2)$ and reverses at the position of the transition for $U(1)$	Evidence that there is no phase transition. The same reading on large lattices

Sources and reproduction

Item	Kind	Source / tools
The split into β-monotonicity and the renormalisation comparison, (4)(5), the partial results, the expansion of the heat-kernel trace and the two comparisons	This book's study	The record of this book's study (the idea of splitting the band into β-monotonicity and the renormalisation comparison, the examination of β-monotonicity, the one-loop check). The computations of (4) and (8) were checked in this chapter
Strong-Coupling-End Expansion Proposition and the Comparison at the Strong-Coupling End	This book's study (Assumption)	The record of this book's study (the strict form of the comparison on the strong-coupling side, and the section correcting it). The type of the polymer expansion is that of G. Münster's (1981) strong-coupling expansion of the vortex free energy (the bibliographic data and the text of the original source have not been checked)
Table 1	Reading	Numerical values of the ratio of modified Bessel functions I_2/I_1, and numerical roots of $u(\beta') = u(\beta)$ [4]. The numbers were checked in this chapter

Item	Kind	Source / tools
Tables 2–4, the column bias	Reading	The numerical record of this book's study (the test of the renormalisation comparison, the scan of the step function on the symmetric torus, the $U(1)$ control, the addition of $4^4 \to 8^4$ and the merging of seeds, the twist cost of a column). Replica exchange and Bennett's acceptance ratio method
Lemma Reducing the Comparison inside the Band, Corollary Crossing the Band in Finitely Many Steps, calibration	Proved here	This book's study (the reduction lemma from the examination of the band candidates). The proof was checked line by line in this chapter
The four faces, the criterion "distinguish from the U(1) transition", the intermediate goals	This book's study	The record of this book's study (the arrangement of roads for closing the crossover band). Details in 26-10
The claim of Tomboulis and its criticism	Cited	E. T. Tomboulis, arXiv:0707.2179. K. R. Ito, E. Seiler, arXiv:0711.4930 (eq. (3.5)), arXiv:0803.3019 (detailed study). T. Kanazawa, arXiv:0805.2742 (Appendix A). For the history, see 26-08
Tomboulis's short paper of the 1980s	Cited	E. T. Tomboulis, Phys. Rev. Lett. 50 (1983) 885 (title not confirmed; the text has not been read)
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) (the table of sources of 26-01)

Terms used in this chapter — the crossover band, β-monotonicity, the renormalisation comparison, the isometric comparison, the comparison that reshapes the fluctuations, the step function, the four faces, the criterion "distinguish from the U(1) transition", the reduction lemma. Their meanings are in A-02 Glossary.

Next chapter: 26-07 The band seen from the side of time — the statement corresponding to the band on the road of Part 23, and why the first-step wall, the "summit-curvature shortfall wall", is a miniature of the band and not the band.

The band seen from the side of time

> This chapter is this book's own investigation. § 06 is a map of open questions.

What this chapter shows — Part 23 was a road that approaches the mass gap from the side of Langevin time (relaxation of a stochastic dynamics). This chapter states exactly which statement on that road corresponds to the crossover band, and settles the relation between the band and the wall, the "summit-curvature shortfall wall", at which Part 23 actually stops. There are two answers. (1) The four-dimensional goal of Part 23 (the physical-unit target level of 23-15) is a statement uniform over all scales — that the upper side of the diffusive-relaxation-time assumption (the relaxation-time upper-bound assumption) and the one-sided bridge hold with the same constant as $\beta \to \infty$ — and it must cross the band. (2) The first-step wall, the "summit-curvature shortfall wall", is a miniature of the band, but it is not the band. This is because its crossover length is fixed classically by the first conjugate point of the Jacobi equation at the summit (about 7.84 sites), does not depend on $\hbar$, and is finite in lattice units. The length $1/\Lambda$ of the four-dimensional band is born by quantisation and diverges in lattice units. The two walls **are of the same type and differ in content**. We put this into a comparison table of six columns.

Prerequisite chapters — 23-15 § 07 (the three levels of the goal), 23-01 § 06 (the one-sided bridge), 23-09 § 06 (the curvature $\frac{15}{4}$ of one site at the summit, and 4), the short-core step of 23-11 § 05 (convexity up to length 7), 24-06 (paying with rarity), 24-07 (precisely because it is four-dimensional; the three sources of difficulty), 26-06 (the renormalisation comparison and the four faces), 4-07 (dimensional transmutation).

Said first — The level is [Res]. The only proofs set out are the computation of the Jacobi equation at the summit and of the first conjugate point in § 02, the proof of the Hidden-Convexity Proposition in § 03, and the one line of algebra in § 01. The contrast between the two walls is a physical estimate, and the statement that crosses the band is itself an assumption on either road. A note on words: 23-01 § 05 in Part 23 uses the word "band statement" for the statement that C_P lies in a range of width a constant times the square of a length (the range of constant-multiple width = the band). The "band" of this chapter is the "crossover band" of Part 26 (the crossover at $L \sim 1/\Lambda$), a different word. The list of places where "band" is used in other senses in Parts 23 and 24 is in 26-00 Table 0. It is also different from the "window" of Parts 23 and 24 (the site window, the $\hbar$ window, the time window). The convexity constant up to length 7 is written $\kappa_{\mathrm{C7}}(k)$, to distinguish it from the chessboard density κ_{cb} (the same quantity as $\kappa(k)$ of 23-11).

The order of this chapter

1. The statement on the road of Part 23 that corresponds to the band — the physical-unit target level, and uniformity with the same constant
2. The wall of the first step is fixed by a classical length — the Jacobi equation at the top and the first conjugate point
3. The type is the same — different tools at the two ends, the worst-case estimate stops in the middle, and on to averaging tools
4. The content differs — the origin of the crossover length, rare and typical, phase transitions
5. The comparison table of the two walls — six columns (Table 1)
6. A supplement to Part 23 — the road of blocks and mixing — towards the third branch of 23-15
7. What cannot be said
8. The table of the five grades and the dependency graph — Table 2

01

The statement on the road of Part 23 that corresponds to the band — the physical-unit target level, and uniformity with the same constant

Part 23 split the goal of the mass gap on the four-dimensional lattice into three levels (23-15 Table 5): the lattice target level, that the correlation length is finite at each β on the lattice; the physical-unit target level, that the lattice mass measured in physical units, $m_{\mathrm{lat}}(\beta)/a(\beta)$, tends to a positive finite value Δ_∞; and the continuum target level, that the gap is positive in the continuum theory. On the road from the side of time, the physical-unit target level is what corresponds to the band.

The lower half of the physical-unit target level (that the mass does not collapse to 0) follows by one line of algebra from two statements: the upper-bound assumption, the upper side of the diffusive-relaxation-time assumption ((14) of 23-01 § 05), and the one-sided bridge (in the form of the One-Sided Bridge Proposition of 23-01 § 06).

$$\exists\,\beta_0,\ C,\ \kappa>0\ \ \forall \beta\ge\beta_0:\qquad \underbrace{a(\beta)^2\,C_P(\beta)\ \le\ C}_{\xi^2\uparrow},\qquad \underbrace{C_P(\beta)\ \ge\ \kappa\,\xi(\beta)^2}_{\text{one-sided bridge}}\ \ (1)$$

$$\Longrightarrow\qquad \frac{m_{\mathrm{lat}}(\beta)}{a(\beta)}\ =\ \frac{1}{a(\beta)\,\xi(\beta)}\ \ge\ \sqrt{\kappa/C}\qquad (\beta\ge\beta_0)\ \ (2)$$

Proved here Proof. From the two parts of (1), $\kappa\xi^2 a^2 \le a^2 C_P \le C$; taking the reciprocal of the square root of both sides gives (2). ∎ (The same algebra as (16) of 23-01 § 06. C_P is the Langevin Poincaré constant of the Wilson measure, $\xi = 1/m_{\mathrm{lat}}$ the correlation length in lattice units, and $a(\beta)$ the lattice spacing.)

The content of (1) lies in the order of the two quantifiers: C **and** κ **do not depend on** β, that is, it holds with the same constant on all lattices as $\beta\to\infty$. The one-sided bridge can be shown, for continuous time and a finite spatial lattice, as a proposition with named assumptions, but its lifting to the four-dimensional lattice is only a prospect of form Assumption (the row of the fourth step in 24-07 Table 2). The upper-bound assumption is an assumption in four dimensions Assumption.

Why this statement crosses the band Reading (physical estimate). As β grows, the lattice spacing $a(\beta)$ goes to 0, and the correlation length in lattice units grows like $\xi(\beta)\propto e^{3\pi^2\beta/11}$ (the one-loop bookkeeping, 23-03 Table 2). The Wilson measure at one fixed β contains all scales from one lattice step up to $\xi(\beta)$, and the effective coupling at the scale of length L is small for $L\ll 1/\Lambda$, becomes $O(1)$ at $L\sim 1/\Lambda$, and beyond that enters strong-coupling behaviour. $C_P(\beta)$ is a supremum over all test functions, so it includes the slowest relaxation, that is, the behaviour at the longest scales. So showing (1) with a C independent of β includes passing, with the same constant, at every β, through the scales at which the effective coupling becomes $O(1)$ — the crossover band. The goal of the road from the side of space (the uniform slab tension: a tension positive uniformly in L, 26-00) is a statement of the same type. That the two roads arrive at a statement of one type, "the same constant at every scale", is what the confluence of Part 26 means.

This uniformity takes up again something Part 23 already said. The end of 23-15 § 07 wrote that what is missing is not the number of steps but uniformity, holding at every scale with the same constant, and that it appears at the first step in a form uniform as $\hbar\to 0$ and on the four-dimensional lattice in a form uniform as $\beta\to\infty$. The question of this chapter is whether that first-step version is **the same thing** as the four-dimensional band.

02

The wall of the first step is fixed by a classical length — the Jacobi equation at the top and the first conjugate point

First, the place. The first step is the path measure $\mu\propto e^{-S/\hbar}$ of the double well in 0+1 dimensions, with

$$S(z)=\sum_i\Big[\frac{(z_{i+1}-z_i)^2}{2\varepsilon}+\varepsilon V(z_i)\Big],\qquad V(z)=\frac{(z^2-1)^2}{8},\qquad \varepsilon=\tfrac12\ \ (3)$$

(23-04). The wall appears in the conditional measure on the interval (the core) $Q_k=[-x_c,0]^k$ ($x_c=0.89$, Neumann faces) in which the path stays between the summit $z=0$ and the well, and the goal is that its Poincaré constant is uniform in the length k of the core, the end values and $\hbar$ (the core Poincaré uniformity assumption, 23-15 § 02).

The Hessian at the top configuration $z\equiv 0$ is the tridiagonal $\mathrm{tridiag}(-2,\frac{15}{4},-2)$, and the diagonal $\frac{15}{4}$ of one row falls short of the sum of couplings 4 by $\frac14$ (23-09 § 06). We reread this shortfall in the language of the calculus of variations.

The Jacobi equation at the top. Linearising the Euler–Lagrange equation of S at the top $z\equiv 0$ gives, for the displacement u_n,

$$-2u_{n-1}+\tfrac{15}{4}\,u_n-2u_{n+1}=0\ \ (4)$$

For general ε the diagonal is $2/\varepsilon+\varepsilon V''(0)=2/\varepsilon-\varepsilon/2$ and the coupling to neighbours is $1/\varepsilon$, and (4) can be written $u_{n-1}+u_{n+1}=(2-\varepsilon^2/2)u_n$. Substituting $u_n=e^{\pm in\theta}$, the characteristic equation is $2\cos\theta=2-\varepsilon^2/2$, that is,

$$\cos\theta=1-\frac{\varepsilon^2}{4}\qquad (\varepsilon=\tfrac12\ \text{gives}\ \cos\theta=\tfrac{15}{16},\ \ \theta=0.35542\ldots)\ \ (5)$$

and since $|\cos\theta|<1$, the solution is a rotation $u_n=\sin(n\theta)$ ($u_0=0$). Seen through the linearisation $\ddot u=V''(0)u=-\frac12 u$ of the Euclidean equation of motion $\ddot z=V'(z)$ too, the top is not a point of exponential departure but a point of rotation.

The first conjugate point. The position $n=\pi/\theta$ at which the solution starting from $u_0=0$ next becomes 0 is the first conjugate point. At $\varepsilon=\frac12$,

$$\frac{\pi}{\theta}=8.8391\ldots,\qquad \frac{\pi}{\theta}-1=7.8391\ldots\ \approx\ 7.84\ \ (6)$$

Proved here (the numerical value of $\theta=\arccos(15/16)$. As an independent check, running the recursion (4) from $u_0=0,u_1=1$ gives $u_8=0.844>0$, $u_9=-0.164<0$, and the zero of the linear interpolation is 8.837).

The threshold of convexity = the conjugate point. The smallest eigenvalue of the Hessian $\mathrm{tridiag}(-2, \frac{15}{4}, -2)$ at the top of a lump of length k with both ends fixed is

$$\kappa_{\mathrm{C7}}(k) \;=\; \tfrac{15}{4} - 4\cos\frac{\pi}{k+1} \;=\; 8\sin^2\frac{\pi}{2(k+1)} - \frac{1}{4} \quad (7)$$

(the closed formula for the eigenvalues of a tridiagonal matrix), and it is positive only when $\pi/(k+1) > \theta$, that is, $k+1 < \pi/\theta$. From (6), it is positive for $k \le 7$ and negative for $k \ge 8$ Proved here ($\kappa_{\mathrm{C7}}(7) = 0.0545$, $\kappa_{\mathrm{C7}}(8) = -0.0088$. The difference between numerical diagonalisation and (7) is of order 10^{-15} for $k = 5$–10). "Convex up to length 7, broken from 8" (the Short-Core Convexity Lemma of 23-11 § 05) is exactly the discrete version of the Jacobi criterion of the calculus of variations: whether a conjugate point falls in the interval $[0, k+1]$.

This length is classical and does not depend on $\hbar$. $\hbar$ never appears in (4)–(7). What appears is only the form of the action S (the coefficient $1/\varepsilon$ of the kinetic term and the curvature at the top $V''(0) = -\frac{1}{2}$). The limit taken at the first step is $\hbar \to 0$ with $\varepsilon = \frac{1}{2}$ fixed, so the crossover length stays at about 7.84 sites throughout the limit and is finite in lattice units. In units of time it is $\varepsilon \cdot \pi/\theta \approx 4.42$, close to the value $\pi/\sqrt{|V''(0)|} = \pi\sqrt{2} \approx 4.44$ (the half period of rotation at the top) in the limit where time is made continuous. Sending ε to 0 makes the number of sites grow as $1/\varepsilon$, but that is not the limit the problem of the first step takes.

Example. For $\varepsilon = \frac{1}{4}$ it is $\pi/\theta - 1 \approx 16.75$ sites, which in units of time is ≈ 4.44. As a number of sites the length changes with the step; as a physical time it is fixed by the number $V''(0)$ in the action.

A numerical agreement with the mixing length (the mechanism is different). The number of sites over which the influence of the worst end value (the top) drops below 10^{-2} in total variation hardly moves for $\hbar = 10^{-2}$–10^{-5} and is about 9; changing the step to $\varepsilon = 0.35, 0.5, 0.7, 0.9$ gives $12, 9, 7, 5$, numerically close to $\pi/\theta = 12.7, 8.8, 6.3, 4.9$ Reading . But the mechanism is not "staying near the top for half a period". The optimal path as $\hbar \to 0$ starts going down from the top at once and reaches the corner $-x_c$ of the box in a finite number of steps (7 at $\varepsilon = \frac{1}{2}$), and the mixing length is roughly that number of steps plus 2 Reading . Why the two values are close is not known. What this chapter claims about the conjugate point is only the convexity threshold of (7).

03

The type is the same — different tools at the two ends, the worst-case estimate stops in the middle, and on to averaging tools

Setting the two walls side by side shows the same type Reading (physical estimate; a comparison of types, implying no statement on either side).

There are different tools at the two ends. At the first step, short lumps ($k \le 7$) are bounded uniformly in the end values and $\hbar$ by the convexity of (7) (the Short-Core Convexity Lemma Proved here), and inside the wells local Poincaré within the phase (the in-phase local Poincaré condition, 23-06) works. In the four-dimensional band, the expansion works at the strong-coupling end, and the one-loop comparison at the weak-coupling end (26-06 § 03; on the strong-coupling side the line of argument for $u \le u_0 \approx 10^{-80}$, on the weak-coupling side the one-loop level).

In the middle, the estimate of the worst configuration stops. At the first step, every tool that takes the worst configuration as its entrance comes back to the same one-line shortfall $\frac{15}{4} < 4$ Proved here (each row is a two-line computation).

- The one-site Dobrushin influence coefficient: the conditional one-site measure is $\frac{15}{4}$-convex, and with the coupling $1/\varepsilon = 2$ to neighbours the influence from one neighbour is $\le 2/\frac{15}{4} = \frac{8}{15}$, and from two $\frac{16}{15} > 1$.
- The influence of a block of length 7: by Brascamp–Lieb and an entrywise comparison of M-matrices, the influence from the left end to the right end is $2\,(M_7^{-1})_{71} = 2 \cdot 2^6/\det M_7 \approx 1.18 > 1$ (7×7 for $M_7 = \mathrm{tridiag}(-2, \frac{15}{4}, -2)$, $\det M_7 = 1770735/16384 \approx 108.08$).
- The Otto–Reznikoff criterion: it needs the positivity of $\mathrm{tridiag}(-2, \frac{15}{4}, -2)$, which fails on the infinite chain with lower end $-\frac{1}{4}$.

In the four-dimensional band, the road that estimates with the worst frame plays the same role. The tension that comes out of the slab-confinement estimate $\sup_\Phi \theta \le \tanh(\beta L^2)^{\ell^2}$ (25-04) is $\sigma \ge -\ln\tanh(\beta L^2) \approx 2e^{-2\beta L^2}$, which gets worse superexponentially in L ($2e^{-80}$ at $\beta = 2.5$, $L = 4$). Structural inputs alone (reflection positivity, duality, the transfer matrix) hold in the same form for $\mathbb{Z}_2$ and $U(1)$, and cannot cross the band (26-04). Each side has a "worst single point": at the first step the face $x \equiv 0$ where every site is at the top, in four dimensions the place where the plaquettes are at the centre element -1.

Moving to averaging tools, there is a prospect that uniformity comes out. At the first step, the correlation length of the transfer operator gets shorter as $\hbar$ is made smaller (from 1.68 to 0.48 as $\hbar = 0.1$ goes to 3×10^{-4}) Reading , and the influence of the worst end value also vanishes over a length almost independent of $\hbar$ (§ 02). This is because the rarity of configurations near the top ($P(z > -0.5) \approx 10^{-252}$ at $\hbar = 3 \times 10^{-4}$) Reading is paid automatically inside the transfer operator ("paying with rarity" of 24-06 § 06). On the four-dimensional side, if L is fixed, frame decorrelation is replaced by the single chessboard density κ_{cb} (an averaged quantity) (26-05). Estimates across scales are entrusted to one of the four faces (26-06 § 07) Assumption .

A first-step version of the averaging tool has been proposed down to a concrete form (§ 06). We set out, with proof, the uniqueness proposition at its entrance.

Hidden-Convexity Proposition. For any $\varepsilon > 0$, $c > 0$, $l \geq 1$ and end values $a, b \in [-c, 0]$, the action S (with the V of (3) and both ends fixed at a, b) has exactly one local minimum on the box $[-c, 0]^l$, and it is the global minimum.

Proved here Proof. Put $z_i = -\sqrt{\rho_i}$ ($\rho_i \in [0, c^2]$). (i) $(z_{i+1} - z_i)^2 = \rho_{i+1} + \rho_i - 2\sqrt{\rho_i \rho_{i+1}}$, and the geometric mean $\sqrt{xy}$ is concave on $[0, \infty)^2$, so $-\sqrt{\rho_i \rho_{i+1}}$ is convex. (ii) The end term $(z_1 - a)^2 = \rho_1 + a^2 - 2|a|\sqrt{\rho_1}$ is also convex, since $-\sqrt{\rho_1}$ is convex (likewise at the right end). (iii) $V(z) = (z^2 - 1)^2/8 = (\rho - 1)^2/8$ is strictly convex in ρ with second derivative $\frac{1}{4}$. Hence $\tilde{S}(\rho) := S(-\sqrt{\rho})$ is strictly convex on the convex set $[0, c^2]^l$ with $\nabla^2 \tilde{S} \geq (\varepsilon/4) I$, and its local minimum is unique and global. $\rho \mapsto -\sqrt{\rho}$ is a homeomorphism from $[0, c^2]^l$ to $[-c, 0]^l$, so the local minima correspond to each other. ∎

What the Hidden-Convexity Proposition says is the uniqueness of the minimum, not a Poincaré inequality. The Dirichlet form is written in the flat metric of z, so convexity in the coordinates ρ does not give Bakry–Émery as it stands. Writing the measure in ρ attaches $\prod_i (2\sqrt{\rho_i})^{-1}$ to the density, and the concavity of its logarithm breaks convexity only for $\rho_i \lesssim \sqrt{2\hbar/\varepsilon}$ (a thin layer very close to the summit) (comparing $\frac{\varepsilon}{4\hbar}$ with $\frac{1}{2\rho^2}$). One can read the non-convexity of the "summit-curvature shortfall wall" as moving from what looked, in the coordinate z, like lumps at the summit of length 8 or more, to rare thin layers site by site in the coordinate ρ Assumption (the argument that pays for this layer with rarity has not been built).

04

The content differs — the origin of the crossover length, rare and typical, phase transitions

Even with the same type, three points that decide the weight of the wall differ Reading (physical estimate).

(i) The origin of the crossover length. The crossover length of the first step is the first conjugate point (about 7.84 sites), a classical length fixed only by numbers written in the action S (the coefficient of the kinetic term and the curvature at the top). Throughout the limit $\hbar \to 0$ it stays finite in lattice units (§ 02). The length of the four-dimensional band is $1/\Lambda$, and the action has no ruler of length (in four dimensions the coupling is dimensionless, 24-07 § 01). The length is born by quantisation, in the one-loop bookkeeping in the form $\Lambda = \mu \exp(-1/(2b_0 g^2(\mu)))$ (dimensional transmutation, 4-07) Cited , and in lattice units $1/(a\Lambda)$ diverges like $e^{3\pi^2 \beta/11}$ as $\beta \to \infty$. All derivatives of $\exp(-1/g^2)$ vanish at $g = 0$, so this length appears at no order of the expansion in the coupling.

(ii) Whether bad configurations are rare or typical. At the first step, configurations near the summit are rare (the 10^{-252} of § 03), and the currency of "paying with rarity" of 24-06 works. In the four-dimensional band, the bad configurations (thick centre vortices, fractional instantons) are the mechanism of confinement itself, and are typical. The rate of rarity runs out in the band. This is the same point as the row of the source "no expansion linking scales" in 24-07 § 05 (in the infrared, configurations for which the Gaussian approximation fails become typical, and there is no currency of rarity).

(iii) Whether a phase transition is possible. The core of the first step is a one-dimensional nearest-neighbour chain over a compact one-site space. The transfer operator is a positive integral kernel, so there is no phase transition that would prevent estimates uniform in the length of the chain Assumption (a fact about the principal eigenvalue of positive integral kernels; the same bibliography as [001] of 23-15 Table 1, the original still to be obtained). At fixed β in four dimensions, an estimate uniform in the volume has the same weight as "there is no phase transition", and $U(1)$ actually has a transition inside the band ($\beta \approx 1.01$). An argument that closes the band must distinguish itself from $U(1)$: this is the criterion "distinguish from the U(1) transition" (26-06 § 08).

One decisive difference follows from (i). At the first step, if one chooses a **finite** block (length 12) longer than the crossover length, a two-stage road stands: analysis around a single minimum inside the block, and strong mixing between blocks (§ 06). In four dimensions the crossover length $1/\Lambda$ diverges in lattice units, so there is no finite block longer than it. **So we cannot expect the tools of the first step to cross the band as they are.** The "dimensionless coupling = no classical length" of 24-07 § 01 is the reason for this difference.

A supplement. Through the reduction lemma of 26-06 § 06, the inside of the band becomes "an estimate uniform in the volume at fixed β", a statement of the same form as the wall of the first step (an estimate uniform in the length k at fixed $\hbar$). Three things are common: a fixed parameter, uniformity in the volume, and passing through a non-convex region. That the weights still differ is because of (ii) and (iii). And after the reduction, the difficulty of crossing scales gathers at the seam between the upper end of the band and ultraviolet control (ibid. § 06). At the first step there is no flow of the coupling (among the sources of difficulty, the source "dimensionless coupling" is absent), so there is nothing corresponding to this seam.

05

The comparison table of the two walls

Table 1 gathers § 02– § 04 into six columns. The label of each cell is written inside the cell. Comparisons of type (readings across rows) are all estimates Reading .

Table 1 The wall of the first step and the four-dimensional band (columns (1)–(6))

	(1) Place and object	(2) Crossover length and its origin	(3) Tools that work at the two ends	(4) Tools that stop in
The first-step wall, the "summit-curvature shortfall wall"	The path measure of the double well in 0+1 dimensions ($\varepsilon = \frac{1}{2}$, $\hbar \to 0$). The Poincaré constant of the conditional measure on the core $[-x_c, 0]^k$, uniform in k, the end values and $\hbar$ (the core Poincaré uniformity assumption)	The first conjugate point at the summit, $\pi/\theta - 1 \approx 7.84$ sites, $\cos\theta = 1 - \varepsilon^2/4$. A classical length fixed by the action alone, independent of $\hbar$, finite in lattice units Proved here	Short lumps: convexity for $k \le 7$ (the Short-Core Convexity Lemma, $\kappa_{C7}(7) = 0.0545$) Proved here . Inside the wells: local Poincaré within the phase (the in-phase local Poincaré condition; labels per row in 23-06)	Curvature at the wc the rows of Dobrusl of length 7 (≈ 1.18),), Brascamp–Lieb a back to $\frac{15}{4}$ against 4 worst point is the fa
The four-dimensional band	The vortex free energy of the slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ ($n_{12} = 1$) of the four-dimensional $SU(2)$ lattice. $\ln(1/\theta(L,L)) \ge cL^2$ uniformly in L (26-01)	$1/\Lambda$. The action has no length (the coupling is dimensionless); it is born by quantisation (dimensional transmutation) Cited . Diverges in lattice units as $\beta \to \infty$	The strong-coupling expansion ($u \le u_0 \approx 10^{-80}$, part of the bookkeeping sketched) Assumption . The one-loop comparison (with reshaped fluctuations, $\beta - \beta' = 0.258$) Proved here (one-loop level)	The worst frame: su gets worse superexp 04). Structural inpu for $\mathbb{Z}_2$ and $U(1)$ (26- comparison goes ag freedom (26-06). Th centre element -1

How to read Table 1. Reading columns (3)–(5) across, the type is the same (different tools at the two ends, the worst-case estimate stops in the middle, then averaging tools); comparing columns (2) and (6) down, the content differs (whether the origin of the length is classical or quantum, whether a phase transition is possible). The answer to the question "can the summit-curvature shortfall wall be written in the same words as the band?" is: **the type can be written in the same words, and the content (the origin of the crossover length) differs**.

06

A supplement to Part 23 — the road of blocks and mixing

The following road has been proposed as a candidate for supporting the core Poincaré uniformity assumption without going through the LSI constant A_W of the site window (23-09). As the third branch, "the road of blocks and mixing", of the fork of Part 23 (23-15 § 02), its table of rows and dependency graph are in the subsection on the road of blocks and mixing in 23-15. Here we write only the skeleton needed for the relation to the band.

1. **Block LSI** Assumption : on blocks of length $\ell_B \ge 12$ (avoiding the conjugate point ≈ 7.84 and the length 8 at which convexity breaks), uniform in the end values and $\hbar$. The entrance is the Hidden-Convexity Proposition (§ 03) Proved here ; the curvature at the minimiser for the worst end values $\lambda_*(12) \ge 0.255$ Reading (interval arithmetic for the minimisers at the four corners remains); and Witten's Morse theory together with the Menz–Schlichting decomposition, which derive from a single non-degenerate minimum that there are no exponentially small eigenvalues (the form at the corners of a Neumann box not yet checked).
2. **Strong mixing** Assumption : in three stages — the optimal path as $\hbar \to 0$ reaches the corner of the box within 7 steps from every starting point (the record is a 50-digit KKT certificate, the Hidden-Convexity Proposition and Topkis monotonicity; Reading until it is checked whether this is interval arithmetic with outward rounding), concentrates within a width $\approx 0.8\sqrt{\hbar}$, and has a Doeblin overlap in the boundary layer at the corner — one gets $\mathrm{TV}(n) \le (1-\delta)^{\lfloor n/10 \rfloor}$ with δ independent of $\hbar$. The second and third stages are along the lines of the constrained Laplace method.
3. **The standard theorem for one-dimensional chains** Assumption : from block LSI and strong mixing, an LSI uniform in the length of the chain (of the Zegarlinski and Martinelli–Olivieri type; the version for compact continuous spins not yet checked). Poincaré follows from LSI and reaches the core Poincaré uniformity assumption.

Where it stands. Even if the road of blocks and mixing closes, the first step does not close. The rows common to the two roads (the tube, the reduced kink gas, the cross term, the extension of the in-phase local Poincaré condition and others: the 13 common rows of 23-15 Table 1) remain on the road of blocks and mixing too. If a mild dependence on $\ln(1/\hbar)$ remains in the mixing length, the power statement ($\ln CP / \ln \xi \to 2$) is expected to be unaffected, but rows that need the constant to be uniform in $\hbar$ are affected (details in the subsection on the road of blocks and mixing in 23-15).

The relation to the band. The road of blocks and mixing is the first-step version of the stage "move to averaging tools" in the type of § 03. And as in § 04 (i), its key was that there is "a finite block longer than the crossover length". The four-dimensional band has no such finite block. Even if the road of blocks and mixing succeeds at the first step, we cannot say it gives a tool that crosses the band. In the form carried over to four dimensions (a box criterion with fixed boundary + strong mixing, the face $\delta(L_0)$), that "strong mixing" is precisely where $U(1)$ breaks Reading (physical estimate).

07

What cannot be said

(a) Neither half of the statement (1) that crosses the band has been shown in four dimensions. (2) is one line of algebra on top of (1).

(b) The comparison of the types of the two walls (§ 03, § 04, readings across the rows of Table 1) is an estimate, and we do not say that results at the first step imply anything in four dimensions. Only the form of the statement lifts (23-03 § 06).

(c) All we claim about the conjugate point is that the convexity threshold of (7) is fixed by the first conjugate point. The reason for the numerical agreement with the mixing length is not known.

(d) The rows of the road of blocks and mixing, apart from the Hidden-Convexity Proposition, are "Assumption" and "Reading". We do not say that the road of blocks and mixing closes the first step.

(e) That a one-dimensional chain has no phase transition is a statement resting on bibliography whose original has not been obtained.

(f) That four dimensions "has no finite block" is not a claim that the road of blocks is impossible in four dimensions. It means that the form of the first-step tools does not carry over as it is.

08

The table of the five grades and the dependency graph

Table 2 The rows of this chapter (the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Basis	Scope
1	(1) $\Longrightarrow$ (2): $m_{\mathrm{lat}}/a \geq \sqrt{\kappa/C}$	Proved here	§ 01 (the same algebra as (16) of 23-01)	$\beta \geq \beta_0$
2	The one-sided bridge $CP \geq \kappa\xi^2$	Proved here	23-01 § 06	Continuous time, finite spatial lattice, the ground-state assumption, the stationary path-measure assumption, the metric comparison assumption
3	Lifting the one-sided bridge to the four-dimensional lattice	Assumption	24-07 Table 2	—
4	The upper-bound assumption: $a^2CP \leq C$, with C independent of β	Assumption	23-01 § 05	Four dimensions, $\beta \geq \beta_0$
5	Showing (1) with the same constant includes crossing the band	Reading	§ 01 (physical estimate)	—
6	The Jacobi equation at the summit (4) and $\cos\theta = 1 - \varepsilon^2/4$	Proved here	§ 02	$V = (z^2-1)^2/8$
7	The first conjugate point $\pi/\theta - 1 = 7.839\ldots$	Proved here	§ 02 (two independent counts)	$\varepsilon = \frac{1}{2}$
8	(7): $\kappa_{C7}(k) > 0 \iff k+1 < \pi/\theta \iff k \leq 7$	Proved here	§ 02, the short-core step of 23-11	$\varepsilon = \frac{1}{2}$, summit configuration
9	The crossover length does not depend on $\hbar$ and is finite in lattice units as $\hbar \to 0$	Proved here	§ 02 (no $\hbar$ in (4)–(7))	$\hbar \to 0$ with ε fixed
10	The numerical closeness of the mixing length ≈ 9 and π/θ (the mechanism is the time to reach the corner)	Reading	§ 02	$\hbar = 10^{-2}$–10^{-5}, four values of ε
11	The worst-configuration tools (Dobrushin, blocks, Otto–Reznikoff) come back to $\frac{15}{4}$ against 4	Proved here	§ 03	$\varepsilon = \frac{1}{2}$
12	The worst-frame estimate gets worse superexponentially in L	Proved here	25-04	With N and the lattice spacing fixed
13	The Hidden-Convexity Proposition (uniqueness of the minimum)	Proved here	§ 03	End values $a, b \in [-c, 0]$, box $[-c, 0]^l$
14	The non-convexity moves, in the coordinate ρ, to a thin layer $\rho \lesssim \sqrt{2\hbar/\varepsilon}$ (a reading)	Assumption	§ 03	—

Row	Statement (one line)	Grade	Basis	Scope
15	$1/\Lambda$ is born through dimensional transmutation and diverges in lattice units	Cited	4-07, 24-07 § 01	One-loop bookkeeping
16	The one-dimensional chain of the core has no phase transition	Assumption	§ 04 (positive integral kernel; the original still to be obtained)	Compact one-site space
17	The two walls are of the same type and differ in content (Table 1)	Reading	§ 03– § 05 (physical estimate)	—
18	The road of blocks and mixing: block LSI + strong mixing $\Longrightarrow$ the core Poincaré uniformity assumption	Assumption	§ 06, the third branch of 23-15	$\varepsilon = \frac{1}{2}$, core $[-x_c, 0]^k$
19	Four dimensions has no finite block longer than the crossover length, and the tools of the first step do not carry over as they are	Reading	§ 04, § 06 (physical estimate)	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  (1) ⟹ m_lat/a ≥ √(κ/C)
 2  PRV  One-Sided Bridge Proposition                    continuous time, finite spatial lattice
 3  ASM  lifting the bridge to four dimensions ⟸ 2
 4  ASM  upper-bound assumption (C independent of β)     four dimensions
 5  RDG  showing (1) with one constant crosses the band ⟸ 1, 3, 4
 6  PRV  Jacobi equation at the summit, cos θ = 1 − ε²/4
 7  PRV  first conjugate point π/θ − 1 ≈ 7.84 ⟸ 6          ε = ½
 8  PRV  convexity threshold k ≤ 7 = conjugate point ⟸ 6, 7
 9  PRV  crossover length independent of ħ, finite ⟸ 6, 7  ħ → 0 with ε fixed
10  RDG  numerical closeness of mixing length ≈ 9 and π/θ
11  PRV  worst-configuration tools come back to 15/4 vs 4 ⟸ 8
12  PRV  worst frame gets superexponentially worse in L (25-04)
13  PRV  Hidden-Convexity Proposition
14  ASM  non-convexity moves to thin layers in ρ ⟸ 13
15  CIT  1/Λ from dimensional transmutation, diverges in lattice units
16  ASM  the one-dimensional chain of the core has no phase transition
17  RDG  same type, different content ⟸ 9, 11, 12, 15, 16
18  ASM  road of blocks and mixing ⟹ core Poincaré uniformity ⟸ 13
19  RDG  no finite block in four dimensions ⟸ 9, 15, 18
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The "Proved here" nodes of this chapter are the one line of algebra, the linear computations at the summit (the Jacobi equation, the first conjugate point, the convexity threshold, the classical length, the worst-configuration tools), and the Hidden-Convexity Proposition; the statements that cross the band (the upper-bound assumption in four dimensions, and showing it with the same constant) are "Assumption" and "Reading". There is no node for the mass gap.

What this chapter can say

What is said	What is not said
Proved here That the lower half of the physical-unit target level follows in one line from the upper-bound assumption and the one-sided bridge with the same pair of constants	The upper-bound assumption and the bridge in four dimensions. The physical-unit target level itself
Proved here That the convexity threshold of the first step (length 7) is fixed by the first conjugate point $\pi/\theta - 1 \approx 7.84$ of the Jacobi equation at the summit and does not depend on $\hbar$. That the worst-configuration tools come back to $\frac{15}{4}$ against 4. The Hidden-Convexity Proposition	The core Poincaré uniformity assumption itself. Why the mixing length is close to the conjugate point
Reading That the two walls are of the same type, and differ in the origin of the crossover length (classical vs quantum), rare vs typical, and whether there is a phase transition (Table 1)	Implications of first-step results for four dimensions. A tool that crosses the band
Assumption That the road of blocks and mixing (block LSI + strong mixing) is a candidate for supporting the core Poincaré uniformity assumption	Proofs of each stage of the road of blocks and mixing (apart from the Hidden-Convexity Proposition)

Sources and reproduction

Item	Kind	Source / tools
The three levels of the goal, the upper-bound assumption, the one-sided bridge	This book	23-15 Table 5, 23-01 § 05, 23-01 § 06

Item	Kind	Source / tools
The Hessian at the summit and $\frac{15}{4}$ against 4, the Short-Core Convexity Lemma	This book	23-09 § 06, 23-11 § 05
The Jacobi equation, conjugate points (4)–(7)	Proved here	This book's study (the section on linearisation at the top in the record of proposals for the wall of Part 23). $\theta = \arccos(15/16)$ and π/θ are numerical, and were checked in this chapter by two independent checks, the sign change of the recursion and diagonalisation of the tridiagonal matrix (a computation of a few seconds)
The values of the three worst-configuration tools	Proved here	The same record (the section on worst-case analysis). $2(M_7^{-1})_{71}$ was recomputed in rationals in this chapter (1.184, against the rounded value 1.19 of the record)
Hidden-Convexity Proposition	Proved here	The same record (the section on the proof of the Hidden-Convexity Proposition). The proof was checked line by line in this chapter. Arguments of the Benguria–Brezis–Lieb and Brezis–Oswald type (originals not checked)
The correlation length of the transfer operator, the mixing length, the optimal path, $\lambda_*(12)$	Reading	The same record (the numerical section), and the arranged version of the road to a uniform LSI
The standard theorems of the road of blocks and mixing	Assumption (candidate for Cited)	The LSI for one-dimensional chains of B. Zegarlinski and F. Martinelli–E. Olivieri, Witten's Morse theory of B. Helffer–F. Nier and D. Le Peutrec, the decomposition of G. Menz–A. Schlichting (none of the originals checked yet)
Dimensional transmutation, the dimensionless coupling, the three sources of difficulty	Cited	The source tables of 4-07 and 24-07
Each cell on the four-dimensional side	This book	25-04, 26-04, 26-05, 26-06

Terms used in this chapter — the crossover band, the diffusive-relaxation-time assumption (the upper-bound assumption), the one-sided bridge, the Jacobi equation, conjugate point, hidden convexity, block LSI, strong mixing, the road of blocks and mixing, the criterion "distinguish from the U(1) transition". Their meanings are in A-02 Glossary.

Next chapter: 26-08 A map of the roads tried and stopped — a list of the roads tried and stopped, and why they stopped. The two points that work in four dimensions are in 24-07 Because it is four dimensions.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band
the ledger of exclusions, why they stopped, the selection criteria · Kiichi, Shiori and Rin

A map of the roads tried and stopped

> This chapter is a map of open questions. § 03 is established knowledge; § 04 is this book's own investigation.

What this chapter shows — We gather into one ledger the roads that were tried in order to cross the crossover band, and that stopped. The ledger has three layers: thirty roads (§ 02); roads discarded, or limited in scope, in the study of the band chapters (§ 05); and three ideas that turned out to be no more than restatements of roads in the ledger (§ 06). Each row carries "why it stopped" and its origin, and the label of the reason is set per row. For rows that rest on literature, only literature checked against the original carries [Cited]; literature not checked is marked "not yet checked". Three roads are looked at in detail: that the average over local containers stays at $R/\ln R$ in four dimensions (§ 03); the claim of Tomboulis (2007) by Migdal–Kadanoff interpolation, and the criticism that it does not distinguish U(1) (§ 03); and that the comparison principle with the twisted T^2 fails numerically (§ 04). Finally we set out the selection criteria that a new candidate should pass first (§ 07).

Prerequisite chapters — 26-00 (the breakdown of the band), 26-01 (the correct form of the band), 26-02 (the lateral-widening inequality), 25-03 (containers), 25-04 (slab confinement), 26-06 (the renormalisation comparison), 23-10 § 04 (the seven types of breakage).

Said first — This page is a map of the record and sets out no new theorem. "Stopped" means "it was found that this set of inputs cannot cross the band"; it does not mean that the parts of that road are of no use elsewhere (slab confinement, for example, is a theorem that holds for every β once the lattice spacing and T^2 are fixed). Most of the reasons for stopping are judgements at the level of physical estimates; negations with proofs are few. The labels are set per row to keep that distinction.

The order of this chapter

1. How to read the ledger — what "stopped" means and how labels are set
2. Thirty roads — why they stopped, and their origins (Table 1)
3. Two roads that had stopped in the literature — the average over local containers, and Migdal–Kadanoff interpolation

4. The comparison principle with the twisted T^2 — true at one loop, false numerically
5. Roads discarded, or limited in scope, in the band chapters (Table 2)
6. Three that were only restatements (Table 3), and the questions of type (Table 4)
7. The selection criteria and calibration against the literature (Table 6)
8. The table of the five grades and the dependency graph

01

How to read the ledger — what "stopped" means and how labels are set

The study in this book tried many roads in search of a tool that crosses the band. Every road stops in the end at the same single point: **there is no non-perturbative principle showing that, at intermediate coupling, disorder becomes stronger as the scale is enlarged**. Setting the ways of stopping side by side, three lessons come up again and again.

- Tools that work only in a perturbative regime (weak or strong coupling) cannot cross the band.
- Local operations cannot change topological quantities such as the twist sector.
- The sources of monotonicity (correlation inequalities, reflection positivity) work in the opposite direction to the mechanism of confinement (disorder).

The purpose of the ledger is not to repeat the same road. When a new candidate is thought of, we first check whether it is a restatement of something in this ledger (§ 06 gives examples).

The columns of each row are as follows.

- **Road**: a one-line summary of the idea tried.
- **Why it stopped**: where the road became unable to cross the band.
- **Label**: the label of the reason for stopping. Proved here means the reason has a proof on paper (set out in a chapter of this book); Cited means it rests on literature checked against the original; Computer-assisted is computer assistance; Reading is a numerical reading; "Reading (physics)" is a judgement at the level of a physical estimate. Even when the reason is an estimate, the judgement not to re-examine the road does not change. But a road stopped for a reason that is an estimate can be reopened if there is a new input.

- **Origin**: the record of this book's study, or the literature. Literature checked against the original is marked [Cited]; literature not checked is marked "not yet checked".

A note on words. In this part, following the convention of 26-00 § 02, we write "band", and distinguish it from the "window" of Parts 23 and 24 (the site window, the ħ window, the time window).

02

Thirty roads — why they stopped, and their origins

Numbers 1–28 in Table 1 are roads that this book's study tried in search of a principle that crosses the band; 29 and 30 are roads found to have already stopped in the literature when that study was re-checked by another hand. The labels are labels of "why it stopped".

Table 1 Thirty roads (labels are labels of why it stopped; "this book's study" is the record of the study in this book)

#	Road	Why it stopped	Label	Origin
1	Reflection positivity, chessboard estimates	The inequality points the wrong way, and only one side (the lower bound) of the flux free energy comes out	Reading (physics)	This book's study. The chessboard itself is in 26-05
2	Continuous monotonicity in the box size L (the Feynman–Hellmann derivative)	There is no correlation inequality for $SU(2)$ that fixes the sign of the difference	Reading (physics)	This book's study
3	Checking finitely many L by interval arithmetic	In four dimensions the size of the computation is out of reach. In the hierarchical model an interval-arithmetic certificate could be made (551 sectors, passed on re-running all sectors; a result for the hierarchical model only)	Reading (physics: a judgement of scale), Computer-assisted (hierarchical model only)	This book's study
4	Projection to $\mathbb{Z}_2$ and GKS (the splitting $SO(3) \times \mathbb{Z}_2$)	The premise of the splitting was rejected by four-dimensional data	Reading	This book's study
5	The infrared bound (Fröhlich–Simon–Spencer)	The wrong direction. It is the inequality on the side that shows a massless phase	Reading (physics)	Fröhlich–Simon–Spencer (not yet checked)

#	Road	Why it stopped	Label	Origin
6	Proof by contradiction extending the result for a small T^2 continuously in L	There is no quantity that excludes a fixed point, and it comes back to the band itself	Reading (physics)	This book's study
7	Kramers–Wannier duality	It only restates the band as a one-variable inequality (the magnetic flux free energy is smaller at scale $2L$ than at L)	Reading (physics)	This book's study (the duality itself not yet checked)
8	Local entropy and finite-energy arguments	Topological conservation: a local closed surface in $\mathbb{R}^4$ meets a plane an even number of times, so local operations do not change the twist sector	Reading (physics)	This book's study
9	Slab confinement	With the lattice spacing and T^2 fixed, the area law holds for every β. A bound uniform in N is the band itself, and the tension goes to 0 in the continuum limit	Proved here (the theorem with N fixed)	25-04
10	GKS for the $\mathbb{Z}_2$ effective model of centre vortices	Removing the frustration also removes the mechanism of confinement	Reading (physics)	This book's study
11	The Nishimori structure	**Withdrawn** (annealed and quenched were confused; negated numerically)	—	This book's study
12	The data-processing inequality	No channel independent of the sector can be built. It becomes a detector that tells sectors apart, but not a driving force that produces disorder	Reading (physics)	This book's study
13	Group averaging over thick vortices	It works on a small T^2, but on $\mathbb{R}^4$ it comes back to the problem of the tension of the vortex sheet	Reading (physics)	This book's study
14	Renormalising the worst flip cost	It becomes monotonically non-increasing, but decrease is not guaranteed	Reading (physics)	This book's study
15	Percolation of vortex sheets	Stochastic domination needs a worst-case lower bound, and reduces to strong coupling	Reading (physics)	This book's study

#	Road	Why it stopped	Label	Origin
16	Stability of the gap, Lieb–Robinson bounds	Works only inside the perturbative radius, and reduces to strong coupling	Reading (physics)	This book's study
17	Convexity in the coupling of the sheet	Only the cost of thin sheets is visible	Reading (physics)	This book's study
18	Adaptive averaging over the shape of the sheet	A rewriting of a partition of unity; a tautology	Reading (physics)	This book's study
19	Volume independence at large N_c	The band comes back, changed into the effective volume $N_c L$	Reading (physics)	This book's study
20	Positivity of the coefficients of the character expansion	It follows automatically from reflection positivity, and carries no new information	Reading (physics)	This book's study
21	Extracting monotonicity from the hierarchical model (ratio ordering, TP_2 and the like)	False in general	Reading	This book's study
22	Single-site Bakry–Émery curvature	The single-site gap is always positive; the wall lies in the influence between plaquettes	Reading (physics)	This book's study
23	Monotone quantities from strong subadditivity, a lattice version of the a-theorem	The lattice-only version does not distinguish phases. The version that does needs rotation invariance, which is available only after crossing the band (circular)	Reading (physics)	This book's study (Casini–Huerta-type arguments not yet checked)
24	Curvature of the space of gauge orbits, renormalisation of a membrane gas, tools for sharp phase transitions	The curvature correction is $O(1)$ and cannot cancel the negative $O(\beta)$ of the Hessian of the action, reducing to strong coupling / four-dimensional membranes have no screening mechanism / the surface representation of $SU(2)$ has weights that change sign, and no FKG	Reading (physics)	This book's study

#	Road	Why it stopped	Label	Origin
25	Spin and the index theorem (counting soft modes)	At one loop it is a restatement of asymptotic freedom, and the non-perturbative part comes back to the band	Reading (physics)	This book's study
26	Renormalisation of curvature (Bakry–Émery curvature and O'Neill's formula, the curvature-dimension condition of the quotient)	Negative curvature is not preserved by coarse-graining. Following the effective one-link weight on a one-dimensional chain by convolution, the minimum of the curvature gets worse at the first step ($SU(2)$: $-6 \to -14$; $U(1)$: $-4 \to -8$)	Reading	This book's study
27	The comparison principle with the twisted T^2 (for all loops)	True at one loop, false non-perturbatively by numerics (§ 04)	Reading	This book's study (§ 04)
28	The multiscale Bakry–Émery criterion (the Polchinski equation)	It can pass non-convex regions, but scale-by-scale control of the renormalised potential is the band itself. The version for compact groups is also not in place	Reading (physics)	Bauerschmidt–Bodineau (not yet checked)
29	Averaging over local vortex containers	The identity is known, and with the container length fixed it reaches only $R/\ln R$ in four dimensions (§ 03)	Cited	Mack–Petkova (1980), Kovács–Tomboulis hep-lat/0108017. 25-03 § 06
30	Monotonicity of the vortex free energy by Migdal–Kadanoff interpolation	The key inequality ((3.5) of Ito–Seiler; the interpolation-parameter commonality inequality) has not been shown outside strong coupling. The argument does not distinguish $U(1)$ (§ 03)	Cited	Tomboulis arXiv:0707.2179, Ito–Seiler arXiv:0711.4930, Kanazawa arXiv:0805.2742

N_c in Table 1 is the number of colours, a different quantity from the size N of T^2_N. Number 11 is kept here as a single line only. Its content was withdrawn, so it is not used elsewhere in this book.

Note that most labels in Table 1 are "Reading (physics)". Most of the reasons for stopping are not counterexamples or negative theorems but estimates of the form "this road comes back to the band itself at this point". The rows with proofs are number 9 (slab confinement is a theorem with N fixed, and is not uniform in N) and, in Table 2,

the stopped road "isometric comparison", the stopped road "structural inputs only", the stopped road "flip certificate" and the stopped road "supremum estimate over frames" (the scope is in the brackets of each row's label). Numbers 29 and 30 of Table 1 are the "Cited" rows that had stopped in the literature. The "Computer-assisted" of number 3 is a result on the hierarchical model and says nothing about the four-dimensional lattice (the hierarchical model is a different thing from four dimensions in that Migdal–Kadanoff decimation is exact there).

03

Two roads that had stopped in the literature — the average over local containers, and Migdal–Kadanoff interpolation

This section summarises existing knowledge (two roads that had stopped in the literature). The introduction is in 17-02 § 05.

Numbers 29 and 30 in Table 1 are roads that, after this book's study had tried them itself, were found to have already stopped in the literature. Both were checked against the text of the original [Cited].

The average over local vortex containers — only up to R/ln R in four dimensions

The identity that, with the frame (the outside of the container) fixed, the signs of the sectors inside the containers become independent and the Wilson loop is bounded by the product of the biases of the containers, was the Container Identity of 25-03 § 03. This is the "vortex container" identity of Mack–Petkova (1980), written in eqs. (18)–(22) of Kovács–Tomboulis (hep-lat/0108017). The only differences are that we wrote the sectors with the sum ensemble and took the average over the frame rather than the maximum; this book claims no novelty.

§ 3 of the same paper states the limit of this road in four dimensions. The length of a container passing through a point inside a rectangular loop with sides $T \times R$ ($T > R$) is of order R^2, and to keep the cost of the container constant the width of its cross-section must be fattened to order $\sqrt{\ln R}$. Only about $RT/\ln R$ independent containers fit inside the loop, and what comes out is

$$V(R) \;\geq\; \text{const} \cdot \frac{R}{\ln R} \qquad \text{(Kovács–Tomboulis, eq. (25))} \quad (1)$$

and no more [Cited]. A linear potential (the area law) needs a single fat container the size of the loop. This is exactly the global form of Tomboulis–Yaffe (26-01 § 04), the same condition as crossing the band.

In one line, the reason it stopped is: **the road of laying out containers of fixed length independently does not reach the area law on** $\mathbb{R}^4$. On a slab with a small T^2, the

length of the container (the area of the vortex sheet) stops at the area of T^2, so this obstacle does not appear (25-03 § 06). The obstacle comes back when T^2 is widened — that is, in the band.

Monotonicity of the vortex free energy by Migdal–Kadanoff interpolation

Tomboulis (arXiv:0707.2179) claimed confinement at every coupling for four-dimensional $SU(2)$. The flow of the argument is as follows [Cited].

1. By Migdal–Kadanoff-type decimation (lattice spacing $a \to ba$), build upper and lower bounds on the partition function ((3.4) and (3.7) of the original).
2. Interpolate their coefficients with a parameter t, and obtain a representation in which decimation becomes an equality at some value t^* ((3.12), (3.16), (3.35)). Build the same representation in the twisted sector ((4.19), (4.26)).
3. **The key step**: that the same t^* can be taken in the untwisted and the twisted sectors ((5.14)). This is to be derived from the inequality $A \geq A^+$ ((5.15)), and it is stated that showing it at strong coupling is enough.
4. In conclusion, the true vortex free energy is at most the vortex free energy of the decimated system ((5.22)). The decimated system flows to strong coupling, so the strong-coupling expansion gives $F \sim L\, e^{-\hat{\sigma} A}$, and the Tomboulis–Yaffe inequality gives the area law.

Against this there are two criticisms [Cited].

- **Ito–Seiler (arXiv:0711.4930)**: that t^* can be taken in common for the two sectors reduces to their inequality (3.5), $A(\alpha) > A^+(\alpha)$, which has not been shown outside strong coupling. Further, the Migdal–Kadanoff flow goes to strong coupling **regardless of whether the group is abelian or non-abelian**, so if the argument went through as it stands, $U(1)$ would also confine at every coupling. Four-dimensional $U(1)$ has a non-confining phase, so an input that distinguishes groups is needed somewhere.
- **Kanazawa (arXiv:0805.2742, Appendix A)**: the continuity argument in the interpolation parameter does not hold (a counterexample is given). It is not shown which step of the argument fails for $U(1)$. For any compact connected group the Migdal–Kadanoff flow goes to strong coupling, so the renormalisation side cannot distinguish $U(1)$ from $SU(2)$, and **all the non-abelian content goes into the existence of t^***.

Footnote 10 of the original itself writes that "for abelian systems, inequalities of type (5.15) follow from the Griffiths inequalities". Together with the remaining steps this

would show confinement for $U(1)$, a sign that the argument does not distinguish groups.

In one line, the reason it stopped is: **a renormalisation that does not distinguish groups cannot be an argument that closes the band**. This is an example in the literature of the criterion "distinguish from the U(1) transition" of 26-06 § 08 (show that $SU(2)$ has no transition, distinguishing it from the transition that $U(1)$ has inside the band, at $\beta \approx 1.01$). The lateral-widening inequality of this book (26-02 § 02) restricts the comparison to the form "change only the geometry, within one theory and one coupling", and has a form that fails for $U(1)$ (26-01 § 03, 26-09 § 02). But the difficulty of showing it is of the same type, and it is still needed across the band.

Relatedly, Theorem 1 of Kanazawa (arXiv:0805.2742) is an identity relating the ratio of twisted partition functions on lattices of different sizes to a ratio at shifted couplings, but it also holds for $U(1)$, so it says nothing about confinement [Cited]. Another paper of Ito–Seiler (arXiv:0803.3019) is not yet checked.

04

The comparison principle with the twisted T² — true at one loop, false numerically

This is number 27 of Table 1. **The comparison principle** is the claim that for every planar loop C

$$\langle W(C)\rangle_{\mathbb{Z}^4} \;\le\; \langle W(C)\rangle_{\mathbb{Z}^2\times T^2_N,\; n_{12}=1} \tag{2}$$

holds (the right-hand side is the twisted slab, 25-04 § 02). If it held, then together with slab confinement (the area law for every β with N fixed) it would give the area law for every β on $\mathbb{Z}^4$. That is why it was worth trying.

True at one loop [Reading]. Comparing with the heat kernel of fluctuations around the background, the inequality holds at each time. The colour weight of the sum over images is $c(n) = 4\cdot\mathbf{1}[n\in(2\mathbb{Z})^2] - 1$, and we checked this numerically to machine precision (a numerical check at the one-loop level; no proof on paper is set out).

False non-perturbatively [Reading]. By Monte Carlo we compared the twisted slab on the lattice $16\times16\times N_T\times N_T$ (the $\mathbb{R}^2$ directions periodic with 16, T^2 with N_T) with 16^4 ($SU(2)$, Wilson action).

Test of the comparison principle (plaquette expectation values; parentheses give the statistical error in the last digit)

β	N_T	**Twisted slab**	16^4	**Reading**

β	N_T	Twisted slab	16^4	Reading
2.1	2	0.52952(67)	0.53494(40)	The slab is lower: 7σ taking errors from both sides, 5.5σ in an independent rerun. $W(2,2)$ is also lower
1.8	2	0.43966(56)	0.44129(18)	The slab is lower (about 2.7σ)
2.4	2	0.6275(10)	0.6300(2)	The slab is lower (about 2.4σ)
2.6	6, 8	Signs that the twisted slab overshoots 16^4 downwards (2.5σ and 1.9σ in the plaquette)		Undetermined

Taking C in (2) to be a single plaquette, the right-hand side is the plaquette of the slab and the left-hand side the plaquette of 16^4. At $\beta = 2.1$, $N_T = 2$ the slab is 7σ lower, so (2) fails for small loops. This can be read as the frustration of the twist (the cost of forcing a twist into a small T^2) lowering the plaquette [Reading] (physics). The overshoot on the weak-coupling side ($\beta = 2.6$) lacks statistics and is not settled.

Further, at $N = 1$ it contradicts the scaling behaviour of asymptotic freedom [Reading] (physics). The slab theorem gives $\sigma_{\mathrm{tw}}(1) \geq -\ln\tanh\beta \approx 2e^{-2\beta}$, but if the scaling behaviour of asymptotic freedom is accepted, $\sigma_\infty a^2 \propto e^{-(6\pi^2/11)\beta} = e^{-5.38\beta}$, and at large β (2) points the other way.

In one line, the reason it stopped is: **an ordering that holds at one loop breaks down non-perturbatively, starting from small loops**. Two criteria come out of this (§ 07): the criterion "not a rewording of one loop" (it is a non-perturbative estimate) and the criterion "not broken by small loops" (it does not fail at short scales, or it has a form that claims only large loops). The asymptotic form for large loops only ($\sigma_{\mathrm{tw}}(L) \leq \sigma_\infty$) has not been refuted, but there is no mechanism to show it, and it hits the same wall as number 2 of Table 1 (there is no inequality for $SU(2)$ that fixes the sign of the difference).

One positive observation also remains [Reading] (physics). At weak coupling the twisted T^2 is close to infinite volume already from small N_T, and approaches it from the opposite side to the untwisted case. This fits the picture of adiabatic continuity, but there are signs that the approach is not monotone, a fact unfavourable to tools based on monotonicity.

05

Roads discarded, or limited in scope, in the band chapters

In the study of Parts 25 and 26 there are, apart from Table 1, roads that were discarded and roads whose usable scope was limited. Table 2 gathers them. The first five (1–5) are "roads discarded", the rest "roads limited in scope".

Table 2 Roads discarded, or limited in scope, in the band chapters (labels are labels of why it stopped)

#	Road	Why it stopped	Label	Chapter
1	**Isometric comparison**: isometric comparisons (the identity map, group translation, gauge transformation, decimation)	In a comparison that does not change the shape of the fluctuations, the contribution of the one-loop spin term $+4$ is exactly 0, and only $\beta' \geq \beta + 0.023$ (the anti-asymptotic-freedom direction) comes out	Proved here (one-loop level)	25-07, 26-06 § 03
2	**Structural inputs only**: structural inputs only (reflection positivity, duality, the transfer matrix, the centre identities)	They hold as they are for $\mathbb{Z}_2$ and $U(1)$ too, so they cannot distinguish groups (the negative lemma)	Proved here (in the form "from this set of inputs alone")	26-04
3	**Deterministic squeeze**: getting the area law by a squeeze between deterministic maps	The remainder (relative entropy) of the Transport Identity (the Gibbs variational principle) grows with the area of the vortex sheet, so the form $e^{-\sigma A}$ does not come out. It can be used for the qualitative goal $\Delta F(L, L) \to 0$	Reading (physics)	26-10 § 02
4	**Glassy squeeze**: a squeeze by the $\mathbb{Z}_2$ glass in the upper half of the band	For $\beta \gtrsim 2$ the mean effective $\mathbb{Z}_2$ coupling is 1.1–1.9, far above the pure $\mathbb{Z}_2$ transition point 0.44, and the glass is on the side of the ordered phase. The squeeze is loose	Reading	26-09 § 08, 26-10 § 03
5	**Flip certificate**: using flip certificates (flipping containers) and simultaneous flips to test the band	The flip probability a of a one-step kernel levels off at about 1/2, and $\theta \leq 1 - a$ gives only $\sigma \geq c/L^2$ (a weak form that $U(1)$ also satisfies)	Proved here ($\theta \leq 1 - a$), Reading (the levelling off of a)	25-06, 26-01 § 02
6	**Supremum estimate over frames**: bounds using the supremum over frames $\sup_\Phi$	Of the form $\tanh(\beta L^2)^{\ell^2}$, getting worse superexponentially in L. For uniformity only forms averaged over the frame can be used	Proved here	25-04 § 02
7	**Dobrushin-type criterion**: using the usual Dobrushin, Dobrushin–Shlosman and van den Berg–Maes criteria with frames	The continuous field follows the boundary, and the influence coefficients are close to 1	Reading (physics)	(literature not yet checked)

#	Road	Why it stopped	Label	Chapter
8	**β-monotonicity from easy parts**: taking β-monotonicity as the easy component, band = β-monotonicity + the renormalisation comparison	β-monotonicity turned out to be a problem of the same weight as the monotonicity in β of the string tension in lattice units (a correction)	Reading (physics)	26-06 § 02
9	**One-directional lateral widening**: restricting lateral widening to one direction only	Doubtful in view of the behaviour of the spatial string tension at high temperature (lattice measurements by Bali et al., not yet checked); restricted to the square shape only	Reading (physics)	26-02 § 02
10	**Transport by Wilson flow**: using the Wilson flow to transport vortex insertions	Negative numerically. The form recast as forward transport (Moser-type) is open	Reading	25-06
11	**Conformal frame**: the conformal frame $H^2 \times S^2$ (turning scale into position)	Negative on examination: hyperbolic nesting does not increase the number of containers, and the core stays in the same place	Reading (physics)	26-10 § 04
12	**Strong-coupling expansion only**: covering the band with the strong-coupling expansion alone	The expansion about flat surfaces stops at roughening ($\beta_R \approx 1.9$, not yet checked)	Reading (physics)	26-06 § 04

06

Three that were only restatements, and the questions of type

When thinking of a new candidate, we first check that it is not a restatement of something in the ledger. Table 3 lists three ideas that, when the band candidates were examined, were not made candidates but judged to be restatements of the ledger.

Table 3 Three ideas that were only restatements

#	Idea	What it was a restatement of	Label
1	**Approach of the holonomy to Haar**: that the law of the holonomy of large loops approaches the Haar measure (the boundary holonomy centralises under the Weyl measure)	The decay of the part odd under the centre (half-integer representations) is the Wilson area law itself, a restatement of the ring made by the column decomposition and the Tomboulis–Yaffe inequality. The even part (adjoint representations) has a perimeter law and does not help to distinguish groups	Reading (physics)

#	Idea	What it was a restatement of	Label
2	**Anomaly of the centre and the θ term**: consistency of the mixed anomaly between the centre and the θ term	A kinematic constraint (the criterion "dynamical" of § 07). At $\theta = 0$ it forces nothing	Reading (physics)
3	**Success in the hierarchical model**: taking success in the hierarchical model (a model in which Migdal–Kadanoff decimation is exact) as evidence for the band	In the hierarchical model $U(N)$ also confines, so it is no test of the criterion "uses non-commutativity essentially"	Reading (physics)

The study of detours in Part 23 (23-10 § 04) extracted seven types from failures on the double well in 0+1 dimensions. The individual detour plans do not overlap directly with four-dimensional candidates, but the questions of type apply to four-dimensional candidates as they are. Table 4 lists those questions.

Table 4 The seven types of breakage, and the questions to put to a new candidate

Type	One line	Question to put to the candidate
1	There is one source (the curvature at one site at the top)	On the side of the worst configuration, what is being paid?
2	Structural factors that enter as ratios have a ceiling	Does it change the defect set, or does it only move the factor of the ratio?
3	Paying for the same place twice	Have the places paid been drawn in the same figure?
4	Confusing currencies (the speed of exit and rarity)	Does the step that pays for rarity go through entropy?
5	The address of the difficulty	Which quantity does the difficulty rest on?
6	The mismatch between the direction of fluctuation and the soft direction	Is a vector being collapsed to a scalar?
7	Transfer	Is the sum an invariant (does another column become heavier by as much as the lightened column)?

The candidate rules of Part 24 (Table 3 of 24-08 § 02) can also be used here, not as exclusions but as a record of "which root they do not work on". No candidate has "works" written in the column of the root that connects the ultraviolet and the infrared (source: no expansion linking scales, of 24-07), and the "coupling window" (a word of Part 24, 24-06) that appears in the rarity class closed by one step of coarse-graining (rule of axiomatising scale + rule of the rarity class) was a rewording of that root. Anyone who says a band candidate works on the lack of an expansion linking scales must name an input different from every candidate.

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07
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The selection criteria and calibration against the literature

Turning the reasons for stopping inside out gives candidate conditions that a tool crossing the band should satisfy. Table 6 sets those conditions side by side with the ways of stopping they came from. All are candidates for necessary conditions, not sufficient conditions that guarantee crossing the band if satisfied [Reading] (physics). Especially heavy are the four "uses non-commutativity essentially", "not a rewording of one loop", "crosses the smallest band" and "distinguish from the U(1) transition".

Table 6 The selection criteria (with the ways of stopping they came from)

Criterion	Content	Way of stopping it came from
Stands on the lattice alone	Holds on the lattice alone (does not rely on rotation invariance)	Number 23 of Table 1
Distinguishes phases	Distinguishes phases (fails in a massless phase, or the limiting values differ)	Number 23 of Table 1
Does not rely on positivity	Does not rely on positivity or FKG	Number 24 of Table 1 (the surface representation of $SU(2)$ has weights that change sign)
Uses non-commutativity essentially	**Uses non-commutativity essentially**. A principle that gives the same conclusion in models with a massless phase, such as four-dimensional $U(1)$ or two-dimensional $O(2)$, is wrong	Number 30 of Table 1, "structural inputs only" of Table 2, "success in the hierarchical model" of Table 3
Compatible with asymptotic freedom	Compatible with asymptotic freedom	The contradiction at $N = 1$ in § 04
Handles topological sectors	Can handle topological sectors (twists)	Number 8 of Table 1
Not a rewording of one loop	Is **a non-perturbative estimate, not a rewording of one loop**	Numbers 25 and 27 of Table 1
Dynamical	Is dynamical (uses the action and the flow of the coupling). Kinematic constraints alone (duality, reflection positivity, anomalies, positivity of the partition function) cannot reach, since a non-confining phase can also exist without contradiction	"Structural inputs only" of Table 2, "anomaly of the centre and the θ term" of Table 3
Passes through non-convex regions	Can pass through non-convex (negatively curved) regions. Tools based only on convexity, log-concavity or lower bounds on curvature cannot be used in the band	Number 26 of Table 1

Criterion	Content	Way of stopping it came from
Crosses the smallest band	**Can cross the smallest band**: in two-dimensional asymptotically free models (the $O(N)$ model at large N, the Gross–Neveu model), with the ultraviolet cutoff removed. A monotone quantity that cannot cross this cannot cross four dimensions either	Calibration against the literature (below)
Not broken by small loops	Does not fail for small loops (short scales). Or has a form that claims only large loops	§ 04
Non-isometric linearisation	The linearisation of the comparison map is not isometric in the fluctuations	"Isometric comparison" of Table 2
Distinguish from the U(1) transition	**Shows that $SU(2)$ has no transition, distinguishing it from the transition that $U(1)$ has inside the band** ($\beta \approx 1.01$)	Number 30 of Table 1, 26-06 § 08

Behind the criterion "crosses the smallest band" is the fact that even for the two-dimensional $O(3)$ model, exponential decay at all temperatures at the lattice level (Polyakov's conjecture) is unsolved (as stated in the lecture notes of Peled–Spinka, arXiv:1708.00058; not yet checked). A tool that shows a uniform gap in four dimensions must at least have the power to show that.

Calibration against the literature. Where is the current frontier on each side of the band? Only literature whose checking is complete is marked [Cited].

- The strong-coupling side: Cao–Nissim–Sheffield (arXiv:2505.16585) showed, for $U(N)$ lattice Yang–Mills, the area law for all N at β below a $\beta_0(d)$ depending only on the dimension d [Cited]. The same authors, in arXiv:2509.04688, check the mass-gap condition of Durhuus–Fröhlich (1980) by methods of stochastic dynamics and obtain the area law in the 't Hooft regime ($\beta \lesssim c_d N$) [Cited]. Both are results on the strong-coupling side and do not reach β inside the band. This book's Lean also contains an area law only inside the strong-coupling window, and a vortex floor whose rate per unit area does not depend on the transverse size (20-06; the lattice spacing is fixed, and the prefactor remains). This does not reach the band either.
- The universality of the band: there is a paper that states explicitly that, for every version of the Gross–Neveu model, there is no proof of chiral symmetry breaking in the limit with the ultraviolet cutoff removed (Fabbri–Goller arXiv:2606.13075, not yet checked). The infrared analysis of two-dimensional models at large N (Kopper–Magnen–Rivasseau 1995, Kopper 1999) is with an ultraviolet cutoff (not yet checked).
- Computer-assisted renormalisation: a result showing with computer assistance the flow near the high-temperature fixed point in two-dimensional tensor renormalisation (Ebel–Kennedy–Rychkov arXiv:2506.03247, not yet checked) can be

seen as one of the current frontiers. It is far from the band of four-dimensional $SU(2)$.

In summary, **within what we have examined, there is no proof of mass generation in an asymptotically free theory in the limit with the ultraviolet cutoff removed**. The existence of the continuum limit of four-dimensional Yang–Mills was shown by the construction published on 2026-11-10 (at the level of the committee's assessment; Volume 2), but it does not show the mass gap. The band is not a hole peculiar to this book's road, but the unsolved core common to asymptotically free theories. This sentence, however, is a statement about what we have examined, and includes literature not yet checked.

A table that rearranges the reasons for stopping by type of argument is Table 2 of 16-11.

08

The table of the five grades and the dependency graph

The labels of each row of Tables 1–3 are placed in the tables. The rows this page sets out itself are the following fourteen.

Table 7 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The Container Identity with the frame fixed is the container identity of Mack–Petkova	Cited	Kovács–Tomboulis eqs. (18)–(22)	Differing only in the sum ensemble and the averaged form	—
2	Independent containers of fixed length give only up to $V(R) \gtrsim R/\ln R$ in four dimensions	Cited	Kovács–Tomboulis eqs. (23)–(25)	$\mathbb{R}^4$, independent containers	—
3	The key step of Migdal–Kadanoff interpolation reduces to the interpolation-parameter commonality inequality and has not been shown outside strong coupling	Cited	Ito–Seiler arXiv:0711.4930	The argument of Tomboulis arXiv:0707.2179	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
4	The Migdal–Kadanoff flow goes to strong coupling regardless of the group, and the renormalisation side does not distinguish $U(1)$ from $SU(2)$	Cited	Kanazawa arXiv:0805.2742, Appendix A	Any compact connected group	—
5	The comparison principle holds at one loop	Reading	§ 04 (the heat-kernel inequality at each time, numerically)	One-loop level	—
6	The comparison principle fails at $\beta = 2.1$, $N_T = 2$	Reading	The table in § 04	$16^2 \times 2^2$ against 16^4, 7σ (independent rerun 5.5σ)	—
7	The comparison principle at $N = 1$ contradicts the scaling behaviour of asymptotic freedom	Reading (physics)	§ 04	As long as the scaling behaviour of asymptotic freedom is accepted	—
8	Slab confinement holds for every β with N and the lattice spacing fixed, and is not uniform in N	Proved here	25-04	N fixed, lattice spacing fixed	The Lean column of 25-04
9	An isometric comparison gives only the direction $\beta' \geq \beta + 0.023$	Proved here	25-07	One-loop level	—
10	Structural inputs alone cannot distinguish groups	Proved here	26-04	"From this set of inputs alone"	—
11	The reasons for stopping of the remaining rows of Tables 1, 2 and 3	Reading (physics)	This book's study	The conditions of each row	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
12	In the hierarchical model a certificate can be made by interval arithmetic over finitely many sectors	Computer-assisted	Number 3 of Table 1	Hierarchical model only. Says nothing about the four-dimensional lattice	—
13	The frontier on the strong-coupling side is the area law at small β for all N	Cited	Cao–Nissim–Sheffield arXiv:2505.16585, 2509.04688	$\beta \le \beta_0(d)$, $\beta \lesssim c\,dN$	—
14	The thirteen selection criteria (Table 6) are candidates for necessary conditions on a tool that crosses the band	Reading (physics)	§ 07	Candidates for necessary conditions, not sufficient conditions	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  the Container Identity is known                          Kovács–Tombouli
 2  CIT  independent containers reach only R/ln R ⟸ 1              R⁴
 3  CIT  the key step of Tomboulis reduces to (3.5), unproved     outside strong
 4  CIT  the Migdal–Kadanoff flow does not distinguish groups     any compact con
 5  RDG  the comparison principle holds at one loop               one-loop level
 6  RDG  the comparison principle fails at β = 2.1, N_T = 2       7σ, rerun 5.5σ
 7  RDG  comparison principle at N = 1 contradicts asymptotic freedom   physic
 8  PRV  slab confinement (N fixed)                               25-04
 9  PRV  isometric comparison gives β' ≥ β + 0.023                25-07, one loop
10  PRV  structure alone does not distinguish groups              26-04
11  RDG  why the remaining rows stopped                           physical estima
12  CMP  certificate for the hierarchical model                   hierarchical mo
13  CIT  the frontier on the strong-coupling side                 small β
14  RDG  the selection criteria ⟸ 2, 4, 5, 6, 7, 9, 10, 11
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The row of the selection criteria is the only node on the conclusion side, and its label is "Reading (physics)". No row of this page shows either that the band can be crossed or that it cannot. A stopped road is only recorded as "cannot cross with this set of inputs"; a road that adds something to those inputs lies outside the ledger.

What this chapter can say

What is said	What is not said
Cited The average over local containers reaches only $R/\ln R$ in four dimensions. The argument by Migdal–Kadanoff interpolation has an unproved key step and does not distinguish groups	That the claim of Tomboulis (2007) is false (what can be said is that an unproved step remains)
Reading That the comparison principle fails for small loops ($\beta = 2.1$, $N_T = 2$, 7σ, rerun 5.5σ)	That the asymptotic form for large loops only is false
Proved here Slab confinement, the isometric comparison, the negative lemma for structure alone (under the conditions of each chapter)	That roads adding something to these cannot cross
Reading (physics) The reasons for stopping of the remaining roads, and the selection criteria	Proofs of most of the reasons for stopping. That a tool meeting the criteria crosses the band

Sources and reproduction

Item	Kind	Source / tools
The vortex container identity and its four-dimensional limit	Cited	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117. T. G. Kovács, E. T. Tomboulis, hep-lat/0108017 (Vortex waistlines and long range fluctuations), eqs. (18)–(25). Chapter 25-03
The claim by Migdal–Kadanoff interpolation	Cited	E. T. Tomboulis, arXiv:0707.2179 ((3.4), (3.7), (3.12), (3.16), (3.35), (4.19), (4.26), (5.14), (5.15), (5.22) and footnote 10 of the original)
Its criticism	Cited	K. R. Ito, E. Seiler, arXiv:0711.4930 ((3.5)). T. Kanazawa, arXiv:0805.2742 (Appendix A, Theorem 1). Ito–Seiler arXiv:0803.3019 not yet checked
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. T. Kanazawa, arXiv:0808.3442. Chapter 26-01
The frontier on the strong-coupling side	Cited	S. Cao, R. Nissim, S. Sheffield, arXiv:2505.16585, arXiv:2509.04688
The rest of the calibration against the literature	Not yet checked	Fabbri–Goller arXiv:2606.13075, Ebel–Kennedy–Rychkov arXiv:2506.03247, Bauerschmidt–Bodineau (Comm. Pure Appl. Math. 2021), Kopper–Magnen–Rivasseau (1995), Kopper (1999), Peled–Spinka arXiv:1708.00058, Fröhlich–Simon–Spencer, the measurements of the spatial string tension by Bali et al.
Test of the comparison principle	Reading	Numbers from this book's study ($SU(2)$, Wilson action, the twisted slab on $16 \times 16 \times N_T \times N_T$ and 16^4, plaquettes and small Wilson loops). The errors are values re-taken in a re-check (7σ) and an independent rerun (5.5σ)

Item	Kind	Source / tools
The thirty roads, roads discarded in the band chapters, restatements, the selection criteria	This book's study	The record of this book's study (the ledger of exclusions, and the examination of band candidates)

Terms used in this chapter — the ledger of exclusions, why it stopped, vortex container, the comparison principle, Migdal–Kadanoff decimation, the isometric comparison, the selection criteria, the crossover band (the band), β-monotonicity, the renormalisation comparison. Their meanings are in A-02 Glossary.

Next chapter: 26-09 The measured record — the numbers this part cites, on one page.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band
vortex free energy, slab tension, the step function, the column bias · Kiichi, Shiori and Rin

The measured record

This chapter is this book's own investigation. The established knowledge it builds on is in 5-07 and 5-01.

What this chapter shows — We gather on one page the numbers that Part 26 cites. When the small twisted T^2 is widened, the vortex free energy of $U(1)$ rises and that of $SU(2)$ does not (§ 02). The string tension of the slab increases with T^2 and saturates, with no significant overshoot visible (§ 03). The β-dependence of the vortex free energy (§ 04). When the whole lattice is doubled, the $SU(2)$ vortex becomes cheaper and the $U(1)$ vortex costlier (§ 05). The sign of the step function of one step flips for $U(1)$ at the known phase transition $\beta_c \approx 1.01$, and does not flip for $SU(2)$ in the range measured (§ 06). The width needed to keep the column bias small with the outer frame held fixed hardly grows with N for $SU(2)$, and grows for $U(1)$ (§ 07). A diagnosis of the $\mathbb{Z}_2$ glass representation (§ 08).

Prerequisite chapters — 25-01 (the twisted T^2 and flux), 26-01 (the sector bias θ), 26-02 (the lateral-widening inequality), 26-06 (the renormalisation comparison), 5-07 (the Monte Carlo method).

Said first — Every number on this page is a numerical reading [Reading]. The lattices are small, up to 8^4; the points in β and the numbers of seeds (initial values of the random numbers) and frames are few; the errors are statistical only and may come out on the small side. No row is evidence of crossing the band; each is a record of a tendency, for choosing the direction of candidates. Each row states the lattice size, β, the number of seeds or frames, and the error. Rows for which the record does not state the number of seeds are marked "not recorded".

The order of this chapter

1. How we measured, and symbols — the vortex free energy, the sector bias, notes on artefacts
2. Widening the T² — $U(1)$ rises, $SU(2)$ does not (Tables 1–3)
3. The string tension of the slab — it increases and saturates (Tables 4 and 5)
4. The β-dependence of the vortex free energy (Table 6)

5. Doubling the whole lattice (Tables 7–10)
6. The sign of the step function — the $U(1)$ contrast (Table 11)
7. The column bias — the $\ell \times N$ scan (Tables 12–14)
8. A diagnosis of the $\mathbb{Z}_2$ glass (Table 15)
9. Artefacts of small lattices — a list of cautions for reading
10. The table of the five grades and the dependency graph

01

How we measured, and symbols

The lattice. The slab lattice is $(\mathbb{Z}/N_T)^2 \times (\mathbb{Z}/M)^2$; the first two directions (size N_T) are the small T^2, and the remaining two (size M) stand in for the plane $\mathbb{R}^2$. The T^2 carries the 't Hooft twist $n_{12} = 1$ (25-01). A symmetric torus is written N^4. The action is the Wilson action $-\frac{\beta}{2}\sum_p z_p \operatorname{tr} U_p$ ($SU(2)$) and its $U(1)$ version.

The vortex free energy. We write the free-energy difference with the sector in which the flux n_{34} in the planar directions is changed as

$$\Delta F = F(n_{34}=1) - F(n_{34}=0) = -\ln \frac{Z(n_{34}=1)}{Z(n_{34}=0)} \quad (1)$$

and set the sector bias to $\theta = \tanh(\Delta F/2)$ (the same quantity as F_v of 26-01 § 04). The larger ΔF, the costlier the vortex (the harder it is to enter), and θ approaches 1. In the confinement regime one expects $\Delta F \to 0$.

How we measured. ΔF is obtained by replica exchange (exchanging configurations between replicas with interpolated couplings to produce samples that span the sectors) and by Bennett's acceptance-ratio method (BAR). The errors are statistical. At some points the difference between two runs with different seeds is 1–2 times the stated error, so the errors may come out on the small side. The string tension of the slab is read from the Creutz ratio

$$\chi(R) = -\ln \frac{W(R,R)\,W(R-1,R-1)}{W(R,R-1)\,W(R-1,R)} \quad (2)$$

($W(R,T)$ is the $R \times T$ Wilson loop). The Creutz ratio at small R includes the Coulomb term and overestimates the tension, so we read only the tendency, not the absolute value.

Notes on artefacts (in advance). Cautions common to all the tables are listed in § 09. In particular, $N_T = 2$ is the regime of lattice artefacts, 2^4 is too small to see one-loop behaviour, and column-bias values of about 0.1 are at the statistical floor.

02

Widening the T² — U(1) rises, SU(2) does not

We hold the size M in the planar directions fixed and widen only the size N_T of the T^2. This is a test of the sign of the lateral-widening inequality (26-02 § 02).

Table 1 $\Delta F(N_T)$, $M = 8$ ($n_{12} = 1$, replica exchange + Bennett's acceptance ratio method; number of seeds not recorded) [Reading] — the numbers are the same measurement given in 25-08 Table 3. We do not copy them here.

For $U(1)$, widening the T^2 makes ΔF increase sharply, and θ goes $0 \to 0.39 \to 0.78 \to 0.99$. $SU(2)$ goes $0 \to 0.63 \to 0.33 \to 1.51$, much more gently ($\theta$ goes $0 \to 0.31 \to 0.16 \to 0.64$). Under $N_T = 4 \to 8$, $\ln(1/\theta)$ goes $0.93 \to 0.012$ for $U(1)$ (about 1/80), and $1.18 \to 0.45$ for $SU(2)$ (about 1/2.6). In this geometry, however, θ is of order 1 and the ceiling $\theta \leq 1$ weakens the power to discriminate; and $N_T = 8 = M$ for $SU(2)$ is the symmetric torus 8^4 (physical size $L\sqrt{\sigma} \approx 1.5$). $\beta = 1.5$ for $U(1)$ and $\beta = 2.5$ for $SU(2)$ are not matched to the same physical size.

Table 2 $\Delta F(N_T)$, $M = 12$ (remeasured with a larger container) [Reading] — the numbers are the same measurement given in 25-08 Table 4. We do not copy them here.

For $U(1)$ at $N_T = 8$, 24 replicas gave 0.54 ± 0.53 with poor mixing; we give the value with 40. The number of seeds is not recorded.

Table 3 $SU(2)$, $\beta = 2.4$, $M = 8$ [Reading] — the numbers are the same measurement given in 25-08 Table 5. We do not copy them here.

Reading [Reading]. In three settings ($M = 8$ and 12 at $\beta = 2.5$, $M = 8$ at $\beta = 2.4$), the vortex free energy of $SU(2)$ does not increase when the transverse T^2 is widened (except for the symmetric point $N_T = M = 8$ at $\beta = 2.5$, $M = 8$). For $U(1)$ it increases at both values of M. This is a numerical sign of the sign of the lateral-widening inequality. The errors are large, 0.1–0.5, and there are only two points in β. That $N_T = 8$ at $\beta = 2.4$ is smaller than the same point at $\beta = 2.5$ fits with the same lattice being physically larger at $\beta = 2.4$ ($L\sqrt{\sigma} \approx 2.2$ against 1.5).

03

The string tension of the slab — it increases and saturates

One of the necessary conditions for the uniform slab tension (26-02) is that the string tension of the slab $\sigma_{\mathrm{eff}}(N_T)$ does not overshoot much in N_T. We measured the Creutz ratio (2) for $SU(2)$, $\beta = 2.5$, $M = 16$. The four-dimensional value is $\sigma a^2 \approx 0.036$.

Table 4 Creutz ratios of the slab, 400 measurements ($SU(2)$, $\beta = 2.5$, $N_T \times N_T \times 16 \times 16$; number of seeds not recorded) [Reading] — the numbers are the same measurement given in 25-08 Table 6. We do not copy them here.

Table 5 The same setting with five times the statistics (2000 measurements) and $N_T = 10$ [Reading] — the numbers are the same measurement given in 25-08 Table 7. We do not copy them here.

Reading [Reading]. The peak in $\chi(4)$ at $N_T = 6$ seen with 400 measurements (0.089 ± 0.013, about 2.3σ from $N_T = 8$) became 0.061 ± 0.004 with 2000 measurements; it was a statistical fluctuation. $\chi(4)$ increases over $N_T = 4 \to 8$ and nearly saturates at $N_T = 10$ with 0.071 (1.7σ from $N_T = 8$). No significant overshoot is visible. $\chi(4)$ includes the Coulomb term, so it comes out larger than the four-dimensional $\sigma a^2 \approx 0.036$. We read only the tendency in N_T, and β is a single point.

04

The β-dependence of the vortex free energy

Table 6 $SU(2)$, $N_T = 4$, $M = 8$ ($4 \times 4 \times 8 \times 8$, $n_{12} = 1$, replica exchange + Bennett's acceptance ratio method, number of seeds not recorded) [Reading]

β	2.2	2.3	2.4	2.45	2.5	2.55	2.6	2.7
ΔF	0.05 ± 0.05	0.00 ± 0.05	0.31 ± 0.10	0.30 ± 0.13	0.63 ± 0.18	0.66 ± 0.13	1.43 ± 0.19	2.97 ± 0.33

Reading [Reading]. Within the errors, it increases monotonically in β. The reading is that this does not contradict β-monotonicity (26-06 § 02) (β-monotonicity has not been shown for $SU(2)$). For $\beta \le 2.3$ the vortex is nearly free ($\theta \approx 0$), and at $\beta = 2.6$–2.7 it becomes costly sharply. This can be read as the fixed lattice $4 \times 4 \times 8 \times 8$ becoming physically small and entering the dilute regime ($\theta \to 1$). The points at 2.45 and 2.55 are interpolation points added for the fit in § 05.

05

Doubling the whole lattice

We test the renormalisation comparison (26-06 § 05) directly. We measure ΔF on the fine lattice (all directions doubled) and read off, from the curve of the coarse lattice

(Table 6 and others), the β' of the coarse lattice that gives the same value. The prediction of asymptotic freedom is $\beta' < \beta$ for $SU(2)$, and $\beta' \geq \beta$ for $U(1)$, which does not run.

Table 7 Slab $(N_T, M) = (4, 8) \to (8, 16)$ $(n12 = 1)$ [Reading]

Group	β	$\Delta F(4, 8, \beta)$	$\Delta F(8, 16, \beta)$: per seed	Combined	Seeds	Matching β'	$\beta - \beta'$
$SU(2)$	2.7	2.97 ± 0.33	0.65 ± 0.39, 1.31 ± 0.48 (the second run has a minimum exchange acceptance rate of 0.10)	0.92 ± 0.30 (error-weighted mean)	2	≈ 2.57 (2.45–2.65 including the error of the curve)	$\approx +0.13$ ($+0.05$–$+0.25$)
$SU(2)$	2.6	1.43 ± 0.19	-0.57 ± 0.46, -0.41 ± 0.51	-0.50 ± 0.34 (consistent with 0)	2	$\lesssim 2.45$	$\gtrsim +0.15$
$SU(2)$	2.5	0.63 ± 0.18	0.41 ± 0.34	—	1	2.2–2.5	0–$+0.3$
$U(1)$	1.5	0.83 ± 0.11	2.32 ± 0.45 (48 replicas, minimum exchange acceptance rate 0.37)	—	1	> 1.7 (outside the curve)	< -0.2

The coarse-lattice curve for $U(1)$ is $\Delta F(4, 8, \beta') = 0.42, 0.55, 0.83, 0.92, 1.46$ ($\beta' = 1.3, 1.4, 1.5, 1.6, 1.7$). The first run for $U(1)$ (32 replicas) gave -0.18 ± 0.12, but the minimum exchange acceptance rate was 0.03 and the mixing poor, so it is not used.

Reading [Reading]. For $SU(2)$, doubling lowers the vortex free energy ($2.97 \to 0.92$ at $\beta = 2.7$), and the matching β' is smaller than β. For $U(1)$ it rises ($0.83 \to 2.32$, about 3σ), and $\beta' > \beta$. The direction of the comparison separates between the two groups. The $+0.13$ at $\beta = 2.7$ is somewhat smaller (about 1.5σ) than the one-loop 0.258 and than the roughly 0.23 expected from the known values of $a\sqrt{\sigma}$. The record notes that the reversed direction for $U(1)$ may also include artefacts of the small lattice.

Next, on the symmetric torus N^4, we look at the two steps $2^4 \to 4^4$ and $4^4 \to 8^4$.

Table 8 $SU(2)$, $\Delta F(N^4, \beta)$ on the symmetric torus ($n12 = 1$, one seed each) [Reading]

β	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75	3.0
2^4	0.034	0.094	0.213	0.573	1.39	2.71	4.29	5.95	7.88
4^4	0.01	−0.01	0.01	0.01	−0.03	0.50	2.46	5.10	7.33

The errors are 0.01–0.08 at 2^4 and 0.03–0.26 at 4^4. The β' of the single step $2^4 \to 4^4$ (fitting the 4^4 values to the 2^4 curve) is as in Table 9.

Table 9 The single step $2^4 \to 4^4$ for $SU(2)$ [Reading]

β	≤ 2.0	2.25	2.5	2.75	3.0
β'	< 1.0 (outside the curve)	1.72	2.20	2.62	2.93

β	≤ 2.0	**2.25**	**2.5**	**2.75**	**3.0**
$\beta - \beta'$	> 0.75–1	0.53	0.30	0.13	0.07

At every readable β (1.75–3.0), $\beta - \beta' > 0$; that is, the step function of one step has no fixed point. The motion per step grows towards strong coupling; that it falls to 0.07 on the weak-coupling side is seen as an artefact of 2^4 being too small for the one-loop regime.

Table 10 $4^4 \to 8^4$ for $SU(2)$. ΔF is given per seed and as their simple mean (the error is half the difference between seeds; for three runs, the mean and the spread) [Reading]

β	$\Delta F(4^4)$: per seed → combined	$\Delta F(8^4)$: per seed → combined	Seeds (8^4)	$\beta - \beta'$
2.1	—	-0.04 ± 0.17	1	— (≈ 0)
2.3	—	-0.04 ± 0.10	1	$\gtrsim 0.1$
2.5	2.46 (1 run)	0.48 ± 0.48	1	$\approx +0.25$ ($+0.19$–$+0.5$)
2.6	3.96 ± 0.13 (1 run)	—	—	—
2.7	4.96, 4.47 → 4.72 ± 0.25	3.57, 2.65 → 3.11 ± 0.46	2	$+0.16$ ($+0.13$–$+0.19$)
2.75	5.10, 5.26 → 5.18 ± 0.10	—	—	—
2.8	5.33, 5.87 → 5.60 ± 0.27	—	—	—
2.9	6.88, 6.23 → 6.56 ± 0.33	5.68, 3.52, 4.84 → 4.68 ± 0.63	3	$+0.20$ ($+0.12$–$+0.29$)
3.0	7.33, 7.83 → 7.58 ± 0.25	5.38, 6.17 → 5.78 ± 0.40	2	$+0.18$ ($+0.14$–$+0.25$)

Reading [Reading]. For $4^4 \to 8^4$ too, $\beta - \beta' > 0$ at every β measured, with a mean of about $+0.2$. This is close to the one-loop asymptotic value 0.258 and to the roughly 0.23 expected from the known values of $a\sqrt{\sigma}$. At some points the difference between seeds is 1–2 times the stated error (at $\beta = 2.9$, 8^4 gives 5.68 and 3.52), and we read the errors as quite likely on the small side. The reading with added seeds came closer to the asymptotic behaviour than the single-seed reading ($+0.08$–$+0.13$). It also fits the reading of the 8×16 slab at $\beta = 2.7$ (Table 7, $+0.13$).

06

The sign of the step function — the U(1) contrast

We carried out the same $2^4 \to 4^4$ scan for $U(1)$. Four-dimensional $U(1)$ has a phase transition at $\beta_c \approx 1.01$ from the confining phase to the Coulomb phase.

Table 11 $U(1)$, $\Delta F(N^4, \beta)$ on the symmetric torus and the single step $2^4 \to 4^4$ ($n_{12} = 1$, one seed each, errors not recorded) [Reading]

β	**0.6**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**1.4**	**1.6**
2^4	0.14	0.62	1.15	1.80	2.46	3.10	4.18	5.23

β	0.6	0.8	0.9	1.0	1.1	1.2	1.4	1.6
4^4	0.00	0.00	0.05	1.07	2.71	3.68	4.46	5.85
$\beta - \beta'$	> 0.3	> 0.3	> 0.3	$+0.12$	-0.04	-0.11	-0.05	< 0

Reading [Reading]. The sign of $\beta - \beta'$ for $U(1)$ flips between $\beta = 1.0$ and 1.1. This is the position of the known phase transition $\beta_c \approx 1.01$. In the confining phase $\beta' < \beta$, and in the Coulomb phase $\beta' > \beta$. For $SU(2)$, the same scan gave $\beta - \beta' > 0$ at every β measured (Tables 9 and 10). The sign of the step function of one step works as a diagnostic that tells whether there is a phase transition. This is the numerical face of the criterion "distinguish from the U(1) transition" (26-06 § 08), which an argument that closes the band must satisfy.

What this sign test can say is only that, on lattices up to 8^4 and at the β points of the table, the sign for $SU(2)$ did not change. It is not evidence that $SU(2)$ has no phase transition.

Relatedly, there are two readings [Reading] concerning the intermediate goal of seeing $\Delta F(L, L) \to 0$ at any rate (26-06 § 08). For $U(1)$ ($\beta = 1.2$), $2^4 \to 4^4$ gives $3.10 \to 3.68$, flat, which does not contradict the picture of heading towards the classical half-flux value in the Coulomb phase. For $SU(2)$ ($\beta = 2.75$), it falls, $5.95 \to 5.10$.

07

The column bias — the ℓ × N scan

We measure the column face $f(\ell)$ (26-10 § 02). The outer frame of the column $T^2_N \times \ell \times \ell$ is taken from an equilibrium configuration and held fixed; for each frame, the conditional free energy $\Delta F_B = \ln Z_B(+)/Z_B(-)$ when a sheet of n_{34} is put inside the column, and the bias $\theta_B = \tanh(\Delta F_B/2)$, are obtained by replica exchange + Bennett's acceptance ratio method, and we take $\mathbb{E}|\theta_B|$ over the frames. It is this $\mathbb{E}|\theta_B|$ that enters the Chessboard Estimate (26-05). $SU(2)$ is at $\beta = 2.3$, and $U(1)$ at $\beta = 1.3$ in the Coulomb phase.

Table 12 Varying ℓ at $N = 4$ (numbers of frames in parentheses) [Reading]

ℓ	$SU(2)$, $\beta = 2.3$: $\mathbb{E}\lvert\theta_B\rvert$	$U(1)$, $\beta = 1.3$: $\mathbb{E}\lvert\theta_B\rvert$
2	0.93 ± 0.06 (5)	—
3	0.58 ± 0.17 (4)	0.95 ± 0.05 (4)
4	0.23 ± 0.04 (12)	0.61 ± 0.21 (4)
5	0.07 ± 0.03 (12)	0.49 ± 0.09 (12)
6	0.09 ± 0.04 (4)	0.41 ± 0.06 (12)

ℓ	$SU(2)$, $\beta = 2.3$: $\mathbb{E}\lvert\theta_B\rvert$	$U(1)$, $\beta = 1.3$: $\mathbb{E}\lvert\theta_B\rvert$
8	—	0.25 ± 0.05 (8)

For $\ell \geq 5$, the column bias of $SU(2)$ is 1/5 to 1/7 of that of $U(1)$. $U(1)$ decreases slowly (about $\ell^{-1.4}$ over $\ell = 5 \to 8$), while $SU(2)$ drops to 1/3 over $\ell = 4 \to 5$. The values of $SU(2)$ for $\ell \geq 5$, however, are close to the floor made by the per-frame statistical errors (which bias $|\theta_B|$ positive), and at these sizes one cannot decide whether it is $e^{-\sigma\ell^2}$ or a power. The sign can change from frame to frame (for $SU(2)$ at $\ell = 4$, the mean of ΔF_B is -0.05). In frames where the holonomy around the block is close to -1, the sector with the sheet is favoured.

Table 13 Varying N at $\ell = 5$ (numbers of frames in parentheses) [Reading]

N	$SU(2)$, $\beta = 2.3$	$U(1)$, $\beta = 1.3$
4	0.07 ± 0.03 (12)	0.49 ± 0.09 (12)
6	0.20 ± 0.05 (6)	0.97 ± 0.01 (6)
8	0.39 ± 0.05 (5; the sixth frame stopped partway at the time limit)	—

With ℓ fixed, the bias approaches 1 with N for both groups. While the bias is small, it grows roughly in proportion to N^2 ($\Delta F_B \propto N^2 f(\ell)$), and the proportionality coefficient for $SU(2)$ is about 1/7 of that for $U(1)$. The difference between the groups should appear in how the ℓ needed to keep θ small grows, so we varied both ℓ and N.

Table 14 The $\ell \times N$ scan: $\mathbb{E}\lvert\theta_B\rvert$ (numbers of frames in parentheses) [Reading]

	$\ell = 4$	**5**	**6**	**7**	**8**	**10**
$SU(2)$, $\beta = 2.3$, $N = 4$	0.23 (12)	0.07 (12)	0.09 (4)	0.11 (4)		
$SU(2)$, $N = 6$	0.43 (4)	0.20 (6)	0.15 (4)	0.08 (4)		
$SU(2)$, $N = 8$	0.43 (4)	0.38 (5)	0.13 (4)	0.29 ± 0.06 (3)		
$U(1)$, $\beta = 1.3$, $N = 4$	0.61 (4)	0.49 (12)	0.41 (12)		0.25 (8)	0.27 (4)
$U(1)$, $N = 6$		0.97 (6)	0.98 (4)		0.51 (4)	0.45 (4)

Reading [Reading]. For $SU(2)$, at every one of $N = 4, 6, 8$, the bias falls to the statistical floor (about 0.1) at $\ell \approx 6$–7, and the needed ℓ hardly grows with N. For $U(1)$, even at $N = 4$ it stays at 0.27 at $\ell = 10$, and at $N = 6$ it is 0.45 even at $\ell = 10$; the needed ℓ grows with N, and the fall is slower. The 0.29 at $N = 8$, $\ell = 7$ comes from only 3 frames (cut off at the time limit); that it is larger than the 0.13 at $\ell = 6$ we take as a fluctuation due to the small number of frames (all 3 frames have $\Delta F_B > 0$, 0.43–0.85). This does not contradict the expectation ($\ell \sim \sqrt{\ln N}$ for $SU(2)$, $\ell \propto N$ for $U(1)$), but the lattices are small and the frames few, so the form of the growth cannot be determined.

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08
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A diagnosis of the Z_2 glass

Writing $U = \sigma V$ ($\sigma \in \mathbb{Z}_2$, V a section of $SO(3)$), the twisted sector becomes "a $\mathbb{Z}_2$ random-plaquette gauge model on an $SO(3)$ background" (26-10 § 03). The effective $\mathbb{Z}_2$ coupling per plaquette is $K_p = (\beta/2)|\mathrm{tr}\, U_p|$, and the frustration (a $\mathbb{Z}_2$ monopole) is a cube c for which $\prod_{p \in \partial c} \mathrm{sign}\, \mathrm{tr}\, U_p = -1$.

Table 15 $SU(2)$, periodic 8^4, heat bath with 150 thermalisation sweeps and 100 measurements (number of seeds not recorded, errors not recorded) [Reading]

β	$\langle P \rangle$	Fraction with $\mathrm{tr}\, U_p < 0$	$\langle K \rangle$	Fraction with $K < K_c = 0.4407$	Frustrated cubes	p_{eff}
1.0	0.243	0.297	0.455	0.494	0.481	0.209
1.5	0.363	0.204	0.741	0.282	0.452	0.162
2.0	0.501	0.109	1.120	0.149	0.349	0.091
2.3	0.602	0.051	1.437	0.081	0.216	0.045
2.5	0.653	0.029	1.660	0.051	0.140	0.027
2.7	0.686	0.018	1.870	0.034	0.093	0.017

p_{eff} is the frustration density ρ of the cubes, converted to a probability of independent sign flips by $(1 - 2p_{\mathrm{eff}})^6 = 1 - 2\rho$. K_c is the transition point of pure $\mathbb{Z}_2$ gauge theory, and the value to compare with is the Nishimori multicritical point $p_c \approx 0.11$ of the four-dimensional random-plaquette gauge model (checking not yet done).

Reading [Reading]. For $\beta \gtrsim 2.0$, the mean coupling $\langle K \rangle$ is 3–4 times K_c, and $p_{\mathrm{eff}} < 0.11$. The $\mathbb{Z}_2$ glass is likely on the side of the ordered phase. For $\beta \lesssim 1.5$, $p_{\mathrm{eff}} \geq 0.16$, and soft plaquettes ($K < K_c$) are 28–49%: the disordered side. The conversion to independent signs is crude (soft plaquettes and frustration are correlated), so the boundary is a rough guide. From this diagnosis, the glass road is limited to the lower half of the band (the stopped road "glassy squeeze" of 26-08 § 05).

09

Artefacts of small lattices — a list of cautions for reading

- **$N_T = 2$ (and 2^4) is the regime of lattice artefacts.** At $N_T = 2$ one side of the T^2 is two lattice sites, the frustration of the twist is large, and it is far from the continuum picture (a gas of fractional instantons). That the $\beta - \beta'$ of the single step $2^4 \to 4^4$ falls to 0.07 on the weak-coupling side (Table 9) is also seen as due to 2^4 being too small for the one-loop regime. Values at $N_T = 2$ are kept as "a reference for the direction".

- **The ceiling of θ.** In geometries where θ is of order 1 ($M = 8$ of Table 1), $\theta \leq 1$ holds the ratio down and weakens the power to discriminate between the groups.
- **The statistical floor.** The column bias $\mathbb{E}|\theta_B|$ is biased positive by the per-frame statistical errors. Values of about 0.1 are close to the floor, and one cannot decide between an area exponent and a power (Tables 12 and 14).
- **The Coulomb term.** The Creutz ratio at small R includes the Coulomb term and overestimates the tension (Tables 4 and 5).
- **Poor mixing.** Runs with a small minimum replica-exchange acceptance rate (replicas 24 and 32 for $U(1)$, the 0.10 of the second $SU(2)$ run) fluctuate strongly in value. Runs we did not use are noted under the tables.
- **Number of seeds and errors.** There are 1–3 seeds. At some points the difference between two runs with different seeds is 1–2 times the stated error, so the errors are quite likely on the small side.
- **How β is matched.** The β of $U(1)$ and of $SU(2)$ are not matched to the same physical size. What we compare is only the direction (sign) within each group.
- **A withdrawn measurement.** We have withdrawn reading the measurement that flips containers simultaneously as "the numerical face of the band". The flip probability of the one-step kernel levels off at about 1/2, and what comes out is only the weak form, which $U(1)$ also satisfies (the stopped road "flip certificate" of 26-08 § 05). The measurement itself is in 25-08.

10

The table of the five grades and the dependency graph

All the rows of this page are numerical readings. The label is "Reading", with " (physics)" added for readings that include a physical estimate.

Table 16 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	Widening the T^2 increases ΔF for $U(1)$ and not for $SU(2)$	Reading	Tables 1–3	$M = 8, 12$, $\beta = 2.4$, 2.5 ($SU(2)$), $\beta = 1.5$ ($U(1)$), $N_T \leq 8$	—
2	The slab $\chi(4)$ increases with N_T and saturates, with no significant overshoot	Reading	Tables 4 and 5	$\beta = 2.5$, $M = 16$, 2000 measurements, $N_T \leq 10$	—
3	ΔF increases in β	Reading	Table 6	$4 \times 4 \times 8 \times 8$, $\beta = 2.2$–2.7	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
4	Doubling the whole lattice: $\beta' < \beta$ for $SU(2)$, $\beta' > \beta$ for $U(1)$	Reading	Tables 7–10	$(4,8) \to (8,16)$, $2^4 \to 4^4$, $4^4 \to 8^4$, 1–3 seeds	—
5	$\beta - \beta'$ for $4^4 \to 8^4$ averages about $+0.2$, close to the one-loop 0.258	Reading	Table 10	$\beta = 2.5$–3.0, 1–3 seeds	—
6	The sign of the single step for $U(1)$ flips between $\beta = 1.0$ and 1.1; for $SU(2)$ it does not flip in the range measured	Reading	Tables 9 and 11	$2^4 \to 4^4$, one seed	—
7	The ℓ that keeps the column bias small hardly grows with N for $SU(2)$, and grows for $U(1)$	Reading	Tables 12–14	$N \le 8$, $\ell \le 10$, 3–12 frames	—
8	The $\mathbb{Z}_2$ glass is on the ordered side for $\beta \gtrsim 2$ and on the disordered side for $\beta \lesssim 1.5$	Reading (physics)	Table 15	8^4; the conversion to independent signs is crude	—
9	The measurements 1–7 do not contradict the directions of the lateral-widening inequality, the renormalisation comparison and the criterion "distinguish from the U(1) transition"	Reading (physics)	§ 02–§ 07	Tendencies on small lattices. Not evidence	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
1  RDG  widening T²: U(1) rises, SU(2) does not                M = 8, 12
2  RDG  slab tension increases and saturates                   β = 2.5, 2000 m
3  RDG  ΔF increases in β                                      4×4×8×8
4  RDG  doubling test: SU(2) β' < β, U(1) the opposite ⟸ 3     1–3 seeds
5  RDG  β − β' ≈ +0.2 for 4⁴ → 8⁴ ⟸ 4                         1–3 seeds
6  RDG  sign of the step function: U(1) flips at β_c ⟸ 4       2⁴ → 4⁴
7  RDG  ℓ × N of the column bias                               N ≤ 8, ℓ ≤ 10
8  RDG  diagnosis of the Z₂ glass                              8⁴
9  RDG  consistency of directions ⟸ 1, 2, 4, 6, 7              tendency
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The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. This page has no "Proved here", "Cited" or "Assumption" rows. Nor does it have nodes for the statements that cross the band (the uniform slab tension, the lateral-widening inequality, the renormalisation comparison); those are placed as assumptions in the ledger of conditions of Part 27 (27-02).

What this chapter can say

What is said	What is not said
Reading That on small lattices the directions of $SU(2)$ and $U(1)$ separate in every test: widening the T^2, doubling the lattice, widening the column	Behaviour on large lattices. Determining the values. The lateral-widening inequality and the renormalisation comparison themselves
Reading That the sign of the step function for $U(1)$ flips at the position of the known transition, and that for $SU(2)$ it does not flip in the range measured	That $SU(2)$ has no phase transition
Reading (physics) That the $\mathbb{Z}_2$ glass is on the ordered side in the upper half of the band	The position of the phase boundary of the glass (the conversion is crude)

Sources and reproduction

Item	Kind	Source / tools
All the tables	Reading	The record of numbers from this book's study (numerical computations of vortex insertion: the global flux free energy, the string tension of the slab, the β-dependence, tests of the renormalisation comparison, scans of the step function on the symmetric torus, the $U(1)$ contrast, the added $4^4 \to 8^4$ runs and the combining of seeds, the twist cost of a column, the diagnosis of the $\mathbb{Z}_2$ glass), and the numerical section of the survey of roads that would close the crossover band. Replica exchange and Bennett's acceptance ratio method, heat bath, Creutz ratios
The phase transition $\beta_c \approx 1.01$ of $U(1)$	Reading	Known numerical lattice values (5-06)
The Nishimori multicritical point $p_c \approx 0.11$	Checking not yet done	Numerical literature on the four-dimensional random-plaquette gauge model
The one-loop 0.258	Proved here	Chapter 25-07 (one-loop level)

Terms used in this chapter — the vortex free energy ΔF, the sector bias θ, the Creutz ratio, replica exchange, Bennett's acceptance ratio method, the step function, the column bias $\mathbb{E}|\theta_B|$, the $\mathbb{Z}_2$ glass, frustration. Their meanings are in A-02 Glossary.

Next chapter: 26-10 Candidate new viewpoints — into inequalities for which local quantities these tendencies might be grown.

Candidate new viewpoints

> This chapter is a map of open questions. It marks the boundary between the parts that close and the parts that do not.

What this chapter shows — We have narrowed things down to this point: crossing the middle of the crossover band needs one inequality that uses non-commutativity (26-06 § 07). This chapter sets out candidates for the local quantity in which to write that one inequality. For the four faces — the softness of the twist direction κ_W, the flux mixing of the box $\delta(L_0)$, the twist cost of the column $f(\ell)$, the pressure margin — we write the local quantity that carries each, why we expect it to hold for $SU(2)$, where it breaks for $U(1)$, and whether there are numbers (§ 02). We then give the $\mathbb{Z}_2$ glass representation, restricted to the lower half of the band (§ 03), the conformal frame $H^2 \times S^2$, which turned out negative on examination (§ 04), and three new candidates — reduction by the compactness of the band, the log-concavity of the Weyl measure and the height–azimuth decomposition, and the local step function of the column (§ 05).

Prerequisite chapters — 26-06 (the renormalisation comparison, the list of the four faces, the reduction lemma), 26-08 (stopped roads and the selection criteria), 26-09 (the measured record), 25-03 (containers), 26-01 (the correct form of the band).

Said first — The candidates on this page are all **at the level of physical estimates**. None of them closes the band, and none gives a theorem at any β in the band. Paper proofs are set out only for the elementary facts that serve as footholds for the candidates (the transport identity, the Jensen sandwich, the density of a projection, and so on), each labelled "Proved here". We do not give numbers for the chances that a candidate succeeds (such numbers are only an aid to judgement, not numbers with a basis).

The order of this chapter

1. How to read the candidates — the checkpoints a candidate must pass
2. The four faces of the band in the middle (Table 1)
3. The $\mathbb{Z}_2$ glass — restricted to the lower half of the band
4. The conformal frame $H^2 \times S^2$ — negative on examination
5. Three new candidates — reduction by compactness, the Weyl measure, the step function of the column
6. Five other candidates (Table 2)
7. What cannot be said

8. The table of the five grades and the dependency graph

01

How to read the candidates — the checkpoints a candidate must pass

The candidates on this page are set out after passing the selection criteria of 26-08 § 07 as checkpoints. In particular, we name the following four for each candidate.

- **Does it use non-commutativity essentially?** A candidate that gives the same conclusion for $U(1)$ is wrong. This is why we write "where it breaks for $U(1)$" for each candidate. A table that widens the opponents of the test beyond U(1) is in 26-12.
- **Is it not a rewording of one loop?** A quantity that at one loop becomes asymptotic freedom itself is not enough on its own.
- **Does it not fail at small scales?** Claims are put in a form restricted to outside the lattice artefacts, such as $\ell \geq \ell_{\min}$.
- **Does it distinguish from the transition of $U(1)$ within the band ($\beta \approx 1.01$)?**

For each candidate we write (i) the local quantity that carries it, (ii) why we expect it to hold for $SU(2)$ (the place where non-commutativity enters), (iii) where it breaks for $U(1)$ and $\mathbb{Z}_2$, (iv) the check against the ledger (26-08), (v) whether there are numbers. The labels are "Proved here" for elementary facts, "Reading" for numbers, and "Reading (physics)" for estimates.

02

The four faces of the band in the middle

There are four candidates for the local quantity in which to write the non-abelian inequality that crosses the band. We call them "faces" in the sense that it suffices to show any one of them, uniformly in the scale, in a form that fails for $U(1)$. For every face, a comparison across scales (for example $f(2\ell, 2N, \beta) \leq K f(\ell, N, \beta')$) is a local version of the same difficulty as the global renormalisation comparison, and does not become easier. What changing the face gains is that the object of the proof becomes "a statement about a finite region with its boundary held fixed".

Table 1 The four faces (expectations and reasons are physical estimates; numbers are readings)

Face	Local quantity that carries it	Why we expect it to hold for $SU(2)$	Where it breaks for $U(1)$, $\mathbb{Z}_2$	Numbers
Softness of the twist direction κ_w	The variance of the force along the generator of the twist, within a band of width w (adapted stiffness)	Decreases with w. The soft direction of Nielsen–Olesen; a non-perturbative version of the one-loop spin term $+4$	$U(1)$ has no spin term, so $\kappa_w \geq c\beta$. $\mathbb{Z}_2$ has no continuous path and can use only thin sheets (2β per unit area)	None

Face	Local quantity that carries it	Why we expect it to hold for $SU(2)$	Where it breaks for $U(1)$, $\mathbb{Z}_2$	Numbers
Flux mixing of the box $\delta(L_0$)	The total variation distance of the law of the inner cube when the centre flux at the boundary of the box is changed	Goes to 0 exponentially with L_0	In the Coulomb phase of $U(1)$ it stays at $O(\beta)$	None
Twist cost of the column $f(\ell)$	The conditional free energy of the twist and the bias $\mathbb{E}\lvert\theta_B\rvert$ of the column $T^2_N \times \ell \times \ell$ with its outer frame held fixed	Of type $e^{-\sigma\ell^2}$	$U(1)$ is of type $\pi^2\beta/\ell^2$, $\mathbb{Z}_2$ is $\approx 2\beta$	Yes (26-09 § 07)
Pressure margin	The pressure difference p $\infty - pG,L$ between the system with the twist pattern and the system without it (the exponent of the chessboard density κ cb)	A positive margin remains at intermediate β	$U(1)$ does not produce a small κ_{cb}	None

The softness in the twist direction $\kappa_{\mathcal{W}}$ — joining sectors by transport

The foothold is an identity for the free energy given by a map that carries one sector to the other. Take a measurable bijection T (Jacobian J_T) from the configuration space of sector 0 to that of sector 1; let μ_0, μ_1 be the Gibbs measures of the two sectors and $\Delta F = -\ln(Z_1/Z_0$). Then

$$\Delta F \;=\; \mathbb{E}_{\mu 0}\left[S \circ T - S - \ln J_T\right] \;-\; \mathrm{KL}\big(T_{\#}\mu 0 \,\big\|\, \mu 1\big) \quad (1)$$

holds [Proved here].

Proof. By the change of variables, $Z_1 = \int e^{-S(Tx)} J_T(x)\, dx$, so setting $G := S \circ T - S - \ln J_T$ we have $Z_1/Z_0 = \mathbb{E}_{\mu 0}[e^{-G}]$. The measure $\tilde{\mu} \propto e^{-G}\mu_0$ is μ_1 pulled back by T^{-1}, and $\mathrm{KL}(\mu_0 \|$ $\tilde{\mu}) = \mathbb{E}_{\mu 0}[G] + \ln \mathbb{E}_{\mu 0}[e^{-G}]$. Since T is a bijection, $\mathrm{KL}(\mu_0 \| \tilde{\mu}) = \mathrm{KL}(T_{\#}\mu_0 \| \mu_1)$. Rearranging gives (1). ∎

The first term of (1) is "the action cost of the map", and the second is the remainder. If the map T is deterministic, the remainder (the relative entropy) grows with the area of the vortex sheet, so an area law of type $e^{-\sigma A}$ does not come out [Reading] (physics) (the stopped road "deterministic squeeze" of 26-08 § 05). The goal of the transport road is therefore not the area law but the qualitative $\Delta F(L, L) \to 0$ (the intermediate goal of 26-06 § 08).

(ii) Why we expect it to hold for $SU(2)$ [Reading] (physics). The "adapted" map that carries sectors along the generator of the twist is a triangular change of variables with $J_T = 1$: a map that preserves volume while changing the shape of the fluctuations. It is the non-perturbative version of the principle of 25-07 — that what hides the antiscreening $+4$ is not

preserving volume but not changing the shape of the fluctuations. κ_w is the size of the fluctuations of the action cost of that map, and we expect it to decrease with w if the soft direction of Nielsen–Olesen works within a band of width w.

(iii) Where it breaks. $U(1)$ has no spin term, and $\kappa_w \geq c\beta$ remains. $\mathbb{Z}_2$ has no continuous path joining the sectors and can use only thin sheets (2β per unit area). **(iv) The ledger**: there is a risk of hitting item 25 of Table 1 (spin and the index theorem are a restatement of asymptotic freedom at one loop); the dividing line is whether the non-perturbative part can be cut out as an estimate of the variance κ_w. **(v) Numbers**: none.

The mixing of box magnetic flux $\delta(L_0)$ — a finite-box criterion

A candidate for a gauge version of a Dobrushin–Shlosman-type finite-size criterion. The form of the goal is "if $\delta(L_0, \beta) \leq \varepsilon_d$, then the area law"; the criterion itself has not been written down [Reading] (physics). It does not use correlation inequalities (GKS), so it does not hit the obstacle that $SU(2)$ has no GKS (items 2 and 4 of Table 1).

For $SU(2)$ we expect $\delta(L_0)$ to go to 0 exponentially with L_0. In the Coulomb phase of $U(1)$, in the Villain model in the Gaussian approximation, the relative entropy stays at $O(\beta)$ independent of L_0 (a count at the level of estimates) [Reading] (physics). Caution: if the ordinary Dobrushin criterion is used with a frame, the continuous field follows the boundary and the influence coefficient becomes nearly 1 (the stopped road "Dobrushin-type criterion" of 26-08). $\delta(L_0)$ is a quantity that changes only the centre flux precisely to avoid this following. No numbers.

The twist cost of a column $f(\ell)$ — the same quantity as the Wilson loop of the box

Given the skeleton Σ (the outer frame of the columns), the interiors of the columns are conditionally independent, so

$$|\theta| \;\leq\; \mathbb{E} \prod_B |\theta_B(\Sigma)| \qquad (2)$$

holds [Proved here] (the same framework as the Container Identity of 25-03 § 03, with no new content). What is new is how $f(\ell)$ is read. The electric flux in a box with its outside held fixed is **the same quantity as the area law of the Wilson loop of the box** [Reading] (physics). This closes a loop: the Tomboulis–Yaffe inequality (from θ to the Wilson area law, 26-01 § 04) and (2) (from the Wilson area law of the box to θ).

Two things are missing to go round the loop (open): the comparison between periodic and fixed boundary conditions, and the loss of $\ln N$. The estimate is that even if the loss remains, $\ln(1/\theta(L, L)) \geq cL^2/\ln L$ comes out, and through Tomboulis–Yaffe this gives $\langle W \rangle \leq 2 \exp(-cA/\ln A)$ (decay faster than the perimeter law) [Reading] (physics).

There are numbers (26-09 § 07) [Reading]. On $T_4^2 \times \ell \times \ell$ with the outer frame held fixed, $\mathbb{E}|\theta_B|$ for $SU(2)$ ($\beta = 2.3$) at $\ell \geq 5$ is 1/5 to 1/7 of that for $U(1)$ ($\beta = 1.3$), and the ℓ needed to keep

θ small hardly grows for $SU(2)$, staying at 6–7 for $N \le 8$, while it grows for $U(1)$. At these sizes, the difference between fixed and periodic boundary conditions does not seem to matter. The values for $SU(2)$, however, are close to the statistical floor, and $e^{-\sigma \ell^2}$ cannot be told apart from a power.

The pressure margin — a computer-assisted squeeze

We write the chessboard density as $\kappa_\Lambda = \exp(|B|(p_{G,\Lambda} - p_\Lambda))$ (p_G is the pressure of the system with the twist pattern, $|B|$ the volume of a tile). From above: if the pattern is reflection-symmetric, the periodic transfer matrix is positive, and the monotonicity of Schatten norms gives $p_{G,\Lambda} \le p_{G,L}$ (Λ a multiple of L). From below: $p_\Lambda \ge p_\infty$, together with a lower bound p_{lb} on the infinite-volume pressure taken from a dual certificate of the loop-equation bootstrap (the method of Kazakov–Zheng; checking not yet done). Combined, $\kappa_\Lambda \le \exp(|B|(p_{G,L} - p_{\mathrm{lb}}))$ becomes uniform in Λ, and an explicit area law for the tension comes out at fixed β — that is the estimate [Reading] (physics).

This face does not cross the band (uniformly in the lattice spacing). What it could give is confinement at fixed β for four-dimensional $SU(2)$ at intermediate coupling (for example $\beta = 1.0$). In the range we searched, no proof of that was found. Combined with the reduction of § 05 (the candidate "reduction by compactness of the band"), a result of this kind at fixed β becomes a component of the band as it stands. $\beta = 1.2$ for $U(1)$ should not produce a small κ, and can be used as a control. No numbers.

03 The Z_2 glass — restricted to the lower half of the band

Writing an element of $SU(2)$ as $U = \sigma V$ ($\sigma \in \mathbb{Z}_2$, V a section of $SO(3)$), the partition function of a twisted sector can be written $Z_n = \int d\omega\, Z_n(\omega)$. $Z_n(\omega)$ is the partition function of a $\mathbb{Z}_2$ gauge theory with couplings $\beta J_p(\omega)$ fixed by the $SO(3)$ background ω ($J_p = \frac{1}{2}\operatorname{tr} V_p$, with random signs) and twist n. Setting $X(\omega) = \ln Z_0(\omega)/Z_1(\omega)$ and $P_n(d\omega) \propto Z_n(\omega) d\omega$, we have

$$\mathbb{E}_{P_1}[X] \;\le\; \Delta F \;\le\; \mathbb{E}_{P_0}[X] \qquad (\Delta F = \ln Z_0/Z_1) \tag{3}$$

[Proved here].

Proof. The left side follows from $Z_0/Z_1 = \int Z_1(\omega) e^{X(\omega)} d\omega / Z_1 = \mathbb{E}_{P_1}[e^X] \ge e^{\mathbb{E}_{P_1} X}$ (Jensen). Likewise the right side from $Z_1/Z_0 = \mathbb{E}_{P_0}[e^{-X}] \ge e^{-\mathbb{E}_{P_0} X}$. ∎

The right-hand side of (3) is the twist free energy of a $\mathbb{Z}_2$ random-plaquette gauge model quenched in the $SO(3)$ background. This representation alone exists in the same form for $U(1)$, so it does not satisfy the criterion "distinguish from the U(1) transition". The distinction between groups enters **dynamical quantities**: the frustration of the background ($\mathbb{Z}_2$ monopoles) and the density of soft plaquettes [Reading] (physics).

Result of the diagnosis (26-09 § 08) [Reading]. On 8^4, for $\beta \gtrsim 2.0$ the mean of the effective $\mathbb{Z}_2$ coupling is 1.1–1.9, well above the transition point 0.44 of pure $\mathbb{Z}_2$, and p_{eff}, the frustration converted to independent signs, is 0.017–0.09, below the Nishimori multicritical point 0.11. **In the upper half of the band, the glass is on the side of the ordered phase**. The $SO(3)$ field already distinguishes the sectors ($|X|$ is large), and the sandwich (3) becomes very loose. Confinement there can be read as carried by the $SO(3)$ part (thick centre vortices). This fits the physical picture that centre vortices are thick at weak coupling [Reading] (physics).

For $\beta \lesssim 1.5$, $p_{\text{eff}} \geq 0.16$, soft plaquettes are 28–49%, and the glass is on the disordered side. **The glass road is therefore restricted to the lower half of the band**. We have discarded it for the upper half (the stopped road "glassy squeeze" of 26-08 § 05). A candidate use in the lower half is a variant of the mixing of box magnetic flux: condition the outside on the $SO(3)$ part only, flip only the centre flux at the boundary (the $\mathbb{Z}_2$ sign), and measure the total variation distance $\delta_c(L_0, \beta)$ of the inner law. The $SO(3)$ frame does not move, so the obstacle of the continuous field following the boundary (Dobrushin-type criterion) is avoided. The only coupling left is the $\mathbb{Z}_2$ glass. In the upper half of the band δ_c should also be close to 1, and this variant belongs to the work of pushing up the lower end of the band (26-06 § 04) [Reading] (physics).

04

The conformal frame H² × S² — negative on examination

The space $\mathbb{R}^4$ with one straight line removed is conformal to $H^2 \times S^2$ [Proved here]: with r the distance from the line and t the coordinate along the line,

$$dt^2 + dr^2 + r^2 d\Omega^2 \;=\; r^2\Big[\frac{dt^2 + dr^2}{r^2} + d\Omega^2\Big] \quad (4)$$

and the bracket is the product of the hyperbolic plane H^2 (the upper half-plane model) and the unit sphere. If a 't Hooft line is placed on the line, every sphere surrounding the line carries $\mathbb{Z}_2$ flux 1 (an exact analogue of $n_{12} = 1$). And the sphere $\{t\} \times S^2_r$ corresponding to a point $(t, r) \in H^2$ winds an odd number of times with a loop C in the half-plane if and only if (t, r) lies inside the region enclosed by C [Proved here] (the intersection of C with the three-dimensional ball spanned by the sphere is the intersection of the segment $\{t\} \times [0, r)$ with C). This avoids the topological obstacle that stopped the road using local closed surfaces in $\mathbb{R}^4$ (item 8 of Table 1: a closed surface meets a plane an even number of times) by using the line as "the outside". Near the line ($r \to 0$) is the region of small spheres (weak coupling), far away ($r \to \infty$) is the infrared: **scale turns into position**.

The promise went only this far; on examination the result was negative [Reading] (physics).

- A container at height r with hyperbolic size 1 has Euclidean size of order r and carries a sphere S^2_r. This is the band quantity $\theta(r, r)$ itself (with S^2 in place of T^2).

- Hyperbolic nesting: the hyperbolic area of a unit container at height $2r$ (about $[-r, r] \times [r, 3r]$) is $2r\,(1/r - 1/(3r)) = 4/3$, and only 1–2 unit containers at height r fit inside it [Proved here] (computation of an integral). Hyperbolic area does not change under shape-preserving dilation, so nesting does not increase the number of sub-containers; what comes out is about $\theta(2r) \le \theta(r)^{1.3}$, not the exponent 4 of planar nesting.
- The hyperbolic area of a loop in the half-plane (from height d to $d+\ell$, length T) is $T(1/d - 1/(d+\ell)) \approx T/d$, and a uniform θ_0 independent of height gives only the perimeter law. The area law needs $\theta \le e^{-c(r\Lambda)^2}$ for unit containers at height r: the same condition as the band.

Conclusion: the conformal frame puts renormalisation into the language of space as "change of scale = translation in H^2", but kinematics alone does not give the area law, and the core (the growth of disorder through asymptotic freedom) remains in the same place. Its value lies in the clarity of a framework that states the $\mathbb{R}^4$ stage (16-09) without going through slabs. The relation between loops in the measure with a 't Hooft line and loops in $\mathbb{R}^4$ without the line is open, and there is a precedent of the comparison principle failing (26-08 § 04).

05

Three new candidates

We examined eight candidates for the band, and put three of them here (the rest are in § 06). The reasons for the choice: the reduction by compactness of the band is a reduction that can be written reliably on paper, and it moves "where the band is really hard"; the candidate "log-concavity of the Weyl measure" is a two-dimensional checkpoint that can be tried more cheaply than four dimensions, and is one of the few forms that does not fall into the type in which the source of monotonicity and the mechanism of disorder point in opposite directions (the lesson of 26-08 § 01); the candidate "local step function of a column" can be tried directly with the existing numerical tools for the column, and lets us look at the criterion "distinguish from the U(1) transition" directly.

Reduction by compactness of the band

(i) In one line. The band is a bounded closed interval $I = [\beta_a, \beta_b]$ in β, so the renormalisation comparison within the band holds by lowering β' only a little from β, and the band can be crossed in finitely many steps. All that is needed is "confinement at fixed β (two-sided asymptotics with errors of perimeter order)" at each β in the band.

This is set out with a proof as the Lemma Reducing the Comparison inside the Band of 26-06 § 06 [Proved here] (under the two-sided area law at fixed β, the uniform positivity and continuity of the rate, $N \ge N_1$, $M \ge M_0$). Here we write only its meaning as "a viewpoint".

(ii) The local quantity that carries it. Two-sided estimates of the vortex free energy in large volume at fixed β. As local quantities, the area rate $\sigma_N(\beta)$ of the column bias at $\ell \gg \xi(\beta)$,

and the error p of perimeter order.

(iii) Where non-commutativity enters [Reading] (physics). The lemma itself does not depend on the group. Non-commutativity enters entirely through the assumption (confinement for $SU(2)$ within the band). This candidate is a reduction, not a tool, and serves to move the place of the difficulty. After the reduction, what remains within the band is a problem at fixed β (estimates uniform in the volume, non-convex large fields, absence of a phase transition), and the difficulty across scales gathers at the upper end of the band (where the correlation length $\xi(\beta)$ grows sharply in lattice units) and at the seam with ultraviolet control.

(iv) Where it breaks. For $U(1)$ ($\beta \in (1.01, 2.7)$) and $\mathbb{Z}_2$ (on the weak-coupling side of the transition) the assumption is false, so no conclusion follows either. The criterion "distinguish from the U(1) transition" is carried by the assumption.

(v) Check against the ledger. Unlike item 6 of Table 1 (a proof by contradiction extending continuously from a small T^2), the only continuity used is the continuity in β of finite-volume quantities. It is in the same spirit as the idea of considering the step function of the continuum-limit coupling on a bounded interval, but here we stop at the lattice β and do not pass through the continuum limit.

(vi) Relation to known results. On the strong-coupling side, a statement of exactly this form exists at the level of a strand (the Strong-Coupling-End Expansion Proposition of 26-06 § 03). A proof of confinement for the four-dimensional $SU(2)$ lattice at $\beta \approx 1$ within the band was not found in the range we searched (the front line on the strong-coupling side is at small β, 26-08 § 07). Chatterjee (arXiv:2006.16229; abstract checked, text not yet checked) gives a result of the form: if the centre is non-trivial and correlations decay exponentially under arbitrary boundary conditions, then centre symmetry is not broken, and if it is not broken there is confinement. This could be an entrance to qualitative confinement at fixed β, but it does not give the perimeter-order precision of the two-sided area law at fixed β.

(vii) What is needed next. A road that gives the upper side of the two-sided area law at fixed β for each β by finite-box criteria (the mixing of box magnetic flux, the pressure margin), and a road that gives the lower side (that vortices are not too cheap = an upper bound on the string tension) by a variational trial vortex. Both are problems at fixed β, and boxes of size about $\xi(\beta)$ suffice, but at the upper end of the band $\xi(\beta)$ exceeds 10 in lattice units, and finite-box computations in four dimensions stop on scale (item 3 of Table 1). No numbers.

The log-concavity of the Weyl measure, and the height–azimuth decomposition

(i) An elementary fact [Proved here]. The density of the uniform measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ projected onto one coordinate t is

$$\rho_n(t) \;\propto\; (1-t^2)^{(n-3)/2}, \qquad t \in [-1,1] \quad (5)$$

Proof. The slice at height t is a sphere S^{n-2} of radius $\sqrt{1-t^2}$, whose area is proportional to $(1-t^2)^{(n-2)/2}$, and the width (length along the sphere) of the band of height dt is $dt/\sqrt{1-t^2}$. Multiplying gives (5). ∎ Hence for $n \geq 3$, ρ_n is log-concave (flat at $n = 3$), and at $n = 2$ it is log-convex (the arcsine distribution). For $SU(2) \cong S^3$ ($n = 4$), the density of $t = \frac{1}{2}\operatorname{tr} U$ is $\propto \sqrt{1-t^2}$ (the Weyl measure); for $U(1) = S^1$ it is $\propto 1/\sqrt{1-t^2}$.

In the two-dimensional $O(3)$ model, writing the spin as $s = (\sqrt{1-t^2}\cos\varphi,\ \sqrt{1-t^2}\sin\varphi,\ t)$, the uniform measure on S^2 is $dt\,d\varphi/(4\pi)$ ($n = 3$ of (5)), and $s_i \cdot s_j = \sqrt{(1-t_i^2)(1-t_j^2)}\cos(\varphi_i - \varphi_j) + t_i t_j$, so the model of the azimuths φ conditioned on the heights t is an XY model with couplings $J_{ij} = \beta\sqrt{(1-t_i^2)(1-t_j^2)} \geq 0$ [Proved here]. There is no frustration.

(ii) The local quantity that carries it. The effective azimuthal coupling $J_{\text{eff}}(b)$, measured in the coarse variables of a $b \times b$ block, and the fluctuations of the height.

(iii) Why we expect it to hold [Reading] (physics). The conditioned azimuth model is ferromagnetic, so the Ginibre inequality for XY (checking not yet done) can be used in the right direction: lowering the couplings lowers the correlations. What exponential decay needs is also an upper bound on correlations, so **the source of monotonicity and the mechanism of disorder point the same way**. The disorder comes from the fluctuations of the height, that is, from the measure (5) with $n \geq 3$. $O(2)$ has no height.

(iv) Where it breaks. $O(2)$: there is no height, and the azimuths remain XY with coupling β (the KT phase). Four-dimensional $U(1)$: there is no degree of freedom for the direction of the class (S^2). $\mathbb{Z}_2$: there is no continuous azimuth.

(v) Check against the ledger, and risks. Unlike items 22 (one-point Bakry–Émery) and 26 (renormalisation of curvature) of Table 1, it uses not curvature but the log-concavity of a projection and a conditional Ginibre inequality. Unlike item 10 of Table 1 (GKS for $\mathbb{Z}_2$), the conditioned model has no frustration. There are two risks. (a) There is an argument of Patrascioiu–Seiler (not yet checked) for the opposite conclusion (that $O(3)$ may have a KT-type phase) through embeddings of the equatorial direction, and this candidate stands at the site of that dispute. (b) At the level of a single site, the typical values of J_{ij} exceed the KT threshold, and the comparison says nothing. If it works, it works only at the block level, where there is no direct Ginibre comparison, and there is the risk of the transfer type of detour.

(vi) To four dimensions. One needs a representation in which the remaining $U(1)$, conditioned on n in $U = t + i\sqrt{1-t^2}\,n \cdot \sigma$ ($n \in S^2$ the direction of the class), is ferromagnetic, but this has not been constructed. Showing exponential decay at all temperatures for the two-dimensional lattice $O(n)$ ($n \geq 3$) (Polyakov's conjecture) is unsolved (the criterion "crosses the smallest band" of 26-08 § 07).

(vii) Numbers. None. The form of the first test is fixed: measure $J_{\text{eff}}(b)$ for the two-dimensional $O(3)$ ($\beta = 1.5, 1.7$), the control $O(2)$ (the KT phase and the disordered phase),

and the auxiliary $O(4)$ (whose height density $\sqrt{1-t^2}$ is the same as for $SU(2)$); if J_{eff} stays above the KT guide even at sizes at which correlations of $O(3)$ have already decayed, discard this candidate. If J_{eff} falls with b for $O(2)$, the measurement is lowering the coupling mechanically, and the result is not used. A block-level Ginibre comparison does not exist as a theorem, so even if it passes it remains a watch on a necessary condition.

The local step function of a column

(i) In one line. Replace the renormalisation comparison of the global ΔF by a comparison of column quantities with the outer frame held fixed:

$$f(2\ell, 2N, \beta) \;\le\; K \cdot f(\ell, N, \beta')^{c}, \qquad \beta' < \beta, \;\; \ell \ge \ell_{\min} \quad (6)$$

(K and c independent of ℓ and N). **(ii) The local quantity that carries it.** The conditional twist free energy ΔF_B and the bias $\theta_B = \tanh(\Delta F_B/2)$ of the column $T^2_N \times \ell \times \ell$ (the outer frame held fixed at an equilibrium configuration). Two quantities are compared: the median over frames of ΔF_B (with sign) and $\mathbb{E}|\theta_B|$.

(iii) Why we expect it to hold [Reading] (physics). In the existing numbers (26-09 § 07), the ℓ needed to keep θ small hardly grows with N for $SU(2)$, and grows for $U(1)$. Non-commutativity enters where the vortex can localise in a lump of the size of T^2 (a fractional instanton), because its action, by the Bogomolny bound, does not depend on the size of T^2 (25-02).

(iv) Where it breaks. In the Coulomb phase of $U(1)$ the flux spreads uniformly and the column bias decreases only as a power (about $\ell^{-1.4}$ over $\ell = 5 \to 8$), so the local β' should come out at or above β. For $\mathbb{Z}_2$ the sheet tension remains, likewise.

(v) Check against the ledger. Unlike item 27 of Table 1 (comparison of boundary conditions and geometries), it compares the same object at two scales. Nor does it use the correlation inequalities of item 2 of Table 1. Because of the criterion "not broken by small loops", small ℓ (3 or less) may fail through lattice artefacts, so the claim takes the form $\ell \ge \ell_{\min}$. It is a local version of the step function of the coupling defined with twisted boundary conditions (the method of de Divitiis et al.; not yet checked).

(vi) Numbers. There are as yet no numbers measuring (6) itself. The form of the test is fixed: fine side $(N, \ell) = (4, 4)$, coarse side $(2, 2)$ (and, if time allows, $(4, 6)$ and $(2, 3)$), with $SU(2)$ at $\beta = 2.3$, and $U(1)$ in the confining phase at $\beta = 0.9$ and in the Coulomb phase at $\beta = 1.3$. The prediction: $\beta' < \beta$ for $SU(2)$; for $U(1)$, $\beta' < \beta$ at $\beta = 0.9$ and $\beta' \ge \beta$ at 1.3 (the sign flips with the phase). If the sign does not separate between the two $U(1)$ points, this observable does not distinguish phases, and the candidate is lowered a grade. $(2, 2)$ has large artefacts, so it is treated as a sign until confirmed with $(4, 6)/(2, 3)$.

(vii) Difficulty. Showing (6) is a local version of the same difficulty as the global renormalisation comparison (the opening of § 02). What is gained is that the object of the

proof becomes "a statement about a finite region with its boundary held fixed", and that the assumption of the reduction by compactness of the band (the two-sided area law at fixed β) can be written in the language of columns.

06

Five other candidates

Of the eight candidates examined, we give the remaining five in one line each, with the reason none was given priority over the three of § 05. All are labelled "Reading (physics)"; those that include an elementary identity have "Proved here" added in their row.

Table 2 Five other candidates

Candidate	One line	Why it was not given priority	Label
Confinement as sign cancellation	Under the twist-summed measure P (weight $Z_0(\omega) + Z_1(\omega)$), $\theta = \mathbb{E}_P[\tanh(X/2)]$. Even if the glass is in the ordered phase in the upper half of the band ($\lvert X\rvert$ large), θ is small if the signs of X cancel	The distinction between groups appears in the growth as N is made large, so it is close to a restatement of the same quantity as the column face $f(\ell)$. Numerically it needs the ratio of partition functions of a disordered $\mathbb{Z}_2$ gauge theory for each ω, which is large in scale	Proved here (the identity only rewrites the definition $(Z_0 - Z_1)/(Z_0 + Z_1)$ of θ in terms of ω), Reading (physics)
Centre-only influence coefficients	The $\delta_c(L_0, \beta)$ of § 03. Condition the outside on the $SO(3)$ part only, and flip only the centre flux at the boundary	In the upper half of the band δ_c should also be close to 1: for the lower half of the band only (on the side of the work of pushing up the lower end of the band)	Reading (physics)
Diffusion of the sign of the flux by the flow of $\mathbb{Z}_2$ monopoles	The sign flux through each plane changes only by the parity of the number of closed $\mathbb{Z}_2$ monopole curves crossing between planes. The flux mixes through the winding density of monopoles	A $\mathbb{Z}_2$ version of the $U(1)$ monopole picture; the criterion "uses non-commutativity essentially" is weak. There is a dispute over whether confinement survives with monopole-suppressed actions (not yet checked)	Reading (physics)
Determinant in self-dual coordinates	In each sector, the difference between sectors of the Jacobian when the measure is viewed as a push-forward to the anti-self-dual part	A large risk of hitting item 25 of Table 1 (spin and the index theorem are a restatement of asymptotic freedom at one loop). The lattice has no self-dual decomposition	Reading (physics)

Candidate	One line	Why it was not given priority	Label
A quantitative version, uniform in N, of "discrete 1-form symmetry is not broken" for the slab reduced to two dimensions	Read slab confinement as a lattice version of a statement of the type that discrete 1-form symmetry is not spontaneously broken in two dimensions, and make its quantitative version (tension $\geq$ the density of walls of the cheapest sector) uniform in N	As a road, this is exactly the road of increasing N in the semiclassics of a small T^2 (of the same weight as the adiabatic continuity conjecture of Tanizaki–Ünsal); only the reading is new. The dimension conditions in the literature on generalised symmetries are not yet checked	Reading (physics)

07

What cannot be said

- (a) None of the candidates closes the band. For none of them have we shown a theorem at any β in the band.
- (b) The reduction by compactness of the band is a reduction, and we do not say it has made the band easier. The assumption of two-sided area laws inside the band contains confinement at intermediate coupling on the four-dimensional $SU(2)$ lattice (unsolved).
- (c) Even if the two-dimensional test of the log-concavity of the Weyl measure passes, it implies nothing for Polyakov's conjecture or for four dimensions. A block-level Ginibre comparison does not exist as a theorem.
- (d) We do not say that two-dimensional results lift to four dimensions. Only the form of the statement lifts.
- (e) The $SU(2)$ expectations of the four faces are all estimates. Only the column face $f(\ell)$ has numbers, and those are tendencies on small lattices.
- (f) The negative verdict on the conformal frame is a judgement at the level of estimates, that "kinematics alone does not give the area law"; it does not rule out roads that add other inputs in that frame.
- (g) We do not say that the candidates on this page are not combinations of known facts. The lemma of the reduction by compactness of the band and the log-concavity of the projection in the Weyl-measure candidate may be written somewhere.

08

The table of the five grades and the dependency graph

The candidates on this page are all "Reading (physics)". Paper proofs are set out only for the elementary facts that serve as footholds for the candidates.

Table 3 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The free-energy identity (1) for a bijection carrying one sector to the other	Proved here	§ 02 softness in the twist direction	A measurable bijection with a well-defined Jacobian	—
2	For a deterministic map the remainder grows with the area of the sheet, and the area law does not come out	Reading (physics)	Softness in the twist direction in § 02, "deterministic squeeze" of 26-08	—	—
3	The column decomposition $\lvert\theta\rvert \le \mathbb{E}\prod_B \lvert\theta_B(\Sigma)\rvert$	Proved here	Twist cost of a column in § 02, the framework of the Container Identity of 25-03	Conditional independence with the skeleton held fixed	—
4	Column bias: at $\ell \ge 5$, $SU(2)$ is 1/5 to 1/7 of $U(1)$, and the needed ℓ does not grow for $SU(2)$	Reading	26-09 § 07	$\beta = 2.3$ ($SU(2)$), 1.3 ($U(1)$), $N \le 8$, 3–12 frames	—
5	The $SU(2)$ expectations of the four faces, and where they break for $U(1)$ and $\mathbb{Z}_2$	Reading (physics)	§ 02 Table 1	Estimate	—
6	The Jensen sandwich (3) of the $\mathbb{Z}_2$ glass	Proved here	§ 03	The $U = \sigma V$ representation	—
7	The glass is on the ordered side in the upper half of the band; the glass road is restricted to the lower half	Reading (physics)	§ 03, 26-09 § 08	8^4; the conversion is crude	—
8	$\mathbb{R}^4 \setminus \text{line} \cong H^2 \times S^2$ (conformal), and the parity criterion for spheres and loops	Proved here	§ 04 (4)	—	—
9	The hyperbolic nesting area $4/3$, the hyperbolic area of a half-plane loop $\approx T/d$	Proved here	§ 04	Computation of an integral	—
10	In the conformal frame kinematics alone does not give the area law, and the core remains in the same place	Reading (physics)	§ 04	Negative on examination	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
11	Reduction by compactness of the band: the comparison within the band reduces to two-sided area laws at fixed β	Proved here	The Lemma Reducing the Comparison inside the Band of 26-06 § 06	Two-sided area law at fixed β, uniform positivity and continuity of the rate, $N \geq N_1$, $M \geq M_0$	—
12	The projected density (5), and the height–azimuth decomposition of $O(3)$ (XY with couplings $J_{ij} \geq 0$)	Proved here	§ 05 log-concavity of the Weyl measure	—	—
13	The mechanism of the log-concavity of the Weyl measure: the source of monotonicity and the mechanism of disorder point the same way	Reading (physics)	§ 05 log-concavity of the Weyl measure	No block-level comparison exists as a theorem	—
14	The prediction (6) of the local step function of a column: $\beta' < \beta$ for $SU(2)$; for $U(1)$ the sign flips with the phase	Reading (physics)	§ 05 local step function of a column	$\ell \geq \ell_{\min}$. No numbers for (6)	—
15	The identity $\theta = \mathbb{E}_P[\tanh(X/2)]$ of the candidate "confinement as sign cancellation"	Proved here	§ 06 Table 2	A rewriting of the definition of θ	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  the Transport Identity (1)
 2  RDG  remainder of a deterministic map grows with area ⟸ 1   physics
 3  PRV  the column decomposition                              framework of 25-03
 4  RDG  numbers for the column bias                           26-09 §07
 5  RDG  expectations of the four faces ⟸ 1, 2, 3, 4            physics
 6  PRV  the Jensen sandwich (3) of the glass
 7  RDG  the glass road is restricted to the lower half ⟸ 6     8⁴, physics
 8  PRV  conformal equivalence and the parity criterion (4)
 9  PRV  the hyperbolic nesting area
10  RDG  the conformal frame is negative ⟸ 8, 9                 physics
11  PRV  reduction by compactness of the band                  26-06 reduction lem
12  PRV  projected density (5), height–azimuth decomposition
13  RDG  mechanism of the log-concavity of the Weyl measure ⟸ 12   physics
14  RDG  prediction (6) of the local step function of a column ⟸ 3, 4, 11   physic
15  PRV  θ = E_P[tanh(X/2)]
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The conclusion-side nodes (the expectations of the

four faces, the scope of the glass road, the negative result for the conformal frame, and the mechanism and prediction of two of the three new candidates) are all "Reading (physics)", and the "Proved here" nodes are either elementary facts serving as footholds for the candidates, or a reduction whose proof is in another chapter. This page has no node for a statement that crosses the band.

What this chapter can say

What is said	What is not said
Proved here The elementary facts that serve as footholds for the candidates: the transport identity, the column decomposition, the Jensen sandwich of the glass, the conformal equivalence and the parity criterion, the projected density and the height–azimuth decomposition, $\theta = \mathbb{E}_P[\tanh(X/2)]$	That estimates crossing the band follow from them
Proved here That the comparison within the band reduces to two-sided area laws at fixed β (the Lemma Reducing the Comparison inside the Band of 26-06, under the assumption of two-sided area laws inside the band)	A proof of the assumption of two-sided area laws inside the band at any β. That the band has been made easier
Reading (physics) For the four faces and the three candidates, why we expect them to hold for $SU(2)$ and where they break for $U(1)$	An estimate uniform in the scale for any face or candidate
Reading (physics) That the $\mathbb{Z}_2$ glass road is restricted to the lower half of the band; that the conformal frame does not give the area law by kinematics alone	That the glass road pushes up the lower end of the band in the lower half

Sources and reproduction

Item	Kind	Source / tools
The four faces, the glass representation, the examination of the conformal frame	This book's study	The record of this book's study (the survey of roads that would close the crossover band, proposals of mechanisms from the physical point of view)
The three new candidates and the five others	This book's study	This book's study (the examination of candidates for the band). The Lemma of the reduction by compactness of the band is in 26-06 § 06
Numbers for the column bias and the glass diagnosis	Reading	26-09
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. Chapter 26-01
The action of fractional instantons	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166. Chapter 25-02
An entrance to confinement at fixed β	Abstract checked	S. Chatterjee, arXiv:2006.16229 (text not yet checked)

Item	Kind	Source / tools
Literature not yet checked	Checking not yet done	Kazakov–Zheng (bootstrap for lattice Yang–Mills), Patrascioiu–Seiler (the argument for a KT-type phase of $O(3)$), the Ginibre inequality for XY, de Divitiis et al. (the step function with twisted boundary conditions), the dispute over monopole-suppressed actions, literature of Coleman–Mermin–Wagner type for generalised symmetries, the Dobrushin–Shlosman criterion

Terms used in this chapter — the four faces, the adapted stiffness κ_w, the flux mixing of the box $\delta(L_0)$, the twist cost of the column $f(\ell)$, the pressure margin, the $\mathbb{Z}_2$ glass, frustration, the conformal frame, the compactness of the band, the Weyl measure, height and azimuth, the local step function of the column, the crossover band (the band). Their meanings are in A-02 Glossary.

Next chapter: 26-11 Ultraviolet control — what it means to close at infinitely many steps — the foundation under the two roads. What closing in one step means, and where the series stopped.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band ultraviolet control, guide to Part 15 · Kiichi, Shiori and Rin

Ultraviolet control — what it means to close at infinitely many steps

> This chapter is a guide. § 01 is a map of open questions.

What this chapter shows — When the lattice spacing a is sent to 0 and the number of renormalisation steps grows without bound, what does it mean to show, at every step and without the constants degrading, that "one step returns to the same form"? The body of this chapter has been expanded into the first half (15-00 to 15-10) of Part 15 "A record of attempts at ultraviolet control". This page keeps only the five lines of what this chapter shows, the statement of the decomposition of the ultraviolet wall, and a guide for each section.

Prerequisite chapters — 7-05 (the sequence of renormalisation group steps), 7-07 (four dimensions: Bałaban and MRS), 26-00 (confluence), 27-02 (the ledger of conditions).

Said first — This chapter deals with the "ultraviolet control" row of the ledger of conditions, a different task from the crossover band (the band). It deals only with the side of scales finer than the band. The continuum limit in a periodic cubic box was shown by the construction published on 2026-11-10 (at the level of the committee's assessment; Volume 2). What this chapter and Part 15 deal with is ultraviolet control on a different road set up by this book's study (small twisted boxes, and a form uniform in the lattice spacing), and that has not been shown. A construction with twisted boundary conditions and elongated boxes, and a statement in a form uniform in the lattice spacing, are not in the published construction either. What Part 15 gives is a statement of what has to be shown, line-by-line calculations, elementary lemmas, and numbers measured in the smallest systems; each of them only restates a part of ultraviolet control, or splits ultraviolet control into other conditions.

The order of this chapter

1. What this chapter shows — five lines
2. The decomposition of the ultraviolet wall — a class that closes with finitely many parameters, and three remainders (the statement only)
3. Guide to Part 15 — the order of reading

00

What this chapter shows

- Ultraviolet control means showing, at infinitely many steps and without the constants degrading, that the effective density "returns to the same form in one step". The number of steps is $K = \log_L(L_0/a) \to \infty$, and if the sum $\sum_k \delta_k$ of the one-step losses δ_k is finite uniformly in K, the constants survive Proved here (15-00).
- In four dimensions the coupling is dimensionless, so the sum of the losses of size g_k^2 per step diverges like $\ln K$ Proved here (arithmetic under the one-loop flow). That part must be absorbed exactly into the flow of the coupling, and the rest confined to a sum of finite size or to a form that contracts transversely (15-01, 15-03).
- For a group-valued field the measure of the fluctuations depends on the background, and that dependence η_k is the hard point. In the type where the law of the fluctuations does not depend on the background (a product of heat kernels, one per edge), the chain of one-step estimates for flat fields goes through unchanged on the group, giving $\eta = 0$ Proved here (one step only, without the Wilson action and without large fields; 15-07).
- In the smallest system, with the angle of the single coarse plaquette fixed, the slope $\sum f^2 = \beta'/\beta$ of the invariant curve and the transverse contraction $\lambda = \sum f^4 / \sum f^2$ are determined by a single projection, and $\lambda \le \max f^2$ Proved here (at tree level restricted to commuting configurations, one step seen through the marginal distribution of one coarse plaquette; 15-05). The source (the coefficient of θ^4 poured in at each step) is a number measured at one loop Reading .
- Parameters are best counted as "a one-parameter attracting class plus contracting shapes"; as a class that keeps the ratio of the quadratic couplings at 2, it does not close in one step Proved here , Reading ($SU(2)$, four in a row, leading term; 15-06). Bałaban's four-dimensional theorem assumes the smallness of the effective coupling, the theorem that removes this assumption is unpublished, and the one-step remainder including large fields (the net one-step bracket of the coupling) is not in the original sources Cited (15-02).

01

The decomposition of the ultraviolet wall — a class that closes with finitely many parameters, and three remainders

Take a family of weights $\{w_\theta\}$ indexed by finitely many parameters $\theta \in \Theta \subset \mathbb{R}^m$, and suppose one blocking step T

$$T\big(w_\theta\,(1+\rho)\big) = w_{\Phi(\theta)}\,\big(1 + r_\theta[\rho]\big) \quad (1)$$

takes the form (Φ is the flow of the parameters, $r_\theta[\rho]$ the remainder). We put a statement of the form that ultraviolet control follows, uniformly in the number of steps, from the following four Assumption (the form of the statement only; for details see 15-03 § 02).

- Class-closure condition: the number of parameters is finite independently of the step, and the class consists of the one-parameter attracting class of the one-step map and shape coordinates that contract at each step (the evidence in the smallest systems is in 15-06).
- Coupling-flow remainder condition: $\beta_{k+1} = \beta_k - 8b_0 \ln L + O(1/\beta_k)$.
- Shape remainder condition: of the type in which, with transverse contraction $\lambda < 1$ and one-step remainder $\varepsilon(\beta)$, the deviation is bounded by $\sup_j \varepsilon(\beta_j)/(1-\lambda)$ (15-03 § 04).
- Definition and continuity of the effective coupling: continuity within the domain of definition.

That the condition that the coupling stays small follows from the net one-step bracket of the coupling and the definition and continuity of the effective coupling is the Coupling-Smallness Induction Lemma Proved here (15-03 § 03; the net one-step bracket of the coupling written in the variable β is the coupling-flow remainder condition), and the upper bound on the deviation is the arithmetic of a geometric series Proved here. What has not been shown is that the class-closure condition and the coupling-flow remainder condition including large fields hold for the real four-dimensional step; these are the four open rows (15-08).

02

Guide to Part 15 — the order of reading

There are three reading orders (15-00 § 04): the shortest is 15-00 → 15-03 → 15-09, the line of calculation is 15-04 → 15-05 → 15-06, and the line of the wall is 15-02 → 15-07 → 15-08.

What this chapter can say

Can say	Cannot say
This page is a summary and a guide. What can be said is in the tables of each chapter of Part 15	Ultraviolet control in the form of this book's road (no chapter of Part 15 shows it; the continuum limit in a periodic cubic box was shown by the published construction)
Assumption The decomposition of the ultraviolet wall (the form of the statement)	That the class-closure condition, the coupling-flow remainder condition and the shape remainder condition hold for the real four-dimensional step

Sources and reproduction

The sources are in the source table of each chapter of Part 15 (the bibliography of the original sources is in the source table of 15-02).

Words used in this chapter — ultraviolet control, the number of steps, the same form, the three conditions of ultraviolet control, the decomposition of the ultraviolet wall, the class-closure condition, the coupling-flow remainder condition, the shape remainder condition, the continuity of the coupling, the attracting class, the transverse contraction. Their meanings are in A-02 Glossary.

Chapter to read next: 26-12 A sieve for arguments about the band — seven test models.
To read alongside: 15-00 Entrance — what ultraviolet control is.

A sieve for arguments about the band — seven test models

This chapter is a map of what is unsolved. § 05 is this book's study (computations on small lattices).

What this chapter shows — Arguments that would close the band have so far been tested against the criterion "distinguish from the U(1) transition" (26-06 § 08). This chapter widens the test models to seven: U(1) at weak coupling, Z_2 at weak coupling, finite non-abelian groups (the quaternion group, the binary icosahedral group and others), the adjoint-Higgs limit (which becomes the U(1) Wilson action), the first-order transition line of the SU(2) mixed fundamental + adjoint action and its endpoint, the bulk transition of SU(N), and the centreless group G_2. For each model we write which types of argument it rules out, and we give a table that sorts this book's Lean theorems and the candidate statements for the band into "true, false, not defined" for the seven models. In each cell we keep apart what was checked by this book's computations or Lean, what was checked in the literature, and what has not been checked. Finally we list, with their conditions, what was seen in computations on small lattices (counting flat connections of finite groups, the power law on a one-point lattice, the hysteresis of the mixed action, the U(1) constant with the aspect ratio fixed, a trial of the sign term by term). The overall map of ideas is in 16-11.

Prerequisite chapters — 26-04 Structure alone cannot cross, 26-06 Renormalisation comparison, 21-05 The minimal action of a vortex, 5-06 How U(1) differs from the non-abelian case, 5-10 Lattice gauge theory with finite groups.

Said first — This chapter gives no road across the band. There are no proofs in it. The seven models are tests for ruling out arguments, and no new theorem is put forward about any of them. Of the properties of these models, only those this book checked at the relevant place in an original source or a review are marked Cited ; the rest are written as "stated in the literature (not checked by this book)" and are not used as grounds in the text. **The discussion of the endpoint of the mixed action is at a fixed lattice spacing. It does not mean that the continuum SU(2) theory has no mass gap.** The view of widening the test models may itself be known to experts. This book writes it not as a new result but as a sieve used in this book's study.

The order of this chapter

1. How the sieve works — tests in two directions — quantities of confinement and quantities of the gap
2. The seven models — the properties of each, and the arguments it rules out
3. Types of argument, and the models that rule them out (Table 1)
4. Lean theorems and candidate statements for the band × the seven models (Table 2)
5. What was seen in computations on small lattices — five items
6. What this chapter does not say

Labels — Proved here = a hand count whose derivation is in this chapter / Cited = a fact from the literature checked at the relevant place in an original source or a review / Reading = a reading of this book's numbers on small lattices (with lattice, sweeps and seeds) / Assumption = a conjecture or expectation / Lean = a Lean theorem of this book (only inside the strong-coupling window, with the lattice spacing fixed). For details, see "Labels" in 6-00.

Notation. The SU(2) Wilson action is $\beta \sum_p (1 - \frac{1}{2} \operatorname{tr} U_p)$; the couplings of the mixed action are β_F for the fundamental representation and β_A for the adjoint representation; the hopping coupling of the adjoint Higgs is J. ΔF is the vortex free energy of a twisted box, and θ is the vortex ratio (19-02). The slab-tension quantity $s(N) = N^2 \sigma_N$ is the notation of 26-06, and is a different quantity from the one-loop slope s.

Words. The **maximal torus** is the abelian subgroup $\{\mathrm{diag}(e^{i\phi}, e^{-i\phi})\}$ of SU(2) (of the same form as U(1)). The **mixed action** is the Wilson action (the fundamental-representation term) with an adjoint-representation term added. A **first-order transition** is one in which the plaquette average and the like jump as functions of the coupling; a **critical endpoint** is the point where a line of first-order transitions ends. A **bulk transition** is a transition that comes from the lattice action and does not depend on temperature (a zero-temperature transition). It is a phenomenon on the lattice, not physics of the continuum limit. A **flat connection** (a flat configuration, in the wording of 21-07) is a configuration in which the holonomy of every plaquette equals the identity (or the sign of the twist, when there is a twist). The **frozen phase** is the phase of a finite-group gauge theory on the large-β side that sticks close to the flat connections.

01

How the sieve works — tests in two directions

If an argument that would close the band goes through in the same form in another model, and the conclusion is false in that model, then the argument is wrong somewhere or is not enough for the conclusion. This is the sieve. Until now the sieve was U(1) alone: U(1) has a transition inside the band ($\beta \approx 1.01$) to a non-confining phase, so an argument that closes the band must distinguish U(1) (26-06 § 08).

The sieve has two directions Assumption .

- An argument whose core is a **quantity of confinement** (the tension, the vortex ratio, the tension-doubling inequality and so on) must become false in a model that has a non-confining phase.
- An argument whose core is a **quantity of the gap** (the correlation length, mixing in a finite box) must be true in a model that does not confine but does have a gap. A quantity of confinement may be false there. The weak-coupling phase of Z_2 and the centreless group are candidates for this, but this book has not checked in the literature that they have a gap (model 2, model 7).

The mass gap and confinement are different statements (16-09 § 03), and the direction in which the sieve is used is decided by which kind of quantity the core statement is.

02

The seven models

Model 1 — U(1) at weak coupling

Properties. Four-dimensional U(1) lattice gauge theory has a non-confining phase at weak coupling. For the Villain form of the action, it has been shown that the expectation of the Wilson loop has a Coulomb-type lower bound and charges are not confined Cited (Guth 1980, checked in the abstract). Fröhlich–Spencer showed by a different method that four-dimensional U(1) lattice gauge theory has a deconfining transition to a massless (QED) phase; they write mainly in the Villain approximation and state in the text that all the results extend to the Wilson-action model (the extension uses techniques from another paper) Cited (1982, checked in the abstract and § 1 of the preprint). The position of the transition, $\beta \approx 1.01$, is a value from this book's record (26-06 § 04).

Its place on the SU(2) axis. Restricting elements of SU(2) to the maximal torus gives $\frac{1}{2}\operatorname{tr}\operatorname{diag}(e^{i\phi}, e^{-i\phi}) = \cos\phi$, so the SU(2) Wilson action $\beta(1 - \frac{1}{2}\operatorname{tr} U_p)$ is, on the maximal torus, the same expression as the U(1) Wilson action $\beta(1 - \cos\phi_p)$ (with the same β) Proved here . What can be said is only that the expressions for the action coincide; the SU(2) measure is not restricted to the maximal torus. The U(1) model at the same β is in the Coulomb phase for $\beta \gtrsim 1.01$. This book's estimate is that this section of the SU(2) axis is the section that needs an input distinguishing U(1) Assumption .

Arguments it rules out. Arguments using only reflection positivity, the centre and duality identities, and positivity of the transfer matrix (26-04). The renormalisation comparison with the aspect ratio fixed holds on the weak side with a uniform constant for U(1) too, so a comparison statement on the weak side does not distinguish U(1) (§ 05 (d)). What stops U(1) is where the trajectory crosses the section of the transition. However, with a quantification that stretches the aspect ratio, U(1) already fails on the weak side. When the core statement is written, the quantification must be stated explicitly.

Model 2 — Z_2 at weak coupling

Properties. Four-dimensional Z_2 lattice gauge theory has a first-order transition Cited (Creutz–Jacobs–Rebbi 1979, checked in the abstract; the position is the self-dual point $\beta \approx 0.44$, as stated in 26-06 § 04). The weak-coupling side is a non-confining phase; that it has a gap there is stated in the literature (not checked by this book).

Arguments it rules out. Arguments that claim to put a quantity of the gap at the core but use a quantity that becomes false at weak coupling in Z_2. The tension-doubling inequality is false in the non-confining phase of Z_2, since there the slab-tension quantity $s(N)$ goes to 0 (26-06 § 08) Reading . If the statement in the literature that the weak-coupling phase of Z_2 has a gap is correct, then the tension-doubling inequality is not a quantity needed for the gap. Under the same condition, a global logarithmic Sobolev inequality on the torus is expected to fail because mixing between holonomy sectors is slow, so it would not pass as a quantity of the gap Assumption .

Model 3 — finite non-abelian groups

The models. Finite subgroups of SU(2): the quaternion group Q_8 (order 8), the binary tetrahedral group (24), the binary octahedral group (48), the binary icosahedral group (120). All are non-abelian and contain -1 in the centre. A finite-group gauge theory is expected to enter a frozen phase at large β Assumption . The value of β at which it freezes is stated in the literature (not checked by this book).

Arguments it rules out. Arguments whose only input is "non-abelian as an algebra". Finite non-abelian groups are non-abelian, but at weak coupling they do not have the continuous fluctuations of SU(2) (asymptotic freedom). In this book's record there are points

where the covering inequality and monotonicity in β fail for Q_8 (21-01: monotonicity in β in three dimensions without background) Reading . The level of the values in the frozen phase is § 05 (a).

Model 4 — the adjoint-Higgs limit (which becomes the U(1) Wilson action)

The model. To the SU(2) Wilson action we add the term $-J\sum_{x,\mu} n(x)\cdot R(U_{x,\mu})\,n(x+\hat{\mu})$ of a unit vector $n(x)\in S^2$ at each site (adjoint representation) (R is the map from SU(2) to SO(3)).

The form at $J=\infty$ Proved here . As $J\to\infty$ the weight concentrates on configurations with $R(U_{x,\mu})n(x+\hat{\mu})=n(x)$. SO(3) acts transitively on S^2, so a gauge transformation can make $n(x)=e_3$ at every site. The condition then becomes $R(U)e_3=e_3$, that is, U commutes with σ_3, that is, U is an element of the maximal torus. The plaquette products then also lie in the maximal torus, and by the coincidence of the action expressions seen in model 1, the action becomes the U(1) Wilson action (with the same β). The remaining gauge freedom is U(1). That the limiting measure actually concentrates on this set is treated as a formal limit.

As a family. For every value of J, the group is continuous and non-abelian, the centre $\mathbb{Z}_2$ acts trivially on the adjoint field so centre symmetry remains, and the weights are positive Proved here . Reflection positivity is also expected to be kept in this family Assumption .

Arguments it rules out. Arguments that take only "the group is continuous and non-abelian, centre symmetry, positivity of the weights" as input and conclude confinement uniformly for all J in the family. These inputs are the same across the whole family, and $J=\infty$ is the U(1) Wilson action, which has the non-confining phase of model 1 (Fröhlich–Spencer, model 1) Cited . Adding the sign of asymptotic freedom to the inputs is not expected to change this Assumption . The one-loop coefficient of the continuum theory with one real adjoint scalar added (the theory of an adjoint scalar whose length is not fixed), in the normalisation of equation (3) of 4-06 ($16\pi^2 b_0$), is the $22/3$ of pure SU(2) minus $1/3$ for one real scalar (half of a complex scalar), which is 7 and remains positive Proved here . But the limiting theory at $J=\infty$ is itself U(1), and is not asymptotically free. What this model rules out are arguments that conclude confinement uniformly in J from inputs alone that hold for every J in the family.

Model 5 — the first-order transition line of the SU(2) mixed action and its endpoint

The model. The action $\sum_p\left[\beta_F\left(1-\frac{1}{2}\operatorname{tr}U_p\right)+\beta_A\left(1-\frac{1}{3}((\operatorname{tr}U_p)^2-1)\right)\right]$. $\beta_A=0$ is the Wilson action. The adjoint term does not see the sign of the centre.

Properties. In this generalisation of the SU(2) Wilson action (which includes the SU(2), SO(3) and Z_2 models as limits), four-dimensional Monte Carlo has reported that the first-order transition lines of SO(3) and Z_2 meet at a triple point and continue to a critical endpoint Cited (Bhanot–Creutz 1981, checked in the abstract). The position of the endpoint varies in the literature. The apparent endpoint that Bhanot–Creutz saw on small lattices is at about $(\beta_F,\beta_A)\approx(1.5,\,0.9)$ (as quoted in the text of Gavai). Gavai found no first-order bulk transition at $\beta_A=1.25$ on 6^4 to 16^4 lattices, and expects the endpoint to be at $\beta_A\geq 1.25$ Cited (Gavai 1996, checked in the abstract and the text). For a mixed action of the same type for SU(3), it has been reported that approaching the endpoint lowers the mass of the scalar (0^{++}) glueball while the masses of the other glueballs do not change, and that the endpoint is consistent with a critical point at which only the 0^{++} correlation length diverges Cited (Heller 1995, checked in the abstract; the ratio of the tension to the transition temperature is nearly constant). That the scalar mass goes to 0 at the SU(2) endpoint is stated in the literature (not checked by this book). That both sides of the line are confining phases is an inference from the line ending at the endpoint and connecting smoothly to the confining phase on the Wilson axis Assumption .

This book's computation. At $\beta_A=1.5$ and 2.0, strong hysteresis is seen even on a 4^4 lattice (§ 05 (c)) Reading . We do not read the position of the endpoint from this computation.

Arguments it rules out. Arguments that use "there is no transition on the β axis / the step map has no fixed point" over the class of local, reflection-positive, pure-gauge actions. Inside this class there are a line of first-order transitions and a critical endpoint. Those are facts specific to the Wilson axis, and do not extend to the class of actions. Since step 0 of the published construction is not the Wilson action but an effective action with a representation (16-03 § 05), this is not a decorative caution Assumption . Arguments that use "the rate is uniformly positive" over the class of actions cannot be decided by this model: across the first-order line the tension may only jump and remain positive, and what would break it is the gap closing at the endpoint. For the same reason, if at the endpoint the scalar mass goes to 0 in lattice units while the tension remains, then "if there is a tension, a soft argument uniform over the class of actions gives the gap" is false; but since this book has not checked this part for SU(2), we place it as cannot be decided (16-11 Table 3).

Caution. The endpoint is a phenomenon at a fixed lattice spacing. It does not mean that the continuum SU(2) theory has no mass gap. The lesson of this model is that what an argument about the band may use is "being on the renormalised trajectory", and that "being in the class of local, reflection-positive, pure-gauge actions" alone is not enough Assumption .

Model 6 — the bulk transition of SU(N)

Properties. For the plaquette action (Wilson action) of SU(N), it has been reported that the crossover between strong and weak coupling looks like a smooth crossover for SU(4) and is a strong first-order bulk transition for SU(6) Cited (Lucini–Teper–Wenger 2002, checked in the text). That it becomes first-order from SU(5) on, and that both sides of the transition are confining, are stated in the literature (not checked by this book).

Arguments it rules out. That arguments of the same type as for model 5 (arguments relying on the absence of a transition on the Wilson axis) do not extend to general SU(N). Anyone aiming at the mass gap for general SU(N) (A-08 § 06) must pass this sieve too.

Model 7 — the centreless group G_2

Properties. G_2 has a trivial centre, so a "quark" in the fundamental representation is screened by gluons, and the string between static quarks can break. Even so, G_2 gauge theory has been reported to confine (in the sense of the paper) Cited (Holland–Minkowski–Pepe–Wiese 2003, checked in the abstract). That it has a mass gap is stated in the literature (not checked by this book).

Arguments it rules out. Quantities written in terms of centre twists, vortices and the vortex ratio θ are not even defined for G_2 Proved here (there are no centre elements, so there is no twist). Arguments whose core is the tension-doubling inequality or the four faces of inequalities that use non-commutativity do not reach the mass gap for all groups. G_2 needs other quantities (the response to boundary fields and so on) (16-11 § 06).

03

Types of argument, and the models that rule them out

Table 1 Types of argument, and the models that rule them out ("rules out" means that the same inputs are present in that model too, and the conclusion is false or not defined there)

Type of argument (inputs)	Models that rule it out	Reason and label
Only reflection positivity, the centre and duality identities, positivity of the transfer matrix	U(1), Z_2	The same structure holds there too (26-04) Proved here
Only non-abelian as an algebra	Finite non-abelian groups	Non-abelian, but without continuous fluctuations. Failure of monotonicity (21-01, three dimensions, no background) Reading
Concluding confinement uniformly in the family from only: the group is continuous and non-abelian, centre symmetry, positivity of the weights	The adjoint-Higgs family ($J = \infty$ is U(1))	These are the same across the family (model 4) Proved here . The non-confining phase of the U(1) Wilson action Cited (Fröhlich–Spencer)
Being in the class of local, reflection-positive, pure-gauge actions, and "no transition on the axis / no fixed point of the step map"	The first-order line of the mixed action	Inside the class, a line of first-order transitions and a critical endpoint Cited Reading
"The rate is uniformly positive" over the class of actions, or "tension ⇒ gap" by a soft argument uniform over the class of actions	The endpoint of the mixed action (cannot be decided)	Fails only if the gap closes at the endpoint. The SU(3) analogue is Cited ; for SU(2) not checked by this book
No transition on the Wilson axis, for general SU(N)	The bulk transition of SU(N)	Strong first order for SU(6) Cited
Only quantities written with centre twists and vortices	G_2	The quantities are not defined Proved here
Claiming a quantity of the gap while using global mixing	Z_2 at weak coupling (if it has a gap)	Slow mixing between sectors Assumption . The gap of the weak-coupling phase of Z_2 is stated in the literature (not checked by this book)
A lossy renormalisation comparison with the aspect ratio fixed (weak side)	U(1) (does not rule it out on the weak side)	On the weak side it holds with a uniform constant for U(1) too (§ 05 (d)) Proved here Reading . The content of the distinction is pushed into the section of the transition

04

Lean theorems and candidate statements for the band × the seven models

Table 2 has as rows this book's Lean theorems (whether a theorem of the same type holds in the other model) and the candidate statements for the band, and as columns the SU(2) Wilson action (the reference) and the seven models. Labels in the cells: Lean this book's Lean, Proved here a derivation in this chapter, Reading numbers from this book's record, Cited checked in the literature. "—" is a cell this book has not checked, and is not filled in by conjecture. "Not defined" means that the model lacks something needed to define the quantity (a centre twist, a continuous generator).

Table 2 True, false, not defined

Row	SU(2) Wilson (reference)	U(1) weak coupling	Z_2 weak coupling	Finite non-abelian groups	Adjoint Higgs $J = \infty$	Endpoint of the mixed action	SU
Infinite-volume limit, mass gap and area law in the strong-coupling window (20-03, 20-04, 20-06)	True Lean (inside the window)	—	—	—	—	Out of scope (the endpoint is outside the window)	—
Reflection positivity, chessboard, $\theta \geq 0$ (20-01, 21-06)	True Lean	—	—	—	—	—	—
Thin-vortex floor (21-02)	True Lean	—	—	—	—	—	—
Thick-vortex floor (21-03; uses the twisted vacuum)	True Lean	— (the definition does not carry over: there is no twisted vacuum of action 0)	— (the same)	—	— (the definition does not carry over)	—	—
The minimal action of a vortex is positive, $S_N > 0$ (21-05; uses the twisted vacuum)	True Lean	— (the definition does not carry over: there is no twisted vacuum of action 0)	— (the same)	True Proved here (subgroups of SU(2) containing $\mathbf{i}, \mathbf{j}$)	— (the definition does not carry over)	—	—
Tension-doubling inequality	Open	False Reading (26-06 § 08)	False Reading (26-06 § 08)	—	False Proved here Reading ($J = \infty$ is U(1) Proved here, the U(1) values are Reading)	—	—
Softness in the twist direction	Open	—	Not defined Proved here	Not defined Proved here	—	—	—
Mixing of box magnetic flux	Open	—	—	—	—	—	—
Twist cost of a column	Open	—	—	—	—	—	—
Pressure margin	Open	—	—	—	—	—	—
Mixing statement for boxes of physical size	Open (a restatement of the mass gap in a strong form)	—	—	—	—	—	—

Grounds for the cells.

- **The row $S_N > 0$** Proved here. The definition of S_N (21-05) fixes the frame of the block to the twisted vacuum (the configuration of action 0 with the quaternions $\mathbf{i}, \mathbf{j}$ on the links). In finite subgroups of SU(2) containing the quaternions $\mathbf{i}, \mathbf{j}$ (an anticommuting pair) — the four groups of this chapter — the twisted vacuum of the frame is made of $\mathbf{i}$ and $\mathbf{j}$ only, so it can be built as a configuration of the subgroup. Configurations whose interior links are restricted to elements of the subgroup are a subset of the SU(2) configurations, so the lower bound on the action is not smaller than the SU(2) lower bound. The SU(2) lower bound is positive uniformly in the size of the block (the theorem `VortexPos.Svortex_ge_unif` of 21-05). Hence the minimal action of a vortex is positive for the subgroups too (a one-line consequence of the Lean theorem). For abelian groups such as U(1) and Z_2, a single twist cannot be satisfied with action 0 (multiplying the plaquette products over a closed surface leaves only the sign of the twist), so there is no twisted vacuum of action 0 and the definition does not carry over.
- **Expectation under a transferred definition** Assumption. If the definition is transferred to a form without the background twist, the action of the configuration that spreads the measured twist flux π evenly over m^2 plaquettes, multiplied by the number N^2 of transverse positions, is $N^2 m^2 (1 - \cos(\pi/m^2)) \leq N^2 \pi^2 / (2m^2)$, which goes to 0 as the block is made large (Proved here: on the

transferred definition). The same spreading can be done in SU(2) too, inside the maximal torus, once the background twist is removed. The minimal action of a vortex is positive in SU(2) because only the central ± 1 commute with the background twisted vacuum (the quaternions $\mathbf{i}$, $\mathbf{j}$), so the flux cannot be spread continuously (the detour in 21-05).

- **The row of the tension-doubling inequality** Reading . In the Coulomb phase of U(1), $s(N)$ is constant independent of scale, exactly on the borderline; in the non-confining phase of Z_2, $s(N) \to 0$ (26-06 § 08; values from closed-form models and small lattices).
- **The "not defined" cells** Proved here . G_2 has a trivial centre, so quantities written with centre twists (θ, the vortex floors, the tension-doubling inequality, and all of the four faces that use twists or centre magnetic flux) do not exist. Softness in the twist direction uses a derivative along the generator of the twist, so it is not defined for discrete groups (Z_2, finite groups).
- **On the "—" cells.** As far as we have read the statements of this book's Lean theorems (A-04 Table 4g and the pages 21-05, 21-07, 20-06), the proofs of the strong-coupling window, reflection positivity, the chessboard, $\theta \geq 0$ and the thin-vortex floor rely not on the non-commutativity of SU(2) but on other structure of the group (the Dobrushin condition, reflection, the central element -1, compactness), and are expected to carry over to U(1), Z_2 and finite groups Assumption . Theorems that use the twisted vacuum (the thick-vortex floor, the minimal action of a vortex) do not carry over to U(1) and Z_2, definitions included. We have not transferred the proof of each theorem to the other models and checked it, so in the table these are "—". The estimate of the behaviour of the four faces for U(1) is in 26-06 § 07 Table 5 (a physical estimate).

What can be read from Table 2 Assumption . So far the only row whose value differs between the reference column and a model column is the tension-doubling inequality (false for U(1) and Z_2, Reading). The row $S_N > 0$ is true for finite non-abelian groups (Proved here), and its definition does not carry over to U(1). That the facts specific to SU(2) in this book are all facts about the action of configurations is in 16-11 § 03. If we set the criterion that a core candidate is chosen from statements that satisfy both "false for U(1) at weak coupling" and "false at a point of the type of the mixed-action endpoint (if the gap closes at the endpoint; for SU(2) not checked by this book)", then filling in the "—" of Table 2 becomes the next piece of work as it stands.

05

What was seen in computations on small lattices

All are floating-point computations on small lattices, and the values from one run go only as far as "seen" or "did not fail in the trial". The scripts are from this book's study, and were run again for this chapter (the results are written below).

(a) Counting flat connections of finite groups (the 1^4 lattice)

Setting. The 1^4 lattice (one site, four links, six plaquettes). In the frozen limit ($\beta \to \infty$) the partition function is proportional to the number of flat configurations. The untwisted side Z_0 is the number of quadruples of link elements that all commute with one another. The twisted side Z_1 is the number of quadruples with $[U_1, U_2] = -1$ (commutator -1) and the other five pairs commuting.

Hand count Proved here (Z_1 of Q_8). In $Q_8 = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, the anticommuting pairs (U_1, U_2) are two non-central elements on different axes: $6 \times 4 = 24$ ways. U_3, U_4 commute with both U_1 and U_2, so they are the central ± 1: $2 \times 2 = 4$ ways. So $Z_1 = 96$.

Table 3 Numbers of flat connections (full enumeration; no seeds) Reading

Group	Order	Z_0	Z_1	$\ln(Z_0/Z_1)$
Q_8	8	736	96	2.037
Binary tetrahedral group	24	5856	96	4.111
Binary octahedral group	48	18816	384	3.892
Binary icosahedral group	120	76320	480	5.069

On running it again, Z_0 and Z_1 for the four groups agreed with the values above. Reading: in the frozen phase, the free energy of one twist is of the type of a constant independent of L (values seen on 1^4 only).

(b) The power law on a one-point lattice (SU(2), 1^4)

Hand count Proved here . The action is $S = \sum_{\mu<\nu}(1 - \frac{1}{2} z_{\mu\nu} \operatorname{tr} P_{\mu\nu})$, with twist $z_{12} = -1$. If the set of minimal-action configurations is a manifold of dimension m with non-degenerate transverse Hessian, then $Z(\beta) = \int e^{-\beta S}$ behaves as $Z \sim C\beta^{-(12-m)/2}$ (12 dimensions for four links), and $\beta\langle S\rangle = -\beta\, d\ln Z/d\beta \to (12-m)/2$. Twisted side: the minimum has U_1, U_2 anticommuting ($U_1 = i n_1 \cdot \sigma$, $U_2 = i n_2 \cdot \sigma$, $n_1 \perp n_2$: 3 dimensions) and $U_3, U_4 = \pm 1$. Since $m = 3$, $\beta\langle S\rangle \to 9/2$. Untwisted side: configurations in which the four links lie in one maximal torus (2 dimensions for the choice of torus + 4 angles: 6 dimensions), giving $\beta\langle S\rangle \to 3$. This family jumps in dimension at the central points, so a logarithmic correction is expected to enter from there Assumption . The ratio is $Z_1/Z_0 \sim c\beta^{-3/2}$ (with a logarithmic correction).

Computation Reading (Metropolis, 4000 chains × 3000 sweeps, the first 1000 sweeps discarded, one seed, errors only from the statistics of block averages).

β	**Twisted** $\beta\langle S\rangle$	**Untwisted** $\beta\langle S\rangle$
30	4.545(3)	2.862(2)
300	4.498(3)	2.897(2)
3000	4.497(2)	2.860(3)

Reading: the twisted side matches $9/2$ at $\beta = 300$ and 3000 ($\beta = 30$ gives 4.545), and the untwisted side is 0.10–0.14 below 3. The difference of power $-3/2$ was seen. The coefficient of the logarithm is not determined by this computation (the value at $\beta = 3000$ is lower than at 300, and there is a suspicion that thermalisation in the flat directions is slow). Rerun: $\beta = 300$ was run with the same seed, and agreed at 4.4977(33) and 2.8970(19).

(c) Hysteresis of the mixed action (SU(2), 4^4)

Setting Reading . The action of model 5, 4^4 periodic, Metropolis (simultaneous updates by sublattice). β_F is raised $0 \to 2.5$ and then lowered, 80 sweeps at each point, from a cold initial configuration. One seed for each β_A (two only for $\beta_A = 2.0$). We look at the maximum of "down − up" for the plaquette average $\frac{1}{2}\operatorname{tr} P$.

β_A	**Maximum of down − up (at that** β_F**)**	**Reading**
0	0.023 or less (everywhere)	No hysteresis. 0.49–0.52 at $\beta_F = 2.0$, 0.65 at 2.5
1.0	0.079 (1.25), 0.063 (1.5)	A size hard to distinguish from fluctuations
1.5	0.317 (1.0), 0.165 (1.25)	Strong
2.0	0.413 (0.5, 0.75), 0.205 (1.0). With another seed 0.516 (0.5), 0.430 (0.75), 0.311 (1.0)	Strong

Differences of the opposite sign appear in the same runs with sizes up to 0.08 ($+0.076$ at $\beta_A = 2.0$, $\beta_F = 0$, and so on), so differences of 0.08 or less are hard to distinguish from fluctuations in this computation. Reading: at $\beta_A = 1.5$ and 2.0, strong hysteresis was seen even on 4^4. The weak difference at $\beta_A = 1.0$ may be a finite-size effect of 4^4 (of the type of Gavai's re-examination, which reported the first-order transition disappearing on larger lattices). We do not read the position of the endpoint from this computation. Hysteresis also appears with slow dynamics, so neither the mass nor the tension at the endpoint comes out of this computation. Rerun: the four values of β_A were run with the same seeds, and all the values agreed.

(d) The U(1) constant with the aspect ratio fixed

Hand count Proved here . Using the slab approximation for the Coulomb phase of U(1) (equation (4) of 26-03, read with $e^2 = 1/\beta$, vortex area M^2 and transverse area N^2), $\theta(N, M, \beta) \approx 2\exp(-M^2/(2\beta N^2))$, and fixing the aspect ratio $r = M/N$, we get

$$\frac{\theta(2N, 2M, \beta)}{\theta(N, M, \beta')} = \exp\Big(\frac{r^2}{2}\Big(\frac{1}{\beta'} - \frac{1}{\beta}\Big)\Big)$$

which does not depend on N. Putting $\beta' = \beta - c$, $1/(\beta - c) - 1/\beta = c/(\beta(\beta - c))$ decreases in β, so uniformly over all $\beta \geq \beta_1$

$$K = \exp\Big(\frac{r^2 c}{2\beta_1(\beta_1 - c)}\Big) \quad (1)$$

gives the form of the renormalisation comparison $\theta(2N, 2M, \beta) \leq K\theta(N, M, \beta - c)$ (on the approximate formula).

Computation Reading (a numerical scan of the approximate formula: 4000 points in $\beta \in [\beta_1, 10^4]$, $N = 2, 4, 8, 16, 64$, $r = 1, 2, 4$; no seeds). The upper bound of the scan agreed with equation (1). Examples: with $c = 0.258$, $\beta_1 = 3$, $K = 1.016$, 1.065, 1.285 for $r = 1, 2, 4$. With $c = 0.258$, $\beta_1 = 10$, 1.0013, 1.0053, 1.0214. Running it again gave the same values.

Reading: on the weak side, the comparison with the aspect ratio fixed holds uniformly for U(1) too, in the form "runs to strong coupling at any speed c you like", and uses no non-commutativity at all. $K(r)$ grows without bound with r, so with a quantification that fixes the transverse size and stretches the longitudinal one, U(1) already fails on the weak side. The approximate formula drops the effect of monopoles (of order $e^{-c\beta}$ at finite β), so we have not checked whether the same constant holds for true U(1).

(e) A trial of the sign term by term (SU(2), 1^4)

Question. When each plaquette is given its own coupling β_p, does ΔF (on the background $z_{12} = -1$, with and without the measured twist z_{34}) increase in each β_p? We look at the sign of $\partial\Delta F/\partial\beta_p = \langle E_p\rangle_{z_{34}=-1} - \langle E_p\rangle_{z_{34}=+1}$ ($E_p = 1 - \frac{1}{2} z_p \operatorname{tr} P$) Proved here (the derivative of $\Delta F = \ln Z_{z_{34}=+1} - \ln Z_{z_{34}=-1}$).

Computation Reading (Metropolis, 3000 chains × 800 sweeps, the first 200 sweeps discarded, β_p uniform random in $[0.2, 3.0]$ for the 6 plaquettes; 6 sets without the adjoint term and 2 sets with adjoint $\beta_A = 1.0$, 48 differences in all). All 48 are positive, and the smallest difference is 0.029 (a set with one β_p at 0.2). The statistical ratios are 58–1700 (errors from block averages, which may come out small because of autocorrelation, but the signs do not move). Rerun: the first 2 sets (seed 1) were run, and all the values agreed.

Reading: 1^4 is a special lattice in which the six plaquettes are joined at one point, and β_p goes only up to 3, so we cannot say that "monotonicity term by term holds". The contrast with the failure of a similar monotonicity for the finite group Q_8 (21-01: monotonicity in β in three dimensions without background, under conditions different from this trial) is worth keeping on record. The next lattice to try is an elongated box (16-11 § 05).

06

What this chapter does not say

- It gives no road across the band. There are no proofs in it.
- The discussion of the endpoint of the mixed action is at a fixed lattice spacing, and does not mean that the continuum SU(2) theory has no mass gap. The position of the SU(2) endpoint varies in the literature, and this book has not checked that the scalar mass goes to 0 at the endpoint.
- We say neither true nor false for the "—" cells of Table 2.
- Of the properties of the seven models, those written as "stated in the literature (not checked by this book)" are not used as grounds for claims in the text.
- We draw no conclusions about large lattices or the continuum theory from values on small lattices.

What this chapter can say

Can say	Cannot say
Lean The reference column of Table 2 (this book's theorems, inside the strong-coupling window, with the lattice spacing fixed)	Theorems of the same type in the other models (the "—" cells)
Proved here That the expressions of the SU(2) and U(1) Wilson actions coincide on the maximal torus. That the adjoint Higgs at $J = \infty$ becomes the U(1) Wilson action (a formal limit). That with one real adjoint scalar the one-loop coefficient is 7 (positive). The row S_N (positive for finite subgroups; the definition does not carry over to abelian groups). The cells not defined for G_2 and discrete groups. $Z_1 = 96$ for Q_8. The count of powers on 1^4. The U(1) constant (1) with the aspect ratio fixed (on the approximate formula)	A check of the concentration of the limiting measure. A uniform constant for true U(1). $S_N = 0$ under the transferred definition
Cited The non-confining phase of U(1) (Villain action; the extension to the Wilson action in the text of Fröhlich–Spencer). The first-order transition of Z_2. The first-order line and critical endpoint of the mixed action, and the re-examination of the endpoint position (expected at $\beta_A \geq 1.25$). That the scalar mass drops near the SU(3) endpoint. The strong first-order bulk transition of SU(6). That G_2 has a trivial centre and its string can break	The mass at the SU(2) endpoint, the bulk transition of SU(5), the gap of Z_2 at weak coupling and of G_2, the freezing β of finite groups
Reading The numbers of flat connections of finite groups (full enumeration). $\beta\langle S\rangle$ of SU(2) on 1^4 (4000 × 3000, one seed). The hysteresis of the mixed action on 4^4 (80 sweeps per point, one seed for each β_A, two for $\beta_A = 2.0$). The 48 signs term by term (3000 × 800)	The position of the endpoint. That monotonicity term by term holds

Sources and reproduction

Item	Kind	Source or tool
The non-confining phase of U(1) (Villain action)	Cited	A. H. Guth, "Existence proof of a nonconfining phase in four-dimensional U(1) lattice gauge theory", Phys. Rev. D 21 (1980) 2291 (checked in the abstract)
The transition of U(1) to a massless phase, and the extension to the Wilson action	Cited	J. Fröhlich, T. Spencer, "Massless phases and symmetry restoration in abelian gauge theories and spin systems", Commun. Math. Phys. 83 (1982) 411 (checked in the abstract and § 1 of the preprint IHES/P/81/40; written mainly in the Villain approximation, with the statement in the text that all the results extend to the Wilson-action model; the extension uses techniques from another paper)
The first-order transition of four-dimensional Z_2	Cited	M. Creutz, L. Jacobs, C. Rebbi, "Experiments with a gauge-invariant Ising system", Phys. Rev. Lett. 42 (1979) 1390 (checked in the abstract)
The first-order lines, triple point and critical endpoint of the mixed action	Cited	G. Bhanot, M. Creutz, "Variant actions and phase structure in lattice gauge theory", Phys. Rev. D 24 (1981) 3212 (checked in the abstract)
The quotation of the apparent endpoint $(1.5, 0.9)$, and the re-examination on larger lattices (no first-order transition at $\beta_A = 1.25$; the endpoint expected at $\beta_A \geq 1.25$)	Cited	R. V. Gavai, "A study of the bulk phase transitions of the SU(2) lattice gauge theory with mixed action", Nucl. Phys. B 474 (1996) 446, arXiv:hep-lat/9603003 (checked in the abstract and the text)
The scalar mass approaching the endpoint of the SU(3) mixed action	Cited	U. M. Heller, "SU(3) lattice gauge theory in the fundamental–adjoint plane and scaling along the Wilson axis", Phys. Lett. B 362 (1995) 123, arXiv:hep-lat/9508009 (checked in the abstract)
The bulk transition of SU(N) (smooth for SU(4), strong first order for SU(6))	Cited	B. Lucini, M. Teper, U. Wenger, "The deconfinement transition in SU(N) gauge theories", Phys. Lett. B 545 (2002) 197, arXiv:hep-lat/0206029 (checked in the text)
That G_2 has a trivial centre and its string can break; confinement	Cited	K. Holland, P. Minkowski, M. Pepe, U.-J. Wiese, "Exceptional confinement in G(2) gauge theory", Nucl. Phys. B 668 (2003) 207, arXiv:hep-lat/0302023 (checked in the abstract)

Item	Kind	Source or tool
The formula for the one-loop coefficient	Proved here	One real adjoint scalar put into equation (3) of 4-06
Statements in the literature that this book has not checked: the gap of the weak-coupling phase of Z_2, the freezing β of finite groups (Petcher–Weingarten 1980), the scalar mass at the SU(2) endpoint, the bulk transition of SU(5) and the phases on both sides, the gap of G_2	—	Stated in the literature (not checked by this book). Not used as grounds for claims in the text
This book's Lean theorems	Lean	The declarations of 20-06, 21-02, 21-03, 21-05, 21-06, 21-07 (`VortexPos.SN_pos`, `TwistBlocks.thick_vortex_floor`, `TwistedWilson.thin_vortex_floor`, `TwistBlocks.theta_nonneg`, `TwistFlat.no_flat_twist11` and others). The statements were checked in A-04 Table 4g and on the pages 21-05, 21-07 and 20-06
The position of the U(1) transition, the position of the Z_2 transition, the values of the tension-doubling inequality in the three phases, the estimates for the four faces	Reading	26-06 § 04, § 07, § 08
(a)–(e) of § 05	Reading	Scripts from this book's study (numpy floating point, full enumeration and Metropolis). Run again for this chapter: (a) all, (b) $\beta = \mathbf{300}$, (c) all plus one other seed, (d) all, (e) the first 2 sets

Words used in this chapter — sieve, quantity of confinement, quantity of the gap, maximal torus, adjoint Higgs, mixed action, first-order transition, critical endpoint, bulk transition, centreless group, flat connection, frozen phase, twisted vacuum, hysteresis, aspect ratio, sign term by term. Their meanings are in A-02 Glossary.

Chapter to read next: 27-00 Entrance — the promise of this part. The overall map of ideas is in 16-11.

2026-10-15 · chapter Yang–Mills and the mass gap Part 27 Closing — conditional statements entrance, promise, map of four chapters · Kiichi, Shiori and Rin

Entrance — the promise of this part

> This chapter is a guide (the entrance to this part).

This part is a record of the study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — The promise on how Part 27 is written, and the differences that remain even when the ledger of conditions is empty. The figure of the two roads meeting at the band, shown again, and a map of 27-01 to 27-04.

Prerequisite chapters — 16-09 (the table of the seven stages a proof must contain). You can read this part without having finished Parts 23 to 26.

01

The promise of this part

Part 27 writes conditionally until the conditions disappear.

- The statement is split arrow by arrow, and each arrow carries a label. We call something a "theorem" only for arrows whose paper proof has been placed in this book. That is why the title of 27-01 is "the conditional statement".
- The conditions are placed one row at a time in the ledger of conditions (27-02). The ledger is written with the facts as they stand after the publication of 2026-11-10. The continuum limit in a periodic cubic box was shown by the published construction (at the level of the committee's assessment; Volume 2), but the form this book's road needs (twisted boundary conditions, elongated boxes, a form uniform in the lattice spacing) is not there, so the row of ultraviolet control remains. A row disappears when its paper proof has been placed in the book and, if it has a Lean entry, the re-check of that entry is done. As research advances, rows disappear, but they can also split and increase.
- Even if every row of the ledger disappears, the difference from the problem statement remains: the axioms on $\mathbb{R}^4$ (the infinite-volume limit, the recovery of rotations, non-triviality; the infinite-volume limit (along a subsequence) from a sequence of periodic boxes, and the reconstruction, were shown by the published construction, but the reduction from the family of twisted slabs on this book's road

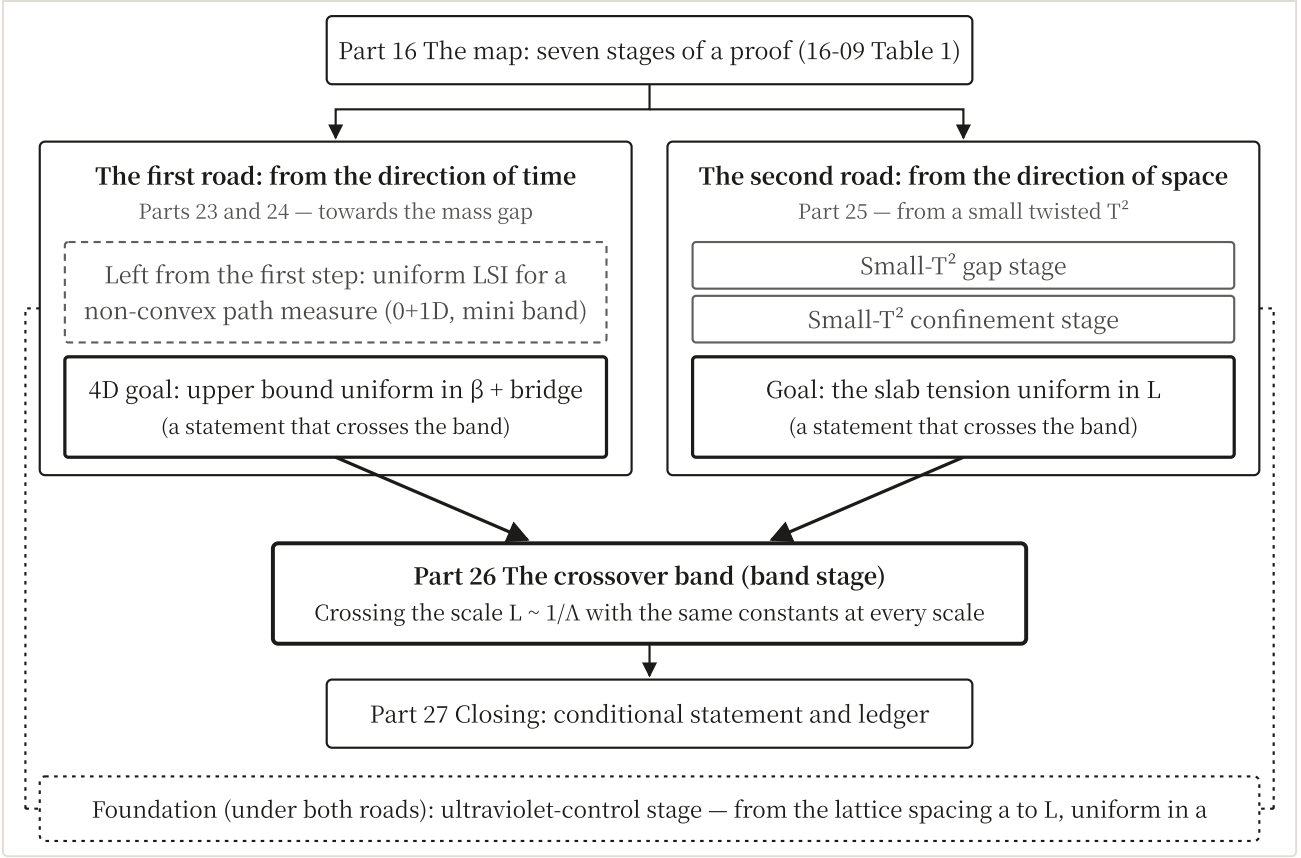
is open), all compact simple groups (existence was shown by the published construction for any compact simple group; the vortex tools cannot be used for groups without a centre), the arrow from the area law to the mass gap (tying the two roads together), and the verification and peer review of the whole. These four (the limit to R^4 [split into four rows in 27-02 Table 3], tying the two roads together, general groups, verification and peer review; seven rows with the breakdown) are placed in a separate frame outside the ledger (27-02 § 05). An empty ledger does not mean that the problem is solved.

- This part has no chapter called "solved".

02

The two roads meet at the band (shown again)

Figure 1 16-09 Figure 1, shown again. On both roads, the arrow flowing into the band leaves only from the row of the four-dimensional goal. The ultraviolet-control stage lies under both roads. Part 27 corresponds to the one box below the band



03

A map of the four chapters

Table 1 The chapters of Part 27

Chapter	Contents (one line)

Chapter	Contents (one line)
27-01 The conditional statement	Under assumptions, to the mass gap and confinement. The label on each arrow, and the arrow still left blank (from the area law to the gap)
27-02 The ledger of conditions	The conditions, one row each. The state in the record, the label, what is needed for it to disappear, the dependency diagram, and the differences that remain outside the ledger
27-03 The table of questions	A cross-index that looks up the questions of Parts 23 to 26 by chapter and item number
27-04 What is not claimed	A cross-index that looks up "what is not claimed" in Parts 23 to 26 by chapter and item number

Sources

Figure 1 is the same figure as in 16-09. The grounds for the four items of the promise are in 27-02 § 01, § 05 and § 06.

Next chapter to read: 27-01 The conditional statement.

2026-10-15 · chapter Yang–Mills and the mass gap Part 27 Closing — conditional statements
the conditional statement, a label for each arrow, the open arrow · Kiichi, Shiori and Rin

The conditional statement — the mass gap and confinement, arrow by arrow

> This chapter is a map of open questions. § 03 is this book's own investigation.

What this chapter shows — Under the assumptions of ultraviolet control, uniform acceptance of insertion, the base point at small T^2, frame decorrelation, the uniform slab tension, the uniform slab gap and tying the two roads together, which arrows the mass gap and confinement pass through. The gap comes out of the small-T^2 gap stage and the transfer matrix, and confinement out of the small-T^2 confinement stage and the band stage; we put a label on every single arrow. The only arrows whose paper proof we could place in the book are the algebra of the doubling iteration (the Doubling-Iteration Theorem) and slab confinement at fixed lattice spacing (the other arrow at fixed lattice spacing, the Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3), has a proof with citations in the paper, and its label in the book is Assumption). The arrow from the area law to the mass gap is open, and even the statement of what should be added has not been settled.

Prerequisite chapters — 16-09 (the table of the seven stages a proof must contain), 27-00 (the promise of this part), 25-04 (slab confinement), 25-05 (the insertion identities), 26-02 (reduction to a single inequality), 26-05 (the chessboard). The names and contents of the assumptions are in A-06 § 06, and the present state of each assumption is in 27-02 Table 2 (the rows of the ledger of conditions) and Table 3 (the differences that remain outside the ledger); this chapter refers to the assumptions by the names of the ledger rows.

Said first — This chapter shows neither the mass gap nor confinement. We do not title it "the conditional theorem" because many of the arrows that make up the statement have no paper proof. Only the arrows whose paper proof we could place in the book are called "theorems"; the rest are listed as labelled "arrows". Even if every row of the ledger of conditions disappears, a difference from what the problem statement requires remains (§ 05).

Order of this chapter

1. The form of the statement — the assumptions, the two conclusions, the two levels (Table 1)
2. The gap side — the small-T^2 gap and the transfer matrix (two arrows)
3. The confinement side — the arrows of small-T^2 confinement and of the band, the Doubling-Iteration Theorem
4. Tying the two roads together — the open arrow tying the two roads together, and the statement of the road from the direction of time
5. What cannot be said, and the differences that remain even when the ledger is empty
6. The label of each arrow, and the dependency graph — Table 2

01

The form of the statement — the assumptions, the two conclusions, the two levels

The statement of this chapter is not a single "if ... then" but a bundle of arrows. There are two conclusions.

- **The mass gap**: the spectrum of the transfer matrix (the Hamiltonian) above the vacuum is $\{0\} \cup [\Delta, \infty)$ with $\Delta > 0$ (4-11).
- **Confinement**: the expectation value of a large Wilson loop decreases exponentially in the area (the area law, 5-08).

The two are different statements. There is no known road that gets the gap from the area law, and "showing confinement also gives the gap" is a misconception (16-08). So this chapter derives the two conclusions each through its own bundle of arrows, and at the end places, still open, the arrow that joins the two on the same theory.

We also split the levels into two. The **theory on a small $\mathbf{T^2}$** is the theory on $\mathbb{R}^2 \times$ a small twisted T^2 (side L, 't Hooft twist $n_{12} = 1$), with the limit of lattice spacing $a \to 0$ taken. It is not what the problem statement requires but an intermediate target. The **theory on $\mathbf{R^4}$** is the theory on $\mathbb{R}^4$, reached from the theory on a small T^2 through the stage of widening the T^2 (the band) and the stage of the infinite-volume limit.

Table 1 lists the assumptions used. Their labels are all Assumption: rows without a proof. The names are taken from the contents.

Table 1 The assumptions placed by the statement of this chapter (the numbers are those of the dependency graph in § 06; the ledger column gives the row of 27-02; the stages are the seven stages a proof must contain, of 16-09)

No.	Ledger	Name	Contents (one line)	Stage	Label
1	Ultraviolet control	Ultraviolet control	From the lattice spacing a up to the scale L, a multiscale analysis uniform in a holds: locality and analyticity of the small-field effective action (the small-field analyticity condition), rarity of large fields (the large-field rarity condition), and the effective coupling stays in $(0, \gamma]$. The continuum limit in a periodic cubic box was shown by the published construction (Volume 2), but a construction with twisted boundary conditions or elongated boxes is not there (within what we searched). Ultraviolet control here is the still open condition that the same construction holds under twisted boundary conditions	Ultraviolet-control stage	Assumption
2	Uniform acceptance of insertion	Uniform acceptance of insertion	Under the small-field frame, the mean of $\tanh(W/2)$ for the dynamically driven vortex insertion is at most $1 - \delta$, uniformly in a (25-06)	Small-T^2 confinement stage	Assumption
3	Base point at small T^2	Base point at small T^2	At a small physical L_0, $\theta(L_0; T^2_{L_0}) \le 1 - \varepsilon_0$ (θ is the sector bias of the container, 25-03). The expected size is $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$ (a one-loop estimate). In the outline it follows from ultraviolet control and uniform acceptance of insertion	Small-T^2 confinement stage	Assumption

No.	Ledger	Name	Contents (one line)	Stage	Label
4	Frame decorrelation	Frame decorrelation (form with L fixed)	The correlations between the frames of containers decay, and the Wilson-loop estimate $\mathbb{E}\prod\theta$ can be split into the product of the individual θ at the loss of a constant	Small-T^2 confinement stage, band stage	Assumption
5	Small-T^2 gap (form uniform in the lattice spacing)	Decay of block observables (form with L fixed)	The correlations of block observables decay exponentially at a rate $m \geq c/L$ uniform in a (the form placed by the research programme for the theory on a small T^2)	Small-T^2 gap stage	Assumption
6	A form inside positivity of the 't Hooft tension (the place of the input to the reduction is the renormalisation comparison)	Lateral-widening inequality	With the container fixed, widening the transverse T^2 by a factor s increases θ by a factor of at most Cs^2 (strong form), or a form in which the total loss is finite (weak form). The assumption when the band is stated in the form of the paper on the reduction of the band (26-02)	Band stage	Assumption
7	The weak form of positivity of the 't Hooft tension	Uniform slab tension	The string tension $\sigma_{\mathrm{eff}}(L)$ of the slab $T_L^2 \times \mathbb{R}^2$ is positive uniformly in L	Band stage	Assumption
8	Uniform decay of block observables	Uniform slab gap	The correlations of block observables decay exponentially, uniformly in the lattice spacing and in the slab size L (the premise of the arrow of the R^4 gap)	Band stage	Assumption
9	Limit to R^4 (outside the ledger)	The limit to $\mathbb{R}^4$	The infinite-volume limit, the OS axioms, the recovery of rotation invariance, uniqueness of the limit, non-triviality	R^4 stage	Assumption
10	Tying the two roads together (outside the ledger)	Tying the two roads together	That the gap road (carrying the small-T^2 gap stage through the band) and the confinement road (from the small-T^2 confinement stage and the band stage to $\mathbb{R}^4$) give both conclusions on the same theory on $\mathbb{R}^4$. The statement of what it suffices to assume has itself not been settled	Small-T^2 gap stage, small-T^2 confinement stage, R^4 stage	Assumption (statement not settled)

We state the correspondence with the ledger first. In the ledger, frame decorrelation (form with L fixed) corresponds to the row of frame decorrelation, and the uniform slab gap to the row of uniform decay of block observables; the lateral-widening inequality and the uniform slab tension are treated as forms inside the row of positivity of the 't Hooft tension. The non-abelian input to the reduction to the band is placed in the ledger in the row of the renormalisation comparison. The row of reflection positivity of z is a closed row (its proof is placed in 26-04 § 03; the algebra of the last step is also closed in Lean (ReflPos)), and in this chapter it is used inside the chessboard arrow. The limit to R^4 and tying the two roads together are differences outside the ledger; they are not work of removing assumptions but work of carrying the conclusion as far as the problem statement. We put them in the table in this chapter so that they can be named as premises of arrows.

We split the "frame decorrelation" of Table 1 into two forms. Frame decorrelation (form with L fixed) is the form for splitting the container estimate on the confinement side, and decay of block observables (form with L fixed) is the form placed by the research programme on the gap side of the theory on a small T^2. They are assumptions of the same family, but as sentences they are different, and decay of block observables (form with L fixed) contains the quantity of the conclusion (the rate m) (§ 02). θ is the sector bias of the container $\theta(R; T_L^2)$ (side of the container R, side of the transverse T^2 L), a quantity different from the θ of A-06 Table 6 (the exponent of accumulation of the shortfall).

02

The gap side — the small-T² gap stage and the transfer matrix

The arrow of the small-T² gap (theory on a small T²). Under ultraviolet control and decay of block observables (form with L fixed), the theory on $\mathbb{R}^2\times$ a small twisted T^2 exists (as a subsequential limit of the distributions of block observables), is reflection positive and invariant under translations of $\mathbb{R}^2$, and its transfer matrix satisfies $\|T_t\| \le e^{-mt}$, $m \ge c/L$ on the orthogonal complement of the vacuum Assumption . This is the form put forward as a proposition by the research programme for the theory on a small T², and the proof is at the stage of an outline. The conclusion corresponds to the ledger row of the small-T^2 gap.

To say precisely what this arrow contains: the transfer-matrix step is an abstract lemma of the vessel stage. It is the general fact that, for a self-adjoint contraction on a real inner-product space, if the correlations decay at rate m on a dense set then the operator norm is at most e^{-m}; it is independent of the physical measure. So what carries the content of the mass gap inside the arrow of the small-T² gap is the part "decays at rate $m \ge c/L$" of the assumption decay of block observables (form with L fixed). Decay of block observables (form with L fixed) contains the quantity of the conclusion, and the arrow of the small-T² gap is most accurately read as "an arrow that restates uniform decay of block correlations as a gap of the transfer matrix". That the twist removes the zero modes (the adjoint field has no zero modes, 25-01) is a reason to expect decay of block observables (form with L fixed) to hold, not a proof of it.

The arrow of the R⁴ gap (theory on R⁴). The gap $m \ge c/L$ of the arrow of the small-T² gap approaches 0 as $L \to \infty$, so as it stands it is not a gap on $\mathbb{R}^4$. Carrying it to $\mathbb{R}^4$ needs the mass gap of the slab to be uniform in L (the uniform slab gap), and further uses, inside the infinite-volume limit (the limit to R⁴), that uniform exponential decay passes to a gap of the transfer matrix of the limiting theory Assumption . Here too, the content of the gap lies in the band assumption (the uniform slab gap). The phrase "the gap comes from the small-T² gap stage and the transfer matrix" describes the case in which the gap of the small-T² gap stage could be carried through the band uniformly in L, and the carrying is the problem of the band itself (26-00).

03

The confinement side — the small-T² confinement stage and the band stage

On the confinement side there are two arrows with a paper proof at the stage of fixed lattice spacing; the arrows at the stage of sending the lattice spacing to 0 and at the stage of the band rest on assumptions.

Two arrows at fixed lattice spacing. (a) The arrow of slab confinement: with the lattice spacing and the size N of the T^2 (in lattice units) fixed, the area law holds for every β, and the tension is bounded by $\sigma \ge -\ln\tanh(\beta N^2) \approx 2e^{-2\beta N^2}$ Proved here (25-04). The tension is set by the cost of the surface of a thin $\mathbb{Z}_2$ vortex, and approaches 0 in the continuum limit. (b) The arrow of the fractional-instanton rate (the Confinement Lower Bound by Fractional Instantons): on the lattice $T^2_N \times T^2_M$ with fixed $N \ge 2$, block side $\ell \ge 2$, $\beta \ge 1$ and $M \in 2\ell\mathbb{N}$, the bias θ of the n_{34} flux satisfies $-\ln\theta \ge c(N,\ell)\,\beta^{-a} e^{-\beta(S_*(N,\ell)+\varepsilon(\beta))} M^2$, $\varepsilon(\beta) = O(\sqrt{\ln\beta/\beta})$ Assumption (no proof is placed in this book; a proof with citations is in the paper: it cites the Chessboard Estimate and the Morse–Bott lemma, and the rigidity constants are shown only to exist. The paper is a draft and has not been through outside checking. 25-02 § 07 (e)). S_* is the minimal action of a fractional instanton inside a frame frozen to the vacuum, and the statement says that the rate is set by fractional instantons, not by the $2N^2$ of a thin vortex. In the ledger this is the

confinement-direction row of the upper end of the renormalisation comparison. The Reverse Estimate at a Fixed Plane (Theorem B of Paper 3) of the same paper is the bound with the size M of the plane fixed. Neither is uniform in N, and neither says anything about the continuum limit.

The arrow of the small-T² area law (theory on a small T²). Under ultraviolet control and uniform acceptance of insertion, Wilson loops in the $\mathbb{R}^2$ plane of the theory on $\mathbb{R}^2\times$ a small twisted T^2 have an area law uniform in a Assumption . The expected form of the tension is $\sigma \geq c' e^{-F/g^2(L)}/L^2$ (F is expected to be the fractional-instanton action $4\pi^2$). Among the parts of the assembly, the insertion identities and the estimate of the mean are closed on paper and in a finite version in Lean (25-05, 25-06), and in the outline the large-deviation step reduces to a chessboard estimate of the same type as the large-field rarity condition of ultraviolet control. The base point at small T² corresponds to the conclusion of this arrow written in the language of the container's bias.

The arrow of the reduction of the band (the form of the reduction of the band). Under the base point at small T², frame decorrelation and the lateral-widening inequality, $\theta(R;T^2_R)\leq e^{-cR^2}$ Assumption (the conditional theorem of the paper on the reduction of the band). The proof in the paper on the reduction of the band is a sketch, and the part "frame decorrelation splits the product at the loss of a constant" is not written as a numerical inequality. In this book we rewrite the assumptions in the form of numerical inequalities, and then put the remaining algebra on paper as the Doubling-Iteration Theorem.

Doubling-Iteration Theorem

Let θ be a function giving a value in $[0,1]$ to each pair of positive lengths (R,L). Let $R_0>0$, $R_k=2^k R_0$, and suppose that for positive constants K_1,K_2, for every $k\geq 0$,

$\theta(2R_k;T^2_{R_k})\leq K_1\,\theta(R_k;T^2_{R_k})^4$ (nesting), $\theta(R_{k+1};T^2_{R_{k+1}})\leq K_2\,\theta(R_{k+1};T^2_{R_k})$ (lateral widenin

holds. Put $K=K_1K_2$, $t_k=\theta(R_k;T^2_{R_k})$. If $K^{1/3}t_0<1$, then for every k

$$\theta(R_k;T^2_{R_k})\;\leq\;K^{-1/3}\exp\bigl(-c\,R_k^2\bigr),\qquad c=\frac{\ln\bigl(1/(K^{1/3}t_0)\bigr)}{R_0^2}>0. \tag{2}$$

Proved here Proof. Since $2R_k=R_{k+1}$, applying the two parts of (1) in turn gives $t_{k+1}\leq K_2\,\theta(2R_k;T^2_{R_k})\leq K_1K_2\,t_k^4=Kt_k^4$. Putting $u_k=K^{1/3}t_k$ gives $u_{k+1}\leq u_k^4$, and by induction $u_k\leq u_0^{4^k}$. Since $4^k=(R_k/R_0)^2$, we have $u_0^{4^k}=\exp(-(R_k/R_0)^2\ln(1/u_0))$, and (2) follows. ∎ Lean Doubling (`Doubling.iterate`) (assumptions: $c\geq 0$, $t_k\geq 0$, $t_{k+1}\leq c^3t_k^4$; conclusion: $c\,t_k\leq(c\,t_0)^{4^k}$), Doubling (`Doubling.decay`) (the same assumptions and $c>0$; conclusion: $t_k\leq(c\,t_0)^{4^k}/c$). Read $c=K^{1/3}$.

The starting value $K^{1/3}t_0<1$ is made from the base point at small T² with a single big jump. If, in the nesting with $T^2_{L_0}$ fixed, $\theta(sL_0;T^2_{L_0})\leq C_1(1-\varepsilon_0)^{s^2}$, and, by lateral widening, $\theta(sL_0;T^2_{sL_0})\leq C_2s^2\,\theta(sL_0;T^2_{L_0})$ can be said, then the $t_0\leq C_1C_2s^2e^{-\varepsilon_0 s^2}$ at $R_0=sL_0$ becomes smaller than $K^{-1/3}$ for $s^2\gg\varepsilon_0^{-1}\ln(1/\varepsilon_0)$ Proved here (the same algebra).

The Doubling-Iteration Theorem is algebra, and all the content is in (1) and in the two inequalities of the jump. The nesting inequality is exact as a partition inside a container with T^2 fixed (25-03), but the sub-containers share frames and are correlated, so splitting into a product with constant K_1 needs frame decorrelation. The lateral-widening inequality is the assumption of Table 1 itself. That is why the label of the arrow of the reduction of the band is Assumption, and only the part that is the Doubling-Iteration Theorem could be put on paper.

The arrow from the container estimate to the area law (from the form of the reduction to the area law). From $\theta(R;T_R^2) \le e^{-cR^2}$ and frame decorrelation, by the container estimate (the container identity of Mack–Petkova, 25-03 Cited), an area law with tension uniform in L Assumption . The global version that does not use frames uses the ratio of twisted partition functions and the Tomboulis–Yaffe inequality Cited , and in exchange the base point has to be shown globally (26-01).

The chessboard arrow, and the arrow from the two uniformities to the area law (the chessboard form). With the size of the T^2 fixed, from the reflection positivity of z Proved here (a proposition of the paper on the reduction of the band; ReflPos (`ReflPos.refl_pos` , `ReflPos.refl_cs`) is the algebra of the core, 26-04) and the Chessboard Estimate Cited , frame decorrelation is replaced by a single density κ_{cb}, and $|\langle W(C)\rangle| \le \kappa_{\mathrm{cb}}^{n(C)}$ Proved here (a statement with L fixed, 26-05). In this language the band becomes two uniformities, and under the uniform slab tension and the uniform slab gap an area law with tension uniform in L follows Assumption . Reading the uniform slab gap in terms of κ_{cb}, the power in the nesting within a fixed T_L^2 holds with constant 1 for all L (the Nested Supermultiplicativity Lemma of 26-05 Paper), and the content of the uniform slab gap uniform in L moves to the comparison across L, (C $_{\mathrm{cb}}$) — the lateral-widening inequality read in terms of κ_{cb} (the Reduction Read in the Chessboard Density of 26-02).

Confinement in the theory on R^4 follows from the area law of the arrow from the container estimate to the area law or of the arrow from the two uniformities to the area law, together with the limit to $\mathbb{R}^4$ Assumption .

04

Tying the two roads together — the open arrow tying the two roads together, and the statement of the road from the direction of time

The arrow tying the two roads together (open). The gap road of § 02 and the confinement road of § 03 both reach $\mathbb{R}^4$ through assumptions that cross the band (the uniform slab gap, or the lateral-widening inequality and the uniform slab tension). To state both conclusions on the same theory on $\mathbb{R}^4$, we need an arrow saying that the two roads give the same limiting theory and that both hold in that theory. This is tying the two roads together (a difference outside the ledger). Its label is Assumption, but unlike the other assumption rows, there is not yet a sentence saying what it suffices to assume.

Since there is no known road going directly from the area law to the mass gap, the arrow tying the two roads together cannot be written as "an extension of the confinement road". At the level of physical expectation the two are linked (excitations on a string of tension $\sigma > 0$ are heavy, 17-03), but that is not an arrow Reading (a physical estimate). The gap is derived by the road of § 02 (carrying the gap of the small-T^2 gap stage through the band), confinement by the road of § 03, and the arrow tying the two roads together is limited to the role of saying "the two are statements about the same theory".

The statement of the road from the direction of time. The conditional statement of the road of Part 23 has a different form. From the upper side $\xi^2\uparrow$ ($C_P \le K\xi_{\mathrm{RG}}^2$) of the diffusive-relaxation-time assumption and the one-sided bridge (the One-Sided Bridge Proposition, $C_P \ge \kappa\xi^2$), a lower bound on the gap in the physical units of the lattice, $m_{\mathrm{phys}} \ge (\kappa/K)^{1/2}$, follows at each β Proved here (algebra on top of the assumptions, 23-01 § 06). Saying this uniformly as $\beta \to \infty$ is the four-dimensional goal of Part 23 (the ledger row "uniformity of the relaxation-time upper bound and the one-sided bridge"; the physical-unit target level of 23-15 Table 5), and this goal too is a statement that crosses the band. The uniform slab gap of the road from the direction of space and the "$\xi^2\uparrow$ uniform in β, and the bridge" of the road from

the direction of time are uniformity statements of the same type, and neither has been shown inside the band (26-07).

05

What cannot be said, and the differences that remain even when the ledger is empty

None of the arrows of this chapter shows the mass gap or confinement. The only things whose paper proof could be placed in the book are the Doubling-Iteration Theorem (algebra), the arrow of slab confinement at fixed lattice spacing, and the chessboard arrow with T^2 fixed. The arrow of slab confinement says nothing about the continuum limit, and the arrow of the fractional-instanton rate (the Confinement Lower Bound by Fractional Instantons) has no proof in the book (the label is Assumption; a proof with citations is in the paper) and is not uniform in N. The arrow of the small-T^2 gap and the arrow of the small-T^2 area law for the theory on a small T^2 are propositions of the research programme, at the stage of an outline. The band forms, the lateral-widening inequality and the uniform slab tension, contain the quantities of the conclusion (bias, tension) in the form "uniform in L", and the uniform slab gap states decay of correlations in a form uniform in L. The band statement itself carries the content of the conclusion. The statement of this chapter is a conditional statement only insofar as the band is placed among the assumptions.

Even if every row of the ledger disappears, a difference from what the problem statement requires remains: all compact simple groups (for groups without a centre the vortex tools cannot be used), the OS axioms on $\mathbb{R}^4$ with the recovery of rotations and non-triviality, and the arrow from the area law to the mass gap. These are in the separate frame "the differences that remain outside the ledger" of 27-02 § 05 (the limit to R^4 and tying the two roads together of Table 1 also go there). Nor is the ledger a table that only shrinks: as research advances, rows can split and new rows can appear.

06

The label of each arrow, and the dependency graph

We list the arrows in the format of the five grades (23-00 § 01). The label is the label of the arrow, not of its premises. Even for an arrow whose premise is Assumption, the label is Proved here if the inference from the premises to the conclusion is written on paper, and Assumption if it is at the stage of an outline.

Table 2 The label of each arrow ("—" marks rows without machine checking; the Lean column gives only the names of bundles re-checked by the main session)

Arrow	Premises ⟹ conclusion	Label	Grounds	Conditions written in the same row	Machine checking
Arrow of the small-T^2 gap	$1 \wedge 5 \Longrightarrow$ existence of the theory on a small T^2 and the gap $m \geq c/L$	Assumption	Proposition of the research programme for the theory on a small T^2 (outline). The transfer-matrix step is an abstract lemma of the vessel	Decay of block observables (form with L fixed) contains the quantity m of the conclusion. Uniform in a; L is small	—

Arrow	Premises ⟹ conclusion	Label	Grounds	Conditions written in the same row	Machine checking
Arrow of the R^4 gap	Arrow of the small-T^2 gap $\wedge$ 8 $\wedge$ 9 $\Longrightarrow$ the gap on $\mathbb{R}^4$	Assumption	Outline	The uniform slab gap (the ledger row of uniform decay of block observables) is uniform decay of block observables; what the arrow of the R^4 gap adds is the transfer to the limiting theory	—
Arrow of slab confinement	Lattice spacing and N fixed $\Longrightarrow$ area law for every β, $\sigma \geq -\ln \tanh(\beta N^2)$	Proved here	25-04	In the continuum limit the tension is 0. Comparison with earlier arguments (Borgs, Tomboulis–Yaffe type) remains	The Lean column of 25-04
Arrow of the fractional-instanton rate	Fixed $N \geq 2$, $\beta \geq 1 \Longrightarrow -\ln \theta \geq c\,\beta^{-a} e^{-\beta(S_* + \varepsilon(\beta))_M 2}$	Assumption	Confinement Lower Bound by Fractional Instantons (no proof placed in the book; a proof with citations is in the paper. 25-02 § 07 (e))	Not uniform in N. Says nothing about the continuum limit. The paper is a draft	—
Arrow of the small-T^2 area law	1 $\wedge$ 2 $\Longrightarrow$ area law for the theory on a small T^2 (uniform in a)	Assumption	Proposition of the research programme (outline). The parts of the insertion are Proved here in 25-05 and 25-06	The form of the tension $e^{-F/g^2(L)/L^2}$ is an expectation	The Lean column of 25-05 (finite version)
Arrow of the reduction of the band	3 $\wedge$ 4 $\wedge$ 6 $\Longrightarrow \theta(R; T_R^2) \leq e^{-cR^2}$	Assumption	Conditional theorem of the paper on the reduction of the band (the proof is a sketch)	Stated at $L \geq L_{\min}(a)$, where there are no lattice artefacts	—
Doubling-Iteration Theorem	(1) $\wedge$ $K^{1/3} t_0 < 1 \Longrightarrow$ (2)	Proved here	The proof in § 03	Algebra independent of the content of θ	Doubling (`Doubling.iterate`, `Doubling.decay`) (assumptions in § 03)
Arrow from the container estimate to the area law	Arrow of the reduction of the band $\wedge$ 4 $\Longrightarrow$ area law with tension uniform in L	Assumption	The Container Identity Cited (25-03) and frame decorrelation	The global version uses Tomboulis–Yaffe Cited and a global base point	—

Arrow	Premises ⟹ conclusion	Label	Grounds	Conditions written in the same row	Machine checking
Chessboard arrow	Reflection positivity of $z \Longrightarrow$ with T^2 fixed, $\lvert\langle W(C)\rangle\rvert \le \kappa_{\mathrm{cb}}^{\eta(C)}$	Proved here	The proposition of the paper on the reduction of the band and the Chessboard Estimate Cited (26-05)	A statement with L fixed. Transverse reflections are not positive because of the twist	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`) (algebra of the core)
Arrow from the two uniformities to the area law	Chessboard arrow $\wedge$ 7 $\wedge$ 8 $\Longrightarrow$ area law with tension uniform in L	Assumption	Outline	The uniform slab tension contains the quantity of the conclusion	—
Arrow of the R^4 area law	(Arrow from the container estimate to the area law, or arrow from the two uniformities to the area law) $\wedge$ 9 $\Longrightarrow$ area law on $\mathbb{R}^4$	Assumption	Outline	—	—
Arrow tying the two roads together	Arrow of the R^4 gap $\wedge$ arrow of the R^4 area law $\wedge$ 10 $\Longrightarrow$ the mass gap and confinement on the same theory on $\mathbb{R}^4$	Assumption	Open arrow. The statement of tying the two roads together is not settled	The area law alone does not give the gap	—
Arrow of the road from the direction of time	$\xi^2\uparrow \wedge$ one-sided bridge $\Longrightarrow m_{\mathrm{phys}} \ge (\kappa/K)^{1/2}$ at each β	Proved here	23-01 § 06 (algebra on top of the assumptions)	A statement at each β. Making it uniform in β is a statement that crosses the band	The Lean column of 23-01

Dependency graph (the numbers of the premises are smaller than the number of the conclusion; 1–10 are the assumptions of Table 1):

```
 1  ASM  ultraviolet control (three conditions)                          ultraviolet-control stag
 2  ASM  uniform acceptance of insertion                                 small-T² confinement sta
 3  ASM  base point at small T²: θ(L₀;T²_L₀) ≤ 1 − ε₀                    small-T² confinement sta
 4  ASM  frame decorrelation (form with L fixed; container form)         small-T² confinement sta
 5  ASM  decay of block observables (L fixed; contains rate m ≥ c/L)     small-T² gap stage
 6  ASM  lateral-widening inequality (strong / weak form)                band stage
 7  ASM  uniform slab tension                                            band stage
 8  ASM  uniform slab gap                                                band stage
 9  ASM  limit to R⁴                                                     R⁴ stage
10  ASM  tying the two roads together (statement not settled)
11  PRV  reflection positivity of z                                      Lean ReflPos.refl_pos, r
12  CIT  Container Identity (Mack–Petkova), Tomboulis–Yaffe inequality
13  PRV  arrow of slab confinement (lattice spacing and N fixed)         tension → 0 in the conti
14  ASM  arrow of the fractional-instanton rate (fixed N, β ≥ 1)         proof with citations in
15  PRV  Doubling-Iteration Theorem                                      Lean Doubling.iterate, d
16  PRV  chessboard arrow: T² fixed, |⟨W(C)⟩| ≤ κ_cb^n(C) ⟸ 11
17  ASM  arrow of the small-T² gap: m ≥ c/L ⟸ 1, 5
18  ASM  arrow of the small-T² area law (uniform in a) ⟸ 1, 2
19  ASM  arrow of the reduction of the band: θ(R;T²_R) ≤ e^(−cR²) ⟸ 3, 4, 6, 15
20  ASM  arrow from the container estimate to the area law (reduction form) ⟸ 4, 12, 19
21  ASM  arrow from the two uniformities to the area law (chessboard form) ⟸ 7, 8, 16
22  ASM  arrow of the R⁴ gap ⟸ 8, 9, 17
23  ASM  arrow of the R⁴ area law ⟸ 9, 20 (or 21)
24  ASM  arrow tying the two roads together: mass gap and confinement on the same R⁴ theory ⟸
```

The cycle check only looks at whether, for every "⟸", the numbers on the right are smaller than the number on the left. Among the assumption rows, those that contain the quantity of the conclusion are decay of block observables (form with L fixed) (the rate m, in a form uniform in a) and the lateral-widening inequality and the uniform slab tension in the form of positivity of the 't Hooft tension (bias, tension, in a form uniform in L). This is not circular, but it shows that what the arrow of the small-T^2 gap, the arrow of the reduction of the band and the arrow from the two uniformities to the area law add is only a restatement of uniformity and algebra, and that the content lies on the side of these assumptions. The ledger (27-02 § 04) writes its rows in forms that do not contain the quantity of the conclusion. Decay of block observables (form with L fixed), the lateral-widening inequality and the uniform slab tension, which do contain it, are "restatements" of ledger rows rewritten to fit the arrows of this chapter, not the ledger rows themselves. The content of the arrow of the R^4 gap also lies on the side of the uniform decay of correlations in the uniform slab gap. The arrow of slab confinement and the arrow of the fractional-instanton rate are nodes that are not among the ancestors of the arrow tying the two roads together; they record what can be said on paper at the stage of fixed lattice spacing. The arrow of the small-T^2 area law writes the same content as the base point at small T^2 in the language of the theory on a small T^2, and in the graph we kept the base point at small T^2 as a leaf.

What this chapter can say

What is said	What is not said
The doubling iteration (the Doubling-Iteration Theorem): if nesting and lateral widening are assumed in the form of the numerical inequalities (1), then $\theta(R_k;T^2_{R_k}) \le K^{-1/3}e^{-cR_k^2}$ Proved here	The two inequalities of (1) themselves (frame decorrelation and the lateral-widening inequality)

What is said	What is not said
The area law with the lattice spacing and the T^2 fixed (slab confinement) Proved here	Showing, inside this book, the bound on the fractional-instanton rate at fixed N (the Confinement Lower Bound by Fractional Instantons) (the label is Assumption; a proof with citations is in the paper). A bound uniform in N. The tension in the continuum limit
The chessboard estimate with T^2 fixed, $\lvert\langle W(C)\rangle\rvert \le \kappa_{\mathrm{cb}}^{n(C)}$ Proved here	Comparing κ_{cb} across scales (the uniform slab gap)
That the statement has been split into arrows, with a label on each arrow. That the gap and confinement come out of different bundles of arrows	Any of the two gap arrows, the arrow of the small-T^2 area law, the arrows from the reduction of the band to the R^4 area law, and the arrow tying the two roads together (outlines on top of assumptions, or an open arrow). The mass gap, confinement, the continuum limit
That the arrow of the road from the direction of time (a lower bound on the gap at each β) is algebra on top of the assumptions Proved here	The form uniform in β (a statement that crosses the band)

Sources and reproduction

Item	Kind	Source / tools
Names and contents of the assumptions in Table 1	This book	A-06 § 06, 27-02
The form of the arrow of the small-T^2 gap and the arrow of the small-T^2 area law (the research programme for the theory on a small T^2)	Assumption	The research programme as rewritten in this book's study (SU(2) on $\mathbb{R}^2\times$ a small twisted T^2). The propositions are at the stage of an outline
Arrow of slab confinement	Proved here	25-04
Arrow of the fractional-instanton rate	Assumption	Theorem A (the Confinement Lower Bound by Fractional Instantons) and Theorem B (the Reverse Estimate at a Fixed Plane) of the paper on lattice fractional instantons (draft) "Fractional instantons confine on the lattice: a rigorous semiclassical bound for SU(2) gauge theory on a twisted small torus" (A-07). No proof is placed in the book (a proof with citations is in the paper)
Arrow of the reduction of the band, chessboard arrow, the definition of the lateral-widening inequality	Assumption, Proved here	The conditional theorem, the proposition on reflection positivity and the chessboard corollary of the paper on the reduction of the band (draft) "Where the twisted-slab route to confinement stops: a corrected window target, a reduction to transverse widening, and SU(2) versus U(1) numerics" (A-07)
Doubling-Iteration Theorem	Proved here	The proof in the text. The algebra is the Lean bundle `Doubling`
Container Identity	Cited	G. Mack, V. B. Petkova (1980). Checked in the source table of 25-03
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985). Checked in the source table of 26-01
Arrow of the road from the direction of time	Proved here	23-01 § 06

Terms used in this chapter — the conditional statement, arrow, the theory on a small T^2 and the theory on R^4, ultraviolet control, uniform acceptance of insertion, the base point at small T^2, frame decorrelation, the lateral-widening inequality, the uniform slab tension, the uniform slab gap, tying the two roads together, the doubling iteration, the chessboard density κ_{cb}, the open arrow. Their meanings are in the A-02 Glossary and A-06 § 06.

Next chapter to read: 27-02 The ledger of conditions — the present state of each assumption, and what is needed for it to disappear.

The ledger of conditions

This chapter is a map of open questions. § 01, § 02 and § 06 are guide material.

What this chapter shows — A ledger that lists, one row at a time, the conditions placed by the conditional statement of Part 27 (27-01). For each row: the mark of its state in the record, its label in the book, what is needed for it to disappear, the rows it presupposes, and the chapters that treat it. A diagram of the dependencies between rows. And, in a frame separate from the ledger, what remains as a difference from the problem statement even when every row of the ledger has disappeared.

Prerequisite chapters — 16-09 (the table of the seven stages a proof must contain, and the figure of the two roads). The rows of the first road are taken from 23-15 Table 1 and 23-15 § 07.

Said first — The ledger is a list of conditions, not a gauge of progress. We do not count its rows and read off "this much left". This chapter adds no new results. The proofs and counterexamples for the contents of the rows are in the chapters in the right-hand column (including chapters still being written).

Order of this chapter

1. The definition of "a row disappears" — the rule placed at the head of the ledger
2. A copy of the table of stages — the marks of 16-09 Table 1, and the rows of the ledger (Table 1)
3. The ledger — twelve rows and their breakdown (Table 2)
4. The diagram of dependencies between rows — the numbers of premises are smaller than those of conclusions
5. The differences that remain outside the ledger — what remains even when every row has disappeared (Table 3)
6. The day rows increase — the ledger is not a table that only shrinks

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01

The definition of "a row disappears" — the rule placed at the head of the ledger

Rule

A row of the ledger disappears when, and only when, the following two are both in place. (1) A paper proof of the statement of the row has been placed in some chapter of this book (a proof existing in the record is not enough). (2) If the row has a Lean entry, the theorems written in that entry have passed a re-check on the side of this book that reads the axioms again.

The point of this rule is to separate the state of the record from the state of the book. Even a row that is closed in the record (mark ■) is not removed from the ledger until its paper proof has been placed in a chapter of the book. For example, reflection positivity of z was a row closed in the record, and its proof has been placed in 26-04 § 03. The algebra of the last step is also closed in Lean (ReflPos). This row is closed.

One more thing. When a row disappears, the rows that use it as a premise do not disappear automatically. For example, even if the base point at small T^2, frame decorrelation and the renormalisation comparison all disappear, positivity of the 't Hooft tension disappears only when the paper for the reduction from the base point at small T^2, frame decorrelation and the renormalisation comparison to positivity of the 't Hooft tension has been placed in the book (26-02). And even if positivity of the 't Hooft tension disappears, not one of the differences outside the ledger (§ 05) moves.

02

A copy of the table of stages — the marks of 16-09 Table 1, and the rows of the ledger

Table 1 copies the column of marks of 16-09 Table 1 and lists which rows of the ledger bear on each stage. The marks mean the same as in 16-09: they mark the state in the record, and are not a gauge of how far things are proved (16-09 § 02). The contents of the stages and the column of what is known are kept only in 16-09 and are not repeated here.

Table 1 The marks of the seven stages a proof must contain (a copy of 16-09 Table 1) and the rows of the ledger (■ closed in the record / ◐ partial / □ open)

Stage	Mark	Ledger rows
Vessel stage	◐	Reflection positivity of z (writing out the assembly of the vessel is not set up as a row. § 06)

Stage	Mark	Ledger rows
Ultraviolet-control stage	◐	Ultraviolet control (the continuum limit in a periodic cubic box was shown by the published construction; the form needed by this book's road is open)
Small-T^2 gap stage	◐	Small-T^2 gap
Small-T^2 confinement stage	◐	Uniform acceptance of insertion, base point at small T^2 (and the form of frame decorrelation with L fixed)
Band stage	□	In the language of confinement: frame decorrelation, renormalisation comparison, positivity of the 't Hooft tension; in the language of the gap: band comparison (in the language of the gap), uniform decay of block observables. In the language of the time side: uniform gap of the double well (in miniature), uniformity of the relaxation-time upper bound and the one-sided bridge
R^4 stage	◐	Outside the ledger: the four parts of the limit to R^4, tying the two roads together (infinite volume from a sequence of periodic boxes, and the reconstruction, were shown by the published construction)
General-group stage	□	Outside the ledger: general groups (existence was shown by the published construction for every compact simple group)

03

The ledger — twelve rows and their breakdown

Table 2 is the ledger. Rows 1–10 are the rows of the second road (from the direction of space), and the uniform gap of the double well and the uniformity of the relaxation-time upper bound and the one-sided bridge are the rows of the first road (from the direction of time). The rows of the band are set up in two languages. Frame decorrelation, the renormalisation comparison and positivity of the 't Hooft tension are rows in the language of confinement (the vortex free energy, the charged sector with electric flux), and the band comparison (in the language of the gap) and uniform decay of block observables are rows in the language of the mass gap (the neutral sector without electric flux). We put the ledgers of the two roads into one because both roads stop at the same band (16-09 Figure 1). The rows are numbered in an order in which premises come before conclusions (§ 04). We write for the case of the group SU(2), the Wilson action on the lattice, and the 't Hooft twist $n_{12} = 1$ put on T^2.

Table 2 The ledger of conditions (to the right of the row name is the mark = the state in the record. The label is the label in the book, for each claim within the row. "What is needed for it to disappear" is what is needed in addition to (1)(2) of the rule in § 01. The six rows under the renormalisation comparison are its breakdown)

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
1 **Ultraviolet control** ◐	A multiscale analysis uniform in a from the lattice spacing a up to a small physical scale L_0 (locality and analyticity of the small-field effective action, rarity of large fields, the effective coupling staying small). On a small T^2	Assessment The continuum limit in a periodic cubic box was shown by the published construction (at the level of the committee's assessment; Volume 2) / Assumption A construction with twisted boundary conditions or in an elongated box, and a statement in a form uniform in the lattice spacing, are not in the published construction (within the range searched). The ultraviolet control of this row is a condition that is still open: that the same construction stands under twisted boundary conditions	Extending the published construction to the form this book's road needs (twisted boundary conditions, elongated boxes, a form uniform in the lattice spacing) (Volume 2, 16-03, the seam in Table 2 of A-08). On the side of this book's road (Part 15), completing the written-out steps of Bałaban (including the steps the original sources call unpublished), and verifying them. An assumption that depends on asymptotic freedom, which does not apply to U(1) or $\mathbb{Z}_2$ Premises: — Chapters: 7-07, 25-00, 26-11 (summary) → Part 15 (15-00–15-09; the decomposition statement is in 15-03, the four rows that do not close in 15-08) Tables that split this row into small questions: 15-09 Table 1 (the table of ultraviolet questions, with the current state of the answers), 15-08 Table 1 (the four rows that do not close) The small-field contribution to the coupling holds, under conditions, in the same form on the group as in the sources for flat fields; what the sources lack is the explicit quadratic term between blocks and the group version of its local cancellation (15-08 § 06)
2 **Small-T^2 gap** ◐	The twist removes the zero modes, and the gap of the transfer matrix $\geq c/L$ is uniform in a	Proved here the twist part (uniqueness of the vacuum, the gap $4\sin^2(\pi/2N)$ of the two-dimensional lattice Laplacian, $S_{\min} > 0$ in the (1,1) sector; N fixed) / Assumption the form uniform in a	Under ultraviolet control, writing the lower bound on the gap uniformly in a Premises: ultraviolet control Chapters: 25-01

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
3 **Uniform acceptance of insertion** ◐	Under the small-field frame, the Barker-type acceptance rate $b_F = (1 - \mathbb{E}[\tanh(W/2)])/2$ of the vortex insertion dynamically driven by a procedure of length $m(a)$ is at least $\delta/2$, uniformly in a. Said in terms of the mean, the mean of $\tanh(W/2)$ is at most $1 - \delta$ (W is the work of the insertion). "Uniform in the lattice spacing" becomes false unless the length of the procedure $m(a) \propto (L/a)^2 (r/a)^2$ is put into the statement (with the length fixed, the dissipation grows like $(L/a)^2/m$)	Paper the sandwich of acceptance rates: for the Metropolis type $e^{-\Delta F} O \le a_F \le \min(e^{-\Delta F}, O)$, for the Barker type $\frac{1}{2} a_F \le b_F \le \min(a_F, 1/(1+e^{\Delta F}))$. O is the overlap of the path measures of the forward and reverse paths, and $O \ge \frac{1}{2} e^{-D}$ (D is the dissipation). As a corollary, uniform acceptance of insertion is, up to constants, the same as boundedness of the insertion free energy (ΔF bounded uniformly in a) $\wedge$ the uniform lower bound on the path overlap (O bounded below uniformly in a). Uniform acceptance of insertion $\Rightarrow$ base point at small T^2 is one line ($b_F \le 1/(1+e^{\Delta F})$), and uniform acceptance of insertion contains the base point at small T^2 (Lean LatticeGaugeAcceptanceSandwich (`barker_le_of_crooks`, `overlap_sandwich`). The algebra of inequalities with a Crooks-type identity placed as an assumption; the Bretagnolle–Huber step is outside Lean) / Cited linear response of the dissipation (Sivak–Crooks 2012) / Assumption an estimate of the friction $\zeta \le \mathrm{Var}/\mathrm{gap}$ uniform in a, and the remainder of linear response / Reading making the procedure longer lowers the work, and $m \cdot D$ approaches a constant for $m = 64$–256 (the computation record (2026-10-22). The acceptance rates in 25-08 Tables 1 and 2 are Barker-type values. Measurement conditions in 25-08)	A lower bound uniform in a for the uniform lower bound on the path overlap: an estimate of the friction (boundedness of the fluctuation per plaquette of the face, and the gap of the heat bath of the container $\gtrsim (a/r)^2$; on the side of ultraviolet control and the small-T^2 gap, not on the side of frame decorrelation and the renormalisation comparison) and an estimate of the remainder of linear response. Boundedness of the insertion free energy is the content of the base point at small T^2, so removing uniform acceptance of insertion also means attaching a measurable certificate to the base point at small T^2 Premises: ultraviolet control, small-T^2 gap Chapters: 25-06, 25-08

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
4 **Base point at small T^2** ◐	At a small physical length L_0, in the symmetric container $(D_{L_0} \times T^2_{L_0}$, twist $n_{12} = 1)$, $\theta(L_0) \le 1 - \varepsilon_0$ uniformly in a, $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$	Cited the form of the string tension (Tanizaki–Ünsal, arXiv:2201.06166 eq. (2.37), dilute regime, one loop) and the numerics of fractional instantons on the twisted lattice (Bergner–González-Arroyo–Soler, arXiv:2505.10396) / Paper Lemma on the Upper Bound for the Minimal Action of a Symmetric Box: the minimal action of the symmetric box satisfies $S_{\min}(N,N) \le 2N^4(1-\cos(\pi/N^2)) < \pi^2$ for every N (a lattice version of 't Hooft's constant-curvature solution; an abelian configuration is placed explicitly. Lean LatticeGaugeAbelianTwistAction (`totalAction_eq`, `totalAction_lt_pi_sq`). That the minimal action of sector (1,1) is at most the action of this configuration is outside Lean). With the lattice fixed, the rate as $\beta \to \infty$ is $S_{\min}(N,N)$ (the Laplace principle) / Assumption the form uniform in a. It splits into two conditional rows: ultraviolet control around a vortex background (a background of constant curvature, including the extension to non-flat backgrounds) and insertion acceptance in a symmetric container (uniform acceptance of insertion in a symmetric container of size L_0)	The two rows, ultraviolet control around a vortex background and insertion acceptance in a symmetric container. The one-loop estimate of the work in insertion acceptance in a symmetric container becomes a ratio of determinants of the lattice fluctuation operator (explicitly writable) around the constant-curvature background. A lower bound on the action (the lattice BPS bound) is not needed for the base point at small T^2 (it is needed on the upper side of the dilute regime, in the converse direction of the upper end of the renormalisation comparison). Finite volume does not appear in the base point at small T^2 (it is on the side of the upper end of the renormalisation comparison and positivity of the 't Hooft tension). The lattice version is the same semiclassical picture as the upper end of the renormalisation comparison Premises: ultraviolet control, uniform acceptance of insertion Chapters: 25-02
5 **Reflection positivity of z** ■	The ensemble summed over twists is positive under reflections through lattice points in directions 3 and 4. As a corollary, the Chessboard Estimate	Proved here / Lean: ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`) (the algebra of the core on non-negative weights)	— (the proof has been placed in 26-04 § 03, and the same section has the fact that transverse reflections are not positive because of the twist. The algebra of the last step is also closed in Lean (ReflPos). This row is closed) Premises: — Chapters: 26-04, 26-05
6 **Frame decorrelation (the confinement side)** ◐	The correlations between the frames of containers decay uniformly in L. Taking the container to be the whole interior of a block, on a fixed T^2_L the nested inequality $\kappa_{\mathrm{cb}}(sR;L) \le \kappa_{\mathrm{cb}}(R;L)^{s^2}$ holds with constant 1 for all L, R, s	Paper Nested Supermultiplicativity Lemma (nested supermultiplicativity. It uses only the Markov property and the nested form of Mack–Petkova, and does not use reflection positivity. Lean LatticeGaugeNestedBlocks (`nested_supermultiplicative`) is an abstract form on a finite probability space; the Markov property is outside Lean). With L fixed, reflection positivity of z and the chessboard replace it by a single density κ_{cb} (Proved here, L fixed). The chessboard is used only in the step that descends from κ_{cb} to a single container and Wilson loops / Assumption the comparison across L (lateral comparison of the chessboard density)	Writing the reduction of 26-02 with $\kappa_{\mathrm{cb}}(L)$ instead of θ gives $C_D = 1$, and the part uniform in L does not remain as an independent assumption Paper. What remains is the comparison across L (lateral comparison of the chessboard density) $\kappa_{\mathrm{cb}}(R;L') \le C(L'/L)^2\kappa_{\mathrm{cb}}(R;L)$, which is the lateral-widening inequality read in terms of κ_{cb}. There is no nesting sideways (tiling T^2_{2L} with four copies of T^2_L makes the twist even, and transverse reflections are not positive because of the twist), so this comparison sits in the same place as the renormalisation comparison. Split into columns, it becomes the same problem as one of the faces of the middle of the renormalisation comparison (twist cost of a column). If the base point at small T^2 is kept in terms of θ, a one-scale row $1 - \kappa_{\mathrm{cb}}(L_0;L_0) \ge \eta(1 - \theta(L_0;L_0))$ is needed. Placing the paper (the Nested Supermultiplicativity Lemma, and the reduction written with κ_{cb}) in the book Premises: reflection positivity of z (only the descending step) Chapters: 26-02, 26-05

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
7 **Renormalisation comparison** (the core of the band) □	The vortex free energy at side $2N$ and coupling β is at most a constant times that at side N and coupling β', with $\beta' < \beta$. Non-perturbatively, for every β	Six rows of breakdown	An estimate with the error kept within a constant factor, for every β. An argument that closes the band must be able to distinguish U(1) (U(1) has a transition inside the band, at $\beta_c \approx 1.01$). The construction published on 2026-11-10 gives no handle inside the band (it carries every step at weak coupling and stops at the lattice of spacing 1). At the weak end there is a non-perturbative input that "the finer the step, the weaker the coupling", but it is a statement about the coefficients of the effective action, and it works only where the coefficients are very large (coefficients of about 9×10^5 or more, at very weak coupling). It is not the "input that distinguishes U(1)" needed in the middle of the band (Volume 2) Premises: reflection positivity of z, frame decorrelation Chapters: 26-06
Strong-coupling end of the renormalisation comparison ■	The range $\beta \lesssim 10^{-79}$	Assumption (there is a proof in the record. The constants are crude and the bookkeeping is a sketch, so it stays Assumption until the paper is placed in the book)	Writing out the steps, and improving the constants. In the strong-coupling window (β about 10^{-3} or less), this book's Lean has the area law and a vortex floor whose rate per area does not depend on the transverse size (20-06; the lattice spacing is fixed and the prefactor remains; it is not the renormalisation comparison itself) Chapters: 26-06
Lower end of the renormalisation comparison □	Pushing confinement and the comparison up to $\beta \approx 0.3$–1	Assumption	A finite-size criterion by mixing of box magnetic flux, and computer assistance (a certificate with interval arithmetic) The computer-assisted chain for the hierarchical model is a theorem about the Migdal–Kadanoff image itself, and Tomboulis's representative (the image shrunk uniformly) is pushed up in the transverse coordinate by at least (5/3)(1−α) in one step and leaves the tube, so it cannot be used as it stands as a tool for the lower end of the renormalisation comparison either Proved here (the note in 15-04 § 05) Chapters: 26-06, 15-04

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
Middle of the renormalisation comparison □	One inequality that uses non-commutativity. Any one of the four faces (softness in the twist direction, mixing of box magnetic flux, twist cost of a column, pressure margin), uniformly in the scale	Assumption (none of the faces is proved)	A proof in a form that fails for U(1) What is proved in Tomboulis's sandwich (arXiv:0707.2179) is an upper and lower inequality for a single number, the partition function; the step needed for the form of the renormalisation comparison, "representing the partition functions with and without a twist by the same representative (a common interpolation parameter)", is not proved (K. R. Ito and E. Seiler). This step must become false for U(1), and sits in the same place as the middle of the renormalisation comparison Cited Proved here Reading A candidate for the fifth face (the proposal of a one-step inequality with a fixed partner of comparison): the string tension of the one-step Migdal–Kadanoff image does not exceed the string tension of the genuine coarse theory, $\sigma[T_{\mathrm{MK}}w] \le 4\sigma[w]$. In the leading strong-coupling term and in the one-loop asymptotics the directions agree, and it has a form that fails for U(1) at the step where the MK image enters confinement first. Inside the band it is unknown, and it has not been shown at any β Assumption (an idea) On finite boxes the threshold of the proposal of a one-step inequality with a fixed partner of comparison is 1 ($R := \Delta F(\mathrm{coarse}; T_{\mathrm{MK}}w)/\Delta F(\mathrm{fine}; w) \ge 1$. ΔF is dual to σ, and the genuine one step preserves ΔF) Proved here . Over one step of the orbit, $4^4/2^4$ is $R = 1.34(7)$ and $6^4/3^4$ is $R = 1.17(20)$, within the resolution, so no verdict is given. There is neither evidence of failure nor numerical support for the side where it holds. It is a weak test that even the do-nothing map passes, and the failure for U(1) does not appear in a one-step box Reading (15-04 § 05) Chapters: 26-06, 26-10, 15-04
Upper end of the renormalisation comparison ◐	Semiclassics at fixed N and large β	Assumption the confinement direction (the Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3) of the draft paper: at fixed N, $-\ln\theta \ge c\beta - a_e - \beta(S_*+\varepsilon)M^2$. No proof is placed in this book. A proof with citations is in the paper: the general chessboard theorem and the Morse–Bott argument are cited) / Assumption the converse uniform in M, and $S_* = S_{\min}$	A uniform bound in the converse direction (already reduced to a single positivity lemma), and agreement of the minimal actions Chapters: 25-02, 26-06
One-loop end of the renormalisation comparison ■	With the comparison that reshapes the fluctuations, $\beta - \beta' = 0.258$. With the comparison that keeps the shape of the fluctuations, $\beta' \ge \beta + 0.023$ (the anti-asymptotically-free direction)	Proved here (at the one-loop level; not a non-perturbative claim) / Reading the direction of the numbers (SU(2) has $\beta' < \beta$ throughout the band; for U(1) the sign flips at $\beta_c \approx 1.01$. Lattices, seeds and errors in 26-09)	Making it non-perturbative is ultraviolet control and the middle of the renormalisation comparison Chapters: 25-07, 26-09

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
Band in the language of the coupling (mixed derivative) ☐	In a box of side L with a uniform background holonomy (angles α, γ) at the two ends in time (the Schrödinger functional box), the mixed derivative $M = \partial^2 \ln Z / \partial\alpha\,\partial\gamma$, normalised to β at tree level as M_R, satisfies $M_R(2L) \le (M_R(L) - \delta)_+$ on the interval of the band, or $-\Delta \ln M_R / \Delta L^2 \ge \sigma_0 > 0$ (δ, σ_0 independent of the lattice spacing). M drops exactly the terms that depend on one boundary only, and measures the free energy of the electric flux running through from bottom to top (the dual of the vortex of the middle of the renormalisation comparison) Reading . What is measured is the difference D_R over two values of γ. The difference $h := \beta_R - D_R$ from the coupling of the same box, $\beta_R = 4/\bar{g}^2$ (the derivative at the single point $\eta = \pi/4$), is the marker of the one-sided terms Ultraviolet: M_R has the same one-loop running $-8b_0 \ln(L/a)$ as β_R (at one loop on the lattice the coefficient is 1.009 times), and its doubling step is 0.2575, the same as for β_R. Only the constant differs, $M_R - \beta_R = +0.21$; this difference comes entirely from the charged fluctuations and is 0 for $U(1)$ Cited Computed Band: h has the form of a function of $\bar{g}^2$; it turns positive at $\bar{g}^2 \approx 4.5$, has its maximum 0.14–0.18 at 5–10, falls to 0 at 60–140, and is slightly negative (-0.01–-0.03) on the strong-coupling side. All 36 points with $L/a = 3$–8 lie on one curve, pointwise within 2σ Computed Strong coupling: for fixed $L \ge 4$, below the convergence region of the expansion ($\beta < \beta_0(L)$), $h < 0$. The leading term is the one-sided band term $-(8/s)u^{3L} \approx -(16/\pi)L^2 u^{3L}$ ($u = I_2(\beta)/I_1(\beta)$, $s = \sin(\pi/2L^2)$); it does not enter D_R), and the term of the tube joining top and bottom is also negative Paper Lean (StrongCouplingOneSidedTerm, 9 theorems. Assumptions: the remainder $\lvert R\rvert \le C_L u^{3L+1}$, $\sin x > x - x^3/4$, the tree normalisation $\hat{D} \le 3$ of D_R, the coefficients on paper). Convergence of the expansion is Cited (Osterwalder–Seiler). A β_0 uniform in L is not yet done	Assumption (the band inequality is not shown at any β) / ultraviolet Cited Computed / band Computed / strong coupling Paper Lean (convergence Cited)	A non-perturbative proof of the inequality on the interval of the band. Distinguishing $U(1)$ takes the shape of $h(\bar{g}^2)$, not the size of h: for $U(1)$, $h \approx 0$ deep in the Coulomb phase and in the confined phase ($\lvert h\rvert \le 0.023$ at 11 of 12 points), with a peak only at one point inside the width of the transition of the finite box (0.35 at $L/a = 4$, still 0.29(4) at $L/a = 8$); for $SU(2)$ the peak lies on a band of width $\bar{g}^2 \approx 5$–10 Computed Reading . The one-loop ultraviolet difference +0.21 is also 0 for $U(1)$ Computed (the computation record (2026-10-23 to 10-28)) The Schrödinger functional box and the coupling $\bar{g}^2$ are as defined by Lüscher–Narayanan–Weisz–Wolff (hep-lat/9207009) Cited Premises: ultraviolet control (the coupling at the ultraviolet end) Chapters: 26-06, 26-09
8 **Band comparison (in the language of the gap)** ☐	For the dimensionless quantity $x(L) = L\,m_0(L)$, the lowest mass $m_0(L)$ of the neutral sector of the box measured by the size of the box, the doubling map has no fixed point on the interval of the band: $x(2L) \ge x(L) + \delta$, with δ independent of the lattice spacing	Assumption / Paper In the Coulomb phase of U(1), x stays at the photon value $x^* \approx 2\pi$, so the band comparison (in the language of the gap) fails. The ratio-of-rates form $m_0(2L) \ge (\frac{1}{2} - \varepsilon) m_0(L)$ ends up holding on periodic boxes for U(1) ($m_0(2L)/m_0(L) = 1/(2\cos(\pi/2L)) > \frac{1}{2}$), so it is not a form that distinguishes the groups	A proof in a form that fails for U(1). Of the four faces of the middle of the renormalisation comparison, the two that also work in the language of the gap are κ_w (softness in the twist direction) and $\delta(L_0)$ (the total-variation distance inside when the centre flux on the boundary of the box is changed), extended to all boundary conditions; as a new face, one step function of x is added. The phases to be excluded are a massless phase and first-order coexistence; the non-confining phase of $\mathbb{Z}_2$ (which also has a gap) need not be excluded. The Tomboulis–Yaffe inequality holds for U(1) by the same proof and is a tool that descends to the charged sector, so it is not a tool for the band comparison (in the language of the gap) Reading . Since it can be written in a language that does not use the centre, it can be defined even for the groups without a centre among general groups Premises: ultraviolet control (the coupling at the ultraviolet end is placed at the finite value that ultraviolet control hands over) Chapters: 26-06

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
9 **Uniform decay of block observables (the gap side)** ☐	For block observables F, G with vacuum expectation 0 (their supports a distance t apart), $\lvert\langle FG\rangle_{a,L}\rvert \le C_F C_G\, e^{-mt}$, with $m > 0$ independent of a and L. The form of the gap of the small-T^2 gap uniform in L (the task of a uniform gap in 16-07)	Assumption / Paper Uniform decay of block observables gives the mass gap under the limit to R^4 (the content that the arrow of the R^4 gap in 27-01 asks of the uniform slab gap has this form, and frame decorrelation is not enough) / Cited uniform exponential decay under arbitrary boundary conditions gives the area law (Chatterjee 2021, CMP 385, Theorem from Uniform Decay to the Area Law (Theorem 2.4 of Chatterjee 2021); on the lattice)	The band comparison (in the language of the gap), and carrying the small-T^2 gap uniformly in L. Positivity of the 't Hooft tension is the band in the language of confinement, and even if the base point at small T^2, frame decorrelation and the renormalisation comparison disappear, uniform decay of block observables does not move. On the side of time, uniformity of the relaxation-time upper bound and the one-sided bridge is a row of the same type Premises: ultraviolet control, small-T^2 gap, band comparison (in the language of the gap) Chapters: 27-01, 16-07
10 **Positivity of the 't Hooft tension** (the band in the language of confinement) ☐	$\ln(1/\theta(L,L)) \ge cL^2$, with c independent of L (the 't Hooft string tension is positive)	Proved here the reduction from the base point at small T^2, frame decorrelation and the renormalisation comparison (the algebra of the iteration in Lean: Doubling (`Doubling.iterate`)). In the form written with κ_{cb}, the chessboard-density base point and the lateral comparison of the chessboard density, the same iteration goes through with $K = 4C$ (Lean: LatticeGaugeNestedBlocks (`doubling_iterate`)) / Assumption the band itself	The base point at small T^2, frame decorrelation and the renormalisation comparison disappearing, and the paper for the reduction being placed in the book. Positivity of the 't Hooft tension is the band in the language of confinement; the mass gap needs uniform decay of block observables separately. Intermediate milestones: (a) exclusion of a deconfined phase, (b) $\ln(1/\theta) \ge cL^2/\ln L$ Premises: base point at small T^2, frame decorrelation, renormalisation comparison Chapters: 26-01, 26-02
11 **Uniform gap of the double well** (what remains of the first step) ☐	For the path measure of the 0+1-dimensional double well, a gap uniform in k and $\hbar$ for chain length $k \ge 8$ (a uniform logarithmic Sobolev inequality for a measure with non-convex regions). The band in miniature	Assumption . The convex side (length ≤ 7) and the uniqueness of the minimum are closed in the record (the section that places them in the book is a supplement to 23-15)	Closing the rows of 23-15 Table 1 in a form uniform as $\hbar \to 0$, or stitching finite blocks together with strong mixing The side $\hbar \to 0$ (the small-ħ remaining item) reduces to showing, as $\hbar \to 0$, one margin among the inputs of the Six-Node In-Phase Poincaré Proposition, $\hbar\rho - 2\hbar\kappa \ge 0.067$ Paper . The two limiting conditions (the bridge landscape: the global minimum is unique and interior, with $\lambda_{\min} \ge 0.2$ and $\kappa_0 \le 0.1282$; the coarse landscape: $U_0''(t^*) \ge 1.943$) are checked with interval arithmetic, and the limit of the margin is at least 0.758 Computer-assisted (the action gap is in floating point Reading). To overlap with the range of the record ($\hbar \ge 0.005$), the quantitative uniform Laplace method and a lemma on one-dimensional Muckenhoupt asymptotics on the side of $\hbar\rho$ are newly needed Assumption . The value of $\hbar_0$ expected from the quantitative uniform Laplace method is 4×10^{-4}–8×10^{-4} Reading (the computation record (2026-10-22)) For $\hbar \in [0.001949, 0.005]$, the margin of the lattice target level $\hbar\rho - 2\hbar\kappa \ge 0.067$ holds by interval arithmetic on six subintervals Computer-assisted (the arithmetic on the subintervals and the logic of the covering are Lean : LatticeGaugeH1HbarInterval, 15 theorems). The gap to the $\hbar_0$ of the quantitative uniform Laplace method is not yet done; from the next subinterval on, the estimate of the corner tail does not close with this computing machinery, and the machinery has to be replaced (keeping the weights as logarithms, or revisiting the estimate of the corner tail) Reading (the computation record (2026-10-24 to 10-28)) Premises: — Chapters: 23-09 § 06, 23-15, 26-07

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
12 **Uniformity of the relaxation-time upper bound and the one-sided bridge** (the four-dimensional goal, side of time) □	The upper side of the diffusive-relaxation-time assumption (the relaxation-time upper-bound assumption, $CP \le K\xi^2$) and the one-sided bridge, uniform as $\beta \to \infty$ (the physical-unit target level of 23-15 Table 5)	Assumption	Crossing the band in the language of time. Even if the uniform gap of the double well disappears, the uniformity of the relaxation-time upper bound and the one-sided bridge does not move (the uniform gap of the double well is a miniature, not an implication) Premises: ultraviolet control Chapters: 23-01, 23-15 § 07, 26-07

What to read from Table 2. (a) The rows marked ■ in the record are only reflection positivity of z and two rows of the breakdown of the renormalisation comparison (the strong-coupling end and one loop). For reflection positivity of z the proof has been placed in 26-04 § 03, and the algebra of the last step is also closed in Lean (ReflPos), so this row is closed. The two rows of the breakdown have not yet disappeared by the rule of § 01. (b) Non-abelian dynamics enters at the middle of the renormalisation comparison (and at frame decorrelation and positivity of the 't Hooft tension, which gather into it), and this row has a form that correctly fails for U(1) and $\mathbb{Z}_2$. The other rows are structural facts, or rows that depend on ultraviolet control. (c) The weak form of positivity of the 't Hooft tension (the weak form of the lateral-widening inequality, the uniform slab tension) has almost the same content as the band. We read the reduction not as having made the problem easier but as having pinned down the middle of the renormalisation comparison as the place where the non-abelian input is needed. (d) Uniform acceptance of insertion, the base point at small T^2 and frame decorrelation are conditional (◐) rows, but their content has not disappeared; it has moved to other rows. Uniform acceptance of insertion is the same as the conjunction of boundedness of the insertion free energy and the uniform lower bound on the path overlap, and contains the base point at small T^2; the base point at small T^2 splits into the two rows ultraviolet control around a vortex background and insertion acceptance in a symmetric container; and the part of frame decorrelation uniform in L moves, as the lateral comparison of the chessboard density, to the same place as the renormalisation comparison. (e) Even if positivity of the 't Hooft tension, the band in the language of confinement, disappears, uniform decay of block observables remains for the mass gap. That the two roads stop at the same band is right as a type, but they arrive at different sectors.

04

The diagram of dependencies between rows — the numbers of premises are smaller than those of conclusions

Figure 1 writes the dependencies between the rows of the ledger and the rows outside it, split into layers. Arrows point from premise to conclusion, and every row has a larger number than its premise rows. The rows of the ledger are written in forms whose statements do not contain the quantities of the conclusion (the gap on $\mathbb{R}^4$, the area law on $\mathbb{R}^4$) (the assumption rows ultraviolet control, uniform acceptance of insertion, frame decorrelation, the middle of the renormalisation comparison, the band comparison (in the language of the gap), uniform decay of block observables, and uniformity of the relaxation-time upper bound and the one-sided bridge). The assumptions in forms that contain the quantities of the conclusion (the assumption of the arrow of the small-T^2 gap and the forms of the band) are marked explicitly as "restatements" in 27-01 (assumptions of arrows that restate uniform decay of correlations as a gap, and uniform bias and tension as an area law). Positivity of the 't Hooft tension is the band itself and contains a quantity close to the conclusion of the conditional statement. We put positivity of the 't Hooft tension in the ledger to show, in one row, the road to removing it once the reduction (from the base point at small T^2, frame decorrelation and the renormalisation comparison to positivity of the 't Hooft tension) is on paper. There are no exceptions to the numbering rule.

```
Layer 0   1 ultraviolet control ◂                5 reflection positivity of z ■
Layer 1   2 small-T² gap ◂ (← 1)
          6 frame decorrelation ◂ (← 5; only the descending step)
Layer 2   3 uniform acceptance of insertion ◂ (← 1, 2)
          7 renormalisation comparison □ (← 5, 6)
          8 band comparison (in the language of the gap) □ (← 1)
Layer 3   4 base point at small T² ◂ (← 1, 3)
          9 uniform decay of block observables □ (← 1, 2, 8)
Layer 4  10 positivity of the 't Hooft tension □ (← 4, 6, 7)
———————— below this line: outside the ledger (§05) ————————
Layer 5   limiting measure on R⁴ and its positivity ◂ (← ultraviolet control in arbitrary volume)
          rotation invariance □ (← ultraviolet control in arbitrary volume + a new lemma)
          local fields and OS regularity □ (← ultraviolet control in arbitrary volume + the task of local fields)
          uniqueness and non-triviality of the limit □ (← 10, 9)
Layer 6   tying the two roads together □ (← 9, 10, uniqueness and non-triviality of the limit)
          general groups □ (← limit to R⁴)
Layer 7   verification and peer review □ (← all rows)

ultraviolet control in arbitrary volume = ultraviolet control + extensive extension to arbitrary volume + continuity in
(the rows outside the ledger include parts that the published construction showed, in the form from a sequence of period

First road   12 uniformity of the relaxation-time upper bound and the one-sided bridge □ (← 1). The row of the same type
             11 uniform gap of the double well □ is 12 in miniature (not an arrow of implication)
```

Figure 1 Dependencies between the rows (to the right of ← are the premises). The base point at small T^2 and the upper end of the renormalisation comparison stand on the same semiclassical picture, but this is not an arrow of implication, so it is not in the figure. Uniform acceptance of insertion ⇒ base point at small T^2 follows in one line, but the relation is that uniform acceptance of insertion contains the base point at small T^2, so the figure only puts uniform acceptance of insertion among the premises of the base point at small T^2. The premises of uniqueness and non-triviality of the limit are, as written in 16-07, positivity of the 't Hooft tension and tying the two roads together; since the gap side of tying the two roads together has been moved to uniform decay of block observables, we write positivity of the 't Hooft tension and uniform decay of block observables Reading . "Area law ⇒ mass gap" is not set up as an arrow (§ 05).

05

The differences that remain outside the ledger — what remains even when every row has disappeared

Separate frame

Even if all twelve rows of the ledger disappear, four differences from the requirements of the problem statement (Jaffe–Witten 2000, 0-03) remain (the limit to R^4 [split into four rows in Table 3], tying the two roads together, general groups, verification and peer review; seven rows with the breakdown). An empty ledger does not mean that the problem is solved.

Table 3 The differences that remain outside the ledger (the correspondence with the tasks of 16-07 is this book's arrangement)

Row	Difference	Mark	Present state	16-07
Limiting measure on R^4 and its positivity	Of the limit to $\mathbb{R}^4$, the limiting measure, translations, preservation of reflection positivity, the Hilbert space and $H \geq 0$	◐	Assessment Infinite volume from a sequence of periodic boxes (along a subsequence) and the reconstruction were shown by the published construction (the Wightman axioms phase by phase; there is no convergence or uniqueness of the limit; Volume 2) / Assumption The reduction from the family of twisted slabs of this book's road is open. Its conditions are ultraviolet control in arbitrary volume (ultraviolet control extended to arbitrary volume with extensive constants) and continuity in the sources uniform in the lattice spacing and the volume. The rest is known argument (compactness, preservation of inequalities under weak limits, a symmetric contraction semigroup) Cited . The statement of ultraviolet control alone is not enough: no source derives, as a theorem, boundedness uniform in a of the expectations of gauge-invariant observables Cited	Task toward infinite volume
Rotation invariance	Recovery of rotation invariance	☐	It is present within the reconstruction of the published construction (Volume 2). From the family of twisted slabs of this book: it does not follow from invariance under the hypercubic group and continuity. What is needed is a lemma that the limits of regularisations with rotated lattice orientations agree, or a direct estimate that the breaking vanishes as $a \to 0$; neither ultraviolet control nor positivity of the 't Hooft tension promises it. The reflections of reflection positivity of z are positive only in directions 3 and 4, and reflection positivity in directions 1 and 2 comes only from rotation invariance Reading	Task of the remaining axioms
Local fields and OS regularity	Local fields and OS regularity, the linear-growth condition (Osterwalder–Schrader axiom E0′) (Wightman fields)	☐	It is present within the reconstruction of the published construction (Volume 2). From the family of twisted slabs of this book: renormalisation of composite operators and their uniform estimates	Task of the remaining axioms, task of local fields and short distances
Uniqueness and non-triviality of the limit	Uniqueness of the limit, non-triviality, clustering	☐	The published construction has no convergence or uniqueness of the limit. Clustering and a unique vacuum are on the side of decay uniform in the volume (uniform decay of block observables) and of positivity of the 't Hooft tension. Uniqueness is not required on our reading of the problem statement, but it is needed on the road that derives the recovery of rotation invariance from uniqueness. That the two roads give the same theory is also included here	Task of non-triviality
Tying the two roads together	Tying the two roads together: stating both the mass gap and confinement on the same theory on $\mathbb{R}^4$	☐	The area law alone does not give the gap (16-09 § 03). The gap side (the form "carry the gap of the small-T^2 gap through the band to $\mathbb{R}^4$ with the same constants") has been moved to uniform decay of block observables in the ledger, and "the same theory" is included in uniqueness and non-triviality of the limit. What substantively remains is the consistency of scales $0 < \inf m/\sqrt{\sigma} \leq \sup m/\sqrt{\sigma} < \infty$ (along the same sequence $\beta \to \infty$); this is a condition for stating confinement in physical units, and lies outside the requirements of the problem statement (what the problem statement asks for is the mass gap Cited). In three-dimensional U(1) there is an example in which both hold on the lattice and yet the ratio goes to 0, so the condition is not empty Cited (summary)	Task of one scale

Row	Difference	Mark	Present state	16-07
General groups	All compact simple groups	☐	Existence was shown by the published construction for every compact simple group (Volume 2). The vortex tools cannot be used for groups without a centre. On the mass gap: for SU(N), it is expected that the rows can be turned into conditional ones Reading . The places specific to SU(2) are of four kinds, of which two need redoing: the properties of SU(2) = S^3 (the conditional law of one link being one-parameter, the explicit formula of the heat kernel, the Heat-Kernel Fluctuation Rarity Lemma), and -1 lying in the Weyl group (a cubic treatment replacing the evenness of class functions). On top of these comes redoing the numerics of 26-09 for SU(3) (on top of ultraviolet control and the renormalisation comparison, which are open for SU(2)). What gets filled: the twist by the centre $\mathbb{Z}_N$ (for primitive twists the clock-and-shift pair is unique up to conjugation Cited (González-Arroyo, arXiv:hep-th/9807108 § 4.2); the adjoint gap is $4\sin^2(\pi/(NL))$, agreeing with the numerics for SU(3) at $L = 3$–6 Reading (the computation record (2026-10-22)); one loop gives $\beta - \beta' = 11N^2 \ln 2/(12\pi^2)$ Paper), the n-colour version of the colour lemma (Lean ColorAverageN (`colorAverage` , `design_identity`); no assumptions, including the existence of the weighted quartic design; the application to su(N) is on paper), and the SU(N) version of the Tomboulis–Yaffe inequality Cited (Kanazawa, arXiv:0808.3442 Theorem 2; its application to the slabs of this book is not done). All compact simple groups remain open: for groups without a centre (G_2, F_4, E_8), neither the vortex tools nor the arrows that use the centre can be used Cited (16-07 § 07), and for Sp(n) ($n \geq 2$) the centraliser of the pair that absorbs the twist is continuous, so a single twist on T^2 does not remove the zero modes Reading	16-07 § 07
Verification and peer review	Verification	☐	Writing out the whole, Lean (as far as it reaches), expert refereeing, publication. The rules of the prize require refereed publication followed by a period of acceptance	—

The differences outside the ledger are not put in the ledger because they are not conditions of the conditional statement (27-01) but the distance between the conclusion of the statement and the requirements of the problem statement. The rows of the ledger are "work of removing assumptions"; the rows outside are "work of carrying the conclusion as far as the problem statement". If the two were mixed in one table, then on the day the ledger became empty it would read as if the problem had been solved. The gap side of tying the two roads together (crossing the band in the language of the gap) is a condition of the same type as the assumption of the arrow of the R^4 gap in 27-01, so it has been placed inside the ledger as uniform decay of block observables.

What is not set up as an arrow. "Area law ⇒ mass gap" is kept as a physical expectation, read as Reading , and is not made an arrow of Figure 1. From reflection positivity and the area law alone it is false (the direct product of a lattice measure with an area law and U(1) in the Coulomb phase keeps reflection positivity and the area law and has no gap Paper ; no counterexample within simple groups is known). The theorem known on the lattice goes the other way: uniform exponential decay under arbitrary boundary conditions gives the area law Cited (Chatterjee 2021 Theorem from Uniform Decay to the Area Law, Durhuus–Fröhlich 1980). In the language of the transfer matrix, the area law is the gap of the charged sector with electric flux, and the mass gap is the gap of the neutral sector without electric flux.

06

The day rows increase — the ledger is not a table that only shrinks

As research advances, rows disappear, but they can also increase. For example, when the reduction of the band was worked out, the lateral-widening inequality split into two rows, a strong form and a weak form, and it turned out that the strong form is stronger than confinement (it contains monotonicity of the slab tension). When the renormalisation comparison was worked out, two new rows appeared, the strong-coupling end and the one-loop end, and the middle row split into four faces. When tying the two roads together was reviewed, two rows that cross the band in the language of the gap, the band comparison (in the language of the gap) and uniform decay of block observables, appeared inside the ledger, the row of frame decorrelation became the row on the confinement side, frame decorrelation, and the limit to R^4 split into four rows. Writing out the assembly of the vessel (the vessel stage) has no row for now, but a row may appear at the stage of writing it out.

So the number of rows of this ledger, laid out over time, is not a gauge of progress. What to read is the content of each row, and what is needed to remove it.

The current list of premises is in 28-04.

What this chapter can say

What is said	What is not said
The list of conditions of the conditional statement, and for each row its state in the record, its label, and what is needed for it to disappear (Table 2)	The proof of any row. This chapter adds no new results

What is said	What is not said
Proved here the reduction to positivity of the 't Hooft tension, reflection positivity of z, the twist part of the small-T^2 gap, the confinement direction of the upper end of the renormalisation comparison, the one-loop level of the one-loop end of the renormalisation comparison (within the conditions written in the rows)	Positivity of the 't Hooft tension itself, the non-perturbative form of the renormalisation comparison, ultraviolet control, the uniform lower bound on the path overlap, the lateral comparison of the chessboard density, the band comparison (in the language of the gap), uniform decay of block observables
Paper the sandwich of acceptance rates in uniform acceptance of insertion, the Lemma on the Upper Bound for the Minimal Action of a Symmetric Box in the base point at small T^2, the Nested Supermultiplicativity Lemma in frame decorrelation, the reduction of the small-ħ remaining item in the uniform gap of the double well (in the computation record; the paper proofs are not yet placed in the chapters of the book)	That these rows disappear by the rule of § 01
That there are four differences that remain even when every row of the ledger has disappeared (the limit to R^4 [split into four rows in Table 3], tying the two roads together, general groups, verification and peer review; seven rows with the breakdown) (Table 3)	That an empty ledger would mean the problem is solved (this cannot be said; the differences outside the ledger remain)
That, of the limit to R^4, only the limiting measure and its positivity are connected by known arguments, on the condition of ultraviolet control in arbitrary volume	That ultraviolet control in arbitrary volume holds; rotation invariance; local fields and OS regularity; uniqueness and non-triviality of the limit
The arithmetic of the theorem names in the Lean column (the algebra of finite sums and the inequality of the iteration, the inequality of the sandwich of acceptance rates, the action of the abelian twisted configuration, the inequality of nested averages, the identity of the quartic design)	That the Lean column checks the statement of the row itself

Sources and reproduction

Item	Kind	Source / tools
The rows of the ledger, the breakdown, the dependencies	This book	The material of Parts 25 and 26 (the list of tasks up to a proof with its dependency diagram, and the list of assumptions), renumbered so that premises come first. The marks are copies of the state in the record
Rows of the first road, uniform gap of the double well and uniformity of the relaxation-time upper bound and the one-sided bridge	This book	23-15 Table 1, Table 5, 23-09 § 06
The form of the string tension in the base point at small T^2	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166 eq. (2.37) (bibliographic data and the equation in the source table of 25-02)
The sandwich in uniform acceptance of insertion, the Lemma on the Upper Bound for the Minimal Action of a Symmetric Box in the base point at small T^2, the Nested Supermultiplicativity Lemma in frame decorrelation, the small-ħ remaining item of the uniform gap of the double well, the adjoint gap for general groups	Paper, Computer-assisted, Reading	The computation record (2026-10-22). Interval arithmetic only for the two limiting conditions of the uniform gap of the double well
The dissipation in uniform acceptance of insertion	Cited	C. Jarzynski, Phys. Rev. Lett. 78 (1997); G. E. Crooks, Phys. Rev. E 60 (1999); D. A. Sivak, G. E. Crooks, Phys. Rev. Lett. 108 (2012)
The lattice numerics in the base point at small T^2	Cited	G. Bergner, A. González-Arroyo, I. Soler, arXiv:2505.10396
The converse theorem in tying the two roads together	Cited	S. Chatterjee, Commun. Math. Phys. 385 (2021) Definition 2.3, Theorem 2.4 (Theorem from Uniform Decay to the Area Law); B. Durhuus, J. Fröhlich (1980)
The twist and TY for general groups	Cited	A. González-Arroyo, arXiv:hep-th/9807108 § 4.2; T. Kanazawa, arXiv:0808.3442 Theorem 2
The Confinement Lower Bound by Fractional Instantons in the upper end of the renormalisation comparison	Assumption	Draft paper 'Fractional instantons confine on the lattice: a rigorous semiclassical bound for SU(2) gauge theory on a twisted small torus' (2026-10, A-07). The general chessboard theorem and the Morse–Bott argument are cited. A proof with citations is in the paper, and no proof is placed in the book, so the label is Assumption
The reduction to positivity of the 't Hooft tension	Proved here	The conditional theorem of the draft paper 'Where the twisted-slab route to confinement stops: a corrected window target, a reduction to transverse widening, and SU(2) versus U(1) numerics' (2026-10, A-07), and 26-02
The tasks of the differences outside the ledger	This book	16-07 Table 1 and § 07, 16-09
The problem statement	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000)
Lean	Lean	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`), Doubling (`Doubling.iterate`), LatticeGaugeAcceptanceSandwich, LatticeGaugeAbelianTwistAction, LatticeGaugeNestedBlocks, ColorAverageN. What the bundles guarantee is only the algebra of finite sums, the inequality of a real iteration, and the elementary inequalities and identities written in the rows (A-04)

Terms used in this chapter — the ledger, a row disappears, the marks of the state in the record (■◐□), the differences that remain outside the ledger, the renormalisation comparison, the four faces, the band in miniature, the diffusive-relaxation-time assumption. Their meanings are in the A-02 Glossary and the A-06 Index of names.

Next chapter to read: 27-03 The table of questions. The conditional statement itself is in 27-01.

The table of questions (a cross-index)

> This chapter is an index; it points to the relevant sections of each chapter.

What this chapter shows — The questions that the parts of this book left open, gathered into one index. The seventeen questions of Part 23, and the nineteen questions and six candidate rules of Part 24, are looked up by chapter and item number, without moving the items. The questions left by the text of Parts 25 and 26 are given running numbers per table on this page, listed with the section of the chapter they come from and the rows of the ledger of conditions they concern (the rows of 27-02 Table 2, and the differences outside the ledger). The two tables of "the next move" listed by the record of the study are also looked up in the language of the ledger rows.

Prerequisite chapters — 27-02 (the ledger of conditions; the numbers and names of the rows are fixed there). The content of the questions is in the chapters in the right-hand column of each row.

Said first — This chapter adds no new results. Questions are questions, not claims. The status of a question (done, partly done, not yet, new) and "what does not move even if settled" are kept only in the tables of the chapters they come from, and are not copied here. Whichever question is settled, the problem of the mass gap does not move. What moves is one row of the ledger, or one label in one chapter.

Order of this chapter

1. How to read the index — the five sources, the numbering convention, and the correspondence with the ledger rows (Table 1)
2. The questions of Part 23 (Table 2)
3. The questions and candidate rules of Part 24 (Tables 3 and 4)
4. Questions from Part 25 (Table 5)
5. Questions from Part 26 — the assumptions of the reduction of the band, criteria, β-monotonicity and more (Table 6)
6. The next moves listed by the record of the study — two tables looked up by ledger row (Table 7)
7. Looking up from the ledger rows — for each row, the questions that concern it (Table 8)

01

How to read the index — the five sources and the numbering convention

The questions have five sources. Table 1 gives, for each source, how the numbers are assigned and the table that the numbers point to. The numbers are those of the source chapter, used as they are and not renumbered on this page. Parts 25 and 26 do not yet have a chapter gathering their questions, so we pick up sentences in the form of questions from the sections "what cannot be said", "what remains" and "what is lacking" of the text of those parts, and number them on this page. The numbers are plain running numbers per table, without letters, and in the text questions are referred to by short names.

Table 1 The five sources of the questions

Source	Numbers	Count	Table to look in	How the size of a question is written
Part 23 (the first road, first half)	Running numbers 1–17	17	23-17 Table 3	Tools (SAT, paper, python3, interval arithmetic, Lean, literature) and the form of settlement
Part 24 (the first road, second half)	Running numbers 1–19, candidate rules 1–6 and three combinations	19 + 9	24-08 Table 1, Table 2	Kind of work (paper, reading, computation, Lean) and status
Parts 25 and 26 (the second road and the band)	Running numbers 1–16 of Tables 5 and 6	16 + 16	Tables 5 and 6 of this page	As in the sentences of the source sections. No estimate of size is written
Part 15 (ultraviolet control)	Running numbers 1–31	31	15-09 Table 1	As in the sentences of the source chapter (the conditions are in the tables of the source chapter)
"The next move" of the record of the study	Order 1–7 of the record's table, and order of work 1–4	7 + 4	Table 7 of this page	Looked up after replacing them with the language of the ledger rows

The names of the ledger rows are as in 27-02 Table 2 (the twelve rows from ultraviolet control to uniformity of the relaxation-time upper bound and the one-sided bridge, and the six rows of the breakdown of the renormalisation comparison). The differences outside the ledger are the limit to R^{4}, tying the two roads together, general groups, and verification and peer review (27-02 Table 3). The "rows concerned" column of the tables is written with row names.

The column "rows concerned" of the tables gives, as this book's arrangement, which ledger rows' content a question would touch if it were settled. Settling a question does not necessarily make a row disappear. A row disappears only when its paper proof has been placed in the book and, if it has a Lean entry, the re-check is done (27-02 § 01).

02

The questions of Part 23

The questions of Part 23 are "small questions" expected to be settled at the size of an undergraduate thesis. Table 2 lists only a one-line summary and where to look. The form of settlement and "what does not move when settled" are in each row of 23-17 Table 3. The assumption around which the questions of Part 23 turned is the diffusive-relaxation-time assumption (the upper side is the relaxation-time upper-bound assumption, the lower side the relaxation-time lower-bound assumption).

Table 2 The questions of Part 23 (summaries; the full text of each row is in 23-17 Table 3)

Q.	Summary	Tools	Rows concerned	Where to look
1	Does a complete family of $k = 3$ cells exist in $d = 9$?	SAT, Lean	— (the combinatorics of one-link integration)	23-17 Table 3, 23-02 § 02
2	The largest dimension in which a complete family exists, $d_{\max}(k)$	Exhaustive search, paper	— (same as above)	23-17 Table 3
3	The constants of $1 - \lambda_k$ for the Hermite family	Paper, python3	— (worked example)	23-17 § 05
4	The $SU(N)$ version, and the Lean statement, of the saturation of the ratio of functions on the forest	Paper, Lean	Uniformity of the relaxation-time upper bound and the one-sided bridge	23-17 Table 3
5	$CP(\beta)$ in the link variables of two-dimensional lattice Yang–Mills	Paper, python3	Uniformity of the relaxation-time upper bound and the one-sided bridge (control)	23-17 Table 3

Q.	Summary	Tools	Rows concerned	Where to look
6	The best constant of the curvature after one-link integration, $c^{*} \in [21, 27]$	Global optimisation, interval arithmetic	— (the strong-coupling side)	23-17 Table 3, 5-11
7	The coefficient of N^{-3} in the column model, analytically	Paper, python3	Uniform gap of the double well	23-17 Table 3
8	In the column model, the sign of $1/P$ at finite u	python3	Uniform gap of the double well	23-17 Table 3
9	How much does the shortfall c_{2} change with the choice of gauge in the block transformation?	python3	Uniformity of the relaxation-time upper bound and the one-sided bridge	23-17 Table 3
10	C_{P} and ξ^{2} of the two-dimensional $O(3)$ sigma model	python3, literature	Uniformity of the relaxation-time upper bound and the one-sided bridge (an asymptotically free control)	23-17 Table 3
11	C_{P}/ξ^{2} of the three-dimensional $\mathbb{Z}_{2}$ gauge theory	python3	Uniformity of the relaxation-time upper bound and the one-sided bridge (control)	23-17 Table 3
12	Monotonicity in volume of $C_{P}(V)$, and boundary conditions	python3	Uniformity of the relaxation-time upper bound and the one-sided bridge	23-17 Table 3
13	Collect the primary sources on the dynamical exponent of local algorithms for four-dimensional $SU(2)$	Literature	Uniformity of the relaxation-time upper bound and the one-sided bridge	23-17 Table 3
14	Bound from above, by interval arithmetic, the two-point maximal correlation of the medium-core step ($8 \le k \le 14$)	Interval arithmetic, paper	Uniform gap of the double well	23-17 Table 3, 23-15 Table 1
15	Read the power of the prefactor $\hbar^{-p}$ of C_{P} for the core by the capacity of the degenerate point	python3, paper	Uniform gap of the double well	23-17 Table 3
16	A comparison theorem showing the alternating maximal correlation between boxes $c_{\mathrm{alt}} < 1$	Paper	Uniform gap of the double well	23-17 Table 3
17	When a small term that breaks the conservation is added to a dynamics with conservation, the length at which the gap departs from the square of the side	Paper, python3	Uniformity of the relaxation-time upper bound and the one-sided bridge (the smallest model of Part 24)	23-17 Table 3

Reading Table 2. The questions that concern the row of the uniform gap of the double well (what remains of the first step) are the question "the N^{−3} coefficient of the column model", the question "sign of 1/P in the column model", the question "interval arithmetic for the maximal correlation of the core", the question "power of the prefactor for the core and window" and the question "comparison for the alternating maximal correlation of boxes"; the questions opened by "the road of blocks and mixing" in the supplement to Part 23 (the road of blocks and mixing in 23-15) are gathered in Table 6 of Part 26 as the question "two-sided area laws inside the band". The questions concerning uniformity of the relaxation-time upper bound and the one-sided bridge are questions of how to set up the statement of the diffusive-relaxation-time assumption on the side of the four-dimensional lattice; whichever of them is settled, uniformity of the relaxation-time upper bound and the one-sided bridge itself (the upper-bound assumption and the bridge, uniform as $\beta \to \infty$) does not move.

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03
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The questions and candidate rules of Part 24

The questions of Part 24 came out of the study that repairs the form of the statement of the relaxation-time assumption (the length at which conservation breaks, the sector and local versions, the form the one-sided bridge needs, the vacuum measure). Table 3 gives only summaries and where to look; the status (done, partly done, not yet, new) and "what moves and what does not move if settled" are in 24-08 Table 1.

Table 3 The questions of Part 24 (summaries; the full text of each row and its status are in 24-08 Table 1)

Q.	Summary	Size	Where to look (the section the question came from)
1	For the free field, compare the excitation energies of observables supported in the box	Paper	24-04 § 05
2	Monotonicity of the factor that cuts off times, $\varphi(x)/x^2$	Elementary, Lean	24-04 § 03
3	Whether the static ratio catches the slowness, for the sign of the Polyakov loop on a small periodic box	Small computation	24-01 § 02
4	For $U(1)$ on a two-dimensional layer, how the gap decreases as the strip-shaped region is lengthened	Computation	24-02 § 03
5	Primary sources on the autocorrelation of the topological charge in pure $SU(2)$ gauge theory	Reading, computation	24-07 § 07
6	A many-family version of the Two-Projection Lemma, and its condition on maximal correlation	Paper	24-03 § 06
7	$\inf \operatorname{Hess} V_t$ of the one-link Wilson weight convolved with the heat kernel, and the flow of the weight of the bad set	Paper + small computation	24-06 § 07
8	For Wilson's transfer operator, show the time-step comparison assumption ((D) in the paper "one-sided bridge") restricted to functions cut off at low energy	Paper	24-04 § 08
9	For finite-state reversible Markov chains, close in Lean $\tau_{\mathrm{int}} \le C_P$ and the identity of the ground-state transformation	Lean	24-05 § 01
10	Reread the autocorrelations of local modes and topological modes in two-dimensional CP^{N-1} through the eyes of the local version	Reading, computation	24-02 § 07
11	For two-dimensional $SU(2)$, find λ_W and Γ of families of loops at finite β	Computation	24-04 § 06
12	On a four-dimensional box, the values for families of loops the size of the span	Computation	24-04 § 06
13	For the 0+1 double well, sandwich the local ratio by a reduction to one dimension with the position of the kink as coordinate	Paper + small computation	24-04 § 07
14	Add one point $\beta = 7$ for the two-dimensional strip-shaped region, and see whether the slope settles	Computation	24-02 § 05
15	For fields whose ratio lies between the threshold and 1, another argument that sees the mass while staying in the ratio form	Paper	24-04 § 06
16	On Wilson's lattice with discretised time, write the version of the bridge that cuts off times (the In-Window Supremum Proposition and the lower bound in finite volume) under the time-step comparison assumption	Paper	24-04 § 03

Q.	Summary	Size	Where to look (the section the question came from)
17	Whether the Kogut–Susskind Hamiltonian becomes frustration-free when it is split with terms grouped by sector	Paper	24-05 § 04
18	On the CP^{N-1} test bed, add seeds and pairs of N, and read the exponent of the slowness of the topological mode	Computation	24-02 § 07
19	Measure the definition (2) of the length at which conservation breaks with the charge of a four-dimensional subregion	Computation	24-02 § 01

The summaries of Table 3 are shortened from the sentences of the source table. The terms of Part 24 "window factor" and "window version of the bridge" are restated here as "the factor that cuts off times" and "the version of the bridge that cuts off times", so that they are not confused with the band of Part 26 (note on terms: 26-00 § 02). The "strip-shaped region" of the question "U(1) gap on two-dimensional layers" and the question "the single point β=7 of the band" is a width-2 region of a two-dimensional lattice, a different thing from the crossover band. In terms of ledger rows, only the question "reduction to kink positions" touches the uniform gap of the double well; the others are questions that repair the form of the statement of uniformity of the relaxation-time upper bound and the one-sided bridge.

The candidate rules are candidates for the question "if we were to make new mathematics, what would we put as rules?", and each has one question attached (24-08 Table 2). Table 4 lists only the symbol, the name of the rule and where to look. The relation to the three roots of the third step is in 24-08 Table 3.

Table 4 The candidate rules (the one-line candidate and its question are in 24-08 Table 2)

Symbol	Name of the rule	Section where the candidate came from
1	Axiomatising scale	24-07 § 06
2	The algebra of loops	24-07 § 06
3	Conserved quantities and sectors	24-03 § 04
4	The class of measures that pay with rarity	24-06 § 07
5	An axiom of asymptotic freedom	24-07 § 06
6	Curvature rule (Bakry–Émery on the space of orbits)	24-07 § 06
1+4, 3+4, 2+3	Three combinations	The three rows at the bottom of 24-08 Table 2

Seen from the study of Parts 25 and 26, among the candidates the axiom of asymptotic freedom and axiomatising scale sit closest to the problem of the band. In the statements that cross the band, the place where asymptotic freedom enters is narrowed to one (the $\beta' < \beta$ of the renormalisation comparison, 26-06 § 01), and the question of the axiom of asymptotic freedom (what follows from the rule of approaching a free field in the ultraviolet) becomes a question of restating that one place in the language of rules. This is an estimate of position; none of the candidates promises a prospect of crossing the band Reading (a physical estimate).

04

Questions from Part 25

These are the questions that the text of Part 25 (the second road: from the direction of space) left open. The sentence in each row is the sentence of the source section shortened into the form of a question, with its conditions kept in the same row. The group is $SU(2)$, and the small T² carries the 't Hooft twist $n_{12} = 1$. The questions from the three chapters 25-06 to 25-08 (driven insertion, fixed maps, the measured record)

are the five rows at the bottom of the table (from the question "kernels with flip probability near 1" to the question "extending the measurements and their errors").

Table 5 Questions from Part 25 (the rows concerned are rows of the ledger of 27-02; this book's arrangement)

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
1 Reverse direction of the fractional-instanton lower bound	The converse of the Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3): at fixed N, an upper bound by a dilute gas uniform in M (a bound on the string tension from above). In the draft it is reduced to a single positivity lemma	Upper end of the renormalisation comparison	A bound uniform in N, the continuum limit	25-02 § 06, A-07 § 06
2 Agreement of the minimal actions	Agreement of the minimal actions $S_*(N,\ell) \to S_{\min}(N,\infty)$ (the minimal action of a column with its frame frozen to the vacuum, and the minimal action of a free fractional instanton). The numbers agree to five digits (a reading)	Upper end of the renormalisation comparison	Same as above	25-02 § 06
3 Existence of fractional instantons	On the twisted $T^2 \times \mathbb{R}^2$, existence of a solution of topological charge $\frac{1}{2}$ attaining the lower bound $4\pi^2/g^2$ of the action (at present an outline with assumptions)	Base point at small T^2	The rate on the lattice, the band	25-02 § 02, § 07 (c)
4 SU(N) versions of the three facts	The $SU(N)$ version of uniqueness of the vacuum that absorbs the twist, the gap of the adjoint field, and the absence of flat configurations in the (1,1) sector	General groups	Groups without a centre	25-01 § 09 (c)
5 Bias of the outer container	The second term when moving to a fixed sector: that the bias of the outer container is typically small (0 as $M \to \infty$)	Frame decorrelation	A uniform estimate of the container free energy	25-03 § 04, § 08 (b)
6 Estimating the container free energy	Estimate the container free energy ΔF_j uniformly over good frames and uniformly in the lattice spacing (needs a multiscale analysis of the conditional measure with the frame held fixed)	Ultraviolet control, frame decorrelation	The band (the stage of widening sideways)	25-03 § 08 (e)
7 Cost of inserting a thick vortex	With the map of a thick vortex, bound the cost of the insertion independently of N (an improvement of the Slab Confinement Theorem towards the continuum limit; at present an outline, Assumption)	Ultraviolet control, uniform acceptance of insertion	The band	25-04 § 04, § 07 (c)
8 Gap from insertion to the area law	The three things lacking between the insertion estimates and the area law: (i) that the event that insertions are refused simultaneously in all cells is exponentially rare in the number of cells, (ii) that the lower bound on the acceptance rate is uniform in the lattice spacing, (iii) an upper bound on the variance of the insertion ratio	Uniform acceptance of insertion	The band	25-05 § 07
9 U(1) negative control	A negative control that applies the insertion estimates to four-dimensional $U(1)$ (not confining at weak coupling) and checks where they stop working	Middle of the renormalisation comparison (the condition that distinguishes groups)	The estimate for $SU(2)$	25-05 § 07, § 08 (f)

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
10 Oddness of composed insertions	That the map composing insertions in separate cells satisfies the identity "odd under insertion" (at present an assumption of the Many-Cell Second-Moment Estimate, and an assumption in Lean too)	Uniform acceptance of insertion	The identity for one cell	25-05 § 08 (c)
11 Literature check for slab confinement	Comparing the Slab Confinement Theorem with arguments of Tomboulis–Yaffe type and with the area law for spatial Wilson loops at high temperature (Borgs 1985)	— (a question about the label)	The content of the theorem	25-04 § 06
12 Kernels with flip probability near 1	Constructing a kernel whose flip probability a is close to 1. The Flow Lemma $\theta \le 1 - a$ gives only a naive form, because for a one-step kernel a saturates around $\frac{1}{2}$ (a judgement from measurement and the record)	Positivity of the 't Hooft tension (a form usable for testing the band)	Ultraviolet control	25-06 § 06, § 08 (d)
13 Quasi-static limit	The quasi-static limit: that the work W approaches the conditional free energy of the container as the number of driving steps $m \to \infty$ (a proof including an estimate of how the kernel mixes)	Uniform acceptance of insertion	The band	25-06 § 04, § 08 (b)
14 Error of the shaping map	Bound the error of the map that reshapes the fluctuations (the local relative entropy near the vortex surface, and higher loops) at intermediate coupling and for large fields. At one loop it reproduces $\beta - \beta' = 0.258$	Renormalisation comparison (from one loop to non-perturbative)	The statement that crosses the band itself (the middle of the renormalisation comparison)	25-07 § 08 (a)
15 Ultraviolet cost of a fixed map	Whether the extra ultraviolet cost of fixed maps grows in the same way in four-dimensional gauge theory (whether cancellation by Ward identities occurs). The present numbers are three points in a two-dimensional model	Ultraviolet control	The band	25-07 § 06, § 08 (d)
16 Extending the measurements and their errors	Measurements that widen the T^2 beyond the sizes measured (up to about $L\sqrt{\sigma} \approx 2$), and an estimate of the errors of driven insertion and replica exchange (the treatment of autocorrelation)	— (a question about a reading)	Any ledger row (numbers are not proofs)	(b)(c) of 25-08 § 07

Reading Table 5. The reverse direction of the fractional-instanton lower bound and the agreement of the minimal actions are exactly what remains of the upper end of the renormalisation comparison in the ledger; if they are settled, both sides of the rate match on the lattice at fixed N, but the direction of increasing N is the same problem as the middle of the renormalisation comparison. The question "estimating the container free energy" and the question "cost of inserting a thick vortex" are questions that touch the content of ultraviolet control, a wall separate from the band (25-00 § 02). (ii) of the question "gap from insertion to the area law" is a restatement of the row uniform acceptance of insertion.

05

Questions from Part 26 — the assumptions of the reduction of the band, criteria, β-monotonicity and more

These are the questions that the text of Part 26 (the crossover band) left open. The first three rows put into question form the three assumptions of the reduction of the band (26-02 § 02), and the difficulty of the band is concentrated in the third row, the lateral-widening inequality (26-02 § 08). An argument that

closes the band must be able to distinguish $U(1)$ (the criterion "distinguish from the U(1) transition", 26-06 § 08). This condition applies to every row of Table 6.

Table 6 Questions from Part 26 (the rows concerned are rows of the ledger of 27-02; this book's arrangement)

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
1 Non-perturbative proof of the lateral-widening inequality	Show the lateral-widening inequality (strong form $\theta(R;T_{L'}^2) \le C(L'/L)^2\theta(R;T_L^2)$, or the weak form in which the total loss is finite) non-perturbatively across $L \sim 1/\Lambda$, between the dilute regime and the confining regime. It does not follow from structural inputs alone (reflection positivity, duality, the transfer matrix)	Positivity of the 't Hooft tension (the place of the input to the reduction is the renormalisation comparison)	From the limit to R^4 to verification and peer review	26-02 § 02, § 08
2 Uniformity of frame decorrelation	Show frame decorrelation $\theta(sR;T_L^2) \le C_{\mathrm{D}}\,\theta(R;T_L^2)^{s^2}$ uniformly in L. With L fixed it is replaced by the chessboard density κ_{cb}, so the question becomes a comparison of $\kappa_{\mathrm{cb}}(L)$ across scales	Frame decorrelation	The non-abelian input to the band (non-perturbative proof of the lateral-widening inequality)	26-02 § 02, 27-02 Table 2, frame decorrelation
3 Proof of the base point at small T^2	Show the base point at small T^2, $\theta(L_0;T_{L_0}^2) \le 1-\varepsilon_0$, under ultraviolet control and uniform acceptance of insertion (the expected size $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$ is a one-loop estimate)	Base point at small T^2 (premises: ultraviolet control, uniform acceptance of insertion)	The band	26-02 § 02, 27-01 Table 1, base point at small T^2
4 Excluding a deconfined phase	Exclusion of a deconfined phase: $\Delta F(L,L) \to 0$ (at any speed). It rules out the known non-confining phases, but gives no uniform lower bound on the tension	Partway to positivity of the 't Hooft tension	The mass gap, a uniform tension for the area law	26-01 § 06, 26-06 § 08
5 Almost-area-law question	Almost area law: $\ln(1/\theta(L,L)) \ge cL^2/\ln L$ (almost area law; by the Tomboulis–Yaffe inequality, decay faster than the perimeter law)	Partway to positivity of the 't Hooft tension	A uniform tension	26-01 § 06
6 β-monotonicity question	β-monotonicity: $\partial_\beta \Delta F \ge 0$ with the lattice geometry held fixed. Rewritten in terms of electric flux, it is a problem of the same weight as β-monotonicity of the string tension in lattice units, and as far as we searched, not even β-monotonicity of Wilson-loop expectation values has been shown. In the chain of the band it is not needed if the renormalisation comparison is shown directly	Renormalisation comparison (an auxiliary role)	The renormalisation comparison itself	26-06 § 02
7 Z_2 and U(1) on twisted slabs	Checking that, on a slab with twist n_{12}, β-monotonicity for $\mathbb{Z}_2$ and $U(1)$ does not follow from Ginibre's inequality (without the twist it does)	Renormalisation comparison	β-monotonicity for $SU(2)$	26-06 § 01

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
8 Criterion at the lower end	The criterion at the lower end: push confinement and the comparison up to $\beta \approx 0.3$–1, by a finite-size criterion of Dobrushin–Shlosman type using the influence $\delta(L_0, \beta)$ on the inside when the centre flux on the boundary of the box is changed, and by a computer-assisted count of the growth constant of admissible surfaces	Lower end of the renormalisation comparison	Above $\beta \approx 1$ (the middle)	26-06 § 04
9 Bookkeeping at the strong-coupling end	Write out the bookkeeping of the Strong-Coupling-End Expansion Proposition (the interaction of volume polyhedra and surface contours), and make the constant $u_0 \approx 10^{-80}$ larger	Strong-coupling end of the renormalisation comparison	Beyond roughening ($\beta_R \approx 1.9$)	26-06 § 03
10 Four faces of the middle	Show any one of the four faces of the middle (softness in the twist direction κ_w, mixing of box magnetic flux $\delta(L_0)$, twist cost of a column $f(\ell)$, pressure margin) uniformly in the scale, in a form that fails for $U(1)$. As a preliminary step, test κ_w and $\delta(L_0)$ numerically with $U(1)$ as the control	Middle of the renormalisation comparison	Ultraviolet control	26-06 § 07
11 Two-sided area laws inside the band	At each β in the band, the area law on both sides at fixed β (with errors of the order of the perimeter; the assumption of two-sided area laws inside the band). If this can be said, the comparison inside the band shrinks to finitely many steps (the Lemma Reducing the Comparison inside the Band)	Renormalisation comparison	The seam between the upper end of the band and ultraviolet control	26-06 § 06
12 What remains of the road of blocks and mixing	What remains of the road of blocks and mixing for the first step: the second and third stages of strong mixing (the Laplace method with constraints), checking against the original sources the Varadhan-type limit, Witten's Morse theory (corners of the Neumann box) and the standard theorem for one-dimensional chains, and interval arithmetic for the curvature at the minimum $\lambda_*(12)$ and the reach of the optimal path	Uniform gap of the double well	Uniformity of the relaxation-time upper bound and the one-sided bridge (the uniform gap of the double well is a miniature, not an implication)	the road of blocks and mixing in 23-15, 26-07 § 06
13 Convexity threshold and mixing length	Why, in the first step, the convexity threshold (the first conjugate point at the top, ≈ 7.84) and the mixing length (about 9) are numerically close	Uniform gap of the double well	Uniformity of the relaxation-time upper bound and the one-sided bridge	26-07 § 07 (c)
14 Touchstone of the smallest band	Whether the smallest band can be crossed: test a tool claimed to cross the band first on two-dimensional asymptotically free models (the $O(N)$ model at large N, the Gross–Neveu model), with the ultraviolet cutoff still removed (the screening criterion "crosses the smallest band". Exponential decay at all temperatures for the two-dimensional $O(3)$ model is also said to be unsolved; not yet checked)	Positivity of the 't Hooft tension (a touchstone)	The four-dimensional band	26-08 § 07, the question "C_P and ξ^2 of the O(3) σ model"

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
15 Comparison principle for large loops	The asymptotic form of the comparison principle for large loops only, $\sigma_{\mathrm{tw}}(L) \le \sigma_{\infty}$. The form for small loops was broken numerically, but this form has not been refuted (there is no mechanism to show it)	Positivity of the 't Hooft tension	The comparison principle for small loops (broken numerically)	26-08 § 04
16 Testing the three new candidates	Put each of the three new candidates (reduction by compactness of the band, log-concavity of the Weyl measure and a height/azimuth decomposition, local step function of a column) through the tests of the screening criteria	Middle of the renormalisation comparison	None of the candidates closes the band	26-10 § 05, § 07

Reading Table 6. Items 1–3 of Table 6 are the three assumptions of the reduction theorem (26-02 § 03); if all three are in place, the correct form of the band follows. But the three differ in character, and non-abelian dynamics enters only in the question "non-perturbative proof of the lateral-widening inequality" (uniformity of frame decorrelation is replaced by a structural fact once L is fixed, and the question "proof of the base point at small T^{2}" is a statement on the weak-coupling side). From the question "β-monotonicity" to the question "two-sided area laws inside the band" is the breakdown of the renormalisation comparison: the question "criterion at the lower end" is the lower end, the question "four faces of the middle" the middle, and the question "bookkeeping at the strong-coupling end" the strong-coupling end. From the question "touchstone of the smallest band" to the question "testing the three new candidates" are questions coming from the chapters on the roads tried and stopped and on new candidates (26-08, 26-10). The question "what remains of the road of blocks and mixing" and the question "convexity threshold and mixing length" are questions about the first step on the side of time (the first road); even if they are settled, the band does not move (26-07 § 04). The questions of ultraviolet control (the ultraviolet questions) are in Part 15, and 15-09 Table 1 is their index (the fifth source in Table 1).

06

The next moves listed by the record of the study — two tables looked up by ledger row

Among the material of Parts 25 and 26 there are two tables of "the next move", kept as a record of the study. One lists seven rows in decreasing order of the prospect of closing; the other lists four rows in the order of work. Both are tables in the record, not chapters of the book. Table 7 looks up each of their rows in the language of the ledger rows and the chapters of the book. The "amount of work" scale of the record's tables is a number of prospects, not a number with grounds, so it is not copied.

Table 7 The rows of the record's two tables, with the ledger rows and the chapters of the book (the sentences for the content of each row are on the side of the chapters of the book)

Record's table	Order	Content (in the language of the book)	Ledger row	Chapter of the book, question on this page

Record's table	Order	Content (in the language of the book)	Ledger row	Chapter of the book, question on this page
Order of the prospect of closing	1	What remains of strong mixing for the first step (checking the Varadhan-type limit, a uniform form of the Laplace method with constraints)	Uniform gap of the double well	the road of blocks and mixing in 23-15, what remains of the road of blocks and mixing
	2	The logarithmic Sobolev inequality for blocks in the first step (checking Witten's Morse theory and the decomposition against the original sources, interval arithmetic for λ_*)	Uniform gap of the double well	Same as above, what remains of the road of blocks and mixing
	3	Stitching 1 and 2 together (the standard theorem for one-dimensional chains)	Uniform gap of the double well	Same as above, what remains of the road of blocks and mixing
	4	The positivity lemma needed for the converse bound of the paper on lattice fractional instantons (by a low-temperature cluster expansion at one scale)	Upper end of the renormalisation comparison	Reverse direction of the fractional-instanton lower bound, A-07 § 06
	5	Agreement of the minimal actions in the paper on lattice fractional instantons (exponential decay of the minimisers, and cutting off)	Upper end of the renormalisation comparison	The question "agreement of the minimal actions"
	6	Writing out the bookkeeping of the strong-coupling end, and improving the constant u_0	Strong-coupling end of the renormalisation comparison	The question "bookkeeping at the strong-coupling end"
	7	The middle: show the comparison between scales by one of the four faces	Middle of the renormalisation comparison	The question "four faces of the middle"
Order of work	1	Of the faces of the middle, test κ_W and $\delta(L_0)$ numerically with $U(1)$ as the control	Middle of the renormalisation comparison	The question "four faces of the middle"
	2	The upper end: writing out the semiclassical theorem at fixed N	Upper end of the renormalisation comparison	Reverse direction of the fractional-instanton lower bound, agreement of the minimal actions, 25-02 § 06
	3	The lower end: a computer-assisted estimate of the growth constant of surfaces, and formulating the criterion by mixing of box magnetic flux	Lower end of the renormalisation comparison	The question "criterion at the lower end"
	4	Add to the paper on the reduction of the band the scan with step functions and the $U(1)$ control	— (a revision of the paper)	A-07 § 05

None of the rows of the two tables touches the differences outside the ledger. Questions aimed at the differences outside the ledger are in the table of tasks of Part 16 (16-07) and in the unknown column of the table of the seven stages a proof must contain (16-09 Table 1).

07

Looking up from the ledger rows — for each row, the questions that concern it

Table 8 looks up Tables 2–7 again from the side of the ledger rows. The content of each row, its label and what is needed for it to disappear are in 27-02 Table 2; here we list only the numbers of the questions

concerned. A row without questions is a row in which no step small enough to be shrunk into the form of a question is yet in sight.

Table 8 The ledger rows and the questions concerned (this book's arrangement)

Row	Questions concerned
Ultraviolet control	Estimating the container free energy, cost of inserting a thick vortex, the question "ultraviolet cost of a fixed map" (in part), the ultraviolet questions (15-09 Table 1, with the current state of the answers). 15-08 Table 1 (the four rows that do not close) and 15-09 Table 1 are the tables that split this row
Small-T^2 gap	— (the twist part is on paper in 25-01; the form uniform in a depends on ultraviolet control)
Uniform acceptance of insertion	Cost of inserting a thick vortex, gap from insertion to the area law, the question "oddness of composed insertions", the question "quasi-static limit"
Base point at small T^2	The question "existence of fractional instantons", proof of the base point at small T^2
Reflection positivity of z	— (the proof has been placed in 26-04 § 03; this row is closed)
Frame decorrelation	The question "bias of the outer container", estimating the container free energy, uniformity of frame decorrelation
Renormalisation comparison	Strong-coupling end: bookkeeping at the strong-coupling end / lower end of the renormalisation comparison: criterion at the lower end / upper end of the renormalisation comparison: reverse direction of the fractional-instanton lower bound, agreement of the minimal actions / middle of the renormalisation comparison: the question "U(1) negative control", four faces of the middle / testing candidates for the middle of the renormalisation comparison: the question "testing the three new candidates" / outside the breakdown: the question "error of the shaping map", the question "β-monotonicity", the question "Z_2 and U(1) on twisted slabs", two-sided area laws inside the band
Positivity of the 't Hooft tension (the band)	Non-perturbative proof of the lateral-widening inequality (the content), the question "excluding a deconfined phase", the question "almost area law" (intermediate milestones), the question "kernels with flip probability near 1", the question "comparison principle for large loops" (forms usable for testing the band), the question "touchstone of the smallest band" (a touchstone)
Uniform gap of the double well (what remains of the first step)	The question "the N^{−3} coefficient of the column model", the question "sign of 1/P in the column model", the question "interval arithmetic for the maximal correlation of the core", the question "power of the prefactor for the core and window", the question "comparison for the alternating maximal correlation of boxes", the question "reduction to kink positions", what remains of the road of blocks and mixing, convexity threshold and mixing length
Uniformity of the relaxation-time upper bound and the one-sided bridge (the four-dimensional goal, side of time)	Items 4, 5, 9–13 and 17 of Table 2, and items 1–12 and 14–19 of Table 3 (all are questions that repair the form of the statement of this row, not this row itself)
Limit to R^4	—
Tying the two roads together	— (even the statement of what it suffices to assume is not settled, 27-01 § 04)
General groups	The question "SU(N) versions of the three facts" (only the $SU(N)$ part)
Verification and peer review	—

What to read from Table 8. The rows with many questions are the renormalisation comparison, and the uniform gap of the double well and uniformity of the relaxation-time upper bound and the one-sided bridge; the questions on the band itself (positivity of the 't Hooft tension) gather in the non-perturbative proof of the lateral-widening inequality. The questions on excluding a deconfined phase and the almost area law are intermediate milestones weaker than positivity of the 't Hooft tension, and settling them does not make positivity of the 't Hooft tension disappear. The differences outside the ledger have almost no steps that can be shrunk into the form of a question. Tying the two roads together starts, before any question is set, from making a statement of what needs to be shown.

What this chapter can say

What is said	What is not said
Which section of which chapter the questions and candidate rules of Parts 23 and 24 came from (Tables 2–4)	The answer to any question. The status and the full text of "what does not move even if settled" are in the source tables
A list of the questions that the text of Parts 25 and 26 left open, with their source sections (Tables 5 and 6)	Estimates of the size of the questions. Of the sentences of chapters 26-03 to 26-05 and 26-09, only those that overlap with rows of Table 6 have been picked up
The correspondence between questions and ledger rows (Table 8, this book's arrangement)	That settling a question makes a row disappear (the rule for rows disappearing is 27-02 § 01)

Sources and reproduction

Item	Kind	Source / tools
The questions of Table 2	This book	23-17 Table 3
The questions and candidate rules of Tables 3 and 4	This book	24-08 Table 1, Table 2
The questions of Tables 5 and 6	This book	The sections of the chapters in the right-hand column ("what cannot be said", "what remains", "what is lacking", "map")
The record's two tables in Table 7	This book	From the material of Parts 25 and 26, the table of next moves (seven rows in order of the prospect of closing) and the "order of the next work" in the list of tasks up to a proof (four rows). The content of the rows has been replaced with the language of the chapters of the book
Numbers and names of the ledger rows	This book	27-02 Table 2, Table 3

Terms used in this chapter — ledger row, the differences outside the ledger, what does not move even if settled, intermediate milestone, the lateral-widening inequality, frame decorrelation, the base point at small T^2, β-monotonicity, the renormalisation comparison, the four faces, candidate rules, the diffusive-relaxation-time assumption. Their meanings are in the A-02 Glossary and the A-06 Index of names.

Next chapter to read: 27-04 What is not claimed (a cross-index) — the reverse side of the questions: what this book does not show, does not read strongly, and does not call false either.

2026-10-15 · chapter Yang–Mills and the mass gap Part 27 Closing — conditional statements not shown, not read strongly, not called false · Kiichi, Shiori and Rin

What is not claimed (a cross-index)

> This chapter is an index; it points to the relevant sections of each chapter.

What this chapter shows — What this book does not claim, gathered across the parts into one index. The fourteen items of Part 23 (23-16 Table 2) and the sixteen items of Part 24 (24-08 Table 4) are looked up by chapter and item number, without moving the items. From the sections "what cannot be said" of the text of Parts 25, 26 and 27 we pick up items, give them running numbers per table on this page (items 1 onwards of Tables 4 to 6), and list them with the places in the text that support them. Finally, a table of examples in which sentences of Parts 25 to 27 lose their conditions in a one-line summary.

Prerequisite chapters — None. You can start reading anywhere. The three meanings (not shown, not read strongly, not called false) are used as in 23-16 § 01. The numbers of the ledger rows are those of 27-02 Table 2.

Said first — This book does not solve the problem of Yang–Mills existence and the mass gap. This book's study does not construct the continuum limit (the continuum limit in a periodic box, and the reconstruction phase by phase, were shown by the construction published on 2026-11-10; at the level of the committee's assessment; Volume 2). There is no proof of the mass gap, and no proof of confinement in the continuum limit. This chapter only writes that fact out, split into items, and adds no new results.

The order of this chapter

1. How to read the index — the three meanings, and the number of items per part (Table 1)
2. The fourteen items of Part 23 — looking up 23-16 Table 2 (Table 2)
3. The sixteen items of Part 24 — looking up 24-08 Table 4 (Table 3)
4. The items of Part 25 — on the tools of the second road (Table 4)
5. The items of Part 26 — on the band (Table 5)
6. The items of Part 27 — on the conditional statement and the ledger (Table 6)

7. Conditions drop out in summaries — sentences of Parts 25 to 27 and one-line summaries (Table 7)

01

How to read the index — the three meanings, and the number of items per part

"Not claimed" has three meanings (23-16 § 01). Not shown: on that proposition this book shows nothing. Not read strongly: what has been shown carries conditions or assumptions, and read in a form with the conditions removed it becomes a sentence that has not been shown. Not called false: that a method did not reach something is a property of that method, not a claim that the proposition it did not reach is false. Besides these, there are items of "not judged, not counted": rederivations of known results are not counted as results, and the correctness of other documents is not judged.

The third becomes clearer when written in logical form. Writing "P is claimed" as $\vdash P$, $\neg(\vdash P)$ and $\vdash \neg P$ are different sentences, and the "not claimed" of this chapter always has the meaning on the left. The items of Part 26 have one more direction: cases in which a numerical reading points to the opposite of a proposition (the measurement of the comparison principle, item 4 of Table 5). There too, the book does not claim the opposite proposition as a theorem, but places it as a "reading".

Table 1 The items per part (how they are numbered, and how many)

Part	Numbers	Count	Table or section to look in
Part 23 (the first road, first half)	Items 1–14	14	23-16 Table 2
Part 24 (the first road, second half)	Items 1–16	16	24-08 Table 4 (the five items on the candidate rules are at the end of 24-08 § 02)
Part 25 (the second road)	Running numbers 1–13	13	Table 4 of this page (the "what cannot be said" of each chapter)
Part 26 (the crossover band)	Running numbers 1–12	12	Table 5 of this page
Part 27 (closing)	Running numbers 1–6	6	Table 6 of this page

The items of Part 23 apply as they are to Part 24 (24-08 § 04), and in the same way the items of Parts 23 and 24 apply as they are to Parts 25 to 27. In particular, "not reaching something is not evidence that it is false" (items 8 and 12 of 23-16) and "conditions drop out in summaries" (23-16 § 06) are common to all parts.

02

The fourteen items of Part 23 — looking up 23-16 Table 2

Table 2 lists only the gist of each item, its kind, and the main places in the text that support it. The full text of the items and all the supporting text are in 23-16 Table 2. The assumption named by the items of Part 23 is the diffusive-relaxation-time assumption (the upper side is the relaxation-time upper-bound assumption, the lower side the relaxation-time lower-bound assumption), and the gists in Table 2 use this name.

Table 2 The items of Part 23 (gist; full text in 23-16 Table 2)

Item	Gist	Kind	Supporting text (main places)
1	The solution of the problem of existence and the mass gap; a construction of the continuum limit	Not shown	23-01 § 08
2	Even for the first step, the double-well relaxation-time power statement has not been shown. The First-Step Upper-Bound Proposition is algebra on top of four conditions	Not shown	23-05 § 08
3	There is not yet a single $\hbar$ at which all rows of the four conditions hold at once	Not shown	23-12 § 06
4	Computer-assisted results are inequalities inside the range that was computed	Not read strongly	23-08 § 10
5	Readings are used neither as upper bounds nor as proofs of lower bounds	Not read strongly	23-00 Table 1
6	A Lean theorem checks the algebra from assumptions to conclusion	Not read strongly	23-13 § 05
7	From the first step to the second step on, only the prospect that the form of the sentences and the form of the tools lift	Not read strongly	23-03 § 06
8	That a tool did not reach something is a property of that tool	Not called false	the table of scope in 23-10
9	Rederivations of known results are not counted as results. The correctness of other documents is not judged	Not judged	23-14 § 03
10	The record on the side of the four-dimensional lattice also remains assumption	Not shown	23-01 § 05
11	Under the assumption of the one-sided bridge, the upper-bound assumption implies a lower bound on the gap, but the converse is false in general. The diffusive-relaxation-time assumption is not the mass gap itself	Not read strongly	23-01 § 06

Item	Gist	Kind	Supporting text (main places)
12	The "death" of a detour plan does not mean that the remaining proposition is false	Not called false	23-10 § 03
13	The reading that the upper-bound assumption fails in the periodic box is a conditional reading, resting on numbers from the literature and on a conjecture	Not called false	23-01 § 05
14	A proposition with assumption rows left is not put out as a result	Not counted	23-12 § 08

03

The sixteen items of Part 24 — looking up 24-08 Table 4

Table 3 gives the gist of the items of Part 24, their kind, and the main places in the text that support them. The full text is in 24-08 Table 4. What is not claimed about the candidate rules (neither consistency nor a prospect towards the third step is shown, the Boolean-algebra comparison is a metaphor, and five others) is at the end of 24-08 § 02; here we only point to it.

Table 3 The items of Part 24 (gist; full text in 24-08 Table 4)

Item	Gist	Kind	Supporting text (main places)
1	The solution of the problem of existence and the mass gap. The continuum limit, recovery of rotations, the OS axioms and non-triviality are not touched	Not shown	24-00 § 05
2	That the relaxation-time assumption is false (the reading for the periodic box extends the numbers of the originals and a conjecture)	Not called false	24-01 § 06
3	The sector and local versions are candidate definitions, not shown at any β	Not shown	24-03 § 07
4	The one-sided bridge is an inequality on a lattice with continuous time. On Wilson's lattice with discretised time, the time-step comparison assumption ((D) in the paper "one-sided bridge") remains	Not read strongly	24-04 § 08
5	"The gap = the Poincaré constant of the vacuum measure" is an identity, and we do not say it has made the problem easier	Not read strongly	24-05 § 07
6	The definition of the length at which conservation breaks, and the restatement of what to compare with, are candidates	Not read strongly	24-02 § 08
7	The results for hierarchical models and two-dimensional models imply nothing for four dimensions	Not read strongly	24-06 § 09

Item	Gist	Kind	Supporting text (main places)
8	The non-perturbative mechanisms of physics (confinement, instantons, centre vortices, duality) are not used as tools of proof in Part 24	Not shown	24-07 § 08
9	The two points of "because it is four dimensions" are estimates as properties of the critical dimension	Not read strongly	24-07 § 01
10	The "what is not claimed" of Part 23 applies to Part 24 as it is	—	23-16 § 02
11	The local version goes only as far as the definition and the ratio form. The local diffusive-relaxation-time assumption has not been shown for the target gauge theory	Not shown	24-03 § 03
12	The thresholds of the local bridge depend on the choice of reference and on the family of test observables	Not read strongly	24-04 § 06
13	For the 0+1 double well, the local ratio is not 1. Taking it to approach 1 at short distances in four dimensions is an assumption that depends on asymptotic freedom	Not read strongly	24-04 § 07
14	The length at which conservation breaks was measured only on the width-2 region of two-dimensional $U(1)$	Not read strongly	24-02 § 03
15	For $SU(2)$, it is undecided whether the freezing of the topological charge breaks the relaxation-time assumption	Not called false	24-07 § 07
16	The results of the two-dimensional CP^{N-1} test bed imply nothing for four dimensions	Not read strongly	24-02 § 07

Item 8 of Part 24 needs care in reading when set beside Parts 25 and 26. Part 24 did not use centre vortices or instantons as tools of proof, but Part 25 puts vortex surfaces and fractional instantons at the centre of its tools. What it uses, however, are only finite identities on the lattice, estimates at fixed lattice spacing and conditional sentences; it does not bring the physical mechanisms into proofs as they are (items 2, 4 and 5 of Table 4).

04

The items of Part 25 — on the tools of the second road

These items are picked up from the sections "what cannot be said" of the chapters of Part 25 (from the direction of space). The group is $SU(2)$, and the small T^2 carries the twist $n_{12} = 1$. The kind column gives the three meanings (and "not judged, not counted").

Table 4 What Part 25 does not claim, and the sentences of the text that support it

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
1	"Weak coupling persists up to the size of the small T^2" is not a theorem. It is a physical estimate made on the assumption of ultraviolet control, and an estimate uniform in the lattice spacing is nowhere in this book. That the small T^2 and $\mathbb{R}^4$ are connected without a phase transition (adiabatic continuity) is a conjecture and is not used	Not shown	(a)(b) of 25-00 § 08
2	The Slab Confinement Theorem is a statement with the lattice spacing and the size N of the T^2 fixed. The lower bound $-\ln\tanh(\beta N^2) \approx 2e^{-2\beta N^2}$ goes to 0 in the continuum limit, and also vanishes as $N \to \infty$. It says nothing about a tension uniform in the lattice spacing, nor about four-dimensional confinement	Not read strongly	(a)(b) of 25-04 § 07, § 04
3	The Container Identity is a rewriting of the Mack–Petkova identity, and is known. The rewriting in the summed ensemble, the way of moving to a fixed sector, nesting, the insertion identities and the Slab Confinement Theorem are also elementary, and have not been checked against the earlier literature. No novelty is claimed for the results of this part	Not counted	25-03 § 08 (a), 25-04 § 06, 25-05 § 08 (d)
4	None of the insertion identities and inequalities derives confinement. The large-deviation step, the ultraviolet step and the upper bound on the second moment remain	Not shown	25-05 § 08 (a), § 07
5	The count for a dilute gas of fractional instantons (string tension $\propto \Lambda^2(\Lambda L)^{5/3}$, margin $5/3 = 11/3 - 2$) is a physical estimate in the dilute regime $\Lambda L \ll 1$, at the one-loop level. The action values obtained by cooling are upper bounds on the minimal action, not lower bounds	Not read strongly	(a)(b) of 25-02 § 07
6	The Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3) is a statement of a draft, and its label in this book is Assumption. It is not uniform in N, and says nothing about the continuum limit or about the mass gap. The converse bound (uniform in M) is open	Not read strongly	25-02 § 06, § 07 (e)

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
7	The count that, in four dimensions, independent containers give only up to $R/\ln R$ is a count on top of the assumption of the original source (that the free energy of thick containers takes a form uniform in the boundary values), not a four-dimensional theorem. Neither that assumption nor its negation has been shown	Not read strongly	25-03 § 08 (d)
8	The three facts about the twisted T^2 (uniqueness of the vacuum, the gap of the adjoint field $4\sin^2(\pi/2N)$, the absence of flat configurations in the (1,1) sector) say nothing about the lattice spacing $a \to 0$. The gap of the adjoint field is a lower bound on classical fluctuations, not a gap of the quantum effective action	Not read strongly	25-01 § 09 (a)
9	The Lean column covers only finite sums and real algebra. The Haar measure of $SU(2)$, the co-closed sign field, the topology and projections of sectors, and that θ is the bias of a sector are outside Lean	Not read strongly	25-03 § 08 (f), 25-04 § 07 (e), 25-05 § 08 (b)
10	No sentence of this part says anything about widening the T^2 beyond $1/\Lambda$. That stage is the crossover band, and the second road stops there	Not shown	25-02 § 07 (d), 25-09
11	The Driven Insertion Identity and the Flow Lemma do not derive confinement. "For a one-step kernel the flip probability saturates around $\frac{1}{2}$" is a judgement from measurement and the record, not a theorem. The Crooks Relation and the Jarzynski Equality are known	Not read strongly	(a)(d)(e) of 25-06 § 08
12	The comparison of isometric maps and reshaping maps is a statement at the one-loop (Gaussian) level. "Upper bounds come only from variation and explicit construction" is a judgement about this book's list of tools, not a theorem. That the leftover relative entropy is proportional to the area is an estimate for general maps	Not read strongly	(a)(c)(e) of 25-07 § 08
13	The measured numbers of Part 25 are not proofs. The trend from widening the T^2 does not show the band numerically (it goes only up to about $L\sqrt{\sigma} \approx 2$), and reading the measurement of simultaneous flips as a numerical picture of the band has been withdrawn	Not read strongly	(a)(b)(f) of 25-08 § 07

05

The items of Part 26 — on the band

These items are picked up from the chapters of Part 26 (the crossover band). The band is the interval around the scale $L \sim 1/\Lambda$ at which weak coupling turns into strong coupling; it is a different thing from the "window" of Parts 23 and 24 (26-00 § 02).

Table 5 What Part 26 does not claim, and the sentences of the text that support it

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
1	The band is unsolved. This book does not show, in any form, a statement that crosses the band ($\ln(1/\theta(L,L)) \geq cL^2$ uniformly in L, or an intermediate milestone towards it). Neither the renormalisation comparison nor β-monotonicity has been shown at any β of the band	Not shown	26-01 § 06, 26-02 § 08, 26-06 § 09 (a)
2	The reduction did not make the problem easier. The reduction theorem is a statement on top of three assumptions, and the strong form of the lateral-widening inequality is stronger than confinement (it contains monotonicity of the slab tension). The lemma that shrinks the comparison inside the band to the area law at fixed β is also a reduction, and its assumption contains an unsolved problem	Not read strongly	26-02 § 05, 26-06 § 09 (c), 27-02 § 03
3	Do not read "the renormalisation comparison holds at both ends" with the conditions removed. The strong-coupling end is an outline in the range $u \leq u_0 \approx 10^{-80}$ with part of the bookkeeping only sketched, and the weak-coupling end is a computation at the one-loop level. A non-perturbative weak-coupling comparison has not been shown at any β	Not read strongly	26-06 § 09 (b), § 03
4	The comparison principle (in the form that Wilson loops on $\mathbb{R}^4$ are bounded by loops on the twisted slab) is a statement at the one-loop level and is not used as a non-perturbative tool. With $SU(2)$, the Wilson action and $\beta = 2.1$, the plaquette on the twisted slab $16 \times 16 \times 2 \times 2$ is lower than on 16^4 (7σ with errors taken from both sides, 5.5σ in an independent rerun), so it is broken for small loops. The book places this as a reading that refutes it Reading (not a theorem)	Not read strongly	26-08 § 04

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
5	The numbers of the lattice-doubling test (small lattices up to 8^4, few values of β, 1–3 seeds) are a reading of direction, not evidence that $SU(2)$ has no phase transition to a non-confining phase	Not read strongly	26-06 § 09 (d), 26-00 § 03 (ii)
6	We do not say that β-monotonicity is false (no counterexample has been found; a reading). The four faces of the middle are candidates, and none has an estimate uniform in the scale	Not called false	26-06 § 09 (e)(f)
7	The wall of the first step, the "summit-curvature shortfall wall", is the band in miniature, not the band. We do not say that the results of the first step imply anything in four dimensions. That four dimensions have no "finite block beyond the crossover length" is not a claim that the road of blocks is impossible in four dimensions	Not read strongly / Not called false	(b)(f) of 26-07 § 07, 26-00 § 05
8	We do not say that the road of blocks and mixing closes the first step. Apart from the Hidden-Convexity Proposition, its rows are Assumption and Reading	Not shown	26-07 § 07 (d), the road of blocks and mixing in 23-15
9	"The only place where non-abelian enters" is a phrase with the restriction "within this reduction". The margin $5/3 = 11/3 - 2$ is a physical estimate in the dilute regime at the one-loop level	Not read strongly	26-02 § 08, 26-00 Table 1
10	The negative lemma, that the lateral-widening inequality does not follow from structural inputs alone (reflection positivity, the identities of centre and duality, positivity of the transfer matrix), is a statement of the form "from this set of inputs alone", not a claim that the band cannot be crossed	Not called false	26-00 Table 1, 26-04 § 04
11	None of the candidate new viewpoints closes the band. The reduction by compactness of the band is a reduction, and we do not say it has made the band easier. Nor do we say that the candidates are not combinations of known facts	Not shown	(a)(b)(g) of 26-10 § 07
12	"Stopped" for a road tried and stopped means that its tool could not cross the band, not that the proposition aimed at by the road is false. Roads stopped for reasons of estimate can be reopened if a new input appears	Not called false	26-08 § 01

06

The items of Part 27 — on the conditional statement and the ledger

Table 6 What Part 27 does not claim, and the sentences of the text that support it

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
1	Confinement (the area law) is not the mass gap. There is no known road that gets the mass gap from the area law, and the arrow joining the two on the same theory is open (even the statement of what it suffices to assume is not settled)	Not shown	16-09 § 03, 27-01 § 04, 25-00 § 08 (c)
2	The conditional statement is a statement only insofar as the band is placed among the assumptions. Only the arrows whose paper proof has been placed in the book are called "theorems", and the assumptions in the form of the band (the lateral-widening inequality, the uniform slab tension, the uniform slab gap) carry the content of the conclusion	Not read strongly	27-01 § 05
3	The propositions of the research programme on $\mathbb{R}^2\times$ a small twisted T^2 (the arrow of the small-T^2 gap on the gap side, the arrow of the small-T^2 area law on the confinement side) are conditional and at the stage of an outline. The assumption of the arrow of the small-T^2 gap (block correlations decay at rate $m \geq c/L$) contains the quantity of the conclusion	Not read strongly	27-01 § 02, § 03
4	Even if every row of the ledger disappears, that does not mean the problem is solved. Four things remain: the limit to $\mathbb{R}^4$ (split into four rows in Table 3 of 27-02), tying the two roads together, general groups, and verification and peer review (seven rows with the breakdown)	Not shown	27-02 § 05, 27-00 § 01
5	A row closed in the record (mark ■) does not mean it has disappeared from the book's ledger. A row disappears only when its paper proof has been placed in the book and, if it has a Lean entry, the re-check is done	Not read strongly	27-02 § 01
6	The number of rows of the ledger is not a gauge of progress. Rows can also split and increase	Not counted	27-02 § 06

07

Conditions drop out in summaries — sentences of Parts 25 to 27 and one-line summaries

The error of reading a claim strongly arises, in most cases, at the stage of shrinking a sentence of the text into one line (23-16 § 06). Table 7 sets the conditional sentences of the text of Parts 25 to 27 beside one-liners from which the conditions have been dropped entirely. Every sentence in the right-hand column is a sentence this book has not shown.

Table 7 Sentences of the text (with conditions) and one-liners with the conditions dropped (everything on the right is wrong, or not shown)

Topic	Sentence of the text	One-liner with the conditions dropped
Slab confinement	With the lattice spacing and the side N of the T^2 fixed, $\sigma \geq -\ln\tanh(\beta N^2)$ for every β Proved here (25-04 § 02)	Confinement in lattice gauge theory has been shown
Confinement Lower Bound by Fractional Instantons	At fixed $N \geq 2$, $\ell \geq 2$, $\beta \geq 1$, $M \in 2\ell\mathbb{N}$, a lower bound on the area law with rate $S_*(N,\ell)$ (a statement of the draft; the book's label is Assumption , 25-02 § 06)	It has been shown that fractional instantons cause confinement
The one-loop comparison	With the comparison that reshapes the fluctuations, $\beta - \beta' = 0.258$ (per doubling) Proved here (at the one-loop level, 26-06 § 03)	The renormalisation comparison holds
The doubling test	For $SU(2)$, the $\beta - \beta'$ read off is $+0.13$–$+0.25$ ($4^4 \to 8^4$ and others, $\beta = 2.5$–3.0, 1–3 seeds) Reading	$SU(2)$ has no phase transition inside the band
The reduction theorem	Under the three assumptions of the base point, frame decorrelation and the lateral-widening inequality, the correct form of the band follows Proved here (conditional, 26-02 § 03)	The band has been reduced to a single inequality and become easier
The comparison inside the band	Under the area law on both sides at each β of the band (the assumption of two-sided area laws inside the band), the comparison inside the band shrinks to finitely many steps Proved here (26-06 § 06)	The renormalisation comparison inside the band has been shown
The iteration in Lean	`Doubling.iterate` (assumptions: $c \geq 0$, $t_k \geq 0$, $t_{k+1} \leq c^3 t_k^4$; conclusion: $c\,t_k \leq (c\,t_0)^{4^k}$) is algebra of a sequence of real numbers	The reduction of the band has been checked in Lean

Topic	Sentence of the text	One-liner with the conditions dropped
Reflection positivity of z	The proof is placed in 26-04 § 03. The algebra of the last step is also closed in Lean (ReflPos). This row is closed (27-02 § 01)	The whole proof of reflection positivity of z has been checked in Lean

The error types in Table 7 are the same two as in Table 5 of Part 23. One drops the scope (fixed lattice spacing, fixed N, the one-loop level, small lattices); the other drops the grade (Reading, Assumption, the reach of Lean). In Parts 25 to 27 one more type is added: reading "reduced" as "made easier". A reduction gathers the difficulty into one place; it does not reduce the difficulty.

What this chapter can say

What is said	What is not said
That this book shows nothing about the existence and the mass gap of the problem statement (0-03). Which sentences of which chapters support the items of Parts 23 and 24 (Tables 2 and 3)	The truth or falsity of the claims of the problem statement. Existence of the continuum limit (on the side of this book's study; the published construction showed it in a periodic box). A gap uniform in the volume
The items of Parts 25, 26 and 27, and the places in the text that support them (Tables 4–6)	The truth or falsity of the proposition of each item. Of the sentences of chapters 26-03 to 26-05 and 26-09, only those that overlap with rows of Table 5 have been picked up
Pairs of conditional sentences of Parts 25 to 27 and one-liners with the conditions dropped (Table 7)	That the sentences in the right-hand column of Table 7 are false (only that they have not been shown)

Sources and reproduction

Item	Kind	Source / tools
The three meanings	This book	23-16 § 01
Each row of Table 2	This book	23-16 Table 2 and the chapters in its right-hand column
Each row of Table 3	This book	24-08 Table 4 and the chapters in its right-hand column
Each row of Tables 4–6	This book	The sections "what cannot be said", "what remains" and "the differences that remain outside the ledger" of the chapters in the right-hand column
Table 7	This book	The statements and labels of the chapters in the left-hand column. The numerical conditions are in 26-00 § 03 (ii) and 26-06 § 05

Item	Kind	Source / tools
The problem statement	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000) (the source table of 0-03)

Terms used in this chapter — not shown, not read strongly, not called false, not judged / not counted, the conditions that drop out in a summary, the crossover band, reduction, the differences that remain outside the ledger, the diffusive-relaxation-time assumption. Their meanings are in the A-02 Glossary and the A-06 Index of names.

Next chapter: 28-00 Entrance — the statements to be shown, and sufficient conditions through a representation. To read alongside it, A-07 Papers that came out of this book — the main theorem of each paper, with its label and conditions. To go back to the entrance of Part 27, 27-00.

Entrance — the statements to be shown, and sufficient conditions through a representation

> This chapter is a guide.

What this chapter shows — Part 28 lays out "what would suffice" as a tree of the root and its sufficient conditions. This entrance shows the distinction between the root and the sufficient conditions, how to read the premise labels, and the order of the chapters in this part.

Prerequisite chapters — 19-07 Statements to be shown. The firm theorems are in Parts 20–22 (20-00, 21-00, 22-00).

Said first — This part is a list of premises and implications, and it shows none of the roots. Of the roots, the four on the ultraviolet side (matching, convergence of the full sequence, non-degeneracy, equicontinuity) and the tree of premises that supports them are statements about this book's design (measures pushed forward by a fixed block map). The existence of four-dimensional Yang–Mills theory was shown by the construction published on 2026-11-10, with a different design, and the committee's assessment is that it is proved in Lean (Volume 2; the two designs are compared in Part 14, and the record of this book's ultraviolet attempt is in Part 15). So the roots and the tree on the ultraviolet side are kept as a record of the statements to be shown and the sufficient conditions within this book's design. The roots on the infrared side (uniform decay and the mass gap that follows from it, and the band) remain open, and these are the main subject of Volume 3. The premise labels are as of 2026-11-13; they are updated in A-08 A map to the mass gap.

The order of this chapter

1. The root, and sufficient conditions through a representation (Table 1)
2. Key to the premise labels (Table 2)
3. Symbols for why a form is not fixed (Table 3)
4. The chapters of this part

01

The root, and sufficient conditions through a representation

In this part we call the statements to be shown the "roots" of the tree, and their sufficient conditions the "leaves".

The **roots** are the six statements to be shown, written only with the objects defined in Part 19 (the Wilson measure, the push-forward measure, the vortex ratio, the vessel of reconstruction) (19-07).

The **sufficient conditions** are the premises we place in order to derive the root on the present road. Many of them are statements made through quantities used to write down the theory at coarse scales — effective couplings, large-field activities, background fields, classes — and this book calls such a description a **representation**. Many of the quantities of a representation are not yet defined from the lattice.

Table 1 The root, and the sufficient conditions on the present road

Root (19-07)	Language of the root	Sufficient conditions on the present road (names of rows in 28-04)	Language of the sufficient co
Matching	Expectations under the push-forward measure, β_n	One-step bracket of the coupling and recursion of the activity (definition and continuity of the effective coupling, net one-step bracket of the coupling, large-field contribution to the coupling, recursion of the activity, rarity of supply, moving invariant interval and margin)	Effective coupling u_k, activ supply w_k
Convergence of the full sequence	Push-forward measure $\nu_{n,m}$	Contraction of the one-step map and matching at the endpoint, the estimates of the premise of ultraviolet stability	Coordinates of the one-step class, background field
Non-degeneracy	$\nu_{n,m}$ and the set of small fields	Rarity of two-step events (the plaquette part, the background-invariant part)	The plaquette part in the sa language as the root; the background-invariant part background fields
Equicontinuity	$\nu_{n,m}$ and the family of observables	Equicontinuity of observables	Same language as the root
Uniform decay	Covariances under $\nu_{n,m}$, the vessel	Uniform decay of block observables, of the same form as the root; the seam to the vessel of the mass gap; the rows for fine observables, infinite volume, rotation and non-triviality (from approximation of flowed observables by block variables to the limit is not a free field); band comparison (in the language of the gap) and from the area law to the gap	Flowed observables, rotate lattices, box masses
Band	Vortex ratio θ	Base point, nesting and widening (non-negativity of θ, base point at small T^2, single-scale criterion, question of the cost of a twist-inserting map, frame decorrelation, lateral-widening inequality (strong form)), tension doubling (tension-doubling inequality, monotonicity of the vortex free energy and renormalisation comparison)	θ, sheet tension

The root is read for the sequence fixed by matching. If it is read in a form where the sequence of couplings can be chosen freely, $\beta_n \equiv 0$ alone makes the mass-gap and band parts of the root stand (19-07 § 01). The "Matching" row of Table 1 is not one row alongside the other five; it is the row that fixes how the other five are quantified.

The root does not depend on the representation. If the representation is changed (a different definition of the background, a different norm, and so on), the sufficient conditions change, but the root does not. The list (28-04) writes the root at the top and the sufficient conditions below it, as those "on the present road".

02

Key to the premise labels

A premise label is written as one column of a table. It is separate from the labels used in the text (Proved here, Cited, Computer-assisted, Reading, Assumption — the system of 23-00). When the text writes out the statement of a premise itself, its label in the text is Assumption .

Table 2 Premise labels

Label	Meaning	What it does not guarantee
Theorem	Machine-checked in Lean, and a check that compares the definitions and the statement with a separate pair of eyes has confirmed the content of the statement	That it is the premise in exactly its stated form (the scope is written in the same column)
Theorem (weak form)	The same, but only up to a form weaker than the premise. We say separately whether it pays the column as it stands or is a similar but different statement	That it reaches the form of the premise

Label	Meaning	What it does not guarantee
On paper	There is an argument on paper	That it is free of errors (the check comparing definitions and statement with a separate pair of eyes has not been done)
Form fixed, unproved	The form of the statement is settled. "Conditional" means: for the sequence fixed by matching, and only when the restrictions on threshold and period are imposed	That it holds
Form not fixed	The form of the statement is not yet settled. The reason is given by a symbol (Table 3)	—
Depends on design	Determined by a design choice (28-05)	That the choice is determined uniquely
Literature	Depends on a result in the literature	That this book has re-derived it

03

Symbols for why a form is not fixed

Table 3　Reason symbols

Symbol	Meaning
(a)	Too light: it stands for a trivial object, or because a quantity can be chosen freely
(b)	Too strong: it is false, or there are examples where it is false
(c)	Object undefined: a quantity in the statement is not defined from the lattice
(d)	There are numerical signs against the present form
(e)	There is a form at one scale, but no form across steps
(f)	The order of quantifiers is not fixed

A list sorted by origin is in 28-06.

04

The chapters of this part

Chapter	Contents
28-01 Subsequential limits, and the properties that pass to the limit	The parts that stand without premises, and the parts where premises do the work
28-02 Convergence of the full sequence	Convergence of the full sequence through matching at the endpoint
28-03 From decay to the mass gap, and the reduction of the band	From uniform decay to the mass gap of the vessel; from base point, nesting and widening to the area exponent
28-04 The list of premises — tree and labels	The root and the tree of 57 rows of premises, tables by stage, counts by label, a dictionary of other names
28-05 What depends on design choices	Nine choices, and which theorems and premises depend on which
28-06 Open places, and places where the object does not yet exist	Where the 24 rows whose form is not fixed come from, and what would fix their form. What is not claimed

The three chapters on implications are theorems that write "what follows if the premises hold", with all the assumptions shown. What follows can be weaker than the root (28-01 reaches only subsequential limits), and there are also implications that go beyond the root (28-03 goes from uniform decay to the mass gap of the vessel).

What this chapter can say

Can say	Cannot say
The distinction between the root and the sufficient conditions; the meaning of the premise labels and the reason symbols	None of the root is shown

Sources and reproduction

Item	Type	Source, tools
The tree of the root and sufficient conditions, the labels	Assumption	The inventory of premises in this book's study (2026-11-13). The full text is in 28-04

Words used in this chapter — root, sufficient condition, representation, premise label, reason symbol. Their meanings are in A-02 Glossary.

Next chapter: 28-01 Subsequential limits, and the properties that pass to the limit — reads what follows from the premises, with all the assumptions shown. To start from the list, read 28-04.

Subsequential limits, and the properties that pass to the limit

This chapter is this book's study.

What this chapter shows — Let $\nu_{n,m}$ be the measure obtained by pushing the fine measure μ_n on a box of side $2^n L$ forward to the coarse level m by the block map. For it we read, with all the assumptions shown, the conditional theorem "if many assumption fields hold, then a subsequential limit exists, and the limit has reflection positivity, symmetry and a non-degeneracy lower bound" (Lean: `BoxWalk.chain_end_to_end_box`). The content of this theorem splits in two. (a) Properties that hold uniformly on finite lattices pass to subsequential limits. (b) From the one-step structure for the coupling, the matching point can be taken inside an explicit bracket.

Prerequisites — 19-04 The block map and the push-forward measure, 19-01 The Wilson measure, 19-03 Reflection, 19-07 The statements to be shown.

Said first — This is an implication, and none of its assumptions has been shown on the four-dimensional lattice. The existence of a subsequential limit holds, without assumptions, for any family of probability measures. The reading that joins (a) and (b) ("the fine measure is the Wilson measure at the coupling fixed by the matching point") lies outside the theorem. The quantities put into the assumptions (effective coupling, activity, supply, remainders, background invariants, the sequence of observables) are not defined from the lattice. So the fact that this theorem is closed does not mean that the content of the continuum limit has been shown. The existence of four-dimensional Yang–Mills theory was shown by the construction published on 2026-11-10, with a different design (Volume 2). The implication of this chapter is a record of what this book's design (measures pushed forward by a fixed block map) placed as premises (§ 08).

The order of this chapter

1. The form of the theorem — what is fixed and what is left free — only the block map and the push-forward are fixed by definition

2. The table of assumption fields — Lean names, mathematical wording, and whether they are paid by concrete objects (Tables 1 and 2)
3. The first conclusion — the matching point lies in a bracket — a statement about the family of functions u alone, not connected to the measure
4. The second conclusion — convergence along a subsequence — from compactness, it holds for any family of measures
5. Properties that pass to the limit — reflection positivity, invariance, lower bounds on the mass of closed sets, the modulus of continuity of observables (Table 3)
6. The version with the Wilson measure as the fine measure — the scope in which two fields disappear, and the fields that remain
7. The reading that joins the two lies outside the theorem — joining the sequence of bare couplings to the effective coupling needs a definition of the effective coupling
8. Relation to the conditional implication assembled earlier — its assumptions are statements written with the push-forward measure

---

01

The form of the theorem — what is fixed and what is left free

One road to building a continuum limit goes as follows. On boxes whose lattice spacing is halved step by step (periodic boxes of side $2^n L$, $n = 0, 1, 2, \ldots$) we place fine measures μ_n, and push them forward by the block map to the coarse box of side $2^m L$ (19-04). The limit $n \to \infty$ with the coarse level m fixed is the candidate for "the distribution of the block variables of the continuum limit" on a box of fixed physical size.

We fix notation. $\mathbb{B}$ is the closed unit ball of the quaternions, and E_m is the space of configurations that put a value in $\mathbb{B}$ on each link of the box of side $2^m L$. Since E_m is a finite product of balls, it is a compact metric space. $B_m : E_{m+1} \to E_m$ is the block map, $\mathrm{down}_{n \to m}$ its $n - m$-fold composite, and the push-forward measure is

$$\nu_{n,m} = (\mathrm{down}_{n \to m})_* \, \mu_n \qquad (m \le n) \quad (1)$$

(Lean: `ChainBlockMap.blockMap`, `down`, `vP`). The block map includes a retract that returns the average of path products to the ball with threshold $r_0 > 0$ (19-04). r_0 is a free argument of the theorem (field `hr : 0 < r₀`), and the limit may depend on r_0. In

the box version the family of paths and the weights are also fixed (weights $(1/8, 3/4, 1/8)$, six orderings; Lean: `BoxWalk.boxPaths`, `boxW`).

This much is fixed by definition. Inside the theorem, the fine measure μ_n is an arbitrary family of probability measures. Besides this, the coupling functions u_k, activities z_k, supplies w_k, remainders $r_k^{\S}, r_k^{]}$, plaquette variables $Y_{m,p}$, background invariants $f_{m,x}$, the sequence of observables s_n, the reflection Θ_m, the symmetry actions, the set of small fields C_m and so on all enter as free variables. The assumption fields are conditions on these free quantities.

Assumptions	First conclusion	Second conclusion
The one-step structure for the coupling, rarity, splitting of observables, reflection positivity, symmetry, representation, and others (§ 02) All Assumption . The quantities are not defined from the lattice	For any target value t and step n, a point x with $u_n(x) = t$ lies in an explicit bracket (§ 03) A statement about u alone	Along some subsequence, $\nu_{n,m}$ converges at every level m, and the limit inherits the assumed properties (§ 04, § 05) The convergence itself holds without assumptions

Lean: `ChainBlockMap.chain_end_to_end_blockMap` (version with a general family of paths and weights), `BoxWalk.chain_end_to_end_box` (box version) — machine-checked as an implication (assumptions as in the tables). The two conclusions sit side by side, joined by "and", in a single statement.

02

The table of assumption fields

We list all the assumption fields. The column "Paid?" says whether the field is shown for concrete objects (measures or maps with a definition), or remains an assumption about abstract quantities. In the box version, the continuity of the block map and the consistency of the push-forward ($(B_m)_*\nu_{n,m+1} = \nu_{n,m}$) are paid from the definitions inside the theorem and do not appear as fields (Lean: `ChainBlockMap.continuous_blockMap`, `hcons_blockMap`).

Table 1 Fields on the side of measures and observables ($\nu_{n,m}$ is as in (1). "For large n" means "there is n_0 such that for $n \geq n_0$")

Lean field	Mathematical wording	Paid?

Lean field	Mathematical wording	Paid?
`rarity`	Assumption For each level m, large n and each block i of the region R_m: $\nu_{n,m}$(the plaquette variable $Y_{m,p}$ of block i exceeds the threshold $y_h(m)$, or the background invariant $f_{m,x}$ exceeds the threshold $f_h(m)$ (in Lean this threshold is named `θ`, which has nothing to do with the vortex ratio))$\le n_p(m)P_1(m) + n_{pt}(m)S(m)$. Rarity	Abstract. $Y_{m,p}$ and $f_{m,x}$ are free functions, and the event itself is not defined
`hCc`, `hgood`	Assumption C_m is a closed set and contains the event "in every block, plaquette variable $\le y_h$ and invariant $\le f_h$"	Abstract
`hθc`, `hθp`	Assumption Θ_m is continuous. For large n and F in the class $\mathcal{A}_m$, $\int F(\Theta_m\omega)F(\omega)\,d\nu_{n,m} \ge 0$. Reflection positivity	**There is a version paid with the Wilson measure** (§ 06; the class is a strip of fixed width)
`hsc`, `hsym`	Assumption The symmetry action g is continuous. For large n, $\nu_{n,m}$ is invariant under g	Abstract (remains in the Wilson version too)
`split`, `lattice_vanishing`, `shift_vanishing`	Assumption $\lvert s_n(d) - s_n(d')\rvert \le \tau_n + \Omega(\mathrm{dist}(d,d'))$, $\tau_n \to 0$, $\Omega(r) \to 0$ $(r \downarrow 0)$. Splitting of observables (a split form of equicontinuity)	Abstract. s_n is free
`hcount`, `hdense`, `hbd`	Assumption The set D in the space X of observable placements is countable and dense, and $\lvert s_n(d)\rvert \le K$ (K is one number independent of d and n; its Lean name is `Bd`)	Abstract
`hac`, `ham`, `hbr`, `hbv`	Assumption The symmetry action on X is continuous and preserves D, and the breaking $\lvert s_n(gd) - s_n(d)\rvert \le \varepsilon_b(g,n,d) \to 0$	Abstract
`hpos`	Assumption For large n, $\sum_{a,b} c_a c_b\, s_n(x_{ab}) \ge 0$ (the matrix form of reflection positivity)	Abstract
`HR`	Assumption For each $d \in D$ and large n, $s_n(d) = \int F_d\,d\nu_{n,\mathrm{lev}(d)}$ (F_d is a bounded continuous function at level $\mathrm{lev}(d)$). Representation	Abstract (holds trivially if s_n is *defined* as the expectation under ν)

Table 2 Fields on the side of the coupling (the structure `TransientChainError.CouplingStepsTransientE` and two outside it. The domain is $\{x : u_j(x) \ge u_\star\ (j \le k)\}$)

Lean field	Mathematical wording	Paid?

Lean field	Mathematical wording	Paid?
`flow`	Assumption $u_{k+1} = u_k - b + r^{\mathrm{s}}_k + r^{\mathrm{l}}_k$ (the one-step decrease of the coupling)	Abstract. $u, z, w, r^{\mathrm{s}}, r^{\mathrm{l}}$ are free functions $\mathbb{N} \times \mathbb{R} \to \mathbb{R}$, and $W, \Phi, \mathcal{D}, e$ are free functions $\mathbb{R} \to \mathbb{R}$
`small_net`, `transient`	Assumption $b_{\mathrm{inf}} + \delta_k - e(u_k) \le b - r^{\mathrm{s}}_k \le b_{\mathrm{p}} + e(u_k)$, $\sum_{k<n} \max(-\delta_k, 0) \le D_0$ (the net bracket for small fields, and the sum of transients)	
`large_rem`	Assumption $\lvert r^{\mathrm{l}}_k \rvert \le \mathcal{D}(u_k)\, z_k$ (the large-field remainder)	
`act_rec`, `act_nonneg`, `act_init`, `supply`	Assumption $z_{k+1} \le \rho z_k + A w_k + B z_k^2$, $z_k \ge 0$, $z_0 \le W(u_0)$, $w_k \le \Phi(u_k)$ (the recursion of activities, and the supply)	
`fix_shift`, `DW`, `margin`	Assumption $\rho W(s) + A\Phi(s) + B W(s)^2 \le W(s')$ ($u_\star \le s$, $u_\star \le s' \le s + D_0$), $\mathcal{D}(s) W(s) \le Y$, $e(u_\star) + Y \le b_{\mathrm{inf}}$ (a moving invariant interval, and the margin)	
`ustar_pos`, `he`, `hD`, `hρ`, `hA`, `hB`	Assumption $u_\star > 0$, e is decreasing on $[u_\star, \infty)$, $\mathcal{D} \ge 0$, $\rho, A, B \ge 0$	
`h0`, `hCu`	Assumption $u_0(x) = x$. $x \mapsto u_{k+1}(x)$ is continuous on the domain	Abstract

The fields of the two tables can be satisfied simultaneously Proved here . For example, take the region R_m, the class $\mathcal{A}_m$, P and the index sets of the two symmetry actions to be empty, $C_m = E_m$, X a single point, $s_n(d) = \int F_d \, d\nu_{n,\mathrm{lev}(d)}$, $u_k(x) = x - k$, $b = b_{\mathrm{inf}} = b_{\mathrm{p}} = 1$, the remainders, activities and supplies and $W, \Phi, \mathcal{D}, e, \delta, D_0, Y, \rho, A, B$ all 0, and $u_\star = 1$. With this choice, however, the corresponding conclusions are also trivial. No choice that pays the fields with quantities of real content is known.

Detour: what the one-step structure for the coupling actually says

May be skipped. The structure in Table 2 has seventeen fields, but the content that matters for the first conclusion is that "the one-step decrease $u_k - u_{k+1}$ of u lies in the bracket $[\, b_{\mathrm{inf}} + \delta_k - e(u_k),\ b_{\mathrm{p}} + e(u_k) \,]$", together with numerical conditions on $u_\star, e, \delta, D_0$ alone Proved here . Indeed, if an arbitrary u satisfies this bracket, then setting $r^{\mathrm{s}}_k := u_{k+1} - u_k + b$, $r^{\mathrm{l}} = z = w = 0$, $W = \Phi = \mathcal{D} = 0$, $Y = \rho = A = B = 0$ makes `flow`, `small_net`, `large_rem`, the four on activities and supply, `fix_shift`, `DW`, `hD`, `hρ`, `hA` and `hB` hold. The remaining four — $u_\star > 0$ (`ustar_pos`), e decreasing on $[u_\star, \infty)$ (`he`), the sum of transients $\sum_{k<n} \max(-\delta_k, 0) \le D_0$ (`transient`), and $e(u_\star) \le b_{\mathrm{inf}}$ (the $Y = 0$ form of `margin`) — do not follow from this choice, and remain as conditions on the choice of $u_\star, e, \delta, D_0$. The fields for the large-field remainder, activity and supply give a name to one "way of splitting" that produces the bracket; as long as the quantity that fixes the splitting (the decomposition of the effective action) is not defined from the lattice, they add no content by themselves. Which splitting carries meaning is the subject of the list of premises (28-04) and of the open places (28-06).

03

The first conclusion — the matching point lies in a bracket

The first conclusion has the following form.

$$\forall t \geq u_\star + D_0,\ \forall n,\ \exists x: \quad \begin{cases} t + (b_{\mathrm{inf}} - e(u_\star) - Y)\, n - D_0 \ \leq\ x \ \leq\ t + (b_{\mathrm{p}} + e(u_\star) + Y)\, n, \\ u_n(x) = t, \qquad u_k(x) \geq t - D_0 \ \ (k \leq n). \end{cases}$$

In the intended reading, x is the bare inverse coupling of the fine lattice, $u_k(x)$ the effective inverse coupling after k steps of renormalisation, and t the target value matched at the coarse end. Formula (2) reads: "the bare value x that becomes exactly t after n steps lies in an explicit bracket linear in n, and does not leave the domain at intermediate steps." The bracket closes around the one-loop decrease b because the one-step decrease lies in $[b_{\mathrm{inf}} - e,\ b_{\mathrm{p}} + e]$.

Formula (2) is a statement about the family of functions u and the constants of Table 2 alone. The fine measure μ_n, the push-forward measure $\nu_{n,m}$ and the observables s_n appear neither in (2) nor in the fields used to derive (2) (Table 2). Inside the theorem there is not a single field joining u and μ.

The proof runs in two steps. (i) On the domain, induct on k for the activity and the coupling together, and derive that the one-step decrease lies in the bracket (the large-field remainder is bounded by $\mathcal{D}W \leq Y$ and absorbed into the bracket by `margin`). (ii) From $u_0(x) = x$ and the continuity of u_{k+1}, take x with $u_n(x) = t$ inside the bracket by the intermediate value theorem Proved here . Both are arguments about real functions; the lattice does not appear.

04

The second conclusion — convergence along a subsequence

The second conclusion begins with the following existence statement.

$$\exists\, \varphi \text{ strictly increasing},\ \exists\, (\nu_m)_{m \geq 0} \text{ probability measures}: \qquad \nu_{\varphi(k),m} \xrightarrow[k \to \infty]{} \nu_m \quad (\text{wea}$$

Formula (3) **holds for any family of probability measures μ_n, without using a single assumption field**. The reason is that E_m is a compact metric space, and the space of probability measures on it is compact (and metrisable) in the topology of weak convergence Cited (Prokhorov's theorem, in the case of a compact space). Since the product over countably many levels m is also compact, one subsequence φ common to all m can be taken (Lean: `WeakUltravioletExistence.exists_common_subseq`).

So formula (3) by itself says nothing about what the limit is. The limit may depend on the choice of the subsequence φ. The limit has content reflecting the choice of the fine measure only through the "properties that pass to the limit" of the next section. Convergence of the full sequence (not a subsequence) is not among the conclusions of this theorem. It is the subject of the next chapter, 28-02, and one of the statements to be shown (convergence of the full sequence in 19-07, stated for the sequence of couplings fixed by matching).

The consistency of the push-forward $(B_m)_*\nu_{n,m+1} = \nu_{n,m}$ holds by definition, and since B_m is continuous it passes to the limit: $(B_m)_*\nu_{m+1} = \nu_m$ (Lean: `WeakUltravioletExistence.limit_consistent`). The family of limits is a projective system consistent with the block maps.

05

Properties that pass to the limit

The rest of the second conclusion is that the fields of Table 1 pass to the subsequential limit. All are of one type: "a closed condition that holds uniformly at finite n (for large n) is preserved in the weak limit."

Table 3 Properties for large n, and properties in the limit (the Lean conclusion structures `WeakUltravioletExistence.BlockConclusion` and `ObservableConclusion`)

Assumption (for large n)	Conclusion (in the limit)	Tool that carries it over
`h0c`, `h0p`	`reflection_positive`: for $F \in \mathcal{A}_m$, $\int F(\Theta_m\omega)F(\omega)\,d\nu_m \geq 0$	$F \circ \Theta_m \cdot F$ is bounded and continuous, so the integral converges, and ≥ 0 is a closed condition Proved here
`hsc`, `hsym`	`symmetric`: ν_m is invariant under the symmetry action	The same (an equality of integrals of bounded continuous functions) Proved here
`rarity`, `hgood`, `hCc`	`small_field_mass`: $\nu_m(C_m) \geq c_m$, $c_m = 1 - \lvert R_m\rvert\,(n_p P_1 + n_{pt} S)$	A union bound gives $\nu_{n,m}(C_m) \geq c_m$ for large n. The mass of a closed set does not drop in the weak limit: $\limsup_n \nu_n(C) \leq \nu(C)$ Cited
`HR`	For $d \in D$, $S(d) = \int F_d\,d\nu_{\mathrm{lev}(d)}$ and $s_{\varphi(k)}(d) \to S(d)$	The definition of weak convergence itself Proved here
`hbd`	`bounded`: $\lvert S(x)\rvert \leq K$ for all $x \in X$	A closed condition Proved here
`split`, `lattice_vanishing`, `shift_vanishing`, `hcount`, `hdense`	`continuous`, `modulus`: S extends uniformly continuously to all of X, with a modulus of continuity coming from Ω	Equicontinuity by an $\varepsilon/2$ argument; the limit on the countable dense D is extended to X Proved here

Assumption (for large n)	Conclusion (in the limit)	Tool that carries it over
`hac`, `ham`, `hbr`, `hbv`	`invariant`: $S(gx) = S(x)$	The breaking goes to 0, so equality on D; extended to X by continuity Proved here
`hpos`	`positive`: $\sum_{a,b} c_a c_b S(x_{ab}) \geq 0$	A closed condition Proved here

In Lean the non-degeneracy lower bound c_m is written as a subtraction of values in $[0, \infty]$, and when $|R_m|(n_p P_1 + n_{pt} S) \geq 1$ it is read as $c_m = 0$ (a lower bound without content). The lower bound has content only when the right-hand side of the rarity field is small compared with the size $|R_m|$ of the region.

This whole section shows that the implication on the side of measures goes only as far as "properties that hold uniformly at finite n pass to the weak limit". The properties themselves (rarity, reflection positivity, symmetry, equicontinuity of observables) all sit on the side of the assumptions. Of these, only the two fields in the next section are shown for concrete objects.

06

The version with the Wilson measure as the fine measure

There is a version in which the fine measure is the normalised SU(2) Wilson measure (coupling β_n, 19-01) on the box $(\mathbb{Z}/2^n L)^4$ of side $2^n L$. Here $\beta : \mathbb{N} \to \mathbb{R}$ is an *arbitrary sequence*. In addition, the reflection Θ_m is fixed to be the time reflection $x_0 \mapsto -x_0$ in a plane of lattice sites, and the class $\mathcal{A}_m$ is fixed to be the following $\mathcal{A}_m^{(c)}$.

$$\mathcal{A}_m^{(c)} = \{F \text{ bounded continuous} : \ F \text{ depends only on coarse links with base time in } \{1, \dots, 1$$

Then the reflection-positivity field `h0p` and the reflection-continuity field `h0c` are proved inside Lean and disappear (Lean: `BoxWalk.wilson_h0p`, `continuous_reflCfg`; the former is the assumption-free statement "for any sequence of couplings β and large n, the pushed-forward Wilson measure is reflection positive for the class of strip (4)"). The resulting statement is `BoxWalk.chain_end_to_end_box_wilson2` — machine-checked as an implication (the assumptions are those of Tables 1 and 2 minus `h0p` and `h0c`).

We state the scope of this version precisely.

- **Within the scope in which the two fields disappear, it is confirmed.** The class (4) is non-empty where m is large enough (it contains the function $\cos(\mathrm{Re}\, U_{\ell_0})$ of one link; Lean: `BoxWalk.stripSet_witness`), and the reflection positivity of the limit has content.

- **Read as "the chain has been closed with the Wilson measure", it is conditional.** The one-step structure for the coupling, rarity, the symmetry `hsym`, the representation and the observable fields all remain assumptions.
- The class (4) is a narrow class. The strip width c is one and the same number at every level m, does not widen as m grows, and does not include links in the time-0 plane. It is smaller than the positive-time class $\mathcal{A}_m(P)$ (the whole positive half) of the conditional implication assembled earlier (§ 08). That class is defined by reflection in a symmetric plane of the coarse lattice (for example a plane between lattice sites) and by pull-back to supports on the positive side of the fine lattice, and the reflection itself differs from $x_0 \mapsto -x_0$ in a plane of lattice sites. Moreover the conclusion is "some subsequence, and some limit" for each c, so "one limit is reflection positive for the strips of all c" does not follow from this one statement.

Even in the Wilson-measure version, β_n is an arbitrary sequence, and no relation at all is placed between it and u. This is the subject of the next section.

07

The reading that joins the two lies outside the theorem

So far, the content of the theorem has split in two.

(a) The side of measures	(b) The side of the coupling
Properties that hold uniformly on finite lattices (reflection positivity, symmetry, lower bounds on the mass of closed sets, equicontinuity of observables) pass to subsequential limits. The subsequential limit itself exists without premises.	From the one-step structure for the coupling, a matching point x with $u_n(x) = t$ can be taken inside an explicit bracket linear in n.

The intended reading joins the two by one sentence: "the fine measure μ_n is the Wilson measure at the coupling $\beta_n = 4x_n$ fixed by the matching point x_n (the point taken in (2) for the target t)." This sentence is **not in the theorem**. The Wilson version (§ 06) takes an arbitrary sequence β and places no relation between β_n and x_n. So all that can be read from the theorem is that (a) and (b) hold separately.

In the conditional implication assembled earlier (§ 08), this sentence is placed as a definition of the setting (the fine measure is defined to be the Wilson measure at the β_n chosen by the matching bracket for the bare coupling). Even in that implication,

however, the effective coupling u_k is defined not from the push-forward measure $\nu_{n,m}$ but as a quantity built from the Gibbs expectations (or ratios of partition functions) of separate finite lattices fixed for each step k, and the quantities on the side of the coupling and those on the side of the measure (for example a threshold on the coupling side, and the threshold $y_h(m)$ of the rarity event) are not joined inside the theorem. In the Lean chain theorem this definition is not placed either, and the fine measure is an arbitrary family.

To join them, the effective coupling u_k has to be defined from the lattice. For example, if u_k is defined as a function of some expectation under the Wilson measure at bare coupling x, then $u_n(x) = t$ in (2) becomes a statement about defined objects only: "match the fine Wilson measure so that some quantity at the coarse level takes a fixed value." That form is in 19-07 as one of the statements to be shown, "Matching" (19-07 states all the statements other than matching for the sequence of couplings fixed by matching). A choice remains as to which quantity to match by, and that belongs to the design choices (28-05).

In addition, the other quantities of Tables 1 and 2 are not defined from the lattice either. The effective coupling and the remainders (quantities that come from a decomposition of the effective action), the large-field activity and supply, the background field and its invariants, and the sequence of observables and its representation are all free variables inside the theorem. The fact that this theorem is "closed" is no substitute for defining these quantities from the lattice and showing the fields.

08

Relation to the conditional implication assembled earlier

This theorem rewrites an implication of the same type as the chain theorem of the conditional implication that this book's study assembled earlier (a form that derives weak ultraviolet existence of four-dimensional SU(2) lattice Yang–Mills theory in a small box by a machine-checked chain from per-step premises; 2026-10-30), with the coupling fields later put in a different form (the width of the bracket written with an error function e and transient terms δ_k) and the block map replaced by a concrete map on the box. The Lean of the earlier implication is a statement over an abstract block map. In the earlier implication, the fine measure is defined to be the Wilson measure at the β_n chosen by the matching bracket for the bare coupling (§ 07). Among the assumptions of the earlier implication, the small-field set assumption (at each level there is a closed set C_m with $\nu_{n,m}(C_m) \geq 1 - \eta_m$ for large n), equicontinuity of observables (equicontinuity of the expectations s_n) and the summability-of-increments assumption ($|s_{n+1}(x) - s_n(x)|$ is summable) are, once the fine measure is

given, statements written only with the push-forward measure and its expectations. The remaining assumption (the one-step bracket assumption for the coupling) is a statement about the effective inverse coupling u_k; the definition of u_k is on paper, and in Lean it is a free function. None of these assumptions has been shown in four dimensions, and the main theorem is an implication. The existence of four-dimensional Yang–Mills theory itself was shown by the construction published on 2026-11-10, with a different design (Volume 2; the two designs are compared in Part 14). In this book we first set out the statements that can be written with the push-forward measure alone as "statements to be shown" (19-07), and read the theorem of this chapter as the implication that takes charge of the part about subsequential limits and the properties that pass to the limit.

What this chapter can say

Can say	Cannot say
Implication: if all the fields of Tables 1 and 2 hold, then (2) holds, and the properties of Table 3 hold for the subsequential limit of (3). Lean: `BoxWalk.chain_end_to_end_box` (the box version of `ChainBlockMap.chain_end_to_end_blockMap`) — machine-checked as an implication (assumptions as in the tables)	None of the fields is shown for concrete quantities on the four-dimensional lattice (except the two of § 06). The quantities put into the assumptions are not defined from the lattice
Cited The subsequential limit of (3) exists for any family of probability measures (compactness). Lean: `WeakUltravioletExistence.exists_common_subseq`	That the limit does not depend on the subsequence (convergence of the full sequence; 28-02)
Proved here Formula (2) is a statement about u and the constants of Table 2 alone, and is not connected to the measure	That μ_n is the Wilson measure fixed by the matching point (a reading outside the theorem)
Wilson version: when the fine measure is the Wilson measure (an arbitrary sequence of couplings), `h0p` and `h0c` disappear. Lean: `BoxWalk.chain_end_to_end_box_wilson2`, `wilson_h0p` — confirmed within the scope in which the two fields disappear	That the chain has been closed with the Wilson measure (the remaining fields are assumptions). The class is the fixed-width strip (4), not the whole positive half
Proved here The fields of Tables 1 and 2 can be satisfied simultaneously (by trivial objects)	A choice that satisfies the fields simultaneously with objects of real content

Sources and reproduction

Item	Type	Source, tools
The statement of the implication and the conclusion structures (§ 01– § 05)	Lean	Lean of this book's study: `ChainBlockMap.chain_end_to_end_blockMap`, `BoxWalk.chain_end_to_end_box`, structures `TransientChainError.CouplingStepsTransientE`, `WeakUltravioletExistence.BlockConclusion`, `ObservableConclusion`, `Representation`. The statements were copied by reading the source text
Wilson version (§ 06)	Lean	`BoxWalk.chain_end_to_end_box_wilson2`, `wilson_h0p`, `stripSet`, `stripSet_witness`, `reflCfg`. Reflection positivity on the fine lattice is in 20-01
Compactness of the space of probability measures (Prokhorov's theorem), upper semicontinuity of the mass of closed sets	Cited	Standard facts of measure theory. Bibliography only (unchecked). The Lean uses Mathlib lemmas
Example in which the fields are satisfied simultaneously (§ 02)	Proved here	The choice given in the text
Assumptions of the conditional implication assembled earlier (small-field set assumption, equicontinuity of observables, summability-of-increments assumption, one-step bracket assumption for the coupling; § 08)	—	Record of this book's study (2026-10-30)

Words used in this chapter — implication, push-forward measure, subsequence, weak convergence, projective system, one-step structure for the coupling, matching point, bracket, rarity, set of small fields, non-degeneracy lower bound, equicontinuity, representation, strip class. Their meanings are in A-02 Glossary.

Next chapter: 28-02 Convergence of the full sequence — reads conditions sufficient to remove the subsequence, in the form of a Lipschitz contraction of the one-step map and matching at the endpoint.

Convergence of the full sequence

> This chapter is this book's study.

What this chapter shows — The theorem of the previous chapter gives only subsequential limits. To remove the subsequence, it suffices that the differences of coarse-scale quantities when the lattice is made one step finer be summable. In this chapter we read, with all the assumptions shown, one sufficient condition that bounds those differences — a Lipschitz contraction of the one-step renormalisation map, and matching of the coupling at the coarse end — as the convergence of skew-product orbits matched at the endpoint (Lean: `SkewOrbit.coarse_tendsto` and others). Lean examples also show that the conditions $\varepsilon + \tau < 1$ and $4cc' < (1-\varepsilon-\tau)^2$ are necessary and sufficient for this method, and that matching at the endpoint cannot be removed.

Prerequisite chapters — 28-01 Implication of subsequential limits, 19-04 The block map, 19-07 Statements to be shown (convergence of the full sequence).

Said first — This is an implication. The theorem is a statement about orbits of points in a metric space, and no measure appears in it. The assumed one-step map (a map on the marginal coordinate u and the irrelevant coordinate v) is not defined from the lattice. So convergence of the full sequence for the continuum limit of the lattice does not follow from this theorem. What follows is only a conclusion of the form "if one step of renormalisation has this form and this contraction". The existence of four-dimensional Yang–Mills theory was shown by the construction published on 2026-11-10, with a different design (Volume 2). The implication of this chapter is a record within this book's design (measures pushed forward by a fixed block map) (28-00).

The order of this chapter

1. From a subsequence to the full sequence — what is missing — if the increments are summable, the full sequence converges
2. The form of the one-step map — the marginal direction and the irrelevant directions — the skew-product map, and families of orbits matched at the endpoint
3. The table of assumption fields — Lipschitz contraction, diameter, matching at the endpoint (Table 1)

4. The statement — the Cauchy property at the coarse end and uniqueness of the limit — difference $\leq D\theta^n$; the limit does not depend on the bare values
5. The route of the proof — a two-point boundary-value problem — differences in u propagate from the coarse side, differences in v from the fine side
6. Conditions and rate — necessary and sufficient for this method — eigenvalues of a linear model, and a rate that cannot be improved
7. Matching at the endpoint cannot be removed — an example where, for the same map, removing the matching gives a non-Cauchy sequence
8. The assumed map is not defined from the lattice — what must be defined for the statement to become a statement about the lattice

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01

From a subsequence to the full sequence — what is missing

The subsequential limit of the previous chapter (28-01 § 04) exists for any family of probability measures. Since the limit may depend on the choice of subsequence, that alone does not allow us to say "one continuum limit has been determined". We need convergence of the full sequence (the sequence without passing to a subsequence).

There is an easy sufficient condition for convergence of the full sequence. For a sequence s_n of one coarse-scale quantity (for example the expectation of an observable at level m), if

$$|s_{n+1} - s_n| \leq \delta_n, \qquad \sum_n \delta_n < \infty \quad (1)$$

then s_n is a Cauchy sequence and converges, and every subsequential limit equals that limit Proved here (Lean: `WeakUltravioletExistence.tendsto_of_summable_increments`, `subseq_limit_eq`; the form for a family of observables is `WeakUltravioletExistence.weak_uv_observable_full`). In the conditional implication that this book's study assembled earlier (28-01 § 08), a condition of the form (1) is placed as an assumption (the summability-of-increments assumption). The summability-of-increments assumption assumes that, on a countable dense family of gauge-invariant functions, the step-by-step increments of the values of observables are summable. It is not the same statement as weak convergence of the full sequence of push-forward measures (convergence of the full sequence in 19-07, stated for the sequence of couplings fixed by matching): it is stronger by the summability, and the family of functions is different. The step from that family to all bounded continuous functions is an argument on paper.

The question is where to get the δ_n of (1). We push the two fine measures on lattice $n+1$ and lattice n forward to the same coarse level m and compare them. Since renormalisation maps compose, the side of lattice $n+1$ can be seen as "renormalise once, then follow the same road as lattice n". If the effective action after one step differs only a little from the bare action of lattice n, and that difference contracts over the remaining steps, then the difference at the coarse end becomes small. The theorem of this chapter writes the "contracts" part of this line of argument as a statement about orbits of points.

02

The form of the one-step map — the marginal direction and the irrelevant directions

We think of the effective action as described by two coordinates: the marginal coordinate $u \in \mathbb{R}$ (the inverse coupling) and the irrelevant coordinate v (everything else; a point of a metric space X). The one-step map from scale k ($k = n$ is the fine end, $k = 0$ the coarse end) to $k-1$ is

$$\Phi_k(u, v) = \big(u - b_k + r_k(u, v),\ T_k(u, v)\big) \quad (2)$$

(Lean: `SkewOrbit.step`). b_k is the one-step decrease of the coupling (a number near the one-loop value), r_k its correction, and T_k the map on the irrelevant coordinate. The map depends only on the step k, not on the lattice n one started from.

Why a skew product. If a map contracted in every direction, convergence of orbits would follow in the manner of the contraction mapping principle. The renormalisation map does not contract in the direction u of the coupling (under asymptotic freedom, u just decreases by roughly b at each step). Only the irrelevant direction v contracts, and the two directions affect each other through r_k and T_k. A map of this form is called a skew product.

The orbit of lattice n is the sequence of points that starts from the bare value $w_n(n)$ at scale n and goes down to the coarse end $w_n(0)$ by $w_n(k-1) = \Phi_k(w_n(k))$ (Lean: `SkewOrbit.IsOrbit`). For a family of orbits $(w_n)_n$, we call the following three together "admissible" (Lean: `SkewOrbit.Admissible`).

- Each w_n is an orbit from scale n to 0.
- **Matching at the endpoint**: for all n, the marginal coordinate at the coarse end takes the same value $\bar{u}$: $w_n(0)_u = \bar{u}$.
- The irrelevant coordinate stays, at every step, in a set $\mathcal{B} \subset X$ of diameter at most D.

Matching at the endpoint plays the same role as the first conclusion of the previous chapter (28-01 § 03: there is a bare value that becomes exactly the target value after n steps). The two theorems, however, do not refer to each other's quantities.

03

The table of assumption fields

Table 1 Assumptions for the convergence of skew-product orbits matched at the endpoint (fields of the midpoint version `SkewOrbit.coarse_tendsto_mid`; $v, v' \in \mathcal{B}$, u, u' all real numbers)

Lean field	Mathematical wording	Paid?
`hr`	Assumption $\lvert r_k(u,v) - r_k(u',v')\rvert \le \varepsilon \lvert u-u'\rvert + c\, d(v,v')$	Abstract. r_k, T_k, X are free and not defined from the lattice
`hT`	Assumption $d(T_k(u,v), T_k(u',v')) \le c' \lvert u-u'\rvert + \tau\, d(v,v')$	
`hB`	Assumption The diameter of $\mathcal{B}$ is $\le D$	Abstract
`hε`, `hc`, `hc'`, `hτ`, `hD`	Assumption $\varepsilon, c, c', \tau, D \ge 0$	Numerical conditions
`hετ`, `h4`	Assumption $\varepsilon + \tau < 1$, $4cc' < (1-\varepsilon-\tau)^2$	Numerical conditions (necessary and sufficient for this method, § 06)
`hw` (`Admissible`)	Assumption The family of orbits is admissible (being orbits, matching at the endpoint, staying in $\mathcal{B}$)	Abstract
`[CompleteSpace X]`	Assumption X is complete	Abstract

The general version (`SkewOrbit.coarse_step`, `coarse_cauchy` and others) is given one θ and assumes $\tau < \theta < 1 - \varepsilon$ and

$$q := \frac{cc'/(1-\varepsilon-\theta) + \tau}{\theta} < 1 \quad (3)$$

The θ of this chapter is the contraction rate, and has nothing to do with the vortex ratio θ_M (19-02). Taking the midpoint $\theta = (1-\varepsilon+\tau)/2$, (3) follows from $4cc' < (1-\varepsilon-\tau)^2$ (Lean: `SkewOrbit.midpoint_admissible`).

The Lipschitz fields are assumed for all values of u, but this is not a restriction. If the fields are assumed only on a window $u, u' \ge a$, and we add that the orbits stay in the window ($w_n(k)u \ge a$ for all n, k), the same conclusion follows (Lean: `SkewOrbitProbe.coarse_tendsto_window`, `limit_independent_window`; the original theorem is applied to the map cut off by $u \mapsto \max(u, a)$). This suits situations in which only estimates on the weak-coupling window $u \ge u_\star$ are used.

The assumptions can be satisfied simultaneously, and not only in trivial cases. With $X = \mathbb{R}$, $\mathcal{B} = [-1, 1]$ ($D = 2$), $b = 0$, $r(u, v) = \frac{7}{20}v$, $T(u, v) = \frac{7}{10}v + \frac{1}{20}u$ ($\varepsilon = 0$, $c = 7/20$, $c' = 1/20$, $\tau = 7/10$, $4cc' = 0.07 < 0.09$), one can build an admissible family in which the bare irrelevant coordinate oscillates near $\pm\frac{1}{2}$ and does not converge, while the coarse end converges (Lean: `SkewOrbitProbe.Ex.admissible`, `Ex.coarse_converges`, `Ex.bare_not_cauchy`).

04

The statement — the Cauchy property at the coarse end and uniqueness of the limit

Under the assumptions of Table 1, with $\theta = (1 - \varepsilon + \tau)/2$, the following holds.

$$d\big(w_{n+1}(0)_v,\ w_n(0)_v\big) \le D\,\theta^n \qquad (\text{all } n). \ (4)$$

So the irrelevant coordinate $w_n(0)_v$ at the coarse end is a Cauchy sequence; if X is complete it converges **along the full sequence**, and its limit v_∞ satisfies $d(w_n(0)_v, v_\infty) \le D\theta^n/(1-\theta)$. Moreover, the limits of two admissible families matched at the same $\bar{u}$ coincide. The limit does not depend on the choice of the bare irrelevant coordinates $w_n(n)_v$. A continuous function Ψ of the coarse-end point (u, v) converges along the full sequence to $\Psi(\bar{u}, v_\infty)$.

Lean: `SkewOrbit.coarse_step` (formula (4)), `coarse_cauchy`, `coarse_tendsto`, `coarse_rate`, `limit_independent`, `observable_tendsto`, and the midpoint versions `coarse_step_mid`, `coarse_tendsto_mid`, `limit_independent_mid` — machine-checked as an implication (assumptions as in Table 1).

Let us check what (4) compares. Cutting the orbit of lattice $n+1$ to scales n and below gives an orbit that starts at scale n from "the bare value with one step Φ_{n+1} applied" (Lean: `SkewOrbitProbe.one_step_after`). The orbit of lattice n starts from the bare value at scale n. Both are matched to $\bar{u}$ at the coarse end, so this is a comparison of two admissible orbits of the same length n.

Read in the language of the continuum limit, the statement becomes: "if a coarse-scale observable is a continuous function of (u_0, v_0), the sequence of its expectations converges without passing to a subsequence, and the limit does not depend on the choice of the irrelevant part of the bare action." This reading lies outside the theorem (§ 08).

05

The route of the proof — a two-point boundary-value problem

Compare two admissible orbits w, w' of the same length n, and set $x_k = |w(k)_u - w'(k)_u|$, $y_k = d(w(k)_v, w'(k)_v)$.

Step 1: the one-step inequalities. Taking the difference of the first components of (2) and using `hr` of Table 1 gives $x_k \le x_{k-1} + \varepsilon x_k + c\, y_k$, that is,

$$(1-\varepsilon)\, x_k \le x_{k-1} + c\, y_k, \qquad y_{k-1} \le c' x_k + \tau\, y_k \qquad (1 \le k \le n) \ (5)$$

(the latter is `hT`) Proved here . The boundary values are $x_0 = 0$ (matching at the endpoint) and $y_n \le D$ (the diameter of $\mathcal{B}$). x propagates from the coarse side ($k = 0$) towards the fine side, and y from the fine side ($k = n$) towards the coarse side.

Step 2: a weighted maximum. Set $M := \max_{k \le n} y_k\, \theta^{-(n-k)}$. Accumulating the first half of (5) from $x_0 = 0$, induction gives $x_k \le cM\theta^{n-k}/(1-\varepsilon-\theta)$ (it closes exactly because $\theta/(1-\varepsilon-\theta) + 1 = (1-\varepsilon)/(1-\varepsilon-\theta)$) Proved here .

Step 3: tightening. Putting this into the second half of (5) gives, for $k - 1 < n$, $y_{k-1}\theta^{-(n-k+1)} \le qM$ (q as in (3)). At $k = n$, $y_n \le D$. So $M \le \max(D, qM)$, and from $q < 1$ we get $M \le D$, in particular $y_0 \le D\theta^n$ Proved here (Lean: `SkewOrbit.two_point_bound`, `orbit_compare`).

The use of $q < 1$ in Step 3 is where the contraction condition does its work. Since the u direction does not contract, bounding the difference in u by the difference in v requires the difference in u to be 0 at the coarse end (matching at the endpoint).

06

Conditions and rate — necessary and sufficient for this method

Three facts about the choice of θ have been checked in Lean.

- **Necessary and sufficient**: the existence of a θ satisfying $\tau < \theta < 1 - \varepsilon$ and (3) is equivalent to $\varepsilon + \tau < 1$ and $4cc' < (1-\varepsilon-\tau)^2$ (Lean: `SkewOrbitProbe.theta_exists_iff`). Taking the midpoint loses nothing in the conditions.
- **The window of rates**: with $s = \sqrt{(1-\varepsilon-\tau)^2 - 4cc'}$ and $\theta_\pm = \big((1-\varepsilon+\tau) \pm s\big)/2$, every θ with $\theta_- < \theta < \theta_+$ can be used (Lean: `SkewOrbitProbe.rate_window`). The rate can be improved to just above θ_-.
- **A rate that cannot be improved**: with $\varepsilon = 0$, $c = 1/6$, $c' = 1/3$, $\tau = 1/2$ ($\theta_- = 2/3$, $\theta_+ = 5/6$), there are sequences satisfying (5) and the boundary values with $y_0 \ge (2/3)^n$

$/2$. So no decrease faster than θ_- follows from (5) (Lean: `SkewOrbitProbe.Sharp.sharp`).

$\theta_\pm$ are the eigenvalues of the matrix $\begin{pmatrix} 1-\varepsilon & -c \\ c' & \tau \end{pmatrix}$ of the linear model with the sign of the cross term reversed Proved here (the roots of the characteristic polynomial $\lambda^2 - (1 - \varepsilon + \tau)\lambda + (1-\varepsilon)\tau + cc'$). $4cc' \le (1-\varepsilon-\tau)^2$ is the condition for them to be real.

"Necessary and sufficient" here is **relative to this method of proof (the two-point estimate from (5))**. It is not a claim that convergence of the full sequence itself becomes false when the conditions fail. Formula (5) looks only at the sizes of differences and does not use the sign of the cross terms, so for maps where the sign is known there is room for better estimates.

07

Matching at the endpoint cannot be removed

With the same map as in the example of § 03 ($\varepsilon = 0$, $c = 7/20$, $c' = 1/20$, $\tau = 7/10$, $\mathcal{B} = [-1, 1]$), one can build a family in which every member is an orbit and the irrelevant coordinate stays in $\mathcal{B}$, but the marginal coordinates at the coarse end are not aligned. In that family the irrelevant coordinate at the coarse end is not Cauchy (Lean: `SkewOrbitProbe.Ex.matching_needed`).

What lies behind this is that the matrix of the map $\begin{pmatrix} 1 & c \\ c' & \tau \end{pmatrix}$ has a direction with eigenvalue $21/20 > 1$ (a direction close to the marginal direction). In this example, if u at the coarse end is not aligned, the orbits have a component in this direction, and its size alternates from one n to the next. Matching at the endpoint is the operation that fixes the component in this expanding direction afresh for each n.

So convergence of the full sequence does not follow from "contraction" alone. **How the bare coupling is chosen for each n (matching) is part of the statement.** In the language of the lattice this corresponds to "Matching" among the statements to be shown (19-07). The first conclusion of the previous chapter gives an existence of the same type (a bare value that becomes exactly the target value after n steps) for a different quantity, but the u of the two theorems is not set to be the same thing. There too, u is not defined from the lattice (28-01 § 07). Nor has it been shown, for general maps, that an admissible family exists (including that the irrelevant coordinate stays in $\mathcal{B}$).

08

The assumed map is not defined from the lattice

This theorem is a statement about orbits of points in the metric space $\mathbb{R} \times X$. Measures, lattices and block maps appear nowhere in the statement. To read it as convergence of the full sequence for the continuum limit of the lattice, the following must be fixed outside the theorem.

1. **Coordinates**: describe the effective action at step k by (u, v) — in particular the irrelevant coordinate v and its metric. The effective action (the effective measure at step k) itself is not yet defined from the lattice.
2. **The map**: write one step of renormalisation as a map Φ_k on those coordinates (that it takes the form (2), and that b_k is common to the two orbits).
3. **Contraction**: that the map has the Lipschitz estimates of Table 1 with constants satisfying $4cc' < (1 - \varepsilon - \tau)^2$.
4. **Observables**: that expectations of coarse-scale observables are continuous functions of the coarse-end coordinates (u_0, v_0).

All four are at present on the side of the assumptions. In the list of premises (28-04), the row for the Lipschitz contraction of the one-step map and matching at the endpoint carries the label "Form not fixed (object undefined)", and below it hang premises at the level of the representation, such as an induction uniform in the step and one-step renormalisation of small fields. Which coordinates and norm to choose is a design choice (28-05).

We have not checked whether the Lipschitz contraction follows from existing renormalisation-group estimates (estimates of analytic form in the fields and parameters). This chapter makes no claim about that prospect.

What this chapter can say

Can say	Cannot say
Proved here If the increments are summable, the full sequence converges, and subsequential limits equal its limit. Lean: `WeakUltravioletExistence.tendsto_of_summable_increments`, `weak_uv_observable_full`	That the increments of lattice expectations are summable (the assumption of the conditional implication assembled earlier, the summability-of-increments assumption)

Can say	Cannot say
Implication: under the assumptions of Table 1, the irrelevant coordinate at the coarse end converges along the full sequence, the differences are at most $D\theta^n$, and the limit does not depend on the bare values. Lean: `SkewOrbit.coarse_step`, `coarse_tendsto`, `limit_independent`, `observable_tendsto` — machine-checked as an implication (assumptions as in the table)	That the assumed maps r_k, T_k and coordinates X are defined from the lattice. Convergence of the full sequence for the continuum limit of the lattice
The existence of a usable θ is equivalent to $\varepsilon + \tau < 1$ and $4cc' < (1-\varepsilon-\tau)^2$. Lean: `SkewOrbitProbe.theta_exists_iff`, `rate_window`, `Sharp.sharp`	That convergence of the full sequence is false when the conditions fail (this is a limit of the method, not a negation of the statement)
Removing matching at the endpoint, there is a non-Cauchy family for the same map. Lean: `SkewOrbitProbe.Ex.matching_needed`	How to match on the lattice (which quantity, at which step)
Proved here The u of this chapter and the coupling function u of the previous chapter are separate variables of separate theorems	Setting the two to be the same. That an admissible family (orbits with matching that stay in $\mathcal{B}$) exists for the actual maps
The assumptions can be satisfied simultaneously by a non-trivial example. Lean: `SkewOrbitProbe.Ex.admissible`, `Ex.bare_not_cauchy`	That the actual renormalisation maps satisfy the assumptions

Sources and reproduction

Item	Type	Source, tools
The theorem on skew-product orbits (§ 02– § 05)	Lean	Lean of this book's study: 13 declarations in the namespace `SkewOrbit` (`two_point_bound`, `midpoint_admissible`, `orbit_compare`, `coarse_step`, `coarse_cauchy`, `coarse_tendsto`, `coarse_rate`, `limit_independent`, `observable_tendsto` and the midpoint versions). Machine-checked in Lean 4 (13 declarations). The axioms are the standard three. Kernel re-check done. The statements were copied by reading the source text
Equivalence of the conditions, window of rates, a rate that cannot be improved, necessity of matching, the window version (§ 03, § 06, § 07)	Lean	Namespace `SkewOrbitProbe` (auxiliary declarations added when the definitions and statements were compared with a separate pair of eyes: `theta_exists_iff`, `rate_window`, `Sharp.sharp`, `Ex.*`, `coarse_tendsto_window` and others). The axioms are the standard three (record of the check)

Item	Type	Source, tools
Summable increments give the full sequence (§ 01)	Lean	`WeakUltravioletExistence.tendsto_of_summable_increments`, `subseq_limit_eq`, `weak_uv_observable_full`
Eigenvalues of the linear model (§ 06)	Proved here	Computation in the text (characteristic polynomial of a 2×2 matrix)
Assumption of the conditional implication assembled earlier (the summability-of-increments assumption)	—	Record of this book's study (2026-10-30). 28-01 § 08

Words used in this chapter — convergence of the full sequence, summability of increments, skew product, marginal coordinate, irrelevant coordinate, one-step map, orbit, matching at the endpoint, admissible family, Lipschitz contraction, two-point boundary-value problem, window of rates. Their meanings are in A-02 Glossary.

Next chapter: 28-03 From decay to the mass gap, and the reduction of the band — reads the two implications on the infrared side, and the fact that there is no arrow between them.

From decay to the mass gap, and the reduction of the band

> This chapter is this book's study.

What this chapter shows — We read two implications on the infrared side. (i) If a sequence of measures has uniform exponential decay, the theory reconstructed from the limit measure has a mass gap (Lean: `OSGap.massGap_of_measures_sub`). Inside the strong-coupling window, these assumptions are all paid by concrete Wilson measures. (ii) If base point, nesting and widening hold for the vortex ratio θ, then $\ln(1/\theta)$ grows in proportion to the area (Lean: `BandReduction.band_reduction` and others). Finally we write down that there is no arrow from the conclusion of (ii) to the assumptions of (i).

Prerequisites — 19-03 The vessel of reconstruction and the definition of the mass gap, 20-04 Reconstruction and the mass gap, 19-02 The vortex ratio θ, 26-02 Reduction to one inequality.

Said first — Both are implications. Outside the window, (i) remains an assumption, and there is no concrete sequence of measures (a sequence in which the lattice spacing moves) that pays the assumptions. Also, the conclusion alone cannot exclude the trivial case in which the Hilbert space contains only the vacuum. (ii) is a statement of inequalities about a real function $\theta(R, L)$; neither lattice nor measure appears in it. That the actual vortex ratio satisfies the three assumptions has not been shown. There is no arrow, either in Lean or on paper, that derives the mass gap from the area exponent. Exponential decay uniform in the lattice spacing is not in the construction published on 2026-11-10 either; what it has goes only as far as estimates of polynomial speed at long distances (16-03). The two implications of this chapter are shaped towards the open question on the infrared side.

The order of this chapter

1. The two implications, and what lies between them — decay ⇒ gap; the three assumptions ⇒ area exponent; area ⇒ gap is absent
2. From decay to the mass gap — the statement — reconstruction over the limit measure, and ‖ T − P_Ω ‖ ≤ e^{−m}

3. From decay to the gap — the table of assumption fields — reflection positivity, symmetry, translation, decay (Table 1)
4. Inside the strong-coupling window, everything is paid — relation to the theorem of 20-04, and outside the window
5. The reduction of the band — the statement — at scales doubling each time, the bias shrinks by a fourth power
6. The reduction of the band — the table of assumption fields — base point, nesting, widening (Table 2)
7. There is no arrow from the area exponent to the mass gap — the two roads exist separately

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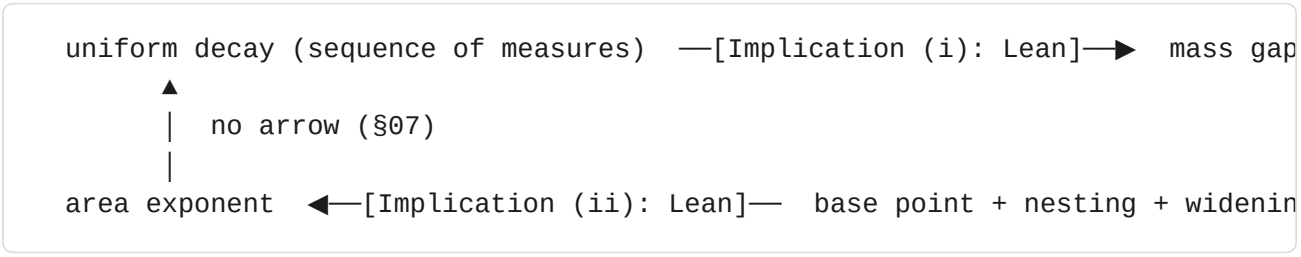
01

The two implications, and what lies between them

On the infrared side (the side of long distances) there are two goals. One is the mass gap (contraction of the transfer operator away from the vacuum); the other is confinement (the vortex ratio and loop expectations decreasing in proportion to the area). The two implications of this chapter take charge of the last step towards each goal.



Both arrows are machine-checked in Lean in the form "if ..., then ...". The assumption sides (uniform decay, the three inequalities) correspond to "mass gap (uniform decay)" and "band" among the statements to be shown (uniform decay and band in 19-07, both stated for the sequence of couplings fixed by matching), and neither has been shown in a form in which the lattice spacing moves. The vertical arrow of the figure, as § 07 explains, does not exist.

02

From decay to the mass gap — the statement

We fix the setting. X is the space of configurations and Y the space of configurations on the positive side. The continuous maps $r, \Theta_r : X \to Y$ are the map that restricts a

configuration to the positive side, and the map that reflects in the mirror and then restricts. $\mathrm{sh} : Y \to Y$ is the translation by one time unit. Observables are bounded continuous functions on Y; among them we use a subspace W that contains the constant 1 and is closed under translation (for example functions that depend only on finitely many time slices). For a measure ν, the pairing and the translation are

$$\langle F, G\rangle_\nu = \int F(\Theta_r x)\, G(rx)\, d\nu(x), \qquad (\tau F)(y) = F(\mathrm{sh}\, y) \quad (1)$$

If the pairing is positive semidefinite and symmetric, and τ is symmetric with respect to the pairing, one can build the Hilbert space obtained by quotienting W by the pairing and completing, the transfer operator T coming from τ, and the vacuum Ω coming from the constant 1 (the vessel of reconstruction; 19-03). A mass gap m means $\|T - P_\Omega\| \le e^{-m}$ (P_Ω is the projection onto the vacuum; 19-03).

Implication (i). Suppose a sequence of probability measures μ_n converges weakly to ν, and the fields of Table 1 hold. Then there is a vessel of reconstruction D over W whose pairing is the pairing (1) of ν, whose translation is τ and whose constant is 1, and D has mass gap m.

Lean: `OSGap.massGap_of_measures_sub` — machine-checked as an implication (assumptions as in Table 1). The conclusion ties the pairing, translation and constant of the vessel to the quantities of ν; it is not a form in which the vessel can be chosen at will.

The route of the proof has three steps. (1) Reflection positivity, symmetry and the symmetry of translation pass to the weak limit because $F \circ \Theta_r \cdot G \circ r$ is bounded and continuous (the same type as 28-01 § 05) Proved here . (2) With these, the vessel over ν can be built. (3) The decay field also passes to the limit, and $\langle \tau^k F, \tau^k F\rangle_\nu - \langle 1, F\rangle_\nu^2 \le C_F e^{-2mk}$ holds for all $F \in W$. Translated into the language of Hilbert space, this says that T^k contracts by e^{-mk} on vectors orthogonal to the vacuum, which gives $\|T - P_\Omega\| \le e^{-m}$ (Lean: `OSGap.Datum.hasMassGap_of_decay`).

03

From decay to the gap — the table of assumption fields

Table 1 Assumptions of `OSGap.massGap_of_measures_sub` ("for large n" means "there is n_0 such that for $n \ge n_0$"; $F, G \in W$)

Lean field	Mathematical wording	Inside the strong-coupling window	Outside the window

Lean field	Mathematical wording	Inside the strong-coupling window	Outside the window
`h`	Assumption $\mu_n \to \nu$ (weak convergence)	All paid by the Wilson measure (§ 04, 20-04)	Abstract (convergence of a sequence in which the lattice spacing moves has not been shown)
`h1`, `hW`	Assumption $1 \in W$, and W is closed under translation		A choice of the class of observables
`hRP`	Assumption For large n, $\int F(\Theta_r x)F(rx)\,d\mu_n \geq 0$ (reflection positivity)		A theorem for measures on the fine lattice (20-01). For pushed-forward measures it depends on the class
`hsym`, `hshift`	Assumption For large n, the pairing is symmetric, and translation is symmetric with respect to the pairing		Abstract
`hdec`	Assumption For each F there is a constant C_F such that for each k and large n, $\int F(\mathrm{sh}^k\Theta_r x)F(\mathrm{sh}^k rx)\,d\mu_n - \left(\int F(rx)\,d\mu_n\right)^2 \leq C_F e^{-2mk}$ (uniform decay)		Form fixed, not shown (needs an m uniform in the lattice spacing)

In the decay field `hdec`, m and C_F do not depend on n. The mass m of the limit theory is a number in units of one time step of the lattice on which the measures μ_n live. To speak of a physical mass in the continuum limit, block observables at coarse scales must decay at a rate independent of the lattice spacing and the box size. That is "mass gap" among the statements to be shown (uniform decay in 19-07, stated for the sequence of couplings fixed by matching), and paying the assumption `hdec` of this implication in that form is open outside the window. The field `h` also assumes weak convergence of the sequence μ_n. When there is only a subsequential limit, as in the previous chapter (28-01 § 04), applying it to that subsequence gives a conclusion "about that limit point", and the limit itself is not determined uniquely.

The conclusion alone cannot exclude the trivial case. If the space obtained by quotienting W by the pairing is only the vacuum (one-dimensional), then $T - P\Omega = 0$ and a mass gap holds for any m. The assumption fields can also be satisfied simultaneously on a one-point space Proved here . A mass gap with content requires saying separately that there are states outside the vacuum (non-degeneracy). The theorem for the strong-coupling window includes non-degeneracy in its statement (20-04).

> 04

Inside the strong-coupling window, everything is paid

In the strong-coupling window $18\tanh|\beta|<1$ ($|\beta|<0.0556$ Reading), the sequence obtained by lifting the normalised Wilson measures on the periodic boxes $(\mathbb{Z}/2(n+1))^4$ periodically to $\mathbb{Z}^4$ converges weakly, without a subsequence, to a single measure ν_β. On top of that, the properties corresponding to the fields of Table 1 (reflection positivity, symmetry, translation, decay) are all shown for concrete measures, the vessel of reconstruction satisfies $\|T-P_\Omega\|\le(18\tanh\beta)^{1/2}$, and there are states outside the vacuum ($0<\beta$; Lean: `InfMain2.wilson_infvol_main2`; the statement gathered into one in 20-04). This is a formalisation of a result of a known type from the strong-coupling expansion Cited (of the type of Osterwalder–Seiler, Ann. Phys. 110 (1978) 440).

Here is what the theorem inside the window does not say. The lattice spacing is fixed at 1. The window is only $|\beta|<0.0556$, and does not reach the side of the continuum limit ($\beta\to\infty$). The mass figure $-\frac{1}{2}\log(18\tanh\beta)$ is only a lower bound.

So the position of implication (i) is as follows. **Inside the window, all the assumptions of the implication are paid, and the conclusion stands as a theorem about concrete Wilson measures. Outside the window, they remain assumptions.** To use it for the mass gap of the continuum limit, one needs to show the fields of Table 1 for a sequence of measures in which the lattice spacing moves (the sequence of fine-lattice measures pushed forward to a coarse level, 28-01), and none of them has been shown.

05

The reduction of the band — the statement

The goal on the side of confinement is that the vortex ratio becomes small in proportion to the area. In the intended reading, $\theta(R,L)$ is the sector bias at a typical frame, for a container of size R in the directions of the plane and a transverse torus of side L, with lengths measured in physical units (26-02 § 01; the definition of the vortex ratio on the lattice is in 19-02). The goal is $\ln(1/\theta(R,R))\ge cR^2$ for a $c>0$ uniform in physical units.

Implication (ii). Let $\theta:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ be an arbitrary function, and assume the structure `Hyp` of Table 2 (non-negativity, base point, nesting, widening). The assumptions outside the structure are $L_0>0$, $s\ge1$, $C\ge0$, $C_D\ge0$, $c\ge0$, $c^3=4CC_D$. Taking doubling scales $R_k=2^k sL_0$, for all k

$$c\,\theta(R_k,R_k)\;\le\;\big(c\,t_0\big)^{4^k},\qquad t_0:=Cs^2\,C_D\,(1-\varepsilon_0)^{s^2}\quad(2)$$

holds. Moreover, if $c \geq 1$ and $0 < c\,t_0 < 1$, then when $\theta(R_k, R_k) > 0$ (the assumptions of `area_exponent` are these three and (2))

$$4^k \ln \frac{1}{c\,t_0} \;\leq\; \ln \frac{1}{\theta(R_k, R_k)} \quad (3)$$

and since $4^k = R_k^2/(sL_0)^2$, this is the form of the area exponent, $\ln(1/\theta(R_k, R_k)) \geq c' R_k^2$.

Lean: `BandReduction.band_reduction` (formula (2)), `area_exponent` (formula (3)), and the parts `BandReduction.jump` (from the base point, one nesting and one widening give $\theta(sL_0, sL_0) \leq t_0$) and `step` (one doubling step, $\theta(2R, 2R) \leq 4CC_D\,\theta(R, R)^4$) — machine-checked as an implication (assumptions as in Table 2).

The core of the proof is `step`. Using nesting with $n = 2$ gives $\theta(2R, R) \leq C_D \theta(R, R)^4$, and using widening with $L = R$, $L' = 2R$ gives $\theta(2R, 2R) \leq 4C\,\theta(2R, R)$; together, $t_{k+1} \leq 4CC_D\,t_k^4$. Setting $c^3 = 4CC_D$ gives $c\,t_{k+1} \leq (c\,t_k)^4$, and induction gives (2) Proved here .

There is also a version that, instead of the base point, puts in a number at one size, $\theta(R_0, R_0) \leq t_0$, and uses widening only "at doubling steps where the bias is already small" (Lean: `BandFiniteSize.doubling`; the assumptions are three in the structure — non-negativity, the $n = 2$ case of nesting, and widening when $\theta(2L, L) \leq \delta$ — and $R_0 > 0$, $C \geq 0$, $C_D \geq 0$, $c \geq 1$, $c^3 = 4CC_D$, $\theta(R_0, R_0) \leq t_0$, $c\,t_0 \leq 1$, $C_D t_0^4 \leq \delta$). The conclusion has the same form as (2).

06

The reduction of the band — the table of assumption fields

Table 2 Fields of the structure `BandReduction.Hyp` (θ is an arbitrary function $\mathbb{N} \times \mathbb{N} \to \mathbb{R}$)

Lean field	Mathematical wording	For the intended θ
`nonneg`	Assumption $\theta(R, L) \geq 0$ for all R, L	For the vortex ratio at fixed lattice spacing, a theorem for all real β when the longitudinal size is even (21-06). The correspondence with the container bias is on paper
`base`	Assumption $\theta(L_0, L_0) \leq 1 - \varepsilon_0$ (base point)	In the form with fixed lattice spacing, the thin-vortex floor $\lvert\theta_M\rvert \leq \tanh(N^2\lvert\beta\rvert)^{M^2}$ is a theorem (21-02). A form uniform in the lattice spacing has not been shown
`nest`	Assumption $\theta(nR, L) \leq C_D\,\theta(R, L)^{n^2}$ for all R, L, n (nesting)	Input on paper (the nesting argument of 26-02). The correspondence with the Lean field is also on paper
`widen`	Assumption For $0 < L \leq L' \leq R$, $\theta(R, L') \leq C\,(L'/L)^2\,\theta(R, L)$ (widening)	Not shown. There is one sufficient condition (direction-wise subadditivity of the vortex free energy) (28-04), but whether it holds is open

This implication is a statement about real inequalities. θ is an arbitrary function, and lattice, measure and lattice spacing do not appear in the statement. That the intended θ (the container bias) satisfies the three assumptions is placed as input on paper and has not been shown. In particular, in the intended reading all lengths are in physical units, and the assumptions must be uniform in the lattice spacing. The floor theorems at fixed lattice spacing (21-02, 21-03) do not give this uniformity.

The assumptions can be satisfied simultaneously Proved here . With $\theta(R, L) = e^{-R^2}$, $C_D = C = 1$ and ε_0 small, nesting holds with equality and widening holds because $(L'/L)^2 \geq 1$. In this example the conclusion is of a form known from the start, and the example only shows that the assumptions are not contradictory. Also, using nesting with $n = 1$ forces $C_D \geq 1$ when $\theta > 0$.

The forms of other statements about the band — the doubling inequality for the tension, monotonicity, one-scale criteria and so on — are laid out with labels in the list of premises (28-04). This chapter does not give them chapters of their own.

07

There is no arrow from the area exponent to the mass gap

There is no arrow, either in Lean or on paper, from the conclusion of implication (ii) (the area exponent) to the assumption of implication (i) (uniform decay of block observables).

"Area law ⇒ mass gap" in its naive form is false in general Proved here . For example, on a two-dimensional lattice, take a gauge theory with an area law and add to it, as a direct product, a decoupled model without a mass gap (for example the critical two-dimensional Ising model; that the gap of the transfer matrix closes at the critical point is in 2-09 § 07). The expectation of a Wilson loop is determined by the gauge part alone, so the area law remains, but the correlations of the Ising part do not decay exponentially, so the theory as a whole has no mass gap (the gap of the transfer matrix of a direct product is the smaller of the two gaps). So deriving decay from the area exponent requires a separate argument that uses the absence of such decoupled parts.

In the opposite direction there is the following result Cited (Chatterjee, Comm. Math. Phys. 385 (2021) 1007, arXiv:2006.16229; 17-02). If centre symmetry is not broken, Wilson loops satisfy $|\langle W(R \times T)\rangle| \leq e^{-V(R)T}$ with $V(R) \to \infty$ (this stops short of saying that the string tension is positive), and if moreover one assumes a strong form of gap — uniform exponential decay under arbitrary boundary conditions — the area law follows. This strong form of decay is not the same as the assumption of implication (i) of this chapter, nor as the mass gap in the Clay sense. The two implications of this book have no arrow in that direction in Lean either.

So on the present map the two roads exist separately.

- **The road of the mass gap**: uniform decay ⇒ mass gap (implication (i)). The assumed uniform decay is shown inside the window and open outside it.
- **The road of confinement**: base point, nesting, widening ⇒ area exponent (implication (ii)). The three assumptions have not been shown in a form uniform in the lattice spacing.

By which road the last arrow to the mass gap is to be drawn (whether, for example, to use the comparison rows that write band quantities in the language of the mass gap) is the subject of the list of premises (28-04) and of the open places (28-06). This chapter does not claim which road goes through.

What this chapter can say

Can say	Cannot say
Implication (i): if a sequence of measures converges weakly and the fields of Table 1 hold, the vessel of reconstruction over the limit measure has mass gap m. Lean: `OSGap.massGap_of_measures_sub` — machine-checked as an implication (assumptions as in the table)	Showing the fields outside the window, for sequences in which the lattice spacing moves. The conclusion alone does not give non-degeneracy (states outside the vacuum)
In the strong-coupling window $18 \tanh \lvert\beta\rvert < 1$, $0 < \beta$, the fields are all paid by concrete Wilson measures, and a mass gap with non-degeneracy holds. Lean: `InfMain2.wilson_infvol_main2` (20-04)	Moving the lattice spacing, outside the window, an upper bound on the mass (only a lower bound)
Implication (ii): from base point, nesting and widening, $\ln(1/\theta)$ grows in proportion to the area at doubling scales. Lean: `BandReduction.band_reduction`, `area_exponent`, `jump`, `step`, `BandFiniteSize.doubling` — machine-checked as an implication (assumptions as in the table)	That the intended θ satisfies the three assumptions (input on paper). Base point and widening uniform in the lattice spacing
Proved here The naive "area law ⇒ mass gap" is false when there is a decoupled part without a mass gap	An arrow deriving uniform decay from the area exponent (neither in Lean nor on paper)

Sources and reproduction

Item	Type	Source, tools
The implication from decay to the mass gap (§ 02, § 03)	Lean	Lean of this book's study: `OSGap.massGap_of_measures_sub`, the vessel `OSGap.Datum`, `HasMassGap`, `OSGap.Datum.hasMassGap_of_decay`. The statements were copied by reading the source text

Item	Type	Source, tools
The theorem for the strong-coupling window (§ 04)	Lean, Cited	`InfMain2.wilson_infvol_main2` (20-04). Source of the type: K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440
The reduction of the band (§ 05, § 06)	Lean	`BandReduction.Hyp`, `band_reduction`, `area_exponent`, `jump`, `step`, `iterate`, `BandFiniteSize.Hyp`, `doubling`. The statements were copied by reading the source text. The reduction on paper is in 26-02
Examples in which the assumptions are satisfied simultaneously (§ 03, § 06)	Proved here	The choices given in the text
A result in the direction from the gap to confinement (§ 07)	Cited	S. Chatterjee, Comm. Math. Phys. 385 (2021) 1007, arXiv:2006.16229

Words used in this chapter — implication, vessel of reconstruction, transfer operator, projection onto the vacuum, mass gap, uniform decay, non-degeneracy, strong-coupling window, vortex ratio, sector bias, base point, nesting, widening, area exponent. Their meanings are in A-02 Glossary.

Next chapter: 28-04 The list of premises — tree and labels — lays out, with labels, the tree whose root is the statements to be shown and whose leaves are the sufficient conditions on the present road, including the assumptions of the three implications.

The list of premises — tree and labels

This chapter is this book's study. The dictionary of § 09 is a guide.

What this chapter shows — A tree with the six statements of 19-07 (the roots) at the top and the 57 rows of sufficient conditions that support them on the present road below, and a table of the 57 rows. For each row we give the name of its content, a one-line statement, the object the statement is about, the label, and, if the form is not fixed, the reason symbol. We also give counts by label, and a table matching these names to the names in the conditional implication assembled earlier and to the names in the ledger of Part 27.

Prerequisite chapters — 28-00 Entrance (key to the labels), 19-07 The statements to be shown. The definitions of the objects are in Part 19 (19-01–19-04).

Said first — This list is as of 2026-11-13, and the labels move. Updates are made in A-08 A map to the mass gap. Many statements in the tables use quantities not yet defined from the lattice (effective coupling, activity, background field and so on). Those rows are marked "Form not fixed" or "Object: no definition", and do not claim that the statement has a meaning. The rows supporting the roots on the ultraviolet side (most of Tables 2 and 3) are a record of sufficient conditions in this book's design (the measure pushed forward by a fixed block map). The existence of four-dimensional Yang–Mills theory was shown, with a different design, by the construction published on 2026-11-10 (Volume 2; the two designs are compared in Part 14). The rows on the infrared side (uniform decay in Table 4, and the band in Table 5) are aimed at questions that remain open.

The order of this chapter

1. How to read the stages and labels — the order of the tables, and seven labels
2. The tree — the root, and sufficient conditions through a representation (Figure 1)
3. Counts by label, and what they tell us (Table 1)
4. Premise of the box continuum limit — the continuum limit in a fixed box (Table 2, 17 rows)
5. Premise of ultraviolet stability — estimates of ultraviolet stability (Table 3, 21 rows)

6. Premises of full-sequence convergence, observables, infinite volume, rotation and non-triviality (Table 4, 7 rows)
7. The band — rows on the vortex free energy (Table 5, 12 rows)
8. Five rows added outside the 57 (Table 6)
9. A dictionary of other names (short version, Table 7)

01

How to read the stages and labels — the order of the tables, and seven labels

The order of the tables. The rows are arranged in four tables, in the order of the stages into which the road to a proof is divided. The number in the left column of each table is a running number within that table. In the text, rows are referred to by the name of their content.

Stage	Table	Rows
Premise of the box continuum limit: the continuum limit in a box of fixed size (premises of the chain of block maps)	Table 2	17
Premise of ultraviolet stability: estimates of ultraviolet stability (step-by-step estimates; the kinds of estimate, in order, are free-field input, constrained Green's functions, background field, small-field expansion, large fields, and induction across steps)	Table 3	21
Premises of full-sequence convergence, fine observables, infinite volume, rotation and non-triviality	Table 4	7
The band (crossover of the vortex free energy; the rows around the band in the ledger of conditions, 27-02)	Table 5	12

The free-field premise (absolute conditions on the free field) is a reference to rows of the premise of ultraviolet stability (bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm), large-field estimate), so it has no rows of its own. The fermion premise is outside the scope of this list.

Labels. A premise label is one column of a table, and is separate from the labels of the text (Proved here, Cited, Computer-assisted, Reading, Assumption) (key in 28-00).

Label	Meaning
Theorem	Machine-checked in Lean, and a check that compares the definitions and the statement with a separate pair of eyes has confirmed the content of the statement

Label	Meaning
Theorem (weak form)	The same, but only up to a form weaker than the premise. The same column says whether it **pays the premise's column as it stands** or is a **similar but different statement** (a different object, a subfamily, a linearisation and so on)
On paper	There is an argument on paper. The check comparing definitions and statement with a separate pair of eyes has not been done
Form fixed, unproved	The form of the statement is settled. There is no proof. "Conditional" means: only when, for the sequence fixed by matching, the restrictions on threshold and period are put into the statement
Form not fixed	The form of the statement is not yet settled. The reason is given by a symbol
Depends on design	A row determined by a design choice (28-05)
Literature	Depends on a result in the literature. Not re-derived in this book

Reason symbols (for rows whose form is not fixed): (a) too light (stands for trivial objects), (b) too strong (false, or there are examples where it is false), (c) object undefined, (d) there are numerical signs against the present form, (e) there is a form at one scale but no form across steps, (f) the order of quantifiers is not fixed.

The "Object" column of the tables says which object the statement is about. Objects with a definition in Part 19 are linked by anchor; those without are marked "no definition".

02

The tree — the root, and sufficient conditions through a representation

The root of the tree is the six statements of 19-07. The root is written with defined objects only. Below it hang the premises (leaves) we place in order to derive the root on the present road. Many of the leaves are statements made through a **representation** (quantities used to write down the theory at coarse scales): effective couplings, activities, background fields, classes.

Figure 1 The tree of the root and sufficient conditions (as of 2026-11-13; labels in parentheses)

```
Matching (19-07 §02)
  └─ sufficient conditions through the effective coupling
       │  · definition and continuity of the effective coupling (design)
       │  · net one-step bracket of the coupling (not fixed)
       │  · large-field contribution to the coupling (not fixed)
       │  · recursion of the activity (not fixed)
       │  · rarity of supply (not fixed)
       │  · moving invariant interval and margin (not fixed)
       ├─ representation level of the premise of ultraviolet stability
       │     · coefficient of the one-step decrease of the coupling (thm, weak
       │     · value of the one-loop coefficient (lit.)
       │     · small-field one-step renormalisation theorem (not fixed)
       │     · induction uniform in the step (not fixed)
       │     · large-field estimate (not fixed)
       │     · large-field gas (not fixed)
       └─ linking the thresholds (not fixed)
── the roots below are for the sequence β_n fixed by matching ──
Convergence of the full sequence (19-07 §03)
  └─ contraction of the one-step map and matching at the endpoint (not fixed)
       └─ · induction uniform in the step (not fixed)
          · small-field one-step renormalisation theorem (not fixed)
          · contraction of irrelevant terms (thm, weak)
          · Constrained Decay Theorem (fixed, vacuum version)
          · zero-order transport of the background (not fixed)
          · one-step background-field lemma (paper) · multistep background fie
          · single-scale expansion tools (paper)
          · gauge fixing step by step (paper) · class and localisation (paper)
Non-degeneracy (19-07 §04)
  ├─ rarity of two-step events: plaquette part (fixed, conditional)
  │     ← itself a statement about defined objects only
  └─ rarity of two-step events: background-invariant part (not fixed)
       · choice of the small-field set and extension of the background invari
       · positivity with background (not fixed)
       · constant of the quadratic form of the background (paper)
       · one-step background-field lemma (paper)
Equicontinuity (19-07 §05)
  └─ equicontinuity of observables (fixed, conditional)
       │  ← itself a statement about defined objects only
       └─ Lipschitz property of the background map
            · positivity with background (not fixed)
            · constant of the quadratic form of the background (paper)
            · contraction of the background map (thm, weak)
Structure (needed by every root)
     · the seven rows from continuity, composition and push-forward consistenc
       of the block map to family of observables and small-field set (thm / th
Free-field foundation
     · bracketing of the propagator (thm, weak)
     · geometry of many-body sums (correlation norm) (thm, weak)
     · geometry of many-body sums (reach constant) (fixed)
     · constrained inverse identity (thm)
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      · single-scale rarity on the fine lattice (thm, weak)
Uniform decay (19-07 §06) = uniform decay of block observables (fixed, conditi
  ├─ seam to the vessel of the mass gap (fixed)
  ├─ infinite volume: ultraviolet estimate uniform in the box size (not fixed)
  ├─ rotation: one-step ultraviolet estimate for the bridge, independence of t
  ├─ fine observables: approximation of flowed observables by block variables
  ├─ non-triviality: the limit is not a free field (not fixed)
  └─ infrared road
        · band comparison (in the language of the gap) (not fixed)
        · from the area law to the gap (not fixed)
Band (19-07 §07)
  ├─ non-negativity of θ (thm, weak)
  ├─ base point at a small box (fixed, conditional)
  ├─ single-scale criterion (fixed, conditional)
  │     ← question of the cost of a twist-inserting map (not fixed)
  ├─ nesting (frame decorrelation) (paper)
  ├─ lateral widening for doubling (fixed, conditional)
  └─ another road
        · tension-doubling inequality (not fixed)
        · monotonicity of the vortex free energy and renormalisation compariso
        · gap of a small twisted box and uniform acceptance of insertion (not
        · from the vortex area exponent to the Wilson-loop area law (lit.)
```

Three things can be read from the tree.

1. **At the level of the root, the forms of the statements are settled** (except for the choice of matching). The root, however, has content only when read for the sequence fixed by matching (if the sequence is taken freely, $\beta_n \equiv 0$ makes the uniform-decay and band parts of the root stand; 19-07 § 01). Of the 24 rows whose form is not fixed, 13 sit at the level of the representation below the root, 8 are rows whose object does not yet exist or which are settled after something upstream, and 3 concern the quantification and truth of band statements (28-06). Changing the representation changes the leaves, not the root.
2. **The plaquette part of non-degeneracy (rarity of two-step events: plaquette part) and equicontinuity (equicontinuity of observables) are premises whose form is, as it stands, a statement about defined objects only**; they do not go through a representation.
3. **The open forms on the side of the band come not from the representation but from the order of quantifiers and from truth** (tension-doubling inequality, monotonicity of the vortex free energy and renormalisation comparison, from the area law to the gap). Their origin differs from the ultraviolet side.

Inside the chain theorem of the present road (28-01), the sequence of bare couplings β_n and the effective coupling are not connected. The Lean theorem takes the sequence of bare couplings arbitrarily. The conditional implication assembled earlier (28-01

§ 08) joins the two by defining the fine measure to be "the Wilson measure at the β_n chosen by the matching bracket for the bare coupling", but the proof itself splits into the coupling side and the measure side. The candidate place in the chain where the two could be joined is linking the thresholds.

03

Counts by label, and what they tell us

Table 1 Counts by label (as of 2026-11-13, 57 rows; the 5 rows added in § 08 are not counted)

Label	Count	Rows
Theorem	3	Continuity, composition and push-forward consistency of the block map, reflection positivity of the fine lattice, constrained inverse identity
Theorem (weak form)	10	Gauge covariance and locality, commuting with the lattice symmetries, reflection positivity at the coarse level, single-scale rarity on the fine lattice, bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm), non-negativity of θ
On paper	8	Symmetries of the measure, family of observables and small-field set, one-step background-field lemma, multistep background field, single-scale expansion tools, gauge fixing step by step, class and localisation, frame decorrelation
Form fixed, unproved	7	Rarity of two-step events: plaquette part, equicontinuity of observables, Constrained Decay Theorem, uniform decay of block observables, base point at small T^2, single-scale criterion, lateral-widening inequality (strong form) (the Constrained Decay Theorem in the version around the vacuum; the other 6 rows conditional)
Form not fixed	24	Net one-step bracket of the coupling, large-field contribution to the coupling, recursion of the activity, rarity of supply, moving invariant interval and margin, rarity of two-step events: background-invariant part, positivity with background, zero-order transport of the background, small-field one-step renormalisation theorem, large-field estimate, large-field gas, induction uniform in the step, contraction of the one-step map and matching at the endpoint, approximation of flowed observables by block variables, ultraviolet estimate uniform in the box size, one-step ultraviolet estimate for the bridge, independence of the infinite-volume limit from the box shape, the limit is not a free field, question of the cost of a twist-inserting map, tension-doubling inequality, monotonicity and renormalisation comparison, gap of a small twisted box and uniform acceptance of insertion, band comparison (in the language of the gap), from the area law to the gap
Depends on design	2	Definition and continuity of the effective coupling, row sums of the lift uniform in the step
Literature	3	Value of the one-loop coefficient, from the vortex area exponent to the Wilson-loop area law, forest interpolation formula

The objects that rows with a theorem label speak about. The 13 rows labelled "Theorem" or "Theorem (weak form)" are statements about one of the following: the Wilson measure on the fine lattice (reflection positivity of the fine lattice, single-scale rarity on the fine lattice, non-negativity of θ), the block map itself (continuity, composition and push-forward consistency of the block map, gauge covariance and locality, commuting with the lattice symmetries), structural properties of the push-forward measure (reflection positivity at the coarse level, for the strip class), the free field, the vacuum background and the linearised blocking (bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm)), or an abstract identity (constrained inverse identity). **There is no theorem giving quantitative estimates for measures at renormalised scales (the push-forward measure $\nu_{n,m}$ or the effective measure at step $k \geq 1$).** For the push-forward measure, what exists is only structural properties carried over from the fine measure by the push-forward.

Rows that pay a column as it stands. Of the 10 rows labelled "Theorem (weak form)", only reflection positivity at the coarse level (for the strip class) pays a premise column of the present road as it stands. The others are similar but different statements — a different object, a subfamily, a linearisation, a one-step form, and so on (distinguished in the label column of the tables).

Where the rows whose form is not fixed come from. Of the 24 rows, 20 include (c) (object undefined) or (e) (no form across steps) among their reasons. The 4 that do not (zero-order transport of the background, tension-doubling inequality, monotonicity and renormalisation comparison, from the area law to the gap) are on the side of "too strong, or there are false examples"; for zero-order transport of the background there is an argument on paper that its present form is false (not yet checked by another pair of eyes).

Rows whose form is fixed but unproved, and choices that make them stand trivially. Of the 7 rows, 6 (rarity of two-step events: plaquette part, equicontinuity of observables, uniform decay of block observables, base point at small T^2, single-scale criterion, lateral-widening inequality (strong form)) have a "fixed form" only for the sequence fixed by matching, with the restrictions on threshold and period put into the statement. If the sequence is taken freely, uniform decay of block observables, base point at small T^2, single-scale criterion and lateral-widening inequality (strong form) stand trivially with $\beta_n \equiv 0$ ($\theta = 0$, and variables separated in time are independent) Proved here . The theorem of uniform decay of block observables in the strong-coupling window (20-04) lies in this direction: it is an example in which the statement can be satisfied, not a check of its content. Non-degenerate small examples exist for the version of the Constrained Decay Theorem around the vacuum (one-dimensional,

scalar, for two of the inputs) and for the form of base point at small T^2 with fixed lattice spacing (the thin-vortex floor, 21-02).

04

Premise of the box continuum limit — the continuum limit in a fixed box

These are the premises for the chain of block maps, in a box of fixed size, to give subsequential limits and the properties that pass to the limit. The implication that uses them is in 28-01. Notation: B is the block map, $\nu_{n,m}$ the push-forward measure, u_k the effective coupling at step k (its definition from the lattice is not fixed), z_k the large-field activity, w_k the supply.

Table 2　Premise of the box continuum limit (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Continuity, composition and push-forward consistency of the block map	B is continuous. The map from step n to m is the composite of single steps. $(B_m)_*\nu_{n,m+1} = \nu_{n,m}$	19-04, push-forward	Theorem: for every family of paths, weights and family of measures	—
2	Gauge covariance and locality	$B(U^g) = B(U)$ $g\|_{\text{coarse sites}}$. $B(U)_b$ depends only on fine links within a finite distance of b	19-04	Theorem (weak form, a similar but different statement): the form for the one-step map. The extension to composites and the reach distance are on paper	—
3	Commuting with the lattice symmetries	Translations, reflections and the hypercubic group commute with B	19-04	Theorem (weak form, groundwork): translations checked; reflection in the time direction in Lean (content not yet checked); reflections in the other axes and the hypercubic group on paper	—
4	Reflection positivity of the fine lattice	For bounded measurable f depending only on links on the positive side, $\int f(\Theta U)\, f(U)\, d\mu_W \geq 0$	19-01, reflection	Theorem: even side length. For the plane of lattice sites, all real β; for the plane through the middle of links, $\beta \geq 0$ and gauge-invariant f (20-01)	—

Number	Name	Statement	Object	Label	Reason
5	Reflection positivity at the coarse level	For F in the positive-time class, $\int (F \circ \Theta)\, F\, d\nu_{n,m} \geq 0$; matrix form $\sum c_a c_b\, s_n(x_{ab}) \geq 0$	19-04, positive side	Theorem (weak form, pays the column as it stands): for the class whose positive side is taken as a strip. The matrix form is on paper. The form taking the whole coarse half as the positive side is not placed (28-05)	—
6	Symmetries of the measure	$\nu_{n,m}$ is invariant under lattice symmetries. The breaking of symmetry of observables $\to 0$	19-04	On paper: the symmetry fields for the pure gauge theory are not in Lean. What Lean has is a version for families of measures with fermions (global gauge transformations and unit translations in the four directions)	—
7	Family of observables and small-field set	D is countable and dense, $\lvert s_n\rvert \leq B_d$, $s_n(x) = \int G_x\, d\nu_{n,\mathrm{lev}(x)}$, C_m is closed and contains the two-step event	19-07	On paper: the inclusion (two-step event $\subseteq$ small-field set) depends on how the background invariants are extended (choice of the small-field set and extension of the background invariants)	—
8	Definition and continuity of the effective coupling	$u_0(x) = x$. $x \mapsto u_{k+1}(x)$ is continuous on $\{u_j \geq u_\star\ (j \leq k)\}$	No definition (u_k)	Depends on design	—
9	Net one-step bracket of the coupling	$u_{k+1} = u_k - b + r_k^s + r_k^l$, $b_{\inf} + \delta_k - e(u_k) \leq b - r_k^s \leq b_p + e(u_k)$, $\sum_k \max(-\delta_k, 0) \leq D_0$	No definition (u_k, small- and large-field remainders r_k^s, r_k^l)	Form not fixed	(c)(e)
10	Large-field contribution to the coupling	$\lvert r_k^l\rvert \leq D(u_k)\, z_k$	No definition (r_k^l, z_k)	Form not fixed	(a)(c)
11	Recursion of the activity	$z_{k+1} \leq \rho z_k + A w_k + B z_k^2$, $z_k \geq 0$, $z_0 \leq W(u_0)$	No definition (z_k, w_k)	Form not fixed	(a)(c)(e)

Number	Name	Statement	Object	Label	Reason
12	Rarity of supply	$w_k \le \Phi(u_k)$ (from an exponential moment of the plaquette variable uniform in the step, $\mathbb{E}\, e^{\sigma Y} \le m_1(\sigma)$)	No definition (w_k, plaquette variables at step k)	Form not fixed	(c)(e)
13	Single-scale rarity on the fine lattice	For $\beta \ge 81$, the probability that all plaquettes of X have $\operatorname{Re} U_p < 0$ is $\le q_1^{\lfloor \lvert X\rvert/6 \rfloor}$, with $q_1 < 1$ independent of the volume	19-01	Theorem (weak form, a similar but different statement, not connected to any column): one scale, $\beta \ge 81$, a ratio to the full partition function (22-01)	—
14	Moving invariant interval and margin	$\rho W(s) + A\Phi(s) + BW(s)^2 \le W(s')$ $(u_\star \le s,\ s' \le s + D_0)$, $D(s)W(s) \le Y$, $e(u_\star) + Y \le b$ inf	Real functions W, Φ, D, e	Form not fixed: W, Φ, D, e are free, and taking them all 0 makes it hold	(a)(c)
15	Rarity of two-step events: plaquette part	For large n, $\nu_{n,m}\big(\beta_m(1 - \operatorname{Re} U_p) > y_h(m)\big) \le P_1(m)$	19-04	Form fixed, unproved (conditional: for the sequence fixed by matching, with the threshold floor and $\eta_m < 1$ put into the statement)	—
16	Rarity of two-step events: background-invariant part	For large n, $\nu_{n,m}\big(f_{m,x} > f_h(m)\big) \le S(m)$	No definition (the background-field invariant f)	Form not fixed	(c)
17	Equicontinuity of observables	$\lvert s_n(d) - s_n(d')\rvert \le \tau_n + \Omega(\operatorname{dist}(d, d'))$, $\tau_n \to 0$, $\Omega(r) \to 0$	19-04 (the family of observables is the one in family of observables and small-field set)	Form fixed, unproved (conditional: for the sequence fixed by matching)	—

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05
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Premise of ultraviolet stability

These are the premises for deriving the representation-level premises of the premise of the box continuum limit (net one-step bracket of the coupling, large-field contribution to the coupling, recursion of the activity, rarity of supply) from step-by-step estimates. Notation: G_k is the family of free-field propagators at step k, P/ω the free propagator (19-04 § 07), $\mathcal{M}$ the one-step map.

Table 3 Premise of ultraviolet stability (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Bracketing of the propagator	For every step k and momentum, $c_0\,P/\omega \preceq G_k \preceq P/\omega$, $c_0 = 3/100$	19-04	Theorem (weak form, a similar but different statement): both the upper and lower sides checked. The object is the symbol recursion (a family of matrices for each momentum): free field, vacuum background, linearisation, weights $(1/8, 3/4, 1/8)$ (15-12)	—
2	Positivity with background	$\lambda_{\min}(H_c(B)) \geq \lambda_0(1 - f_1/f_\star)$	Definition on paper (background field B, constrained Hessian H_c)	Form not fixed: the background field and constrained Hessian are definitions on paper, and three forms stand side by side	(c)
3	Contraction of irrelevant terms	$\mathcal{M}^{17}(\omega G_k) \preceq \tau\,\omega G_{k+17}$, $\tau < 1/170$, uniform in k	19-04	Theorem (weak form, a similar but different statement): the τ of the quadratic form, different from the τ_k of the induction. The one-step estimate at step 0 is conditional (15-12)	—

Number	Name	Statement	Object	Label	Reason
4	Coefficient of the one-step decrease of the coupling	The free-field coefficient b_k converges at rate $(3/5)^k$, and the sum of transients is finite	19-04	Theorem (weak form, a similar but different statement): quadratic functionals of arbitrary data on the constrained covariance of the free field converge at rate $(3/5)^k$. That b_k is such a functional is on paper	—
5	Value of the one-loop coefficient	$b_\infty = 2b_0 \ln 2$	—	Literature: not re-derived for this map	—
6	Contraction of the background map	The sum of one-unit displacements $\sum_j \Delta_j < \infty$ (the free-field part is of the type $\|\psi_l\|^2 \le 0.608^l$)	19-04	Theorem (weak form, a similar but different statement): the free-field part	—
7	Geometry of many-body sums (correlation norm)	Correlation norm of the plaquette field $\Lambda \le 640/9$	19-04	Theorem (weak form, a similar but different statement): free field, linearisation, torus (15-12). The reach constant is in geometry of many-body sums (reach constant)	—
8	Constrained inverse identity	$C = G - GT^*(TGT^*)^{-1}TG$, with upper and lower estimates	Abstract linear maps	Theorem: a finite-dimensional abstract identity and estimates	—
9	Constrained Decay Theorem	The one-step constrained Green's function at step k satisfies, for an exponential weight W, $\|Wx\| \le \frac{11}{2}\|Ww\|/r^2$, with constants uniform in k	Abstract (Lean). The concrete objects are on paper	Form fixed, unproved (the version around the vacuum). The version for a general background has form not fixed (c)	—

Number	Name	Statement	Object	Label	Reason
10	Zero-order transport of the background	$\sup_n 4^n\theta_n < \infty$, $\sum_n 4^n\theta_n^{3/2} < \infty$ (θ_n: the ratio of the sup of background plaquettes in units of step j, divided by coarse units)	Definition on paper (the background as minimiser of the action)	Form not fixed: in its present form, it diverges numerically for the background that minimises the action Reading (the argument on paper that it is false has not yet been checked)	(b)(d)
11	One-step background-field lemma	For small fields, the minimum of the constrained action is unique, analytic, exponentially local, and commutes with coarse gauge transformations	Definition on paper (background field)	On paper	—
12	Multistep background field	Unique and regular on nested small-field regions. The inverse of the linearisation is at most 1 in the action norm, independent of the number of steps	Definition on paper (background field)	On paper	—
13	Row sums of the lift uniform in the step	$\sup \lvert Eg\rvert \le R \sup \lvert g\rvert$, with R uniform in k	Definition on paper (the linear lift)	Depends on design: the choice of gauge and norm. The one-dimensional scalar case is a theorem	—
14	Single-scale expansion tools	Cluster expansion on product white-noise measures, and its convergence	Abstract	On paper	—
15	Forest interpolation formula	—	—	Literature: the formula itself is not in Lean	—
16	Small-field one-step renormalisation theorem	Coarse action = β_{k+1} S_W(background) + local expansion + many-body terms. The many-body terms are exponentially small in distance	No definition (the effective action at step k)	Form not fixed	(c)(e)

Number	Name	Statement	Object	Label	Reason
17	Gauge fixing step by step	Landau gauge at each step. The determinant is block lower-triangular. Estimate of the Haar Jacobian	Definition on paper	On paper	—
18	Large-field estimate	For each step t and each hole γ, $\lvert z_t(\gamma)\rvert \le \exp(-\gamma E\, \beta\, S^\flat(\gamma))$, $S^\flat(\gamma) \ge s_* \lvert\gamma\rvert$, with $s_* > 0$ uniform in step and volume	No definition (activities, coarse class)	Form not fixed	(c)(d)(e)
19	Large-field gas	$\sum_{X \ni x} \lvert z(X)\rvert\, e^{2\lvert X\rvert} \le 1$, and that the partition function of the gas equals the corresponding ratio	No definition (activities)	Form not fixed	(a)(b)(e)
20	Class and localisation	The only local gauge-invariant terms of dimension at most 4 are the constant and $\sum \operatorname{tr} F^2$	Definition on paper (class)	On paper: the algebra of localisation is in Lean; identification with the class is on paper	—
21	Induction uniform in the step	$r_{k+1} \le \tau_k r_k + a_k w_k + s_k$, $w_{k+1} \le b_k r_k + \sigma_k w_k + s'_k$; if the product of the coefficients is below $1/4$, it stays in the ball	No definition (r_k, w_k are real sequences with no definition as coordinates of the class)	Form not fixed	(c)(e)

06

Premises of full-sequence convergence, observables, infinite volume, rotation and non-triviality

These rows have their forms fixed after the forms of the premise of the box continuum limit and the premise of ultraviolet stability are settled. For the rows on fine observables and rotation, the objects needed to write the statements are not yet defined (28-06).

Table 4　Premises of full-sequence convergence, fine observables, infinite volume, rotation and non-triviality (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Contraction of the one-step map and matching at the endpoint	$\lvert r_k(u,v)-r_k(u',v')\rvert \le \varepsilon\lvert u-u'\rvert + c\lVert v-v'\rVert$, $\lVert T_k(u,v)-T_k(u',v')\rVert \le c'\lvert u-u'\rvert + \tau\lVert v-v'\rVert$, $4cc' < (1-\varepsilon-\tau)^2$	No definition (the coordinates u, v of the one-step map)	Form not fixed	(c)
2	Approximation of flowed observables by block variables	Approximate an observable at flow time t by a Lipschitz function of the block variables at step m, with an error $\varepsilon_m(t) \to 0$ uniform in the lattice	No object (flowed observables)	Form not fixed: flowed observables are undefined	(c)
3	Ultraviolet estimate uniform in the box size	The premise of the box continuum limit and the premise of ultraviolet stability hold with constants independent of the box size L	All of those two premises	Form not fixed	(e)
4	Uniform decay of block observables	$\lvert\langle F\cdot G\circ\tau_t\rangle - \langle F\rangle\langle G\rangle\rvert \le C_F C_G\, e^{-mt}$, with m independent of the lattice spacing and the box (exact form in 19-07 § 06)	19-04, 19-03	Form fixed, unproved (conditional: for the sequence fixed by matching, with time differences up to half the period). A theorem in the strong-coupling window (20-04), but in the direction where it stands trivially	—
5	One-step ultraviolet estimate for the bridge	The one-step image from a rotated fine lattice to the standard coarse lattice lies in the ball of the chain	No object (the block map from a rotated lattice)	Form not fixed	(c)
6	Independence of the infinite-volume limit from the box shape	The limits of sequences of tori of different shapes coincide	No definition (the infinite-volume limit)	Form not fixed	(c)
7	The limit is not a free field	(by which quantity to say it is not fixed)	—	Form not fixed	(c)

07

The band — rows on the vortex free energy

In this stage many rows can be written with the vortex ratio θ (19-02) alone. The statements are written in the notation $\theta(R;T_L^2)$ of 26-02 (longitudinal size R, size L of the transverse torus). $m_0(L)$ is the box mass of the box of size L (the gap of the transfer operator), and $\sigma_N(\beta)=\liminf_M(-\ln\theta_{N,M})/M^2$ is the slab tension.

Table 5 Premises of the band (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Non-negativity of θ	If M is even, $0\le\theta_{N,M}(\beta)$ for all real β	19-02	Theorem (weak form, a similar but different statement): the subfamily with M even. The column needs all sizes (21-06)	—
2	Base point at a small box	$\theta(L_0;T_{L_0}^2)\le 1-\varepsilon_0$, uniform in the lattice spacing	19-02	Form fixed, unproved (conditional: for the sequence fixed by matching). The form with fixed lattice spacing is the thin-vortex floor (21-02)	—
3	Single-scale criterion	At a physical size L_*, $\theta(L_*;T_{L_*}^2)<(s^2)^{-1/(s^2-1)}$ (0.63 for $s=2$, 0.7598 for $s=3$)	19-02	Form fixed, unproved (conditional: for the sequence fixed by matching)	—
4	Question of the cost of a twist-inserting map	For a gauge-covariant smooth map T, $\mathbb{E}_+[S\circ T-S-\ln J_T]\le 2\,\mathrm{artanh}(0.7598)$, uniform in the lattice spacing	No object (the map T)	Form not fixed	(c)
5	Nesting (frame decorrelation)	$\theta(nR;T_L^2)\le C_D\,\theta(R;T_L^2)^{n^2}$	19-02	On paper	—
6	Lateral widening for doubling	$\theta(R;T_{L'}^2)\le C\,(L'/L)^2\,\theta(R;T_L^2)$ $(L\le L'\le R)$	19-02	Form fixed, unproved (conditional: for the sequence fixed by matching)	—
7	Tension-doubling inequality	For some $\delta_0>0$ and N_1, $4\sigma_{2N}(\beta)\ge e^{\delta_0}\sigma_N(\beta)$ for all $N\ge N_1$	Definition on paper (slab tension σ_N)	Form not fixed: (d) only for the form uniform in β	(b)(f)

Number	Name	Statement	Object	Label	Reason
8	Monotonicity of the vortex free energy and renormalisation comparison	The free energy of a twist insertion is monotone in β. $s(N,\beta') \le s(2N,\beta)$	19-02	Form not fixed	(b)
9	Gap of a small twisted box and uniform acceptance of insertion	Gap $\ge c/L$ uniform in the lattice spacing. The acceptance rate of driven vortex insertions uniform in the lattice spacing	19-02, 19-03	Form not fixed: the gap operator of small twisted boxes and the insertion map are undefined	(c)
10	Band comparison (in the language of the gap)	For $x(L) := L\,m_0(L)$, $x(2L) \ge x(L) + \delta$, together with stability above the threshold	19-03 (box mass)	Form not fixed: the box mass $m_0(L)$ of the box used (with spatial planes) is undefined	(c)
11	From the area law to the gap	From the band (area exponent) to uniform decay of block observables	—	Form not fixed: the naive form is false in two dimensions	(b)
12	From the vortex area exponent to the Wilson-loop area law	—	—	Literature: the typical-frame version has not been re-derived	—

Among the band rows, the tension-doubling inequality and monotonicity and renormalisation comparison still have a choice of how to quantify the form of the statement (uniformly in β, or at fixed physical size). From the area law to the gap is the arrow joining the side of the mass gap and the side of confinement, and this book does not have it (28-03).

08

Five rows added outside the 57

There are five premises that do not fall under any of the 57 rows. So as not to change the counting, they are kept outside the counts of Table 1. The labels of these 5 rows have not yet passed a check at the same level as the 57.

Table 6　Added premises (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Linking the thresholds	Tie the scale factor β_m of the plaquette variable and the threshold $y_h(m)$ of the two-step event to u_m and the threshold $y_s(u_m)$ on the coupling side (for example under $\beta_m = 4u_m$, $y_h(m) = y_s(u_m)$, or the analytic floor $y_h(m) \le \beta_m(1-\cos\varphi_c)$ (φ_c is a fixed angle that sets the small-field threshold))	No definition (u_m)	Form not fixed: in the conditional implication assembled earlier, too, the theorem does not tie these two together	(c)
2	Choice of the small-field set and extension of the background invariants	That the two-step event is contained in the small-field set. It holds when the background invariants are extended outside $SU(2)$ by values larger than the threshold f_h, and is false when they are extended by 0	Definition on paper (background field)	Depends on design	—
3	Constant of the quadratic form of the background	The three constants of the premise "background estimate" of the implication assembled earlier (constants of the quadratic forms that make up the Lipschitz constant of the background map)	Definition on paper (background field)	On paper	—
4	Geometry of many-body sums (reach constant)	The row sum R of correlations of the plaquette field is finite	19-04	Form fixed, unproved	—
5	Seam to the vessel of the mass gap	The vessel implication (28-03) assumes weak convergence of the full sequence of measures, and that the subspace of observables is closed under time translation. When used along a subsequence, it takes the form "at every limit point"	19-03	Form fixed, unproved (conditional: for the sequence fixed by matching; the same form as the root of full-sequence convergence)	—

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09
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A dictionary of other names (short version)

The same premises go by other names in the conditional implication that this book's study assembled earlier (weak ultraviolet existence in small boxes; 28-01 § 08) and in the ledger of Part 27. We include only what is needed to match them up.

Table 7 A dictionary of other names (short version)

Name of the content	Name in the conditional implication assembled earlier	Ledger (27-02)	Lean
The implication of weak ultraviolet existence in small boxes (the main theorem of the implication assembled earlier)	Main theorem. Its assumptions are the one-step bracket assumption for the coupling, equicontinuity of observables and the small-field set assumption; its conclusions run from the matching bracket for the bare coupling to non-degeneracy (mass of the small-field set)	Ultraviolet control (premise of the box continuum limit; 27-02)	`WeakUltravi`
The chain theorem from per-step premises	The chain theorem (six per-step assumptions). Its set of assumptions differs from that of the Lean box version	—	`BoxWalk.cha` `ChainBlockM`
One step of the coupling (definition and continuity of the effective coupling, net one-step bracket of the coupling)	One-step bracket assumption for the coupling, net one-step bracket of the coupling, quadratic form of the one-step remainder, window condition for the class, definition and continuity of the effective coupling	—	Chain fields
Large-field contribution to the coupling	Large-field contribution to the coupling	—	Chain field
Recursion of the activity	Recursion of the activity, existence of limiting expectations and continuous extension, large-field weight	—	Chain field
Rarity of supply, single-scale rarity on the fine lattice	Rarity of supply, moments and rarity of a single plaquette	—	`FoldCap.rar`
Moving invariant interval and margin	Moving invariant interval and margin, logarithmic threshold and moving interval	—	Chain field
Rarity of two-step events (plaquette part, background-invariant part)	Small-field set assumption, small-field event uniform in the lattice	—	Chain field
Equicontinuity of observables	Equicontinuity of observables, small-field event uniform in the lattice, background estimate	—	Chain field

Name of the content	Name in the conditional implication assembled earlier	Ledger (27-02)	Lean
Reflection positivity (reflection positivity of the fine lattice, reflection positivity at the coarse level)	Support of reflection positivity	Reflection positivity of z (positivity of the ensemble summed over twists; a different thing)	`TorusRP.wil`
Positivity with background	Positivity with background, background estimate	—	—
Large-field gas	Plaquette-level large-field gas	—	—
Induction uniform in the step	Quadratic form of the one-step remainder, window condition for the class	—	`F3Condition`
Constrained Decay Theorem	—	—	`RealSpaceTh`
Contraction of the one-step map and matching at the endpoint	Related to the summability-of-increments assumption (not the same thing; see the note below)	—	—
Uniform decay of block observables	—	Uniform decay of block observables in the ledger	—
Base point at a small box (base point at small T^2)	—	Base point at small T^2 in the ledger	—
Nesting (frame decorrelation)	—	Frame decorrelation in the ledger	—
Lateral widening for doubling (lateral-widening inequality (strong form))	—	Lateral-widening inequality (strong form) in 26-02, and the same row in 27-01	—
Monotonicity of the vortex free energy and renormalisation comparison	—	β-monotonicity, renormalisation comparison (26-06)	—

Name of the content	Name in the conditional implication assembled earlier	Ledger (27-02)	Lean
Gap of a small twisted box and acceptance (small-box gap and acceptance)	—	Small-T^2 gap, uniform acceptance of insertion	—
Band comparison (in the language of the gap)	—	Band comparison (in the language of the gap) in the ledger	—

The summability-of-increments assumption is the assumption that, for a countable dense family of gauge-invariant functions, the step-by-step increments of the values of observables are summable. It is not the same as full-sequence convergence in 19-07 § 03: it is stronger by the summability, and the family of functions is different. Also, the implication assembled earlier defines the fine measure to be the Wilson measure at the β_n chosen by the matching bracket for the bare coupling, whereas the Lean theorem takes the sequence of bare couplings arbitrarily.

Among the ten premises of the implication assembled earlier, support of reflection positivity is reflection positivity. The other nine are the premises other than reflection positivity. The existence of four-dimensional Yang–Mills theory itself was shown, with a different design, by the construction published on 2026-11-10 (Volume 2).

What this chapter can say

Can say	Cannot say
The six roots and the 57 rows of premises are laid out as a tree and in tables by stage, with labels and reason symbols (as of 2026-11-13)	None of the premises is shown in this chapter. The labels follow the record of the checks of that date
The 13 rows with theorem labels are statements about the Wilson measure on the fine lattice, the block map itself, structural properties of the push-forward measure, the free field and linearisation, or an abstract identity	There is no theorem giving quantitative estimates for measures at renormalised scales
Of the 24 rows whose form is not fixed, 20 owe this to an undefined object or to the absence of a form across steps	Answers to the questions that would fix the forms of the open rows (only the questions are in 28-06)
Of the 7 rows whose form is fixed but unproved, 6 are conditional on the sequence fixed by matching	If the sequence is taken freely, 4 of them stand trivially with $\beta_n \equiv 0$. Non-degenerate small examples exist for only 2 rows

Sources and reproduction

Item	Type	Source, tools
The 57 rows of premises, labels, reason symbols, the tree	Assumption	The inventory of premises in this book's study (2026-11-13). The labels come from the Lean sources and from the record of the check that compared definitions and statements with a separate pair of eyes
The assumptions and the ten premises of the conditional implication assembled earlier (Table 7)	—	Record of this book's study (2026-10-30). 28-01 § 08
Rows with theorem labels	Lean	The source tables of each chapter (20-01, 21-06, 15-12, 22-01)
Forest interpolation formula	Cited	Bibliography only (unchecked)
From the vortex area exponent to the Wilson-loop area law	Cited	Bibliography only (unchecked)
Rows of the ledger of conditions, and the rows of uniform decay of block observables and band comparison (in the language of the gap)	—	27-02 The ledger of conditions

Words used in this chapter — premise, root, sufficient condition, representation, effective coupling, activity, supply, background field, class, push-forward measure, two-step event, vortex ratio, slab tension, ledger of conditions. Their meanings are in A-02 Glossary.

Next chapter: 28-05 What depends on design choices — tabulates the rows labelled "Depends on design", and which choices the constants of the other rows depend on.

What depends on design choices

> This chapter is this book's study.

What this chapter shows — One step of renormalisation and the statements of the premises contain places that are not decided by mathematics but chosen and fixed by us. For nine such choices we tabulate the present values and the theorems and premises that would have to be redone if the choice changed. We also confirm here that the theorems of Parts 20 and 21 and the rarity of large plaquettes (22-01) do not depend on the choice of block map.

Prerequisites — 19-04 § 08 Weights as a design choice, 28-04 The list of premises.

Said first — This chapter does not claim which choice makes the premises easier to satisfy. The table of constants for other weights, and the computation of whether overlapping block averages preserve reflection positivity of the push-forward measure, have not yet passed a check and are not included. We only write that there is an issue there. The choices here are choices within this book's design (the measure pushed forward by a fixed block map), and are separate from the design of the published construction that showed the existence of four-dimensional Yang–Mills theory (the two designs are compared in Part 14).

The order of this chapter

1. What a design choice is — fixing things so that a definition is unique
2. The nine choices and their present values (Table 1)
3. Transverse weights, scale factor, pull-back radius — choices in the block map itself
4. Overlap of block averages — the issue with reflection positivity
5. Norm, gauge, definition of the background — choices of representation
6. The way of matching, and the definition of compatibility in the gas
7. Which theorems and premises depend on which choices (Table 2)
8. Other fixings (Table 3)

01

What a design choice is — fixing things so that a definition is unique

To define one step of renormalisation (19-04), one has to settle on one choice of the weights of the family of paths, the scale factor of the blocks, the procedure that returns the average to the group, and so on. The statements of the premises (28-04) also involve decisions such as which norm to measure with and around which background to expand.

These are not determined uniquely by mathematics; we choose and fix them. If a choice changes, the constants and certificates of the theorems that depend on it must be redone, and the statements of the premises themselves change. Among the statements to be shown (19-07), the band can be written with the vortex ratio θ alone and does not depend on the choice of block map. The other five depend, through the push-forward measure $\nu_{n,m}$, on the choice of block map (weights, scale factor, radius), and matching, non-degeneracy and equicontinuity depend further on the choice of observables and sets.

This chapter is a table that makes visible, in one place, which choice affects what.

02

The nine choices and their present values

Table 1　Design choices (as of 2026-11-13)

Choice	Present value	Premises that depend on it (names of rows in 28-04)
Transverse weights of blocking $(\alpha, 1-2\alpha, \alpha)$	$\alpha = 1/8$	The free-field inputs (bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm)), the Constrained Decay Theorem, zero-order transport of the background, row sums of the lift uniform in the step, large-field estimate
Block scale factor	2	The matching bracket, rarity of two-step events: plaquette part, the free-field inputs (from bracketing of the propagator to geometry of many-body sums), zero-order transport of the background, band doubling (single-scale criterion, lateral-widening inequality (strong form), tension-doubling inequality)
Radius r_0 of the radial pull-back	A free parameter $(0 < r_0 \le 1)$	The range of linearisation, large-field estimate

Choice	Present value	Premises that depend on it (names of rows in 28-04)
Overlap of block averages	Overlapping (an average over paths shifted transversely with $(1/8, 3/4, 1/8)$)	Reflection positivity at the coarse level, recursion of the activity, rarity of supply, uniform decay of block observables
Norm	The sup norm and the sum-of-squares norm are mixed from place to place	The Constrained Decay Theorem, zero-order transport of the background, multistep background field, row sums of the lift uniform in the step, induction uniform in the step
Gauge	Landau slice at each step / axial gauge / tree gauge	The Constrained Decay Theorem, row sums of the lift uniform in the step, gauge fixing step by step
Definition of the background	The minimiser of the constrained action	Rarity of two-step events: background-invariant part, positivity with background, zero-order transport of the background, one-step background-field lemma, multistep background field
Way of matching (definition of the effective coupling)	Not fixed (observable, step, target value, choice of solution; two proposed definitions of u_k on paper)	Definition and continuity of the effective coupling, net one-step bracket of the coupling, the domain of every row of the premise of the box continuum limit
Definition of compatibility in the gas	The conditional implication assembled earlier: not adjacent / Lean: not overlapping	Large-field contribution to the coupling, large-field gas

03

Transverse weights, scale factor, pull-back radius — choices in the block map itself

Transverse weights. The paths that build a coarse link are shifted by $-1, 0, +1$ units in a transverse direction, with weights $(\alpha, 1-2\alpha, \alpha)$ (19-04 § 03). At present $\alpha = 1/8$. The constants of the one-step free-field theorem (15-12) were all computed for this value and cannot be used as they stand for other weights: the lower constant $c_0 = 3/100$ of the sandwich on the propagator, the 17-step contraction $\tau < 1/170$, the correlation norm $\Lambda \le 640/9$ of the plaquette field, and so on.

There is an issue as to what becomes better and what worse when the weights change (whether the plaquette field measured in coarse units grows with the step, and how the properties of the constraint change with the weights).

Scale factor. One step halves the lattice spacing. The matching bracket (19-07 § 02) is written in terms of the step number $n = \log_2(1/a_n)$, and the threshold of the single-scale criterion on the band side differs for each scale factor s (0.63 for $s = 2$, 0.7598 for $s = 3$).

Pull-back radius. This is the threshold r_0 of the procedure that returns the average of path products to the unit ball. The linearisation theorem for the block map (15-13) has $0 < r_0 \le 1$ among its assumptions.

Theorems that do not depend on the choices. The theorems of Part 20 (reflection positivity, strong coupling) and Part 21 (vortex floors, Polyakov loops), and the rarity of large plaquettes (22-01), use only the Wilson measure on the fine lattice and twisted partition functions, and do not use the block map. So they depend on none of these three choices (nor on the overlap of § 04) Proved here .

04

Overlap of block averages — the issue with reflection positivity

In the present family of paths, the paths that build a coarse link are shifted transversely, so neighbouring blocks share fine links (an overlapping average).

Reflection positivity (20-01) is a theorem for the Wilson measure on the fine lattice. Whether reflection positivity is preserved for the pushed-forward coarse measure $\nu_{n,m}$ is expected to depend on whether blocks straddle the reflection plane, that is, on whether there is overlap, and **there is an issue here.** The computation has not yet passed a check and is not included.

In the present list of premises, reflection positivity at the coarse level (the row on reflection positivity at the coarse level) is placed in the form of the class whose positive side is taken as a strip (coarse observables depending only on links in the positive half of the fine lattice). In this form it is paid as a statement about the Wilson measure on the fine lattice. The form that takes the whole coarse half as the positive side is not placed among the premises.

Besides reflection positivity at the coarse level, what depends on this choice is rarity across steps (recursion of the activity and rarity of supply; the road that uses the Chessboard Estimate from reflection positivity) and the vessel used to go from uniform decay to the mass gap (uniform decay of block observables).

05

Norm, gauge, definition of the background — choices of representation

These three are choices made not in the block map but in writing down (representing) the theory at coarse scales. They do not appear in the root (19-07), only on the side of the sufficient conditions.

- **Norm.** Whether to measure the plaquette field in the sup norm or in the action (sum-of-squares) norm. The present form of zero-order transport of the background (the row in 28-04) is written in the sup norm, and the reasons (b)(d) for its form not being fixed are tied to this choice. The Constrained Decay Theorem and the multistep background field are written in the sum-of-squares or action norm.
- **Gauge.** Landau slice at each step, axial gauge, tree gauge. Whether the row sums of the lift are uniform in the step (row sums of the lift uniform in the step) changes with the choice of gauge. For the one-dimensional scalar lift, the row sums are a theorem.
- **Definition of the background.** We use the background that minimises the action under the block constraint. The rarity of the background invariants (rarity of two-step events: background-invariant part), positivity with background, and the background-field lemmas (one-step background-field lemma, multistep background field) are written on top of this definition. There is also the road of switching to a different background that can be written by an explicit formula.

06

The way of matching, and the definition of compatibility in the gas

The way of matching. The sequence of bare couplings β_n is fixed by matching the expectation of a coarse observable Y_0 at step m_0 to a target value κ_0 (19-07 § 02). What is chosen is Y_0, m_0, κ_0, and how to pick a solution when there are several. In the language of the premises, one possible road is to make this correspond to the definition of the effective coupling u_k (definition and continuity of the effective coupling). There are two proposed definitions of u_k on paper (read as a renormalised coupling, or as a coefficient of the blocked action). Which is taken changes the meaning of the domain $u_m \geq u_\star$ of every row of the premise of the box continuum limit. The conditional implication that this book's study assembled earlier (28-01 § 08) defines the fine measure to be the Wilson measure at the β_n chosen by the matching bracket for the bare coupling, and the Lean theorem takes the sequence of bare couplings arbitrarily. The premise that ties the threshold on the measure side to the threshold on the coupling side (linking the thresholds) is also settled together with this choice.

The definition of compatibility in the gas. This is the definition of what it means, in the large-field gas, for two large-field regions to be "compatible". The conditional implication that this book's study assembled earlier (28-01 § 08) uses "not adjacent", the Lean form uses "not overlapping", and the two are different. What depends on it is the large-field contribution to the coupling and the large-field gas, both of which have forms not fixed.

07

Which theorems and premises depend on which choices

Table 2 Table of dependencies (● means depends). The top 5 rows are theorems, the rest premises (as of 2026-11-13)

Theorem or premise	Weights	Scale factor	Radius	Overlap	Norm	Gauge	Background	Matchin
Part 20 (reflection positivity, strong coupling)								
Part 21 (vortex floors, Polyakov loops)								
15-12 One step of the free field	●	●						
15-13 Linearisation and the covariance of the push-forward	●	●	●					
22-01 Rarity of large plaquettes								
Reflection positivity at the coarse level				●				
Definition and continuity of the effective coupling, net one-step bracket of the coupling		●						●
Large-field contribution to the coupling								
Recursion of the activity, rarity of supply				●				

Theorem or premise	Weights	Scale factor	Radius	Overlap	Norm	Gauge	Background	Matchin
Rarity of two-step events: plaquette part		●						
Rarity of two-step events: background-invariant part							●	
Free-field inputs (bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm))	●	●						
Positivity with background							●	
Constrained Decay Theorem	●				●	●		
Zero-order transport of the background	●	●			●		●	
Background field (one-step background-field lemma, multistep background field)					● (C2)		●	
Row sums of the lift uniform in the step	●				●	●		
Gauge fixing step by step						●		
Large-field estimate	●		●					
Large-field gas								
Induction uniform in the step					●			

Theorem or premise	Weights	Scale factor	Radius	Overlap	Norm	Gauge	Background	Matchin
Uniform decay of block observables (vessel)				●				
Band doubling (single-scale criterion, lateral-widening inequality (strong form), tension-doubling inequality)		●						

Of the top 5 rows of Table 2, only the two chapters on the free field and linearisation depend on the choices. On the side of the premises, rows that depend on the weights gather in the premise of ultraviolet stability, and rows that depend on the background, norm and gauge gather around the background field and the constrained estimates.

08

Other fixings

These are not choices in the block map or the representation, but fixings that determine the scope of theorems and premises.

Table 3 Other fixings

Fixing	Present value	What depends on it
Threshold for large plaquettes	The rarity theorem uses $\operatorname{Re} U_p < 0$. The premise side uses a threshold $y_h(m)$ for each step	22-01, rarity of supply, single-scale rarity on the fine lattice, rarity of two-step events: plaquette part
Range of β	Strong-coupling windows $18 \tanh \lvert\beta\rvert < 1$ and $24(1 - e^{-4\lvert\beta\rvert}) < 1$; rarity for $\beta \geq 81$	Part 20, 22-01, single-scale rarity on the fine lattice
Reflection plane and positive side	Plane of lattice sites (all real β) / plane through the middle of links ($\beta \geq 0$). At the coarse level the positive side is taken as a strip	20-01, reflection positivity of the fine lattice, reflection positivity at the coarse level
Extension of the background invariants	How values outside $SU(2)$ are set (whether to extend by values larger than the threshold)	The inclusion in the family of observables and small-field set, rarity of two-step events: background-invariant part, choice of the small-field set and extension of the background invariants
Class of observables	Flowed observables (no definition yet)	Approximation of flowed observables by block variables, the rotation premise

Fixing	Present value	What depends on it
Geometry of the twist	Background twist $n_{12} = 1$, transverse size N held fixed	Part 21, every row of the band

What this chapter can say

Can say	Cannot say
The present values of the nine choices and the premises that depend on them are tabulated	Which choice makes the premises easier to satisfy
Proved here The theorems of Parts 20 and 21 and 22-01 do not use the block map, so they do not depend on the choice of block map	—
The constants of the one-step free-field theorem in Part 15 were computed with weight 1/8 and scale factor 2	The constants for other weights (the table has not been checked and is not included)
There is an issue between the overlap of block averages and reflection positivity of the push-forward measure	The answer to that issue (the computation has not been checked and is not included)

Sources and reproduction

Item	Type	Source, tools
The list of choices and the dependencies	Assumption	The inventory of premises in this book's study (2026-11-13)
Definition of the block map and the weights	Lean	Source table of 19-04
Constants of the one-step free-field theorem	Lean	Source table of 15-12
The two definitions of compatibility in the gas	—	The conditional implication that this book's study assembled earlier (28-01 § 08) and this book's Lean bundle

Words used in this chapter — design choice, transverse weights, scale factor, pull-back radius, overlap of block averages, norm, gauge, background field, matching, compatibility in the gas. Their meanings are in A-02 Glossary.

Next chapter: 28-06 Open places, and places where the object does not yet exist — lists the premises whose form is not fixed, sorted by origin.

Open places, and places where the object does not yet exist

> This chapter is a map of open questions.

What this chapter shows — A list that divides into three the 24 rows labelled "Form not fixed" in 28-04, according to where the openness comes from. For each row we write whether what would fix its form is an argument on paper, a small numerical experiment, or a design choice. At the end we place the stages for which the objects needed to write the statements do not yet exist (fine observables, rotation), and what this book does not claim.

Prerequisite chapters — 28-04 The list of premises (row names and labels), 19-07 The statements to be shown.

Said first — This chapter is a list and answers none of the questions. The column "What would fix it" records the kind of question, and does not claim that the question will be settled soon. The content moves, so updates are made in A-08 A map to the mass gap.

The order of this chapter

1. Three origins — Table 1
2. Objects of the representation undefined — 13 rows on the ultraviolet side (Table 2)
3. No object at all, or downstream — 8 rows (Table 3)
4. Order of quantifiers and truth — 3 rows of the band (Table 4)
5. The order of the questions that would fix the forms
6. Stages whose objects are not yet defined
7. What this book does not claim

01

Three origins

There are 24 premises whose form is not fixed (28-04 Table 1). Looking only at the reason symbols ((a) too light, (b) too strong, (c) object undefined, (d) numerical signs against the present form, (e) no form across steps, (f) order of quantifiers not fixed) they look scattered, but divided by origin there are three groups.

Table 1 Origins of the 24 rows whose form is not fixed (as of 2026-11-13)

Origin	Rows	Which rows	Relation to the root
The object of the expression is undefined	13	Net one-step bracket of the coupling, large-field contribution to the coupling, recursion of the activity, rarity of supply, moving invariant interval and margin, rarity of two-step events: background-invariant part, positivity with background, zero-order transport of the background, small-field one-step renormalisation theorem, large-field estimate, large-field gas, induction uniform in the step, contraction of the one-step map and matching at the endpoint	The root (19-07) can be written. Below it, the form of the sufficient conditions passing through the effective coupling, the activity, the background field and the class is not fixed
No object at all, or downstream	8	Approximation of flowed observables by block variables, ultraviolet estimate uniform in the box size, one-step ultraviolet estimate for the bridge, independence of the infinite-volume limit from the box shape, the limit is not a free field, question of the cost of a twist-inserting map, gap of a small twisted box and uniform acceptance of insertion, band comparison (in the language of the gap)	Rows for which an object needed in the statement (flowed observables, the map from a rotated lattice, a map that inserts a twist, the vessel of a small twisted box, and so on) does not exist, or whose form is fixed after the upstream forms are settled
Order of quantifiers and truth	3	Tension-doubling inequality, monotonicity of the vortex free energy and renormalisation comparison, from the area law to the gap	The root for the band can be written. Candidate sufficient conditions have false examples, or the order of quantification is not fixed

The 13 rows of the representation sit at the level of the representation below the root. Changing the representation (a different definition of the background, a different norm, and so on) changes the leaves, but not the root. The openness on the side of the band comes not from the representation but from the quantification and truth of the statements.

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02
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Objects of the representation undefined — 13 rows on the ultraviolet side

Inside the chain theorem of the present road, the effective coupling u, activity z, supply w, remainders, plaquette variables, background invariants and observables are all free variables. Only the block map and the push-forward measure are defined from the lattice (19-04). So the following rows need definitions of their objects before any proof.

Table 2 The 13 rows at the level of the representation, and what would fix their form

Row	Reason	What would fix it	The smallest question
Net one-step bracket of the coupling	(c)(e)	Paper (design)	Define the effective coupling uniquely. If restated in the language of matching (19-07 § 02): which coarse observable to match at which step. Whether the domain $u_m \geq u_\star$ of the other rows can then be written in the same language
Large-field contribution to the coupling	(a)(c)	Small numerical experiment, and design	The present form holds if the activity is chosen freely (light). When a form uniform in volume and step is put into the statement, does that uniformity hold numerically on small lattices for the plain Wilson measure?
Recursion of the activity	(a)(c) (e)	Paper and small numerical experiment	For the measure $\nu_{n,n-1}$ pushed forward by one step only, does pulling back the event "a coarse plaquette is large" to an event on the fine lattice give a form to which the Chessboard Estimate (22-01) applies?
Rarity of supply	(c)(e)	Design	Can the supply be defined as the probability of an event under the push-forward measure (the fine-side image is small, the coarse plaquette is large)? If so, it merges with rarity of two-step events: plaquette part
Moving invariant interval and margin	(a)(c)	Paper, after the definitions of the large-field contribution to the coupling and of rarity of supply	The real functions W, Φ, D, e are free, and it holds if they are all taken to be 0. Do the actual gas coefficients take this form, with the margin numbers holding?
Rarity of two-step events: background-invariant part	(c)	Design (definition of the background)	Whether to make it an event in plaquette variables alone, without background invariants, or to switch to a background that can be written by an explicit formula
Positivity with background	(c)	Design (definition of the background), and paper	Fix the definitions of the background field and the constrained Hessian, and decide which of the three forms standing side by side to take as the premise

Row	Reason	What would fix it	The smallest question
Zero-order transport of the background	(b)(d)	Paper (design)	The present form (measured by the sup over plaquettes) diverges numerically for the background that minimises the action Reading (the argument on paper that it is false has not yet been checked). If the zeroth-order term is written in the action norm, does it close under the same assumptions as the multistep background field?
Small-field one-step renormalisation theorem	(c)(e)	Paper	Is the background field the only undefined object needed to write down the step-0 statement whose input is the Wilson action itself?
Large-field estimate	(c)(d) (e)	Small numerical experiment	Does the lower bound over steps of the action of each hole stay at a positive value proportional to the number of plaquettes as the steps go deeper?
Large-field gas	(a)(b) (e)	Small numerical experiment, and design	The literal form always holds on a fixed lattice (light), while the form with equality on all regions does not hold numerically on small lattices Reading (strong). Fix a form in between that includes uniformity. The choice of the definition of compatibility in the gas (not adjacent / not overlapping) also remains (28-05)
Induction uniform in the step	(c)(e)	Design (the norm of the induction), and paper	When the induction coordinate r_k is defined by coordinates of the class, does the free-field contraction (contraction of irrelevant terms) become a coefficient of the recursion for r_k?
Contraction of the one-step map and matching at the endpoint	(c)	Paper, after the forms of the small-field one-step renormalisation theorem and of induction uniform in the step are settled	In the language of the coordinates of induction uniform in the step, does the contraction condition become a statement in the same coordinates?

Of the 13 rows, 5 (large-field contribution to the coupling, recursion of the activity, rarity of supply, large-field estimate, large-field gas) gather around the treatment of large fields (activities and the gas). Their content (the form of the estimates and the premises on the gas) will not be given chapters in this book until the forms of the statements are settled.

> 03

No object at all, or downstream — 8 rows

Table 3 Rows with no object, or whose form is fixed after the upstream is settled

Row	Reason	What would fix it
Approximation of flowed observables by block variables	(c)	Definitions of the lattice gradient flow and of flowed observables (§ 06)
Ultraviolet estimate uniform in the box size	(e)	Fixed as the uniform version of the premise of the box continuum limit and the premise of ultraviolet stability, after their forms are settled
One-step ultraviolet estimate for the bridge	(c)	Definition of the block map from a rotated fine lattice to the standard coarse lattice (only a sketch on paper)
Independence of the infinite-volume limit from the box shape	(c)	After the definition of the infinite-volume limit (the limit of large boxes)
The limit is not a free field	(c)	The choice of the quantity by which to say it (for example, the limiting value of the same observable at another step m_1, when matched at step m_0)
Question of the cost of a twist-inserting map	(c)	Definition of the map T (the method is not fixed). This row is a sufficient condition for the single-scale criterion, so there is also the road of checking the single-scale criterion directly
Gap of a small twisted box and uniform acceptance of insertion	(c)	Definitions of the operator that measures the gap of small twisted boxes, and of the map for driven vortex insertion
Band comparison (in the language of the gap)	(c)	Definitions of the vessel for boxes with spatial planes, and of the box mass $m_0(L)$

04

Order of quantifiers and truth — 3 rows of the band

Table 4 Rows on the band side whose form is not fixed

Row	Reason	What would fix it
Tension-doubling inequality	(b)(f). (d) only for the form uniform in β	Small numerical experiment. With the transverse size N held fixed and $\beta \to \infty$, there are numbers in which the minimal action of a vortex (21-05) increases with N, a sign pointing against the form uniform in β Reading . Either state it on a sequence at fixed physical size ($N = L_*/a$, $\beta = \beta(a)$), or fix the order of quantifiers

Row	Reason	What would fix it
Monotonicity of the vortex free energy and renormalisation comparison	(b)	On lattices with finite non-abelian groups there are examples where monotonicity fails Reading . Decide what to place as an assumption for $SU(2)$ (the form in 26-06)
From the area law to the gap	(b)	Paper (choice of road). The naive form is false in two dimensions. Whether to fix the last arrow on the side of the mass gap as the single route from band comparison (in the language of the gap) and stability above the threshold, via a uniform lower bound on the box mass, to uniform decay

05

The order of the questions that would fix the forms

Putting first what is settled cheaply (on paper or with small numerics), and, at equal cost, what has more downstream, gives the following order. Items 1–3 are independent of each other.

1. Fix the matching observable Y_0 and step m_0, and define the effective coupling uniquely (definition and continuity of the effective coupling, net one-step bracket of the coupling; paper).
2. Put numbers on the plaquette part of the rarity of two-step events (rarity of two-step events: plaquette part) under the push-forward measure (small numerical experiment). It is a premise written with defined objects only, yet there is no check on small examples.
3. Decide with which background and which norm to state zero-order transport of the background (paper). The definition of the background for rarity of two-step events: background-invariant part is settled at the same time.
4. Whether the recursion of the activity can be stated for the measure pushed forward by one step only (paper and small numerics; after 2).
5. Run the Constrained Decay Theorem through in one dimension from input to output, and check whether the assumptions can be satisfied simultaneously (Lean).
6. Look at the uniformity of the large-field gas in numbers (small numerical experiment).
7. Look at how the large-field estimate depends on the depth of steps (numerics).
8. Run the quadratic part of the induction uniform in the step through in one line (paper; after 1).
9. Look at the shape of the region where the tension-doubling inequality can hold (numerics).

10. Fix the last arrow on the side of the mass gap as a single route (from the area law to the gap; paper).

06

Stages whose objects are not yet defined

For two of the requirements of the problem statement, objects to write the statements about are needed before the forms of the premises can be discussed (19-07 § 08).

Stage	Statement we want to write	Missing object
Fine observables	Convergence in the continuum limit of expectations of point fields and flowed observables	The lattice gradient flow and the definition of flowed observables (not in this book's Lean). The approximation row (approximation of flowed observables by block variables) also has its form not fixed because this definition is missing
Rotation	Agreement of expectations of rotated observables (invariance under the Euclidean group)	Observables on which rotations finer than the hypercubic group act (the fine-observables stage), or one step from a rotated lattice to the standard lattice (the object of the one-step ultraviolet estimate for the bridge)
Infinite volume	Convergence, on windows, of push-forward measures as the box grows	Can be written by adding one definition (restriction to a window in box coordinates, or periodic extension). In the strong-coupling window it can be written by periodic lifting and is a theorem (20-03)
Non-triviality	That the limit differs from the limit of the free field	Can be written, but by which observable to say it is not fixed (the limit is not a free field)

07

What this book does not claim

- **The mass gap itself.** This book shows nothing about the existence of a mass gap for four-dimensional $SU(2)$ Yang–Mills theory (or for any compact simple group). The lower bound on the mass in the strong-coupling window (20-04) is a theorem at one scale with the lattice spacing fixed.
- **Construction of the continuum limit.** This book does not construct the limiting theory as the lattice spacing goes to 0. The existence of four-dimensional Yang–Mills theory was shown, with a different design, by the construction published on 2026-11-10, and the committee's assessment is that it is proved in Lean (Volume 2). The conclusion of the conditional implication that this book's study assembled earlier (weak ultraviolet existence in small boxes in this book's design) goes only as far as

subsequential limits under its premises and the passage of properties of finite lattices to the limit, and many of those premises have forms not fixed (28-01).

- **The band.** This book shows nothing about the statement that spans the strong-coupling side and the weak-coupling side with one constant (19-07 § 07). The vortex floors (21-02, 21-03) are in a form with the lattice spacing and transverse size fixed. The upper bound on the vortex free energy inside the strong-coupling window (20-06) has a rate that does not depend on the transverse size (the prefactor remains), but it does not extend outside the window.
- **Confinement.** Neither a four-dimensional area law outside the strong-coupling window nor a re-derivation of the passage from the vortex area exponent to the area law of Wilson loops (from the vortex area exponent to the Wilson-loop area law) is in this book. The lattice area law inside the strong-coupling window is in 20-06 (a formalisation of a known type; continuum Wilson loops are in neither this book nor the published construction).
- **Non-triviality and rotation invariance.** For these two items of the problem statement, this book has not even fixed the forms of the statements (for rotation, the object does not yet exist; 19-07 § 08).
- **The position of the theorem in the strong-coupling window.** The lower bound on the mass in the strong-coupling window is an example on the side where the statements to be shown stand trivially, not a check of the content of the statements (19-07 § 01).

An index of "what is not claimed" for the earlier parts is in 27-04.

What this chapter can say

Can say	Cannot say
The 24 rows whose form is not fixed are divided into three by origin (representation level 13; no object or downstream 8; quantification and truth 3)	The form of any row is not fixed here
For each row, whether paper, numerics or design would fix its form, and the smallest question, are written down	Answers to the questions. The order of the questions is by cost and amount downstream, not by the prospect of an answer
For the stages of fine observables and rotation, the objects to write the statements about do not yet exist	Definitions of those objects
This book shows nothing about the mass gap itself, the construction of the continuum limit, or the band	—

Sources and reproduction

Item	Type	Source, tools
The division of the 24 rows, the smallest questions, the order of the questions	Assumption	The inventory of premises in this book's study (2026-11-13)
Examples with finite non-abelian groups where monotonicity fails	Reading	Numerical record of this book's study (direct sums on lattices with finite groups). The description in 26-06
That the naive form of "area law to gap" is false in two dimensions	Proved here	The counterexample of 28-03 § 07 (the direct product of a two-dimensional gauge theory with an area law and the critical two-dimensional Ising model; that the gap closes at the critical point is in 2-09 § 07)
Index of what is not claimed	—	27-04

Words used in this chapter — form not fixed, representation, effective coupling, activity, gas, background field, class, order of quantifiers, gradient flow, flowed observable, what is not claimed. Their meanings are in A-02 Glossary.

Next chapter to read: A-01 Table of notation. To read alongside: A-08 A map to the mass gap — updates to this list and progress on the questions are followed there.

Table of notation — one convention for each letter, two meanings separated by name

> This chapter is a guide to notation and terms.

What this chapter shows — The symbols of this book can be looked up by "meaning, chapter of definition, note". The three ways of writing the gauge field, the six ways of writing the coupling, and the normalisation of CP are set out with conversion formulas, tables and moving figures. Different meanings of the same letter (β, Δ, η, σ, Ω and others) are told apart in one table.

Prerequisite chapters — None. As a dictionary, it can be opened from any chapter. The "Chapter" column of each row is the chapter that defined the symbol.

Said first — This chapter is a list of conventions and contains no claims of content. The correctness of formulas is shown in each chapter; here only the arithmetic of conversions and the normalisation of generators were checked on this machine. The names of assumptions are kept separately in A-06 Index of names. This chapter shows nothing about the mass gap.

The order of this chapter

1. How to use this table — the common conventions, and three choices of convention
2. Spacetime and indices — the signature $(-,+,+,+)$ and imaginary time (Table 1)
3. Groups and Lie algebras — normalisation of generators and structure constants (Tables 2 and 3)
4. Three ways of writing the gauge field — where g and i go (Table 4). Detour: why physics likes to attach i
5. Lattice symbols — links, plaquettes, link angles (Table 5)
6. Six ways of writing the coupling — conversions and one-loop coefficients (Table 6, Figure 1, Table 7)
7. Scales, lengths, masses — lattice units and physical units, the symbol $\asymp$ (Table 8)

8. Probability, spectrum, relaxation — three CP and two η (Figure 2, Tables 9 and 10)
9. Different meanings of the same letter — a list for telling them apart (Table 11). Detour: can symbol collisions be eliminated?

01

How to use this table — one convention for each letter

This chapter is a dictionary to look things up in; it need not be read from the front. The tables have four columns, "symbol, meaning, chapter, note", and "chapter" is the chapter that defined the symbol. Check the correctness of formulas in that chapter.

There are four conventions common to all chapters. From Part 3 on, $\hbar = c = 1$. An index appearing twice in the same term is summed. Both $\ln$ and $\log$ are the natural logarithm. When the same letter has two meanings, this is declared at the start of the chapter (the list is in § 09). Beyond that, the appearance of the symbols changes with the following three choices.

$$\underbrace{\text{way of writing the gauge field}}_{\S 04\ \text{three ways}} \times \underbrace{\text{way of writing the coupling}}_{\S 06\ \text{six ways}} \times \underbrace{CP\ \text{normalisation}}_{\S 08\ \text{two ways}} \tag{1}$$

The choices in (1) do not change the physical content, only coefficients and signs. But the same condition looks like a different number. Example. The strong-coupling condition of Shen–Zhu–Zhu, $|\beta_{\mathrm{SZZ}}| < 1/(16(d-1))$, becomes $\beta_{\mathrm{SZZ}} < 1/48$ in four dimensions ($d = 4$), and in this book's Wilson β it is $\beta < 0.0833$ for SU(2) and $\beta < 0.1875$ for SU(3) (§ 06) Computed . The names of assumptions (the diffusive-relaxation-time assumption, the relaxation-time upper-bound assumption, the eight statements of construction and others) are in A-06 Index of names, and the legend of the labels is in 0-01.

02

Spacetime and indices — the signature is $(-,+,+,+)$, and in imaginary time everything becomes $+$

Physical spacetime is Minkowski spacetime, where only the time direction carries a negative sign in the squared distance (2-04, and the existing article One sign is different). Constructive field theory and lattice arguments work in Euclidean space $\mathbb{R}^4$, with time rotated to imaginary values (4-02).

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1), \qquad t = -i\tau, \qquad A_0 = iA_4, \qquad F_{\mu\nu}F^{\mu\nu} \longmapsto \sum_{\mu,\nu=1}^{4} F_{\mu\nu}F_{\mu\nu} \tag{2}$$

(2) gives, from the left, the metric, the rotation (Wick rotation), and the result of rotating. In Euclidean signature the metric is the identity matrix, so upper and lower indices are not distinguished. Example. The light path $-t^2 + |\mathbf{x}|^2 = 0$ becomes $\tau^2 + |\mathbf{x}|^2$ when rotated, positive except at the origin.

Table 1 Spacetime and indices

Symbol	Meaning	Chapter	Note
$x^\mu,\ \mu = 0,\dots,3$	Minkowski coordinates ($x^0 = t$)	2-04	Some textbooks use the opposite signature $(+,-,-,-)$
$x_4 = \tau,\ \mu = 1,\dots,4$	Imaginary time and Euclidean indices	4-02	$\tau = it$
$i,j,k = 1,2,3$	Spatial indices	3-02	Do not mix with the colour indices a,b,c
$E_i,\ B_i$	Colour electric and magnetic fields	3-02	$B_i = \tfrac{1}{2}\varepsilon_{ijk}F_{jk}$; in Euclidean signature $E_{E,j} = F_{4j}$
d	Dimension of (Euclideanised) spacetime	0-03	The problem statement has $d = 4$
$\mathbb{Z}^d,\ a\mathbb{Z}^d$	The integer lattice, the lattice with spacing a	5-01	§ 05

03

Groups and Lie algebras — generators are normalised by $\mathrm{tr}(T^aT^b) = \tfrac{1}{2}\delta^{ab}$

The gauge group G is a compact simple Lie group, and in almost all examples it is $\mathrm{SU}(N)$ (1-04). For generators we take Hermitian T^a in the physics style. Unless the normalisation is fixed, both the structure constants and the coupling constant are off by constant factors.

$$[T^a, T^b] = i\,f^{abc}\,T^c, \qquad \mathrm{tr}(T^aT^b) = \tfrac{1}{2}\,\delta^{ab}, \qquad a,b,c = 1,\dots,N^2-1 \tag{3}$$

In the mathematics style one takes anti-Hermitian $X^a = -iT^a$, giving the form $[X^a, X^b] = f^{abc}X^c$ without i. It is the same f^{abc}. Example. For SU(2), $T^a = \sigma^a/2$ and $f^{abc} = \varepsilon^{abc}$; for SU(3), $T^a = \lambda^a/2$ (half the Gell-Mann matrices). The adjoint Casimir C_A ($f^{acd}f^{bcd} = C_A\delta^{ab}$) is N, the fundamental Casimir C_F ($\sum_a T^aT^a = C_F\mathbf{1}$) is $(N^2-1)/2N$, and the index of the representation $T(R)$ ($\mathrm{tr}_R\,T^aT^b = T(R)\delta^{ab}$) is $\tfrac{1}{2}$ for the

fundamental representation. What enters the one-loop coefficient (4-06) is C_A and $T(R)$.

Table 2 Numerical check of the normalisation of generators Computed

Group	$\max \lvert \operatorname{tr} T^a T^b - \frac{1}{2}\delta^{ab} \rvert$	C_F	C_A	Independent non-zero f^{abc} ($a<b<c$)
SU(2)	0	0.75	2	$f^{123} = 1$
SU(3)	1.1×10^{-16}	1.333333	3	$f^{123}=1$; $f^{147}=f^{246}=f^{257}=f^{345}=\frac{1}{2}$; $f^{156}=f^{367}=-\frac{1}{2}$; $f^{458}=f^{678}=\frac{\sqrt{3}}{2}$

Table 2 was obtained by writing out the matrices, computing $f^{abc} = -2i \operatorname{tr}([T^a, T^b]T^c)$, and checking the residual on substituting back into (3) (at most 1.1×10^{-16}) and that the Casimir is a constant multiple of the identity matrix.

Table 3 Symbols for groups and Lie algebras

Symbol	Meaning	Chapter	Note
G, $\mathfrak{g}$, N	Gauge group, Lie algebra, the size of $\mathrm{SU}(N)$	1-03	Large N is in 17-04
T^a, f^{abc}	Hermitian generators, structure constants	1-04, 3-02	Normalisation is (3)
σ^a, λ^a	Pauli matrices, Gell-Mann matrices	1-04	σ is also used in other meanings (§ 09)
C_A, C_F, $T(R)$	Casimirs and the index of a representation	4-06	Values in the text
χ_r, d_r	Character and dimension of the representation r	5-02	$\chi_r(U) = \operatorname{tr}_r U$
$\Omega(x)$	Gauge transformation (a group element at each point)	3-01	The letter g is kept for the coupling

04

Three ways of writing the gauge field — where to put g, and where to attach i

The same gauge field is written in three ways. The physics chapters "separate Hermitian A from the coupling g", the path-integral and dictionary chapters "absorb g into the field and put $1/g^2$ in front of the action", and the geometry chapters use "an anti-Hermitian 1-form $\mathcal{A}$". The translation is one line.

$$\hat{A} = g\,A, \qquad \mathcal{A} = -i\hat{A} = -ig\,A, \qquad \hat{F} = g\,F, \qquad \mathcal{F} = -i\hat{F} = -ig\,F \tag{4}$$

"Absorbing" is just multiplying by g, and "geometry" is just multiplying further by $-i$. But $-i$ changes the sign of $\operatorname{tr} F^2$. For Hermitian X, $\operatorname{tr} X^2 \geq 0$, while for anti-Hermitian $-iX$ it is ≤ 0, so the geometric action carries a minus sign.

Table 4 Three ways of writing the gauge field (Euclidean action)

	Physics (g placed in D)	**Physics (g absorbed)**	**Geometry (anti-Hermitian)**
Covariant derivative	$\partial_\mu - igA_\mu$	$\partial_\mu - i\hat{A}_\mu$	$d + \mathcal{A}$
Field strength	$\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$	$\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]$	$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$
Gauge transformation	$\Omega A \Omega^{-1} - \frac{i}{g}(\partial\Omega)\Omega^{-1}$	$\Omega \hat{A} \Omega^{-1} - i(\partial\Omega)\Omega^{-1}$	$\Omega \mathcal{A} \Omega^{-1} - (d\Omega)\Omega^{-1}$
Action S_E	$\frac{1}{2}\int \operatorname{tr} F_{\mu\nu}F_{\mu\nu} = \frac{1}{4}\int F^a_{\mu\nu}F^a_{\mu\nu}$	$\frac{1}{2g^2}\int \operatorname{tr} \hat{F}_{\mu\nu}\hat{F}_{\mu\nu}$	$-\frac{1}{g^2}\int \operatorname{tr} \mathcal{F} \wedge *\mathcal{F}$
Chapters that use it	3-02, 3-03, Part 4 (except 4-02; 4-03 uses both)	0-01, 4-02, 18-04	1-07, 3-04, 3-05

The topological charge Q is related to the second Chern number $c_2 = \frac{1}{8\pi^2}\int \operatorname{tr} \mathcal{F} \wedge \mathcal{F}$ by $Q = -c_2$ ((2) of 3-05; the difference in sign is $(-i)^2 = -1$, coming from $\mathcal{F} = -igF$), and the action has the lower bound $S_E \geq 8\pi^2|Q|/g^2$ Known . Example. For SU(2) with $g^2 = 1$ ($\beta = 4$), the action of one instanton is $8\pi^2 = 78.96$, and the weight e^{-S_E} is about 5×10^{-35} Computed .

Detour: why physics likes to attach i

You may skip this. In quantum mechanics observables are Hermitian operators, with real eigenvalues (2-05). Colour charge is an observable, so physics takes Hermitian generators and puts i in the exponent, $e^{i\theta^a T^a}$. On the geometric side, the Lie algebra is the space of tangent vectors at the identity of the group, and $\mathfrak{su}(N)$ is just the anti-Hermitian matrices. One can write e^X, and neither i nor g appears in the curvature. Precisely, the two styles differ only by changing the basis by a factor i, and represent the same connection on the same group. The differences are only the sign of $\operatorname{tr} F^2$ and the i in front of f^{abc}.

05

Lattice symbols — links, plaquettes, Wilson loops

Lattice gauge theory replaces spacetime by a lattice of spacing a and places a group element on each edge (link) (5-01). The key symbols are the Wilson action and its coefficient β.

$$S_W(U) = \beta \sum_P \Big(1 - \frac{1}{N}\operatorname{Re}\operatorname{tr} U_P\Big), \qquad \beta = \frac{2N}{g^2}, \qquad U_P = U_{x,\mu}U_{x+a\hat{\mu},\nu}U^{-1}_{x+a\hat{\nu},\mu}U^{-1}_{x,\nu} \tag{5}$$

(5) sums the deviations from the identity of the product U_P around every small square (plaquette P). $\beta = 2N/g^2$ is the value for which it matches the continuum action, coefficient included, for smooth fields as $a \to 0$ (5-01 § 05) Known . Unless stated

otherwise, this book's β is this Wilson β. $\beta = 4/g^2$ in the SU(2) chapters (23-01, 23-02, 18-05) is the same formula with $N = 2$. A link sees the field only in the form $U_{x,\mu} \approx e^{-iga A_\mu}$. The parallel transport of 1-07 is its inverse, which is why the sign of the exponent of the plaquette is opposite to 1-07 (the trace does not change).

SU(2) links are also written with angles. The study in Part 23 writes $U = e^{i\vartheta\cdot\sigma}$, while the book's (3) gives $U = e^{i\theta^a T^a}$, so they are related by $\theta = 2\vartheta$.

$$U = e^{i\vartheta\cdot\sigma} = e^{i\theta^a T^a} \quad (\theta^a = 2\vartheta^a), \qquad \tfrac{1}{2}\operatorname{tr} U = \cos|\vartheta| \tag{6}$$

Example. For $\vartheta = (0.3, -0.2, 0.5)$, $\frac{1}{2}\operatorname{tr} U$ computed with the matrix exponential and $\cos|\vartheta|$ are both 0.815941 Computed. The difference in how the angle is taken acts as a constant factor on C_P in § 08.

Table 5 Lattice symbols

Symbol	Meaning	Chapter	Note
$a,\ L$	Lattice spacing, number of points on a side of the periodic lattice	5-01	$4L^4$ links, $6L^4$ plaquettes
$U_\ell = U_{x,\mu}$	Group element on the link ℓ	5-01	$U_{x+a\hat{\mu},-\mu} = U_{x,\mu}^{-1}$
$P,\ U_P$	Plaquette and the product around it	5-01	$1 - \frac{1}{N}\operatorname{Re}\operatorname{tr} U_P \in [0, 2]$
μ_β	Wilson measure $Z^{-1} e^{-S_W} \prod dU_\ell$	5-01	dU is the Haar measure
$W(C),\ W_{R\times T}$	Wilson loop $\frac{1}{N}\operatorname{tr} U_C$, rectangle	5-02	T is the length in time
σ	String tension: $\langle W_{R\times T}\rangle \approx e^{-\sigma RT}$	5-02, 5-08	Lattice units. In physical units $\sqrt{\sigma}$
$\vartheta,\ \theta$	SU(2) link angle, $\theta = 2\vartheta$	(6), 23-13	Different from θ of the θ term
$b,\ k,\ \delta(b)$	Factor per step, number of steps, the shortfall of the bookkeeping	23-02	Definitions of the study in Part 23

06

Six ways of writing the coupling — β, β_{Ch}, β_{SZZ}, λ, g^2, α

The same coupling g^2 is written with different letters and different normalisations in different literature. Comparing term by term with the action (5) fixes the conversions (repeated from (8a) of 5-01).

$$\beta = \frac{2N}{g^2}, \qquad \beta_{\mathrm{Ch}} = \frac{\beta}{N}, \qquad \beta_{\mathrm{SZZ}} = \frac{\beta}{N^2}, \qquad \lambda = g^2 N = \frac{2N^2}{\beta}, \qquad \alpha = \frac{g^2}{4\pi} \tag{7}$$

β_{Ch} is Chatterjee's form, writing the action as $\beta_{\mathrm{Ch}} \sum_P \mathrm{Re}\,\mathrm{tr}(I - U_P)$; β_{SZZ} is 't Hooft's convention, writing the inverse coupling as $N\beta_{\mathrm{SZZ}}$ (Shen–Zhu–Zhu, 5-04); λ is the 't Hooft coupling; α is what physics calls the strength of the coupling. The running coefficients also have a way of being written for each variable (Table 1 of 4-06); in this book b_0 is the coefficient for g, and κ that for the Wilson β.

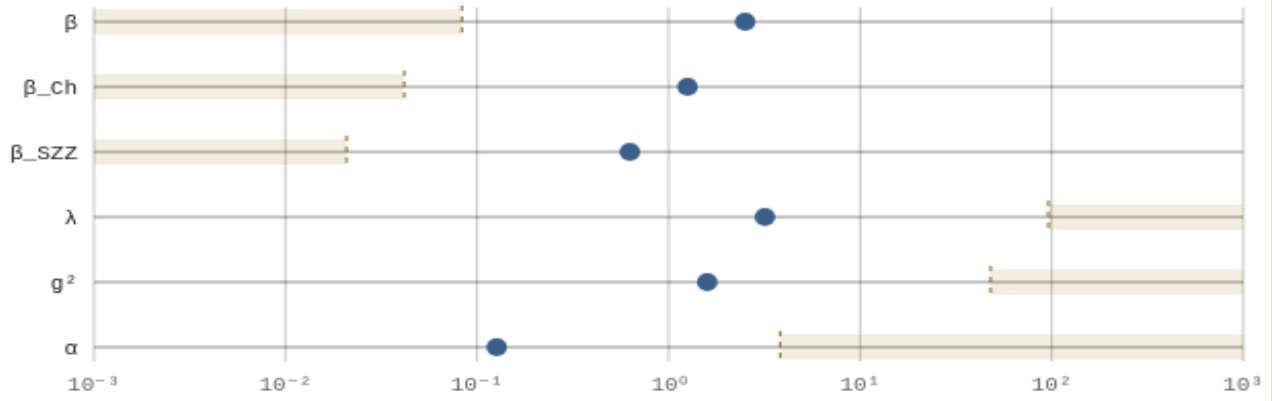
$$\mu \frac{dg}{d\mu} = -b_0 g^3 + \cdots, \qquad b_0 = \frac{11N}{48\pi^2}, \qquad \frac{d\beta}{d\ln\mu} = \kappa = 4N b_0 = \frac{11N^2}{12\pi^2} \quad (8)$$

The right of (8) is the one-loop prediction that the same physics is kept if β is raised by $\kappa \ln 2$ each time a is halved Known . $\kappa = 11/(3\pi^2)$ for SU(2) is the same number as "the slope of the step formula" of the study in Part 23, and also the same number as the value at SU(2) (C_g = 8) of the one-loop slope s = 11C_g/(24π²) of Volume 2 (10-01). Each of them is the known coefficient written in the lattice convention (23-14).

Table 6 Constants for each group (pure SU(N)) Computed

N	dim G	b_0	$2b_0$	κ	$\kappa \ln 2$	$b_0^{(\alpha)} = 11N/12\pi$ (μ² dα/dμ² = −b₀^(α) α², Table 1
2	3	0.04644	0.09288	0.3715	0.2575	
3	8	0.06966	0.13932	0.8359	0.5794	

Figure 1 Coupling converter. Moving the Wilson β with the knob moves the values of the six ways of writing (●) together on logarithmic scales. The vertical dashed line is the edge of the Shen–Zhu–Zhu condition in four dimensions ($d = 4$), $\beta_{\mathrm{SZZ}} = 1/(16(d-1)) = 1/48$, and the coloured band is the side where the condition holds. Switching SU(2) / SU(3) moves the position of the edge.



SU(2), β = 2.512: β_Ch = 1.256, β_SZZ = 0.6280, λ = 3.185, g² = 1.592, α = 0.1267. Outside the condition (edge at β = 0.08333). One-loop increment for halving a: κ ln2 = 0.2575

Computed The numbers in this figure were computed on this machine. The figure computes (7) on the spot (the same formulas as Table 7).

Table 7 Numerical examples of the conversion, and the edge of the strong-coupling condition in four dimensions ($d = 4$) Computed

Case	β	β_{Ch}	β_{SZZ}	λ	g^2	α

Case	β	β_{Ch}	β_{SZZ}	λ	g^2	α
SU(2), $\beta = 2.5$	2.5	1.25	0.625	3.2	1.6	0.12732
SU(3), $\beta = 6.0$	6	2	0.66667	3	1	0.07958
SU(2), edge of the condition	0.08333	0.04167	0.02083	96	48	3.81972
SU(3), edge of the condition	0.1875	0.0625	0.02083	96	32	2.54648

In 't Hooft's convention, β_{SZZ} and λ, the edge of the condition does not depend on N. The SU(2) $\beta = 2.5$ often used in numerical computation and the edge of the condition are about 30 times apart in α. The conditions are to be read in the normalisation of each paper; the conversion here is this book's arithmetic.

07

Scales, lengths, masses — separating lattice units from physical units

The symbols for length and mass come double, in lattice units (numbers with the lattice spacing as 1) and in physical units. If they are confused, "there is a gap on the lattice" and "there is a gap in the continuum" look like the same sentence (4-11).

$$m_{\mathrm{lat}} = \frac{1}{\xi}, \qquad \Delta = m_{\mathrm{phys}} = \frac{m_{\mathrm{lat}}}{a}, \qquad \xi_{\mathrm{phys}} = a\,\xi \tag{9}$$

ξ is the correlation length in lattice units, fixed by the rate of exponential decay of two-point correlations ((6) of 2-08). In the continuum limit, $a \to 0$ and $\xi \to \infty$ are taken together so that the product $a\xi$ stays constant. Example. If $\xi = 20$ and $a = 0.05$, then $\xi_{\mathrm{phys}} = 1$. Halving a needs $\xi = 40$, which at one loop corresponds to raising β by $\kappa \ln 2$. That two quantities are the same up to constant factors is written with the following symbol.

$$A \asymp B \quad \Longleftrightarrow \quad \exists\, 0 < c \le C : \;\; c \le A/B \le C \quad \text{(throughout the range considered)} \tag{10}$$

(10) says that the ratio goes neither to 0 nor to infinity; it is stronger than $\ln A / \ln B \to 1$, which compares only power exponents, and tells apart differences in logarithmic factors (Figure 2 of § 08).

Table 8 Symbols for scales, lengths and masses

Symbol	Meaning	Chapter	Note
H, Ω, $\mathcal{H}$	Hamiltonian, vacuum, Hilbert space	4-11	$H\Omega = 0$
Δ, m	Δ: the constant in the problem statement that bounds the energy of excitations from below. m: the largest value that can be taken as Δ (the lower end of the spectrum)	0-03, 4-11	$m = \inf \sigma(H\|_{\Omega^\perp})$. There is a gap $\Longleftrightarrow m > 0$

Symbol	Meaning	Chapter	Note
ξ, m_{lat}, m_{phys}	Correlation length (lattice units), $1/\xi$, m_{lat}/a	2-08, 18-03	(9)
μ	Renormalisation scale (momentum)	4-06	Same letter as indices and measures (§ 09)
Λ	The scale of dimensional transmutation ($\mu e^{-1/(2b_0 g^2)}$ at one loop)	4-07	Its value depends on the scheme. The ratio $m/\sqrt{\sigma}$ does not
ξ_{RG}, $a := 1/\xi_{\mathrm{RG}}$	The bookkeeping length $b^{k(\beta)}$, and the lattice spacing fixed by it	23-01	Definition of the study in Part 23. Agrees with a of Part 5 under the bookkeeping-length chain

08

Probability, spectrum, relaxation — three C_P and two η

In the second half of the book, the Poincaré constant C_P, which expresses how fast fluctuations are smoothed out, comes to the centre (1-10). It is the supremum of the ratio of the variance to the mean square of the gradient.

$$C_P = \sup_f \frac{\mathrm{Var}_\mu(f)}{\mathcal{E}(f)}, \qquad \mathcal{E}(f) = \mathbb{E}_\mu\Big[\sum_\ell |\nabla_\ell f|^2\Big], \qquad \lambda_1 = \frac{1}{C_P} \tag{11}$$

λ_1 is the gap of the Langevin dynamics that preserves μ. The same letter C_P is used for three quantities. The vacuum C_P is C_P of the vacuum measure at one time ($\Delta = 1/(2C_P^{(\alpha)})$; convention with kinetic term $\frac{1}{2}p^2$, exact for finitely many degrees of freedom; 2-05 § 08, A-06 Table 4); the path C_P is C_P of the path measure (Wilson measure) (C_P of the record of this book's study); and the path C_P measured in a graduation names the normalisation, according to whether the gradient of the path C_P is measured in the link angle ϑ or in the field $A = \sqrt{\beta}\,\vartheta$ (23-01 The line of construction and the diffusive-relaxation-time assumption).

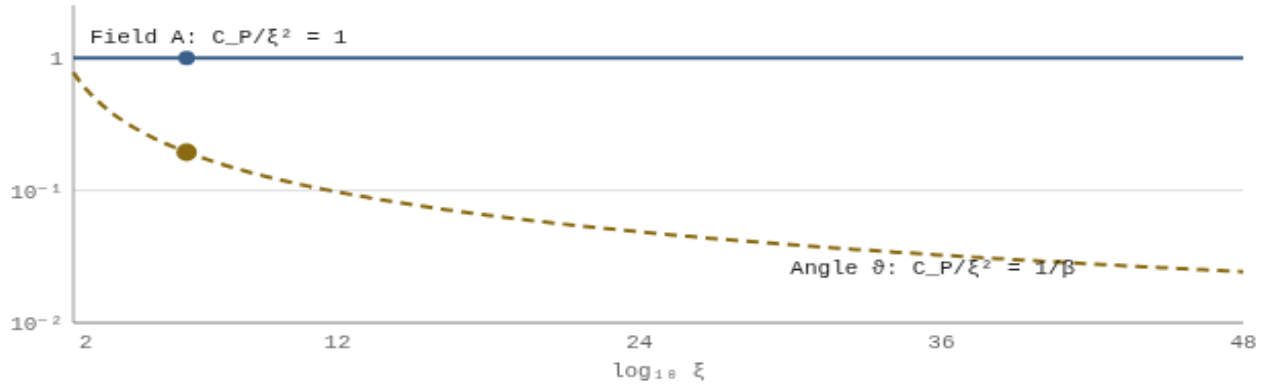
$$\mathcal{E}_A = \frac{\mathcal{E}_\vartheta}{\beta}, \qquad C_P[\mathcal{E}_A] = \beta\, C_P[\mathcal{E}_\vartheta], \qquad \text{in the free approximation } C_P^{(A)} = \xi^2,\ C_P^{(\vartheta)} = \frac{\xi^2}{\beta}$$

The free approximation is the approximation that regards the action as a Gaussian free field keeping only the quadratic terms, and gives $C_P^{(A)} = 1/m^2 = \xi^2$ (m in lattice units). A in (12) is A of § 04 measured in lattice units ($\sqrt{\beta}\,\vartheta^a = -aA^a$; the same field up to sign).

(12) shows that writing "$C_P \asymp \xi^2$" without naming the normalisation can change whether the sentence is true. A sentence comparing power exponents (the relaxation-

time assumption) moves only by the logarithm of β, and is the same sentence in both normalisations. A band statement, that the ratio stays within constant bounds, changes truth value with the normalisation. That is why this book writes band statements with the normalisation, as $(\xi^2)_{\mathrm{band}}[\mathcal{E}_\vartheta]$ (23-01 The line of construction and the diffusive-relaxation-time assumption).

Figure 2 Why the normalisation is named (SU(2), free approximation). Horizontal axis $\log_{10}\xi$, vertical axis C_P/ξ^2 (logarithmic). The solid line is the field normalisation 1, the dashed line the angle normalisation $1/\beta$. β is linked by replacing ξ with the one-loop formula for the bookkeeping length, $\xi \approx e^{3\pi^2\beta/11}$ (up to a constant factor) (asymptotic scaling assumed Physics). Moving ξ with the knob shows the local power exponent $\ln C_P/\ln\xi$ in the readout.



ξ = 10^6.0: β = 5.133, C_P^(ϑ)/ξ² = 0.1948, power exponent 1.8816 (angle) / 2 (field)

Computed The numbers in this figure were computed on this machine. The figure computes (12) and the one-loop formula on the spot (the same formulas as Table 9).

Table 9 C_P/ξ^2 in the angle normalisation and the local power exponent (SU(2), free approximation, replacing ξ by the one-loop formula for the bookkeeping length, $\beta = \frac{11}{3\pi^2}\ln\xi$. Asymptotic scaling assumed Physics)

Computed

ξ	β	$C_P^{(\vartheta)}/\xi^2$	$\ln C_P^{(\vartheta)}/\ln\xi$	$\ln C_P^{(A)}/\ln\xi$
10^2	1.7109	0.5845	1.8834	2
10^6	5.1326	0.1948	1.8816	2
10^{12}	10.2652	0.0974	1.9157	2
10^{48}	41.0609	0.0244	1.9664	2

The ratio in the angle normalisation goes to 0, so $(\xi^2)_{\mathrm{band}}[\mathcal{E}_\vartheta]$ fails in this free approximation. The power exponent $2 - \ln\beta/\ln\xi$ is still 1.97 at $\xi = 10^{48}$, but it goes to 2. This is an example showing a convention of notation, and shows nothing about C_P of the Yang–Mills measure.

There are also two exponents. The dynamical exponent z is the power in $C_P \asymp \xi^z$, and η without qualification is $\eta_{\mathrm{dyn}} = 2 - z$ (18-05 § 02, 2-10). The static $\eta_s = 2\Delta_O - (d-2)$ is the power of correlations fixed by the scaling dimension Δ_O of the operator O, and the two are equal only when the relation $z = 2 - \eta_s$ holds. Example. For Glauber

dynamics of the two-dimensional Ising model, $\eta_s = 1/4$ and $\eta_{\mathrm{dyn}} = -0.1665$ (18-03) Physics .

Table 10 Symbols for probability, spectrum and relaxation

Symbol	Meaning	Chapter	Note
$\mathbb{E}_\mu$, Var_μ, $\langle\cdot\rangle$	Expectation, variance, physical expectation	1-08	$\langle\psi,\phi\rangle$ is the inner product
$\sigma(H)$	Spectrum of an operator	1-10	Different from the string tension σ
CP, λ_1, $\mathcal{E}\vartheta$, $\mathcal{E}_A$	(11)(12)	1-10, 23-01	The three CP: the vacuum C_P, the path C_P, the path C_P measured in a graduation
τ_{exp}, τ_{int}	Exponential and integrated autocorrelation times	2-10, 5-07	For continuous-time Langevin, $\tau_{\mathrm{exp}} = CP$
z, η_{dyn}, η_s	Dynamical exponent, $2-z$, static power	2-10, 18-05	Different from the metric $\eta_{\mu\nu}$

09

Different meanings of the same letter — a list for telling them apart by context

The conventions of mathematics and physics have assigned the same letters to different things, so collisions within one book cannot be avoided. This book keeps the strong conventions as they are and declares the meaning at the start of the chapter. The guide for telling them apart is "subscripts and arguments". Example. $\beta(ba) = \beta(a) - \kappa \ln b + \cdots$ in 18-04 is the Wilson inverse coupling taking the lattice spacing as argument, not the β function $\beta(g)$.

Table 11 Different meanings of the same letter

Letter	Meaning (chapter)	How to tell apart
β	Wilson inverse coupling (5-01; the converted β_{Ch} and β_{SZZ} are in § 06) / inverse temperature $1/k_BT$ (2-08; in 2-07 the length of the loop $\beta\hbar$) / the critical exponent β (2-08 § 06, $1/8$ for the two-dimensional Ising model) / the β function (4-06) / (β) of the three CP (23-01)	Without qualification, the Wilson inverse coupling. The β function carries the argument g. The exponent is written alongside α, γ, ν
Δ	Mass gap (4-11) / Laplacian Δ_{lat} (1-08) / scaling dimension Δ_O (18-05; the scaling dimension of 17-08 is the same) / the decrease per step $\Delta(b)$ and the MCRG shift $\Delta\beta$ (18-04) / coproduct $\Delta(\Gamma)$ (4-05) / the group quantity Δ_G (Adhikari–Cao in 16-01; Δ_G in 16-05 and 16-08 is the mass gap) / difference Δx (2-04)	The scaling dimension carries an operator subscript, the Laplacian carries lat, the coproduct takes a diagram as argument. For Δ_G, see the chapter's declaration
η	Metric $\eta_{\mu\nu}$ (2-04) / $\eta_{\mathrm{dyn}}, \eta_s$ (18-05)	With two indices, the metric

Letter	Meaning (chapter)	How to tell apart
κ	One-loop slope (4-06) / decay rate of the Gaussian chain $1/\xi$ (2-08)	Decay rate only within 2-08
λ	Gell-Mann matrices (1-04) / 't Hooft coupling (17-04) / eigenvalue λ_1 (1-10)	With a colour index, a matrix; with a numeral subscript, an eigenvalue
σ	Pauli matrices (1-04) / string tension (5-02) / spin σ_x (2-08) / spectrum $\sigma(H)$ (1-10)	If the argument is an operator, the spectrum
Ω	Gauge transformation $\Omega(x)$ (3-01) / vacuum (4-11)	With an argument point x, a gauge transformation
θ	Angle of the θ term (3-05) / link angle 2ϑ (§ 05) / gac of the plaquette (5-01)	Multiplied by Q, the θ term
Λ	Lorentz transformation (2-04) / scale of dimensional transmutation (4-07)	If a matrix, the transformation
μ	Spacetime index / renormalisation scale (4-06) / measure μ_β	In subscript position, an index
a	Lattice spacing (5-01) / spacing fixed by bookkeeping $1/\xi_{\mathrm{RG}}$ (23-01) / starting point of a path (sections 01–06 of 2-07; from section 07, the step width) / colour index (3-02)	Superscript, colour. In 2-07 it switches by section
δ	Kronecker δ^{ab} (1-04) / shortfall of the bookkeeping $\delta(b)$ (23-02)	With two indices, Kronecker; with the factor b as argument, the shortfall
P	Plaquette P (5-01) / subscript of the Poincaré constant C_P (1-10)	As a subscript of C, the Poincaré constant
H	Hamiltonian (4-11) / Ising energy function (2-08)	An assumption is the name of a sentence
I	Identity matrix ($\mathrm{tr}(I-U_P)$ in 5-01) / modified Bessel function $I_k(J)=\frac{1}{2\pi}\int_0^{2\pi} e^{J\cos\theta}\cos k\theta\, d\theta$ (sections 06 and 07 of 0-04)	With a subscript k and an argument, the Bessel function
Ghost	The anticommuting Faddeev–Popov fields $c^a, \bar{c}^a$ (4-08 § 03) / states of negative weight in Pauli–Villars regularisation (4-10)	If a field, Faddeev–Popov; if a state of negative inner product, Pauli–Villars
Orientation of links	Parallel transport carries from the start point to the end point (1-07) / the lattice link $U_{x,\mu}$ is its inverse, carrying from $x+a\hat{\mu}$ to x, $U\approx e^{-igaA}$ (5-01 § 02) / numbering of chains: 3-01 uses $U_j=U(j{+}1,j)$, 3-02 uses $U_n=U(n,n{+}1)$	The value of the trace does not depend on the orientation. When comparing signs of exponents, see each chapter's declaration (the correspondence with 1-07 in 5-01 § 05)
T	Generators T^a, index $T(R)$ / length in time (5-02) / temperature (2-08) / transfer matrix T (2-08, 4-11; $T=e^{-aH}$)	Superscript, a generator. Inside a matrix power or a trace, the transfer matrix

Detour: can symbol collisions be eliminated?

You may skip this. Assigning a different letter to every quantity would eliminate collisions, but we do not, for two reasons. One is distance from the literature: writing the string tension with anything other than σ would make the tables of lattice papers unreadable as they are. The other is the number of letters: the number of quantities is far larger than the number of letters in commonly used typefaces. Precisely, a collision does harm only when two meanings stand in the same formula or the same table. For example, the three C_P in Table 3 of 23-01 stand in the same table as the Wilson β, so they are distinguished by name (the vacuum C_P, the path C_P, the path C_P measured in a graduation), rather than by a superscript such as (β) alone.

Table 12 The κ's compared (a copy of 26-05 Table 2): quantities that use the letter κ

Symbol	Meaning	Chapter
κ_{cb}	Chessboard density: the geometric mean, per block, of the product of the container biases of all blocks ((4) of 26-05)	26-05, 26-00 Table 1
κ (the one-sided bridge)	The comparison constant of the One-Sided Bridge Proposition (Theorem 3.1 of the paper) (in the form $CP \geq \kappa \xi^2$)	23-01 § 06
κ_{BE}	The lower bound on the Bakry–Émery curvature	24-09
$\hbar\kappa$, κ_f	An upper bound on the strength of the coupling to neighbouring sites; the crossing strength of Brascamp–Lieb	23-06
$\kappa = 3/1000$	The lower bound outside the defects in the Barta row	23-09
κ (Part 24)	The comparison constant between the Langevin metric and the metric of the kinetic term	24-09
$\kappa_{\mathrm{C7}}(k)$	The convexity constant of a chain of length k (the same quantity as $\kappa(k)$ in 23-11)	26-07
κ_w	Adapted stiffness: the variance of the force along the generator of the twist (width w; one of the candidate quantities for the band)	26-10

Table 13 Other meanings of θ and of "band" (a copy of 26-00 Table 0)

Word	Meaning	Chapters that use it	Relation to the band of Part 26
"Band" in Parts 23 and 24	The "band statement" of 23-01 § 05 and 23-15 (5) (the statement that a quantity lies within a width of constant multiples of the square of a length); ℓ_{band} of 23-09 (a length tied to the band at the summit); the "two-dimensional strip" (a narrow, strip-shaped lattice) of 24-02 § 03, § 05 and 24-09. The "twisted strip" of the two-dimensional model in 25-07 § 06 also has the meaning of a strip-shaped lattice	23-01, 23-09, 23-15, 24-02, 24-09, 25-07	Words with another meaning. Not the band of this part (an interval of scales)

Word	Meaning	Chapters that use it	Relation to the band of Part 26
Three meanings of θ	The threshold θ of 23-09 (θ_B, θ_σ and others; the value that divides configurations into the valley side and the summit side); the sector bias $\theta(R;T^2_L)$ of this part ((4) of 25-01); the vacuum angle θ of Tanizaki–Ünsal (set as $\theta = 0$ in 25-02)	23-09, 25-01, 25-02	Only the sector bias is a quantity of this part. The other two are different quantities

What this chapter can say

Matter	What is said	What is not said
Conventions of notation (Tables 1, 3, 5, 8, 10, 11)	A list collecting in one place the conventions used by the chapters of this book. The definitions are as in the linked chapters	Properties of the quantities each symbol refers to (existence, positivity and so on). These can be said only within the range of the labels in each chapter
Normalisation of generators (3) and structure constants (Table 2)	Checked numerically for SU(2) and SU(3) Computed . Standard convention Known	General G (the list in 1-04)
Three ways of writing the gauge field (Table 4)	That they translate into one another by the substitutions (4). $Q = -c_2$ ((2) of 3-05) Known	—
Conversion of couplings (7) and constants (Tables 6 and 7, Figure 1)	Conversions as this book's arithmetic Computed . b_0 and κ rewrite known one-loop coefficients Known	The content of each paper's conditions (conditions are read in the normalisation of the paper)
Normalisation of C_P ((12), Table 9, Figure 2)	Arithmetic under the free approximation and the one-loop formula (up to constants) Computed	The size of C_P for the lattice Yang–Mills measure. Whether the relaxation-time assumption holds
The mass gap	—	This chapter is only conventions of notation, and shows nothing, on the lattice or in the continuum

Sources and reproduction

Item	Kind	Source / tools
Definition of each symbol (the "Chapter" column of each table)	Chapters of this book	The linked chapters. Primary sources are as in the source tables of those chapters
Definitions of Δ and m in the problem statement	Primary source (text checked)	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official Clay problem description) § 4. As in the source table of 4-11
The Wilson action and $\beta = 2N/g^2$	Known	K. G. Wilson, Phys. Rev. D 10 (1974) 2445–2459. As in the source table of 5-01
The convention for β Ch	Primary source (text checked)	S. Chatterjee, “Yang–Mills for probabilists”, arXiv:1803.01950 (2018) § 3. As in the source table of 5-01
The convention for β SZZ and the condition $1/(16(d-1))$	(1.2)(1.3) of the text checked (the reading of the normalisation is in 5-04)	H. Shen, R. Zhu, X. Zhu, Comm. Math. Phys. 400 (2023) 805–851, arXiv:2204.12737. As in the source table of 5-04
The one-loop coefficient b_0 and its four ways of writing	Known	D. J. Gross, F. Wilczek (1973), H. D. Politzer (1973), D. Tong's lecture notes § 2.4, the QCD chapter of the PDG. As in the source table of 4-06
The three CP, $\mathcal{E}_\vartheta$, $\mathcal{E}_A$, the bookkeeping length	Definitions of the study in Part 23	Source tables of 23-01 and 23-13
η_s, η_{dyn}, z	Physics, the study in Part 23	Source tables of 18-05 and 2-10
The numbers in Tables 2, 6, 7, 9, (6) and § 04	Computed on this machine	Python 3, NumPy, SciPy: `generators.py` (normalisation, commutation relations, Casimirs), `constants.py` (the arithmetic of (7)(8)), `normalisation.py` ((12) and the one-loop formula). Figures 1 and 2 compute the same formulas in plain JavaScript

Next chapter to read: A-02 Glossary (look up the meaning of a term in one to three lines, and jump to the chapter). The names of assumptions are in A-06 Index of assumptions.

Glossary — looking up the words of this book in one line, and jumping to the chapter of definition

This chapter is a glossary; each entry points to the chapter that defines it.

What this chapter shows — The meaning of the main words in this book, in 1–3 lines each. Which word is defined in which chapter, and in what order they build up to reach "the mass gap". How to tell apart words that name different quantities under the same name (β, Δ, η, C_P, "gap").

Prerequisite chapters — None. You may look things up from anywhere, and read the details in the linked chapters. The conventions for symbols are in A-01 Table of notation, the list of assumptions in A-06 Index of names, and the literature in A-03 Guide to the literature.

Said first — The explanations here are summaries; the master copy of each definition is the linked chapter. Some conditions were dropped to fit a single line, so check the hypotheses of theorems in the linked chapters. The mass gap has not been shown (for existence in four dimensions, the committee's assessment is that it is proved in Lean in the construction published in 2026; 6-00). This book does not claim to solve the problem.

Order of this chapter

1. How to use this, and the single road of words — the form of an entry, filtering, twelve words from the group to the mass gap (Figure 1), a table of frequent words
2. Mathematical preparation (1): groups, Lie algebras, geometry — group, SU(2), compact simple group, Lie groups and Lie algebras, connection, holonomy, curvature, gauge transformation
3. Mathematical preparation (2): analysis and probability — Gaussian measure, Lebesgue measure in infinite dimensions, self-adjoint operators and spectrum, variational principle, Poincaré constant
4. Physical preparation — action, states and the Hamiltonian, path integral, imaginary time, correlation length, phase transition, transfer matrix, Langevin and z (Figure 2)

5. Yang–Mills theory: classical, quantisation, axioms — field strength, instanton, orbit space, renormalisation, asymptotic freedom, Λ, gauge fixing and BRST, Gribov, Schwinger functions, OS axioms, mass gap
6. Lattice gauge theory — link variable, plaquette, Wilson action, Elitzur's theorem, Wilson loop, area law and string tension, strong-coupling expansion, continuum limit, lattice correlation-length problem (Chatterjee's Problem 5.1)
7. Constructive field theory — constructive field theory, triviality, cluster expansion, blocking, ultraviolet stability, stochastic quantisation
8. Words for reading the published construction (Parts 6–14) — published construction, assessment, probability kernel, representation, forced activity, shooting, calendar, induction hypothesis, seed, chessboard root, phase measure and others
9. Physical pictures — dual superconductivity, centre vortex, effective string, large N, holography, one scale
10. The words of this book (Parts 16, 18, 23 and 24) — the two columns of settledness, seams, the study with the diffusive-relaxation-time assumption at its axis (the side of the four-dimensional lattice, the model of the first step, the assembly of the first step, computation and checking), the words of Part 24
11. Different quantities under the same name — β, Δ, η, b_0, three C_P, three "gaps", Detour: four uses of the word "dimension"

01

How to use this, and the single road of words

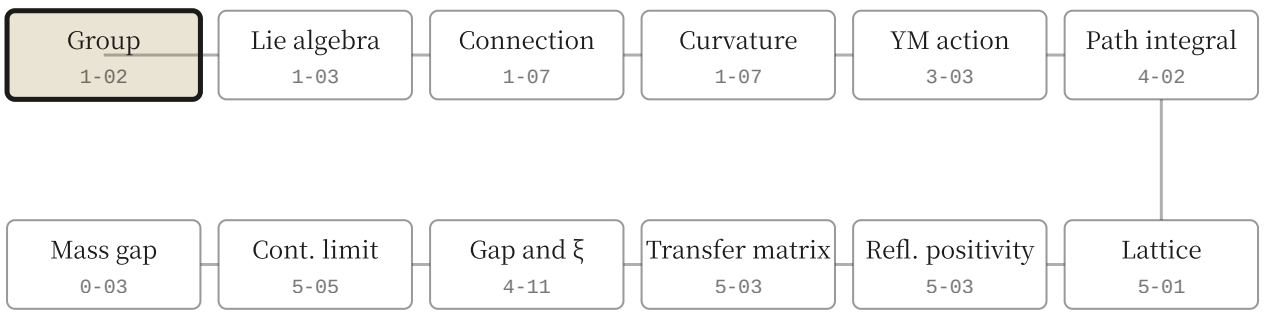
Each entry consists of "the headword", "a meaning in 1–3 lines" and "→ the chapter where it is defined". The order is not alphabetical but follows the parts of the book, so the words needed to read an entry are usually a little above it. To search by word, you can filter with the box below (without JavaScript, use in-page search).

The backbone of this book is one formula. Set the energy of the vacuum of the Hamiltonian H to 0; the distance from it to the next part of the spectrum is called the mass gap (4-11 § 01).

$$\Delta \;=\; \inf\big(\operatorname{spec} H \setminus \{0\}\big) \;>\; 0 \quad (1)$$

To read this formula you need "spectrum", "Hamiltonian" and "vacuum", and to give it meaning for four-dimensional Yang–Mills theory you need everything from groups to lattices and reflection positivity. Figure 1 is that single road.

Figure 1 The single road of words — moving the knob advances a step and shows that word and its chapter of definition below. "Play in order" advances one step at a time. The number in each box is the chapter that uses the word on this single road (the chapters of definition are the links in the entries below)



1. Group — a collection of symmetries (1-02)

Without JavaScript, the twelve boxes of the figure and their chapter numbers can still be read as they are.

Table 1 counts how many chapters the main words appear in. "First chapter" is the chapter where the name first appears, not necessarily the chapter of definition.

Table 1 Number of chapters in which the main words appear (the body text of the 105 chapters other than the appendices; counted excluding script, style, the whole-book contents and the meta line)

Word	Chapters	First chapter
Mass gap	101	0-01
Confinement	56	0-01
Correlation length	54	0-02
Gauge transformation	48	0-01
Asymptotic freedom	47	0-01
Plaquette	46	0-03
Refl. positivity	43	0-02
Wilson loop	43	0-01
String tension	40	0-02
Transfer matrix	38	0-04
diffusive-relaxation-time assumption	26	0-02
Poincaré constant	20	1-10
Dynamical exponent	14	2-08

Computed The numbers in this table were counted on this machine (a small script that counts words in the body text of the drafts; the numbers move as chapters are revised). Of the 26 words counted, we list the top 10 by number of chapters (excluding

"connection", which overlaps with ordinary Japanese) and the 3 words central to Part 23 of this book (relaxation-time assumption, Poincaré constant, dynamical exponent).

02

Mathematical preparation (1): groups, Lie algebras, geometry

Yang–Mills theory is the theory of a mechanism for comparing the orientation of a group from place to place (the connection) and of the mismatch in that comparison (the curvature). The first half of Part 1 prepares the words for this. The key formula is the one line that makes the curvature F from the connection A (in the geometric way of writing; for conversion to the physical way, see A-01).

$$F = dA + A \wedge A \quad (2)$$

Example: if the group is $U(1)$ (complex numbers of absolute value 1), then $A \wedge A = 0$ and $F = dA$, and we are back to the field strength of electromagnetism (2-03). For $SU(2)$, $A \wedge A \neq 0$, and this is where the interaction of the field with itself comes from.

Group

A set with a product, an identity and inverses, whose product is associative. An abstraction of a collection of symmetries (operations that preserve a shape). → 1-02

SU(2), double cover

The group of 2×2 unitary matrices with determinant 1. The same as the quaternions of length 1; its shape is the three-dimensional sphere. It maps two-to-one onto the rotation group SO(3). → 1-02

Compact simple group

A bounded, closed, connected matrix group (a compact connected Lie group) whose Lie algebra is non-abelian and has no ideals other than 0 and the whole (simple in the sense of Lie groups; a finite centre, as in $SU(N)$, is allowed). The problem statement lets the gauge group be chosen arbitrarily from among these. → 0-03

Lie group, Lie algebra

A group whose product can be taken smoothly, and its tangent space at the identity equipped with the commutator $[X, Y]$. The Lie algebra is the "infinitesimal directions" of the group. → 1-03

Connection

A rule for comparing vectors at neighbouring points. The gauge field A is a connection. → 1-07

Parallel transport, holonomy

Carrying a vector along a connection. The change (an element of the group) after going once round a closed path and returning is the holonomy. The prototype of the link variables of the lattice and of Wilson loops. → 1-07

Curvature

The leading part, proportional to the area, of the holonomy of a small loop. Formula (2). The field strength of gauge theory. → 1-07

Gauge transformation (geometry)

Changing the basis at each point by an element of the group. The connection changes as $A \mapsto gAg^{-1} - (dg)g^{-1}$ and the curvature as $F \mapsto gFg^{-1}$. → 1-07

03

Mathematical preparation (2): analysis and probability

Quantised Yang–Mills theory can be written, in imaginary time, as "a probability measure on configurations of the field". A configuration is infinitely many numbers, so we need probability in infinite dimensions and the language of the spectrum of operators. The inequality used most in this book is the Poincaré inequality, which bounds the variance by "how much the value changes when you shake things".

$$\mathrm{Var}_\mu(f) \;\le\; C_P\, \mathcal{E}(f), \qquad \mathcal{E}(f) = \mathbb{E}_\mu\big[|\nabla f|^2\big] \quad (3)$$

Example: if μ is the standard Gaussian distribution on the real line, then $C_P = 1$, with equality for $f(x) = x$ (variance 1, expected squared derivative 1). Known The reciprocal of the best C_P is the gap of the diffusion that has μ as its equilibrium (1-10).

Gaussian measure

A measure determined by its mean and covariance alone. The most basic measure that can be built even in infinite dimensions; constructive field theory often starts from here. → 1-08

There is no Lebesgue measure in infinite dimensions

On an infinite-dimensional separable Hilbert (Banach) space such as ℓ^2, the only measure that is invariant under translations and gives finite values to bounded sets is 0. The reason why the "$\mathcal{D}A$" of the path integral has no literal meaning. → 1-09

Self-adjoint operator, spectrum

The infinite-dimensional version of a Hermitian matrix, and the set of its "eigenvalues" (including continuous parts). The Hamiltonian is self-adjoint and its spectrum is real. → 1-10

Variational principle (Rayleigh–Ritz)

The bottom of the spectrum (the smallest eigenvalue, if there is one) is the infimum of $\langle \psi, H\psi\rangle / \langle \psi, \psi\rangle$. However many trial functions you put in, you only get upper bounds. → 1-10

Poincaré inequality, Poincaré constant C_P

Formula (3), and the best constant that goes in it. Its value depends on the normalisation of $\mathcal{E}$ (what you differentiate by), so each chapter states its normalisation. → 1-10

04

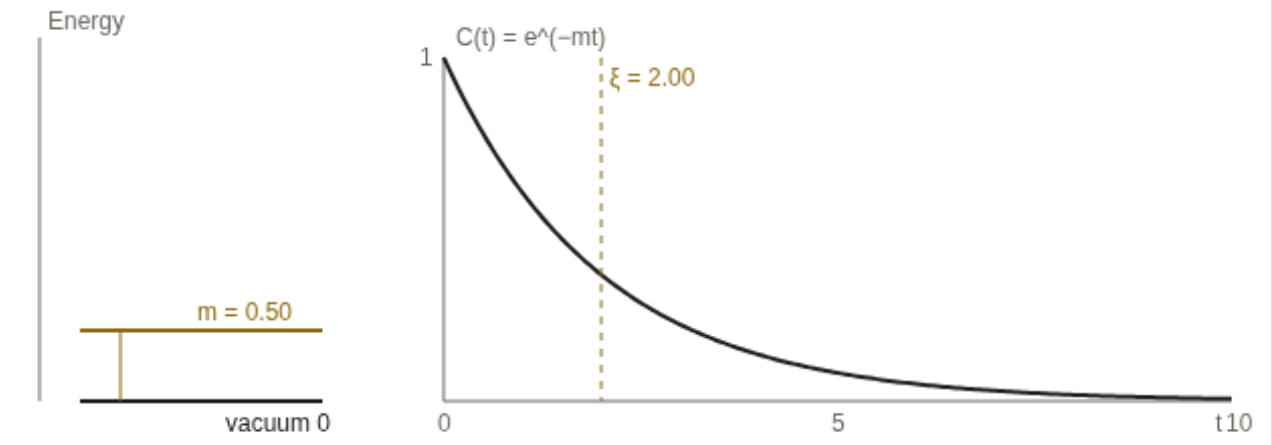
Physical preparation

The words of Part 2 have two faces, quantum mechanics and statistical mechanics. What quantum mechanics calls "the energy gap" and statistical mechanics calls "the length over which correlations decrease" become the same quantity through imaginary time. This is the rereading that runs through the whole book; as a formula, it is the following expansion using the vacuum $|0\rangle$ and the excitations $|n\rangle$ (4-11 § 02).

$$\langle O(t)\,O(0)\rangle - \langle O\rangle^2 \;=\; \sum_{n\geq 1} |\langle n|O|0\rangle|^2\, e^{-(E_n - E_0)t} \;\approx\; C\, e^{-mt}, \qquad \xi = \frac{1}{m} \quad (4)$$

Example: if $m = 0.5$ then $\xi = 2$, and the correlation at a separation of $t = 10$ decreases by a factor of $e^{-5} \approx 0.0067$. Moving the gap m with the knob in Figure 2 changes the spacing of the levels on the left and the rate of decrease on the right together.

Figure 2 Gap and correlation length — changing the gap m with the knob changes at the same time the height of the first excitation on the left and the rate at which the correlation e^{-mt} on the right decreases. The vertical dashed line is t = ξ = 1/m




```
m = 0.50, ξ = 2.00, C(5) = 0.0821, C(10) = 0.0067
```

Table 2 The gap m and the correlation $C(t) = e^{-mt}$ (with the constant C = 1)

m	ξ = 1/m	C(1)	C(5)	C(10)
2.00	0.50	0.1353	0.0000	0.00000
0.50	2.00	0.6065	0.0821	0.00674
0.10	10.00	0.9048	0.6065	0.36788

Computed The numbers in this figure and table were computed on this machine (values of the exponential function; the figure draws the same formula on the spot). The smaller the gap, the farther the correlation persists, and as $m \to 0$ the correlation length becomes infinite. This is what "critical" looks like.

Action, Euler–Lagrange equation

A functional S that assigns one number to a whole path. The classical motion is the path that makes S stationary, and that condition is the Euler–Lagrange equation. → 2-01

State, Hamiltonian

A state is a unit vector in a Hilbert space; the Hamiltonian H is the self-adjoint operator that generates the time evolution e^{-itH} and represents the energy. → 2-05

Path integral

The way of writing quantum theory in which all paths are summed with weight $e^{iS/\hbar}$ (in imaginary time, e^{-S}). → 2-07

Imaginary time, Wick rotation

Substituting $t = -i\tau$. The oscillation e^{iS} becomes the damping e^{-S}, and quantum theory becomes the probability of statistical mechanics. → 2-07

Correlation function, correlation length

The tie between the values at two distant points, and the length ξ when it decreases as $e^{-r/\xi}$. In formula (4), the reciprocal of the gap. → 2-08

Phase transition, critical point, universality

When a parameter is moved, the free energy and other quantities change in a non-smooth (non-analytic) way. If a quantity jumps, it is a first-order transition; if nothing jumps but the correlation length becomes infinite, it is a continuous transition, and that point is a critical point. At a critical point, behaviour that does not depend on the details (universality) appears. → 2-08, 16-11 § 04, 26-12

Transfer matrix

The weight of one step of imaginary time made into a matrix (an operator) $T = e^{-H}$. The ratio of the largest eigenvalue to the next determines how correlations decrease. → 2-07

Langevin dynamics, Glauber dynamics

A noisy gradient flow that has a given measure as its equilibrium (the continuous case), and a stochastic dynamics that rewrites one point at a time (the discrete case). Not the time of the theory but a provisional time for producing samples. → 2-10

Dynamical exponent z, critical slowing down

The exponent when the relaxation time grows as a power of the correlation length, $\tau \sim \xi^{z}$. The phenomenon that the dynamics becomes slower the closer one is to criticality is critical slowing down. → 2-10

05

Yang–Mills theory: classical, quantisation, axioms

These are the words of Parts 3 and 4. The classical theory is the theory of "connections that minimise the square of the curvature", and when it is quantised, one length scale that the classical theory does not have is born. That scale is expressed by Λ, and at one loop it has the following form (4-07).

$$\Lambda = \mu \exp\!\left(-\frac{1}{2b_0\, g^2(\mu)}\right), \qquad b_0 = \frac{11N}{3}\cdot\frac{1}{16\pi^2} \quad (\text{pure } SU(N)) \tag{5}$$

Example: for $SU(3)$, $11N/3 = 11$. As g^2 is made small, Λ approaches 0 in the form e^{-1/g^2}, and does not show up in any power series in g^2. Known

Classical

Field strength

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$. The last commutator is the mark of non-commutativity: the field interacts with itself. → 3-02

Yang–Mills action

$S = \frac{1}{2g^2} \int \operatorname{tr} F_{\mu\nu} F_{\mu\nu}\, d^4x$ (Euclidean, written with g absorbed into A). In four dimensions its value does not change when the scale is changed. → 3-03

Self-dual, instanton

A connection satisfying $F = \pm * F$ (+ is self-dual and − anti-self-dual; the signs of the topological charge are opposite). A finite-action solution that brings the action all the way down to a constant times the absolute value of the topological charge. The representative is the BPST solution (a solution of topological charge 1 with a size around one point). → 3-03

Second Chern number, topological charge

$\frac{1}{8\pi^2} \int \operatorname{tr} F \wedge F$ (in the anti-Hermitian way of writing). An integer that does not change when the connection is changed continuously. → 3-05

Gauge orbit space, Singer's theorem

The space in which connections related by gauge transformations are identified. In the non-abelian case, there is no global way to choose one connection from each orbit continuously. → 3-04

Quantisation

Divergence, regularisation, renormalisation

That loop integrals become infinite at short distances; the cutoff that makes them temporarily finite; and readjusting the couplings so that observables remain finite when the cutoff is removed. → 4-04

Renormalisation group, β function

The rule for how the effective coupling flows when the scale at which one looks is changed. The speed of the flow is the β function $\mu\, dg/d\mu$. → 4-04

Asymptotic freedom

That the β function is negative and the coupling becomes weaker at shorter distances. For pure $SU(N)$, the one-loop coefficient $\mathbf{11}N/\mathbf{3}$ is what matters. → 4-06

Dimensional transmutation, Λ

That in a theory with only a dimensionless coupling, one length scale Λ appears through quantisation. Formula (5). → 4-07

Gauge fixing, Faddeev–Popov, ghost

A condition that narrows the connections representing the same physics down to one, and the determinant that then appears in the integral. The determinant written in terms of fields is the ghost. → 4-08

BRST symmetry

A symmetry, involving the ghosts, that remains in the action after gauge fixing. It is used to check that physical quantities do not depend on the choice of gauge. → 4-08

Gribov copy, Gribov region

That one orbit contains several connections satisfying the gauge condition (a manifestation of Singer's theorem), and the region where the Faddeev–Popov operator is positive. → 4-08

Axioms and definitions

The Wightman axioms, Haag's theorem

The set of conditions that a quantum field theory on Minkowski spacetime should satisfy, and the theorem showing that an interacting field cannot be placed in the same representation as the free field (if the fields at time 0 are unitarily related to a free field and the vacuum is unique and invariant under spatial translations and rotations, then the field is a free field). → 4-09 axioms, Haag

Refl. positivity

The property that the correlation with the time-reflected function is non-negative. With it, a state space with a positive inner product can be built from the Euclidean measure. → 4-10

Schwinger functions

The correlation functions of the field in Euclidean (imaginary) time, $\langle \phi(x1) \cdots \phi(x_n) \rangle$. The Wightman functions continued to imaginary time. → 4-10

The Osterwalder–Schrader axioms, reconstruction

The set of conditions that the Schwinger functions (just above) should satisfy, and the theorem that if they are satisfied, a Hilbert space and a Hamiltonian can be built. → 4-10

Mass gap

Formula (1). That the spectrum above the vacuum is separated by $\Delta > 0$. It can be rephrased as the mass of the lightest particle, but this is a separate condition from the existence of particles. → 4-11

Non-trivial

That the constructed theory is not a free field (Gaussian). One of the requirements of the problem statement. → 0-03

06

Lattice gauge theory

These are the words of Part 5. A lattice replaces spacetime by a mesh of spacing a, and represents the gauge field by elements of the group placed on the edges (links) of the mesh. That gauge invariance is kept exactly is the strength of the lattice. The action is a sum over the small squares (plaquettes), one by one.

$$S_W = \beta \sum_P \Big(1 - \tfrac{1}{N} \operatorname{Re} \operatorname{tr} U_P\Big), \qquad \beta = \frac{2N}{g^2} \quad (6)$$

Example: for $SU(2)$ ($N = 2$) with $\beta = 2.4$, $g^2 = 4/\beta \approx 1.67$. Making β larger makes g smaller, and by asymptotic freedom the lattice spacing a measured in physical units is expected to approach 0 (the one-loop running, 4-07) Physics . Some references use a different normalisation; for conversion, see 5-01.

Link variable, lattice spacing

The element of the group U_ℓ placed on the edge joining neighbouring lattice points (the lattice version of the parallel transport of a connection), and the width of the mesh a. → 5-01

Plaquette, staple

The product U_P of the four edges of a unit square (the holonomy of a small loop), and the product of the three edges with one edge removed. → 5-01

Wilson action

Formula (6). The lattice action that adds $1 - \frac{1}{N} \operatorname{Re} \operatorname{tr} U_P$ for each plaquette; in the naive limit $a \to 0$ it returns to the Yang–Mills action. → 5-01

Elitzur's theorem

On the lattice, the expectation value of a quantity that changes under gauge transformations (such as a component of a single link) is 0 in every volume and every limit. Local gauge symmetry does not break spontaneously. → 5-04

Wilson loop

The trace of the product of links along a closed path. Gauge invariant; it represents the weight for carrying a pair of heavy charges. → 5-02

Area law, perimeter law, string tension

The area law is that the expectation value of a Wilson loop decreases as the exponential of the enclosed area, $e^{-\sigma \cdot \mathrm{area}}$; the perimeter law is that it decreases with the length of the perimeter. The coefficient σ is the string tension. → 5-02

Confinement

That a colour charge cannot be taken out on its own. On the lattice it is often judged by the area law (the static potential growing in proportion to the distance), but there are several formulations. → 5-02

Strong-coupling expansion, character expansion

A method of expanding the weight in characters of the group when β is small. Here the area law and a positive gap can be shown Known (Osterwalder–Seiler 1978). → 5-02

Continuum limit, scaling

The limit $\beta \to \infty$ ($a \to 0$) that keeps quantities in physical units fixed. Scaling is when dimensionless ratios stop depending on a. → 5-05

Glueball

The lightest excitation of pure Yang–Mills regarded as a particle. On the lattice its mass is measured from how correlations decrease. → 5-08

The lattice correlation-length problem, the direction of the lattice version of the mass gap

Chatterjee's (2018) problem. For four-dimensional lattice gauge theory, show that for each β the correlation length $\xi(\beta)$ exists and is finite, and that $\xi(\beta) \to \infty$ as $\beta \to \infty$. If the gap in lattice units stayed uniformly positive, the mass would become infinite in the continuum limit, so the question is posed in this direction. That the gap in physical units is kept is the step beyond (the physical-unit gap statement). → 5-09

07

Constructive field theory

These are the words of Part 7. Constructive field theory is the field that builds interacting field theories as probability measures and proves that they satisfy the axioms. Which dimensions are easy, and where things stop, can be estimated from the "dimension" of the coupling. The Yang–Mills coupling has a power of length.

$$g^2 \sim (\mathrm{length})^{d-4} \quad (7)$$

Example: for $d = 2$, g^2 has the dimension of mass squared, and for $d = 3$ of mass to the first power, and it stops mattering at short distances (super-renormalisable). For $d = 4$ it is dimensionless; it does not weaken by a power of the scale, and moves only logarithmically under the renormalisation group (marginal). In Yang–Mills, by asymptotic freedom, it weakens logarithmically at short distances (4-06). Known Where things stand in each dimension is in 16-01.

Constructive field theory

The field that builds field theories as mathematical objects (measures, operators) and checks the axioms. This book reads it divided into six stages. → 7-01

Triviality

That only a Gaussian (free field) remains when the continuum limit is taken. A result of the type shown for ϕ^4 in four dimensions. → 7-08

Cluster expansion

A technique that splits the measure into a sum of contributions from small lumps far from each other, and shows convergence. Effective at strong coupling and large mass. → 7-04

Blocking, renormalisation group step

One step that coarsens the lattice by a factor b and integrates out the fine fluctuations to get a new action. Repeating it controls the continuum limit. → 7-05

Ultraviolet stability

That, in finite volume, the upper and lower bounds of the final action after repeated blocking do not depend on the lattice spacing. A theorem in three dimensions, a theorem with a hypothesis in four. There are three readings of different strength. → 7-06, 7-07 three readings

Stochastic quantisation, regularity structures

A method of building the measure as the equilibrium of a Langevin equation, and the theory that gives meaning to singular stochastic partial differential equations. → 7-09

08

Words for reading the published construction (Parts 6–14)

These are the words of Volume 2. Volume 2 reads the construction of the existence of four-dimensional Yang–Mills theory published on 2026-11-10 (the Lean code and the committee's assessment). The objects of the published construction are called by

names that carry their meaning, and the identifiers in the code are added as sources. The list of names and symbols is in A-06 § 02, and the conventions for the labels are in 6-00.

Statements and checks

Published construction

The Lean 4 code submitted on 2026-10-28 (yang-mills-group/OpenAI-YM-20261028), and the assessment of the committee that examined it (2026-11-10). This book cites its theorem at the level of "the committee's assessment is that it is proved in Lean". → 6-00

Assessment

The committee's assessment (docs/assessment-1.pdf in yang-mills-group/YM). Part I compares the result of the construction with the problem statement; Part II is the record of the checks. → 6-00

Public theorem, construction theorem, the committee's combined target

The statement in the form of the published question (whose only assumption is that the Lie algebra is simple), a stronger statement that contains it as a projection, and a statement on top of that in which the committee bundled its independent checks into one family. They differ in how much they say. → 6-02

Headline theorems, outside the dependencies

The 17 headline theorems listed by the verification record, and declarations not reached from any of them. Being "inside the dependencies" does not mean being used by the public theorem. → 6-00

Re-checking in the kernel, Comparator

Re-checking the proof in a fresh Lean kernel initialised only with the external libraries, and a tool that matches the statements and definitions against independently written definitions. Neither guarantees that the definitions express the intent of the problem. → 6-02

One step of renormalisation, and the coupling

Probability kernel, predictor, penalty coefficient

One step of renormalisation in the published construction is not a deterministic map but a probability kernel that draws the coarse link with Gaussian-type fluctuations around a central value (the predictor) built from the fine configuration. The strength of the pull is the penalty coefficient c_{pen}. → 9-01, 9-04

Representation

The fixed form in which the density at each step is written: the vacuum term, the main action, regular activities, and the gas of forced activities. The representation is not unique. → 9-06

Regular activity

The sum of the local correction terms in the representation. It vanishes at flat configurations. → 9-06

Forced activity

A correction function with a support that exactly covers the large-field flags and a non-empty "inventory" on it. The whole of it is carried into the representation as a gas. → 9-06, 11-01

Truncated action, flag

The action in which the plaquette terms are cut off linearly above a step-dependent threshold, and the mark of a large field exceeding the threshold. The threshold goes to 0 with the step. → 9-07

Coupling at a step

The coefficient multiplying the term of the normalised curvature squared in the representation at step m. A number that is not determined by the measure alone. → 10-01

Rereading

After one step, rereading the coupling of the representation from the log-ratio of the density at two points, a flat configuration and a test configuration. Strong monotonicity fixes the root uniquely. → 10-03

Shooting

Choosing the bare coupling that brings the coupling at step 0 to the reference value $\bar{\beta}$, by nested intervals of the bands at each step and the intermediate value theorem (the shooting method). Uniqueness is not claimed. → 10-04

Calendar

The bundle of conditions for choosing, in a fixed order, the constants used in the one-step estimates (block side, smoothing exponent, constant of the energy lower bound, rate, entry of the coupling). The conditions uniform in the step are paid here. → 10-05, 11-08

Calibration

Reading the coefficients of insertions of local fields at a reference step and matching them. The inverse of the renormalisation transport is used as input. → 8-04

Multistep estimates

Induction hypothesis, one-step certificate

That the density at every step has a representation of the same form, with a per-step ledger (estimates) attached to its ingredients; and building the representation at step m from that at step $m+1$, with the push-forward identity and the ledger. → 11-00

Seed, region

Each large fine plaquette, each edge far from its prediction and each flag (a seed), and the condition that sorts configurations by the type of their seeds (a region; with hysteresis of some width). → 11-03

Price, mark

The factor per seed (the price), and the part of its allocation left at the exit (the mark; half of the guaranteed lower bound). → 11-05

Strong estimate, weak ledger

The estimate $e^{-\mathrm{rate}\cdot P^2}$ at the exit of one step, and the estimate $e^{-P^{1/5}}$ carried by the induction hypothesis. The strong estimate is converted into the weak ledger and carried. → 11-07

Infinite volume and reconstruction

Slice, thread

The configuration of links within one layer perpendicular to the direction of reflection (a slice), and the set of links in the direction joining adjacent layers (a thread). The split used to build the kernel of the transfer operator. → 12-01

Inside function

A function that is gauge-invariant, measurable and bounded, and depends only on links with both ends inside the strip. The class of functions on which reflection positivity and the chessboard root act. → 12-01

Chessboard root

The size of the expectation of the product of one function copied to all boxes, raised to the power one over the number of boxes. A seminorm satisfying the triangle inequality (not necessarily a norm). → 12-02

Centred tiling

A tiling by boxes whose side is the lattice spacing of step m, with the reference box placed at the centre of the observation point. The size of renormalised observables is measured by this root. → 12-02, 12-03

Column

The arrangement in which a block and its mirror image alternate so as to fill the time circle. A chessboard copy in the single direction of time. → 12-04

Extraction sequence

A pair of a subsequence of cutoffs and a subsequence of volumes. A refinement composes each of the two subsequences. The limit is read as two ordinary limits along this sequence. → 12-05

Window (window function)

In a Ward identity, a smooth function of compact support that equals 1 near one point and whose support contains none of the other points. → 13-01, 12-06

Zero-energy space

The subspace $\ker H$ fixed by the whole time semigroup. That it is one-dimensional is the uniqueness of the vacuum in that phase. → 12-07

Phase measure

The probability measure μ (ν in the assessment) that splits a mixed limiting theory into phases (components). It is built from the Gelfand representation of the commutative algebra of operators compressed to the zero-energy space. → 12-07

Growth condition

That when all pairwise distances of a configuration of points are at least r, the correlations and their derivatives are bounded by a constant times r^{-A}. One form of the bundle of reconstruction axioms. → 12-08

09

Physical pictures

The words of Part 17 are the pictures by which physics, assuming that "the theory exists", explains the mass gap and confinement; most are at the level of Physics . The representative one is the picture in which the colour electric flux is squeezed into a thin tube and behaves as a vibrating string; the potential between heavy charges then takes the following form at long distances R (17-03).

$$V(R) \;=\; \sigma R \;-\; \frac{\pi(d-2)}{24R} \;+\; \cdots \tag{8}$$

Example: for $d = 4$ the second term is $-\pi/(12R)$. The first term is the area law (the string tension), and the second is the zero-point energy of the transverse vibrations of the tube. Physics That the coefficient of the second term does not depend on the details is a result under the assumption that a long tube can be regarded as an effective string.

Dual superconductivity, abelian projection

The picture of confinement in which, when magnetic monopoles condense, electric flux is squeezed into a tube, dually to the way a superconductor squeezes magnetic flux into a tube. Abelian projection is the procedure that partially fixes the gauge, shrinks the group to its abelian part (the diagonal components), and sees monopoles at the singularities of the fixing. → 17-02

Centre vortex

A sheet-like defect across which the phase shifts by an element of the centre of the group. The picture is that if there are many of them in disorder, Wilson loops show an area law. → 17-02

Effective string, Nambu–Goto, Lüscher term

The theory that regards the flux tube as a string weighted by its area, and the $1/R$ term of formula (8) produced by its fluctuations. → 17-03

Large N, 't Hooft coupling, planar diagrams

The limit $N \to \infty$ for $SU(N)$ with $\lambda = g^2 N$ held fixed. Only the Feynman diagrams that can be drawn on a plane remain. → 17-04

Holography, AdS/CFT

The conjecture that the gravitational theory of a certain spacetime is the same thing as a gauge theory on its boundary. The radial direction corresponds to scale. → 17-08

One scale ("a single scale")

The physical premise that every dimensionful quantity of pure Yang–Mills is a constant times a power of Λ. On the lattice it can be written in the form $m^2 \asymp \sigma$. → 17-12

10

The words of this book (Parts 16, 18, 23 and 24)

From Part 18 on, words named within this book appear. The exhaustive list of assumptions is in A-06; here we list only the central set. The quantity $a^2 C_P$ that the study in Part 23 sets up splits into a product of three ratios (17-12; $a = 1/\xi_{\mathrm{RG}}$, $\lambda_1 = 1/C_P$).

$$a^2 C_P \;=\; \underbrace{\frac{a^2}{\sigma}}_{\text{(iii)}} \cdot \underbrace{\frac{\sigma}{m^2}}_{\text{(ii)}} \cdot \underbrace{\frac{m^2}{\lambda_1}}_{\text{(i)}} \qquad (9)$$

Example: with the lattice values for SU(2), $\beta = 2.60$ (the table in 17-12), $m/\sqrt{\sigma} = 3.837$, so factor (ii) is $1/3.837^2 \approx 0.068$. Computed Whether (ii) can be bounded above and below by positive constants independent of β is the question of one scale; the lattice values show the tendency, but are not a proof.

The dictionary and where things stand (Parts 16 and 18)

The two columns of settledness

A way of reading that attaches two grades separately to the correspondence between the physical and the mathematical vocabulary: (a) whether they correspond as definitions, and (b) whether the quantity has been shown to exist in four-dimensional Yang–Mills. → 18-01

Seam

A place where the physical way of digging and the mathematical way of digging do not connect. This book lines up seven of them by type. → 16-05

The study with the relaxation-time assumption at its axis (Part 23)

The side of the four-dimensional lattice

Eight statements of construction

The eight statements into which the problem statement is divided along the road that enters from the lattice (infinite volume, reflection positivity, Wilson loops, physical gap, slowness of fluctuations, recovery of rotations, OS reconstruction, non-triviality). The relaxation-time band at the bookkeeping length is not among the requirements of the problem statement; it is a restatement of the physical-unit gap statement. → 23-01

The bookkeeping length ξ_{RG}

A length fixed by keeping the books on how the coupling decreases at each step that coarsens the lattice by a factor b, and taking the number of steps it takes to fall to a fixed value. We set $a := 1/\xi_{\mathrm{RG}}$ (the definition of the four-dimensional study in Part 23). → 23-01

Relaxation-time assumption

The power statement that the Poincaré constant C_P (the relaxation time of Langevin dynamics) grows with the same power as the square of the correlation length: $\ln C_P / \ln \xi \to 2$. In four dimensions, for the Wilson measure as $\beta \to \infty$. There is no proof. → 23-01

The band statement

The statement "C_P lies within a band of constant multiples of the square of the ruler": $a^2 C_P \asymp 1$. It is moved by logarithmic factors, so in this part it is not called the relaxation-time assumption. → 23-01

Relaxation-time upper-bound assumption, relaxation-time lower-bound assumption

The upper and lower halves of the band statement: the upper-bound assumption is $a^2 C_P \le C$ (an upper bound over all test functions), and the lower-bound assumption is $a^2 C_P \ge c$ (a lower bound for which one test function suffices). The hard one is the upper-bound assumption. Names distinct from the relaxation-time assumption. → 23-01

Band form of the diffusive-relaxation-time assumption $(\xi^2)_{\mathrm{band}}[\mathcal{E}]$

An assumption that compares within a band of constant multiples rather than by powers. Whether the band statement holds depends on the graduation of $\mathcal{E}$ (link angle or field), so it is written with the graduation named. An assumption distinct from the relaxation-time assumption. → 23-01

Relaxation chain, one-scale chain, bookkeeping-length chain

That each of the three factors into which $a^2 C_P$ splits is $\asymp 1$: relaxation and correlation length (the relaxation chain), correlation length and string length (the one-scale chain), string length and bookkeeping length (the bookkeeping-length chain). If all three hold, $a^2 C_P \asymp 1$ follows (the converse does not). → 23-01, 17-12

The one-sided bridge

A proposition that, for the path measure with continuous time, a finite spatial lattice and $T = \infty$, under assumptions such as on the ground state, bounds C_P from below

by a constant times the square of the correlation length, using test functions local in time. It spans only the side of the lower-bound assumption. → 23-01

Topological sectors and the order of volume

In a periodic box, the topological charge Q, which moves between sectors with difficulty, also enters the test functions. Take the reading of the original sources for $SU(3)$ and above that the autocorrelation of Q grows faster than ξ^2 (for $SU(2)$, a single primary source, close to a^{-2}), and apply to it, extended, the conjecture attached to the observation with open boundaries that HMC and Langevin are in the same universality class: then the upper-bound assumption reads as false in a periodic box. In this book's order, where the volume is made infinite first, open boundaries are not expected to be a rescue (an assumption row). → the note to 23-01 § 05

Seventeen roads tried on the four-dimensional lattice

Each is a pair of one question and a set of tools applied to it. The road of comparison with the defect of a submission and the road of a one-dimensional chain of scales are restatements of the original questions. → 23-02

Twenty-one walls of the four-dimensional lattice

The reason, in one sentence, why a road stopped with a tool short of an arrow. The twenty-one walls are divided into five types: the wall of principle, the wall of structure, the wall of tools, the wall of assumptions, and the wall of measurement and literature. → 23-02

Complete family

A set of links that removes exactly one link from every plaquette. If the remaining links are held fixed, the weights of the links of the family separate one by one and can be integrated independently of each other. → 23-02

The shortfall δ(b), the slope κ

The amount by which the window of one lattice step falls short of the continuum $\ln b$ (a number measured in the model of the one-loop kernel), and the slope of the one-loop running of the coupling $\kappa = 11/(3\pi^2)$ (SU(2), $\beta = 4/g^2$; a rewriting of known coefficients). → 23-02

The form of the answer

The statement "under the relaxation-time assumption ($\eta = 0$), the physical-unit gap statement $\Longleftrightarrow \sum_k \delta(b_k) < \infty$". Besides the relaxation-time assumption, it stands on the power form, on a model in which the errors of the steps are additive and

bounded, and on δ being the value of the model of the one-loop kernel (assumption rows). → 23-02

Lower bound on the overlap

The input that, in the representation of the correlation by non-negative weights, the weight of the lightest state is at least q times C(0). With it, a lower bound on the gap follows from correlations at finite k. Without it, spectroscopy ends with an upper bound on the gap (the spectroscopic upper-bound wall). → 23-02

Assumption of projection onto a one-dimensional chain of scales

The assumption that the measure can be projected onto a one-dimensional chain in the scale $s = \ln \ell$. Under it, the two-weight Hardy inequality (a weighted inequality in one dimension whose best constant is bounded above and below by integrals of the two weights) can be used. It cannot be said to be weaker than the relaxation-time assumption. → 23-02

Tube transfer-operator inequality

An inequality for the transfer operator saying that the weight of a tube of perimeter ℓ is at least $\sigma\ell + c/\ell$. If it holds, $m^2 \geq 4c\sigma$. There is no proof. → 23-02

The model of the first step

The staircase (four steps)

Four steps placed from an easy place to a hard place: the path measure of the double well in 0+1 dimensions ($\hbar \to 0$), the Wilson measure of the four-dimensional lattice ($\beta \to \infty$), the continuum limit and the OS axioms, and the mass gap of the limiting theory. This part deals with the first and second steps. → 23-03

Double-well relaxation-time power statement

The statement of the first step, $\ln C_P / \ln \xi \to 2$ ($\hbar \to 0$). It is a statement of the same form as the four-dimensional relaxation-time assumption, differing only in the family of measures, so we call it by the same name on both steps. Distinct from the band statement, which speaks of a band of constant multiples. → 23-03, 23-03 § 01

The five grades (proved here / cited / computer-assisted / reading / assumption)

The labels for the strength of claims in this part: elementary (the proof is placed in the text), a known theorem cited from the original source, computer-assisted, reading (floating point, Monte Carlo, fitting), assumption. One label per sentence. → 23-00

The double well

The potential energy with two valleys, $V(x) = (x^2 - 1)^2/8$. The valleys are at $x = \pm 1$ (curvature 1 at the bottom), and the top at $x = 0$ (curvature −1/2). → 23-04

The path measure

The probability measure that gives paths in imaginary time the weight $e^{-S/\hbar}$. $T \to \infty$ is taken first, then $\hbar \to 0$. The numbers are taken on the lattice with step $\varepsilon = \frac{1}{2}$. There is no gauge field in it. → 23-04

Tunnelling splitting

The difference between the two lowest levels of the Hamiltonian, $\Delta = E_1 - E_0$. In terms of paths, it is determined by the weight $e^{-S_0/\hbar}$ of inserting one kink. → 23-04

Correlation length (of the double well)

The length in imaginary time over which the correlation between two points of a path decreases, $\xi = \hbar/\Delta$. At the leading semiclassical order it grows exponentially, $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$. → 23-04, § 03

Kink

A path that moves from one valley to the other; after the paths that stay in a valley, the path with the smallest action. Its action is $S_0 = 2/3$ (the action of the instanton), and its width is about 2 in imaginary time. → 23-04

Zero mode

The direction $\dot{x}_{\mathrm{cl}}$ with eigenvalue 0 of the fluctuation operator of the kink. It comes from the fact that shifting the kink in the direction of imaginary time does not change the action: one slow degree of freedom, the position of the kink. → 23-04

Diffusion coefficient of the kink

$D = \hbar/S_0$, obtained by projecting the Langevin noise onto the direction of the position of the kink. A statement about the position of one kink, not about the gap $1/CP$ of the whole path. → 23-04

Herring's formula

A formula that writes the difference between the even and odd eigenvalues of the transfer operator directly as a ratio of inner products of eigenfunctions, without finding the two separately and subtracting. It avoids the loss of digits when $\hbar$ is small. → 23-04

Plateau

A flat direction, along which the action hardly changes, made by the stretch between a pair of kinks where the path stays near the top. The curvature of one site

at the top falls short by $\frac{1}{4}$ of the value needed for convexity ($\varepsilon = \frac{1}{2}$). → 23-04

The convex core

The set deep in a valley, $\{x_i \geq a\}$ ($a > 1/\sqrt{3}$). If the path is confined here, $C_P \leq 2/(3a^2 - 1)$ follows, of Bakry–Émery type, uniformly in the length and in $\hbar$. → 23-04

Phase (the phase without kinks)

The measure restricted to configurations in which the path stays on the side of one valley. For example $\mu^{(+)}$, restricted to $x_i \geq 0$ at all sites (reflected at the wall $x_i = 0$). The measure of the in-phase local Poincaré condition. → 23-05

The assembly of the first step

The splitting of the variance

An identity that splits the configuration space into the two valleys and divides the variance into "the variance within each valley" and "the difference of the means of the two valleys". It is the first tool for assembling an upper bound on C_P from the constants of local Poincaré within the valleys and of the transport of the mean difference, but for long paths a single coarse variable is not enough. → 23-05

Local Poincaré

The Poincaré inequality for the measure restricted to one of the parts (a valley, or the phase without kinks). Its constant C_{loc} becomes the in-phase local Poincaré condition. → 23-05

Transport of the mean difference

An inequality that bounds the square of the difference of the means over the two valleys by the Dirichlet form. Its constant C_{md} gives the type of the saddle-tube condition. → 23-05

Four double-well conditions

The four conditions into which the upper side of the first step is divided: the in-phase local Poincaré condition (local Poincaré in the phase without kinks), the saddle-tube condition (transport along the family of flat saddle points; inside the box, the rows of the tube and the core), the box-reduction condition (reduction to the box dynamics), and the gap-inside-the-box condition. Each leaves assumption rows. They are distinct from the relaxation-time assumption, which is a statement on the side of the conclusion. → 23-05

Double-Well Relaxation-Time Upper-Bound Proposition

The conditional proposition that, for the path measure of the double well, assuming the four double-well conditions and the premise rows, $C_P \le K\xi^2 S_0/\hbar$ holds, and through $1 \le R \le K'/\hbar$ the double-well power statement follows. It is algebra on top of the four conditions, and does not say that the first step is closed. → 23-05, 23-03

Scrap

A stretch of the phase without kinks, sandwiched between kink and kink inside the box (of length at most $C\xi$, with values at both ends). The version of the in-phase local Poincaré condition on it is the segment local Poincaré assumption, which requires uniformity also in the end values. → 23-05

Tube

The collection of configurations around a kink, or a kink–antikink pair. The row that bounds the Poincaré inequality of the tube left after the configuration of kinks is held fixed is the saddle-tube Poincaré assumption. The one on a stretch with both ends held in the two valleys is called "the bridge tube". → 23-05, 23-07

Core

The stretch inside the tube where the path stays near the top ($Q_k = (-x_c, 0)^k$, $x_c = 0.89$). The Hessian of the action has negative directions there; bounding its Poincaré constant uniformly in the length k of the core is the row of the core Poincaré uniformity assumption. → 23-07, 23-11

The box dynamics (reduction to the box)

The step that bounds the Poincaré constant of an infinitely long path by the Poincaré constant of the conditional measure on a box of length $C\xi$ with the values on both neighbours held fixed (the box-reduction condition). The Box-Reduction Proposition has the form $C_P \le (1 + 1/n)\, C_{\mathrm{box}}/(1 - c_{\mathrm{alt}})$ and does not use the logarithmic Sobolev inequality. → 23-05, 23-11

Even and odd boxes

The line covered by boxes of length L, neighbours overlapping by l sites, divided into two families by the parity of their numbers. Boxes of the same family are apart from each other, and become independent of each other when the values outside the family are held fixed. → 23-11

Averaging over shifts

Shifting the arrangement of boxes by l sites at a time in n ways, and averaging the inequality for each shift. Since the overlap sites are disjoint across the shifts, the coefficient is only 1+1/n. → 23-05

Maximal correlation

For two σ-algebras, the supremum of the correlation between functions measurable with respect to each. For variables at two points, it can be expressed as the second singular value of the normalised matrix of the joint distribution. → 23-05, 23-11

Alternating maximal correlation

The maximal correlation c_{alt} of the two σ-algebras generated by the values outside the even family and the values outside the odd family. That this is below 1 uniformly in $\hbar$ is the assumption row that remains in the Box-Reduction Proposition (the alternating maximal-correlation assumption for boxes). → 23-05

The lemma of two projections

The Two-Projection Lemma. If the maximal correlation c of two σ-algebras is below 1, the variance is at most $1/(1-c)$ times the sum of the expectations of the two conditional variances. An elementary proof is placed in the text. → 23-05

The Efron–Stein inequality

The inequality that the variance of a function on a product measure is at most the sum of the expected variances when the coordinates are moved one at a time. Used for the inequality on the product of the even boxes. → 23-11

Reduced kink gas

A model that keeps only the slow variables inside the box (the number and positions of the kinks). If it is put into the same type as the Glauber dynamics of the one-dimensional Ising model (the row of the kink-gas gap assumption), the gap takes the form of a sum of the creation and annihilation of pairs and the diffusion of kinks. → 23-05

Cross term

The row that bounds the Dirichlet form of the slow side inside the box (the configuration of kinks Y) by $(1+\chi)$ times the original Dirichlet form (the cross-term uniformity assumption). That χ is uniform in $\hbar$ and in the boundary values is an assumption. → 23-05

Kept site

In the two-scale assembly of the in-phase local Poincaré condition, the site kept every 12 sites (the coarse side). When these are held fixed, the stretches between them become independent of each other, and the marginal on the kept sites becomes a nearest-neighbour chain. → 23-06

Bridge (a stretch with fixed ends)

A stretch with the values at both ends fixed. In the in-phase local Poincaré condition, the 11 sites between kept sites (the fine side); in the saddle-tube condition, a stretch with both ends held in the two valleys and one kink. → 23-06, 23-07

Nesting

The assembly that keeps sites 4 and 8 inside the non-convex 11-site bridge as one more level, splitting it into three small bridges (log-concave) and two sites. → 23-06

Helffer's criterion

The Poincaré version of a tool that judges the gap of a chain from the gap of one site and the strength of the coupling between neighbours. Used in Gershgorin form on the ring of kept sites. → 23-06

Margin

The key quantity of the criterion on the coarse side, $\hbar\rho - 2\hbar\kappa$ (the gap of one site minus twice the coupling to a neighbour). That it is positive is confirmed by interval arithmetic for every pair of end values. → 23-06

Node (of $\hbar$)

A point of $\hbar$ confirmed with computer assistance. The Six-Node In-Phase Poincaré Proposition is a statement at each of the six nodes $\hbar \in \{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$, with a margin of at least 0.067. Distinct from the nodes of the dependency graph. → 23-06

Subinterval

A small interval into which the range of $\hbar$ is cut in order to fill in the $\hbar$ between nodes. Within each subinterval, the movement of the expectation values with $\hbar$ is enclosed in an interval. → 23-06, 23-08

Allowance

In a subinterval, the amount of swelling permitted above the pointwise upper bound (the remainder after subtracting the pointwise upper bound from the upper limit fixed by the margin). If the same amount is allowed for every box, the worst box decides the allowance. → 23-06

Route

The seven lines of attack set up to close the row of the core of the saddle-tube condition (the chain outside the window, the remaining numbers, the sector framework, small assumptions, the one remaining proposition, the seam of the ring, the range of $\hbar$). The seven are called by name, and their order does not express sequence or weight. → 23-07

Site window

The range W of 17 sites around one site inside the core. The logarithmic Sobolev constant A_W of the marginal of its mark is central to the route of the window-constant numerics, the route of the sector framework and the route of the window log-Sobolev inequality. Distinct from the window of $\hbar$. → 23-07

The window of $\hbar$

The range $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ of $\hbar$ handled with computer assistance. The "window" in "the whole window" and "the lower end of the window" is this one; where they could be confused, "site" is added to the site window. → 23-07

The chain outside the window

The chain extending outside the site window. Its contraction ratio $q < 1$ is the condition of the route of the chain outside the window and the route of the seam of the ring; for the core of the bridge, $q_{\mathrm{free}} \le 0.998854904$ became a computer-assisted inequality over the whole window of $\hbar$. Nothing is said outside the window or about $\hbar \to 0$. → 23-07, 23-08

Ring

A short periodic path of length $T \ll \xi$ with a configuration containing one kink–antikink pair. → 23-07

Seam

The place on the ring where the separation ℓ of the pair shrinks and the pair disappears. The configuration there is the plateau at the top itself, and the same difficulty as in the core of the bridge tube appears. → 23-07

Lump

A run of sites close to the top ($|z| < \theta$, $\theta = \frac{1}{2}$), taken as the unit of a sector that does not depend on the sign at the seam. With the rest held fixed, the action of a lump of k sites is convex for $k \le 7$. → 23-07

Outer chain and inner chain

The two kinds of chain that enter the logarithmic Sobolev constant of the site window of the ring. The outer chain is the chain on the side of fixed sign extending outside the window; its contraction is a computer-assisted inequality under conditions such as inside the window and pin ≤ 0. The inner chain is the stage inside the window; its ratio reads 1.24–1.28 and exceeds 1 (a finite coefficient). → 23-07

Boundary stage

A finite stage that appears only for configurations in which the lump touches the edge of the window (pin in $(0, \frac{1}{2})$). Its ratio reads 1.004–1.045, and the computer-assisted row of the outer chain does not cover this pin. → 23-07

Sector framework

The procedure for bounding the gap of the core uniformly in k by paying with the rarity of the places of negative curvature, divided into five stages (sector sum, Gibbs decomposition, componentwise localisation, volume locality, sector composition). What remains as an assumption is not the structure but the size of the constants. → 23-09

Sector

The direct product of boxes for each type of the marks attached to the sites (on the side of the top or not). The variance, the Dirichlet form and the entropy split into sums over sectors, and the number of sectors does not enter the constants. → 23-09

Defect set

The set N_j of configurations in which site j or one of its neighbours is on the side of the top beyond the threshold. Negative curvature exists only inside it. → 23-09

Rarity exponent

The minimum $\ell_{\min}$ of the logarithm of the measure of sets containing a defect. Not a length; it does not depend on the length k of the core. → 23-09

Closing condition

The condition $b(A_W c_H + D_W + \ln 2) < \kappa \ell_{\min}$ under which the right-hand side of the inequality assembled from the five stages of the sector framework is positive. If it is satisfied, the gap of the core is bounded below by a positive number independent of k. → 23-09

Needed value

The upper limit $A_{\max}$ obtained by rewriting the closing condition for A_W. That A_W is smaller than this is the one remaining proposition, the Window Log-Sobolev Threshold Proposition (the route of the window log-Sobolev inequality). The present upper bound lies outside the needed value, and the lower bound from test functions does not exclude the needed value. → 23-09

The three strands

Attacks on the one remaining proposition in three forms that differ in the functional form of the condition: the componentwise form, the scalar form (the Scalar-Strand Proposition), and the Feynman–Kac form. None of them is closed. → 23-09

Scalar-Strand Proposition

The proposition of the scalar strand, with a table of assumptions. With settings such as core length $k = 20$, it bounds the remainder quantities at each of 8 nodes in $\hbar \in [3 \cdot 10^{-4}, 10^{-3}]$. The interval-arithmetic rows have not yet been run. → 23-09

Lifting assumption

The assumption lifting the proxy ratio, in the Scalar-Strand Proposition. The only assumption, which lifts the proxy ratio linearly outside the measured range (the values for lumps of two sites). → 23-09

The common wall

The same place where the three strands stop: at the top, the diagonal of one site, $4 - \frac{1}{4} = 15/4$, does not reach the sum 4 of the couplings to its neighbours. A tool that has convexity as its entrance must pay for this ¼ with the rarity of the plateau, and $\hbar$ enters into the way it is paid. → 23-09

Diagonal dominance

That in each row of a matrix, the diagonal is at least the sum of the absolute values of the off-diagonal entries. For the Hessian of the core it is broken by just ¼ at the top, and on the face where every site is at the top, the smallest eigenvalue of a lump with $k \geq 8$ is negative. → 23-09

Detour plan

A plan that moves one of the two sides of the closing condition, or the form of the inequality itself. The list of detour plans was judged Alive, Conditional, On hold or Dead. → 23-10

The line that killed it

The fact that stopped a detour plan, shrunk to one line. The numbers are readings from a floating-point grid or a transfer matrix. → 23-10

Needed side, form, insufficient side (floor)

The three sides into which the detour plans are divided by direction. The needed side holds plans that make the needed value $A_{\max}$ larger; form, plans that change the functional form of the condition; the insufficient side (floor), plans that reduce the structural loss in the upper bound on A_W. → 23-10

Choice of structure and of tools

The two sources of the constants in the closing condition. The structural quantities (κ, b, $\ell_{\min}$) are decided by the model and by how the defect set is defined; the tool quantities (c_H, ln 2) depend on the choice of the steps of the proof and can be

changed with the same set. What moved the needed value a great deal was the tool side. → 23-09, 23-10

Types of breakage

The reasons that appeared repeatedly in the lines that killed the dead detour plans, gathered into seven (one source, ceiling, paying twice, mixing up the currency, the address of $1/\hbar$, misdirection, transfer). Each type comes with a question to test against a plan. → 23-10

Paying twice

The double-payment type. Paying the same amount again on a set that an earlier step has already paid for. The original Bochner step paid for the negative curvature over the whole core, but that was inside the defect set. A type where fixing it has a large effect. → 23-10

Currency (speed of exit and entropy)

A word of the currency-mix-up type. What the Poincaré inequality measures is the speed of exit; what can pay for the rarity $\mu(B) \le e^{-\ell}$ is the currency of entropy. With only the ratio of the variance to the Dirichlet form, the rarity cancels between numerator and denominator. → 23-10

The address of $1/\hbar$

The type of the address of 1/ħ. If one misjudges which quantity $1/\hbar$ multiplies, a plan merely moves $1/\hbar$ into sectors, marks, comparison constants or the cost of a perturbation. On the plateau at the top, $1/\hbar$ lives not in the direction of the sum but in the radius from the corner of the top. → 23-10

Threshold of convexity

The boundary: the lower bound $\kappa(k) = 8\sin^2\!\big(\pi/(2(k+1))\big) - \frac{1}{4}$ on the Hessian of the action of the core is positive only up to $k \le 7$. A core with $k \le 7$ is bounded by convexity (the Short-Core Convexity Lemma), independently of the end values and of $\hbar$. → 23-11

Three-Range Core-Constant Proposition

The proposition that bounds the Poincaré constant of the core in three ranges by length k: $k \le 7$ by convexity, $8 \le k \le 14$ by the Two-Block Lemma and the maximal correlation of two points, and $k \ge 15$ by applying the reduction to the box to the core. The inputs of the medium-core step and the long-core step are measured values. → 23-11

The wall of conditioning

A wall standing in the direction in which the zero wants to move, because the stretch of the core was conditioned to be "exactly this stretch". In the core on the side of the box whose ends cross the top, this creates bistability. Not physical metastability, but something made by the choice of conditioning. → 23-11

The action barrier

The row of the core action-barrier assumption. The assumption that the Poincaré constant of the core on the box side is bounded in the form $e^{\Delta S_*/\hbar}$ (with a prefactor a power of $\hbar$), and that the barrier is smaller than $2S_0$ ($\Delta S_* < 2S_0$). Under it, the local term of the Gap-Inside-the-Box Proposition vanishes as $\hbar \to 0$. → 23-11

Computation and checking

Computer-assisted

What has been confirmed as an inequality by interval arithmetic (the grade "computer-assisted"). Premises such as the rounding convention are written on the same row. Nothing is said to hold outside the computed range. → 23-08, 23-00

Interval arithmetic

Computation that holds every number as "an interval that is sure to contain the true value". If the upper end of the final interval is smaller than 1 ($q^+ < 1$), then $q < 1$ follows as an inequality. → 23-08

Outward rounding

When computing the ends of an interval, rounding the lower end down and the upper end up. In this part, the result of each operation is moved outward by one to the neighbouring floating-point number. Correct rounding in IEEE 754 is placed in a premise row. → 23-08

Loss of dependence

That intervals forget the ties between quantities coming from the same variable, and widen (evaluating $x - x^2$ naively on [0,1] gives [−1,1]). Fixed by subdividing finely or expanding about the centre. → 23-08, § 04

Positive control, negative control

A positive control asks whether the machine reproduces the answer in a case where it is known in closed form. A negative control asks whether the machine rejects a claim that should not hold. → 23-08, 23-14

Pre-registration

Writing down, before running a computation, the expected range of values, the rate-limiting place and the pass line for the controls, and recording after the result whether they were right. It becomes a record that the pass line was not moved after seeing the result. → 23-08

The integrated table

A table that puts the rows of the five grades for the four conditions of the first step on one sheet, row by row. A map of where there are proofs and where assumptions remain; the proportion of rows does not express progress. → 23-12

The dependency graph

A list that turns the rows of the table into numbered nodes and lines up the premises of each row to the right of "⟸". Which rows the conclusion of the First-Step Upper-Bound Proposition depends on is read by following the premises. → 23-12, 23-05

Node, leaf

A node is one row of the dependency graph. A leaf is a node with no "⟸": a row of assumption, known, computer-assisted or reading. → 23-12

The rule of numbering

The rule that the numbers of the premises to the right of "⟸" are always smaller than the number of the conclusion on the left. If it is kept on every row, the number goes down each time a premise is followed until a leaf is reached, and there is no circle. → 23-12

The required set of leaves

All the leaves that can be picked up by following the premises from the conclusion [111]. The only fork is the single "or" at [103], so there are two sets. → 23-12

The two roads (the core on the box side, the core on the sector side)

The two ways of supporting the fork [103] (the row that the Poincaré constant of the core is uniform). The road of the core on the box side supports it with the Three-Range Core-Constant Proposition and the vanishing of the local term as $\hbar \to 0$; the road of the core on the sector side with a uniform lower bound on the gap of the core in the sector framework and the regularity of the seam. → 23-15

Kinds of work

Six ways of classifying the work needed to close rows of assumption or reading: interval arithmetic, paper (writing a proof), bibliography (checking in the original

source and raising it to "cited"), measurement, another method (out of reach of the present tools), method undecided. → 23-15, § 03

A $\hbar$ that fits all

A $\hbar$ at which every row of the four conditions holds at the same time. Overlaying the ranges of the rows, there is none at present. The double-well power statement ultimately needs every row stated in a form uniform as $\hbar \to 0$. → 23-12

The criterion of cleanness

That the output of `#print axioms` for a Lean theorem is entirely a subset of the three axioms `propext`, `Classical.choice` and `Quot.sound`. It shows that the proof has no holes and that no arbitrary axioms have been added. → 23-13

Putting the conclusion into a definition

The form in which the desired property is written into a definition and that definition is taken as an assumption. The conclusion becomes a copy of the assumption, and the proof closes in one line. It is invisible to the criterion of cleanness. → 23-13

Circle of dependence

The form in which theorem A takes the conclusion of B as an assumption and B takes the conclusion of A. Theorems are checked one at a time, so it is invisible to machine checking; it is checked with the rule of numbering of the dependency graph. → 23-13, 23-12

The four verdicts of reconciliation

The verdicts when the numbers measured in this book's study are compared with the known ones: Agreement, Conditional agreement (how they agree depends on the choice of step, box, convention or form of extrapolation), Disagreement, Not comparable. → 23-14

The words of Part 24

24-00 Entrance

Further considerations

The title of Part 24. A part that considers, in order, the shape of what hinders the relaxation-time assumption, two candidates for restating while keeping the relaxation-time assumption as a premise, two views that do not go through the relaxation-time assumption, and what is at work in four dimensions. Many of its sentences are estimates. → 24-00

The trade of the restatement

Restating the quantity of the problem statement (the gap in Euclidean time) as the quantity of the relaxation-time assumption (the relaxation CP in Langevin time: a quantity defined by a supremum, for which one test function gives a lower bound, and whose upper bound can be assembled from parts). The two are different quantities. → 24-00, 23-01

The estimate label Estimate

A sixth label added to the five labels of Part 23 (proved here, cited, computer-assisted, reading, assumption). Attached to sentences that are opinions offered as material for judgement, with neither proof nor numbers. → 24-00

24-01 A nearly conserved global quantity

A nearly conserved global quantity (one type)

A global quantity Q whose variance is large and whose Dirichlet form is small (it is passed along through fluxes, or changes only in rare configurations), and which does not show up in the gap. Its relaxation time is decided not by the correlation length but by the scale of the events that break Q, and it hinders the upper-bound assumption. That the five examples fall into the same type is a classification (an estimate). → 24-01

The coefficient of breaking of conservation h

The density of the expectation of the square of the sum of the gradients of the charges of subregions (the gradient of the total charge). In the continuum it is 0 by topological invariance; only on the lattice is it non-zero, on configurations with dislocations. What the upper-bound assumption as a whole asks about is this coefficient. This book has no value for the four-dimensional lattice. → 24-01

The transport coefficient d

The density of the expectation of the square of the first moment (the dipole) of the field of gradients of the charge. In the regime where h is small, what the local diffusive-relaxation-time assumption asks about is this coefficient. There is no value for the four-dimensional lattice. → 24-01, 24-03

The boundary-layer coefficient g

The quantity per unit area obtained by subtracting the volume term from the Dirichlet form of the charge cut by a half-space. It enters the boundary correction of the ratio for a sharp box. → 24-01

The crossing length R_*

A length determined by the ratio of the transport coefficient to the coefficient of breaking of conservation (the square of R_* is dλ$_W$/h). The boundary at which the ratio of the charge of a smooth window is of transport type, proportional to R^2, in boxes shorter than this, and of barrier type, saturating at χ_t/h, in longer boxes (an estimate resting on the assumption of cluster decay). → 24-01

Flux through the box

A field running uniformly through the box, which enters the space of tests if the gauge transformations at the boundary sites of the box are left without dividing out. It moves the reference value by a factor of 2.2–2.5 (a reading of the free-field reference numbers). An example where the same type arises not from the dynamics but from the choice of family. → 24-01, 24-03

24-02 The length at which conservation breaks

Three rulers

The candidates to compare CP against, divided into three: the static correlation length ξ, the length at which conservation breaks ℓ_*, and the time for the global quantity to move by one τ_{glob}. They can take different values within the same model. → 24-02

The length at which conservation breaks ℓ_*

The length at which the charge of a subregion begins to relax not by transport through the boundary but by breaking of conservation inside the region. The candidate definition is (2): the smallest R at which the ratio Var/☒ reaches half its saturation value (an estimate). For two-dimensional $U(1)$ on a strip of width 2 with the outer boundary frozen, β = 2–6 and strip length $M \le 256$, the length read from the reading of τ_{int} (an upper bound on the gap) using the fit form $A/M^2 + B$ is not proportional to the correlation length. → 24-02, 24-02 § 03

Tree coordinates and link coordinates

In the Gaussian approximation on the hierarchical lattice, taking the values of the plaquettes as variables gives $CP = 1/\beta$ independent of the number of steps, while taking the link variables makes the gap shrink in inverse proportion to the square of the side. An example where the type of growth of CP changes with the choice of coordinates (the four-dimensional hierarchical lattice is an extension of this book). → 24-02

Conservation of flux

The property that when a link is moved, the values of the plaquettes (the flux) are only passed between neighbours. The source of the slowness of link coordinates. → 24-02

Flux dipole

The quantity X obtained by summing the angles of the plaquettes of the strip with cosine weights. The slowest mode of the Gaussian approximation; the length at which conservation breaks is fixed from the bend in its relaxation. → 24-02, § 04

Wrapping

The event in which the angle of one plaquette crosses $\pm\pi$. A rare event that crosses a barrier of height 2β; it breaks conservation because the group is compact. → 24-02

Topological charge of the strip

The integer obtained by reducing the angles of the plaquettes of the strip to $(-\pi, \pi]$, summing and dividing by 2π. It changes by one at each wrapping. We do not claim it is the same thing as the four-dimensional topological charge. → 24-02

Raw rate and static fluctuation

The rate of relaxation of the topological charge of the strip, split into the ratio of the rate of wrapping events (a dynamical quantity) and the static fluctuation E (a quantity of the measure), which measures how much the charge moves in one event (an estimate of the chain). → 24-02

τ_{glob}

The time for the global quantity to move by one. The form that compares CP with $\max(\xi^2, \tau_{\mathrm{glob}})$ is a candidate estimate set up from readings of the numbers of the two-dimensional strip, and is not shown in any model. → 24-02

Two-dimensional test bed

For the two-dimensional CP^{N-1} model, a comparison of freezing in a periodic box with the local ratio of a box as N is varied. The ratio is a ratio of autocorrelations with the lattice side fixed at $L = 64$, a different quantity from the static ratio of 24-03. The two-dimensional results imply nothing about four dimensions. → 24-02

24-03 Two ways to repair the relaxation-time assumption — the sector assumption and the local assumption

The algebra of time-averaged functionals $\mathcal{D}$

The collection of test functions built with smooth functions from functions at one time averaged over time with weights of bounded support. The CP of Part 24 is

defined as a supremum over this. → 24-03

Window

The time interval $[0, c\ell]$ of c times the ruler of length ℓ (in the version local also in space, a spacetime box of side $c\ell$). ℓ is not fixed by the gap but given from outside (in four dimensions, the bookkeeping length). → 24-03

The local Poincaré constant $C_P^{\mathrm{loc}}(c)$

The supremum of $\mathbf{Var}/\mathcal{E}$ taken only over test functions whose weights have support inside the window. It increases in c, and returns to C_P without assumption as $c \to \infty$. → 24-03

Local relaxation-time assumption ($(\xi^2)\uparrow_c^{\mathrm{loc}}$)

The statement that $C_P^{\mathrm{loc}}(c)$ grows with the same power as ℓ^2, and its upper half $(\xi^2)\uparrow_c^{\mathrm{loc}}(K)$ ($C_P^{\mathrm{loc}}(c)$ is at most K times ℓ^2). It has content different from the relaxation-time assumption as a whole only when stated with one c held fixed. The definition stands on continuous time, a finite spatial lattice, $T = \infty$ and the assumptions of the one-sided bridge. The version for the four-dimensional Wilson measure is a copy of it, with the existence of the infinite-volume measure placed as a premise. → 24-03

The ratio form K_{rel}

The ratio of the local Poincaré constant of a box for the target measure to the value of the massless reference on the same lattice, the same box and the same graduation. The form "bounded by some $K(c)$" is satisfied by the free-field computation for boxes in four spacetime dimensions that are local also in space, and so is empty; hence it is stated with this ratio. → 24-03

The massless reference and Γ

The reference is the local Poincaré constant $C_P^{\mathrm{loc},0}(R)$ of a box for the infinite-volume massless free field, and Γ is its coefficient $C_P^{\mathrm{loc},0}(R)/R^2$. The numbers are readings of the free-field reference, not numbers for the target measure. → 24-03

Gauge quotient reference

The reference for the gauge field measured on functions invariant under gauge transformations at all sites of the box (for a contractible box, functions of the angles of the plaquettes). It does not depend on the parameter of gauge fixing, so this one is used. → 24-03

Sector relaxation-time assumption (upper-bound assumption^{sec})

The statement that, in a box with its physical side held fixed, the Poincaré constant within a sector of topological charge is bounded by the square of the bookkeeping length. It only pushes the obstacle to the relaxation-time assumption into the quantity between sectors; it does not remove it. → 24-03

Sectors and the reflecting form

The sectors cut from the smoothed topological charge at regular values, and the Dirichlet form on them with no boundary condition imposed (the form of Langevin reflecting at the walls). The Poincaré constant of a sector is finite only when the sector is connected. → 24-03

The quantity between sectors C_{trans}

The supremum of the ratio of the variance of the conditional expectation onto the sector number to the Dirichlet form. It gives the sandwich $C_{\mathrm{trans}} \le C_P \le \sup C_P^{\mathrm{sec}} + C_{\mathrm{trans}}$. Its physical reading is the rate of dislocations. → 24-03

Open box and closed box

A box of the conditional measure with the boundary values held fixed (the box of the Box-Reduction Proposition in Part 23) is a closed box; a box of the marginal with the boundary not held fixed (the box of the local relaxation-time assumption) is an open box. → 24-03

24-04 Which form of the relaxation-time assumption the one-sided bridge needs

The family local in time $\mathcal{F}_1$

The family of functionals obtained by integrating functions at one time over time with bounded weights. Under the assumptions of the one-sided bridge and the canonical graduation, the supremum of $\mathrm{Var}\,/\mathcal{E}$ over this family is exactly ξ^2. The bridge uses only this part. → 24-04

The window factor $\varphi(x)$

$1-(1-e^{-x})/x$. The bracket in writing the ratio for the window T as $\kappa\xi^2\varphi(T/\xi) = \kappa T^2 g(mT)$; $\varphi(x)/x^2$ is strictly decreasing (the Window-Factor Monotonicity Proposition). Distinct from the angle of a plaquette φ_p. → 24-04

In-Window Supremum Proposition

The proposition that writes the supremum over the family local in time with the support of the weights restricted to the window $[0,T]$ in closed form, via the root of a transcendental equation (under the assumptions of the one-sided bridge and the canonical graduation). → 24-04

Two forms of the local relaxation-time assumption

The absolute form $(\xi^2)\uparrow_c^{\mathrm{loc}}$ ($C_P^{\mathrm{loc}}(c)$ is at most $K(c)\ell^2$) and the ratio form $(\xi^2)\uparrow_c^{\mathrm{rel}}$ ($K_{\mathrm{rel}}(c)$ is at most K). The constants correspond by $K(c) = K_{\mathrm{rel}}\,\Gamma\,c^2$. → 24-04

Threshold K_*

$K_* = \kappa/(\lambda_W \Gamma)$. Under the local-excitation moment assumption and the infinite-volume ground-state transformation identity, if the ratio K_{rel} is smaller than this, a lower bound on the gap follows (algebra on top of assumptions). Its value changes with the choice of reference. → 24-04

Local-excitation moment assumption

The assumption that there is a local observable supported in the box such that the moment ratio of the spectral measure of the excitation it creates is at most the square of the infinite-volume gap m_∞ plus a box-size correction $\gamma/(c\ell)^2$. It holds for the free field, but is not shown for any interacting model. → 24-04

The band $[K_*, 1)$

For fields whose ratio lies in this range, the local bridge says nothing. The massive free field also lies in this band. Whether there is another argument that sees the mass inside the band is undecided. → 24-04

Colour singlet and the second chaos

Gauge invariants of $SU(2)$ (colour singlets) have no first-order term around the identity, and at the leading order ($\beta \to \infty$), with three independent Gaussian colours, they lie in the second chaos or higher. For this reason the colour-singlet reference is exactly half the value of the gauge quotient. → 24-04

Functionals of winding-number type

Functionals outside the family local in time, such as the functional that integrates with weights the increment of the lifted angle of a free particle on a circle. All of the amount by which the upper-bound assumption is stronger than the gap lies outside this family. → 24-04

24-05 The Hamiltonian lattice and the vacuum measure

The vacuum measure ν

The distribution of the vacuum at one time (the distribution with density the square of the ground state, ψ_0^2). In gauge theory the functions are restricted to gauge-invariant ones. → 24-05

Ground-state transformation

The isometry that sends a function F to $F\psi_0$. With it, the quadratic form of ($H - E_0$) is carried to $\hbar^2/2$ times the Dirichlet form of ν. → 24-05

Gap and Vacuum Measure Proposition

Under the ground-state assumption ((A1) in the paper), the gap can be written with the Poincaré constant of the vacuum measure as $\Delta = \hbar^2/(2C_P(\nu))$, and the supremum is attained by ψ_1/ψ_0. → 24-05

Three Poincaré constants

The C_P of the path measure (a one-sided relation to the gap), the supremum over the family local in time ($\xi^2 = \hbar^2/\Delta^2$), and the constant of the vacuum measure ($C_P(\nu) = \hbar^2/(2\Delta)$). → 24-05

The Kogut–Susskind lattice

The Hamiltonian of a gauge theory with time taken continuous and only space made a lattice (the sum of an electric term and a magnetic term). The quantity of the problem statement is exactly the gap on its gauge-invariant subspace. → 24-05

frustration-free

That, when the Hamiltonian is viewed as a sum of local terms, there is a state that realises the ground states of all the local terms at once. Local-to-global gap theorems need this. → 24-05

Electric–Magnetic Splitting Frustration Proposition

The sum into which the Kogut–Susskind lattice is split as an electric term and a magnetic term is not frustration-free ($g > 0$, finite spatial lattice). For this reason local-to-global gap theorems do not apply to this splitting. → 24-05

The Hamiltonian limit

The order of limits in which the temporal lattice spacing is sent to 0 first. That it leads to the same continuum limit as Wilson's isotropic lattice is an assumption. → 24-05

24-06 Paying with rarity instead of convexity

The ceiling of convexity

On a compact manifold without boundary, the oscillation of a potential satisfying $\operatorname{Hess} V \geq -K\, g$ is at most $KD^2/2$ (the Convexity Ceiling on Groups Lemma). On $SU(2)$ a convex measure cannot leave the range of constant multiples of Haar, and cannot concentrate at the identity at weak coupling. → 24-06

The Bakry–Émery lower bound on curvature κ_{BE}

The constant in $\mathrm{Ric} + \mathrm{Hess}\, V \geq \kappa_{\mathrm{BE}} > 0$. If this holds, log-Sobolev and $C_P \leq 1/\kappa_{\mathrm{BE}}$ follow (a copy of the statement in the original source). A different quantity from the comparison constant κ of the one-sided bridge. → 24-06

Fibre

Where the conditional measure lives on the fine fluctuations (products of many links) when the coarse field is held fixed. Reading the inductive convexity assumption of hierarchical models as Bakry–Émery on the fibres over the group is an estimate. → 24-06

The reversal of convex and non-convex places

For flat ϕ^4, what is non-convex is small fields; on the group, it is the hemisphere far from the identity (large fields). On a closed manifold there is always a non-convex place, but the probability of being there is small. → 24-06

The rate of rarity s

The exponent when a bad set B satisfies $\mu(B) \leq e^{-s}$. The quantity written ℓ in 23-09 and 23-10 of Part 23; in Part 24 the letter was changed because ℓ is used for the ruler. → 24-06

Currency (entropy)

The distinction that the Poincaré inequality is a currency that measures the speed of exit, and that what can pay for rarity is the currency of entropy. A step that pays with rarity goes through an inequality of log-Sobolev type. → 24-06

The Holley–Stroock perturbation

The tool that says that if a measure is changed by a factor e^{-U} with bounded U, the Poincaré constant changes by at most a factor $e^{\operatorname{osc} U}$. The cost is not rarity but the exponential of the oscillation (in this book, a standard form from memory, not checked against the original source). → 24-06

Lyapunov condition

The type that bounds the global Poincaré constant from a drift $LW \leq -aW + b\mathbf{1}_G$ returning from the bad set to the good set and a local Poincaré inequality on the good set. Convexity is used only on the good set. → 24-06

Field-dependent constants

The direction of letting the constants of the convexity assumption at each scale depend on the coarse field (but uniformly below 1). As long as uniformly below 1 is required, it does not get past the ceiling of convexity. → 24-06

The Polchinski flow on the group

The multiscale Bakry–Émery condition along the flow of covariances, carried over to convolution with the heat kernel on the group. A formulation on the group was not found within the range searched. → 24-06

The class of measures that pay with rarity

A candidate class of measures with the rules: locally convex on the good set, the bad set rare, a drift back from the bad set, and rarity paid only through entropy inequalities (an estimate). → 24-06

Subcriticality

The key condition of the solution theory of stochastic quantisation (regularity structures); it holds only when $d < 4$. Four dimensions is exactly critical. → 24-06

24-07 Because it is four dimensions

Critical dimension

The dimension in which the coupling g^2 is exactly dimensionless. For Yang–Mills it is four, and the correction at each change of scale does not shrink by a power but remains. → 24-07

marginal

A term whose coupling is dimensionless and which, by power counting, neither strengthens nor weakens. Four-dimensional ϕ^4 and Yang–Mills are both marginal, but the directions of their flows are opposite. → 24-07

Infrared asymptotic freedom

That the coupling approaches 0 at long distances (four-dimensional ϕ^4). The asymptotic freedom of Yang–Mills is at short distances, the opposite direction. → 24-07

Dislocation

A small instanton, the size of the lattice spacing, passing through the lattice. The smoothed topological charge on the lattice changes only at such moments. → 24-07, 24-01

The four hurdles

A table that lines up, for each of the four steps of Part 23 (the 0+1 double well, the four-dimensional lattice, the continuum limit and the OS axioms, the mass gap of the limit), the hurdle, the known point reached, whether it is in reach, and the kind of cost. Whether it is in reach and the cost are estimates. → 24-07

Three sources of difficulty

The sources of the difficulty of the third step: the source of a dimensionless coupling (all scales act with the same weight), the lack of an expansion linking scales (there is no expansion linking the ultraviolet and the infrared), and the lack of global gauge fixing (non-abelian gauge invariance; there is no global gauge fixing, and large fields cannot be controlled by convexity). → 24-07

Ultraviolet stability (as used in Part 24)

That on a finite-volume lattice, the effective action on the unit lattice stays bounded after arbitrarily many repetitions of the block-spin transformation. Cited as a known point reached in the third step. → 24-07, the entry above

The window of coupling

The range that remains between the currency of rarity on the ultraviolet side (the rate of rarity is proportional to the effective coupling and runs out at the confinement scale) and the currency of curvature on the strong-coupling side, where neither is enough. A restatement of the second source, the lack of an expansion linking scales (an estimate). → 24-07, 24-06

Dual superconductivity (as treated in Part 24)

The mechanism that explains confinement by the condensation of monopoles. The condensation has not been shown and the projection depends on the choice of gauge, so it is not used as a tool of proof. → 24-07, the entry above

Centre vortex (as treated in Part 24)

The mechanism by which vortices taking values in the centre carry the string tension. The evidence is lattice numerics, and there are neither estimate inequalities nor statements about the limit, so it is not used as a tool of proof. → 24-07, the entry above

24-08 Questions, and what is not claimed

Candidate rules (six, and three combinations)

Candidate systems that remove one assumption of an existing framework and put an alternative rule in its place: the axiomatisation of scale, the algebra of loops, conserved quantities and sectors, the class of measures that pay with rarity, the axiom of asymptotic freedom, the rule of curvature; and the rule of axiomatising scale + the rule of the rarity class, the rule of conserved quantities and sectors + the rule of the rarity class, the rule of the loop algebra + the rule of conserved quantities

and sectors. All are candidates, and not even their consistency has been checked. → 24-08

Status of the questions (done, partly done, not yet, new)

The status column of the table of questions in Part 24. "Done" means that the text has answered the question conditionally; "new" is a question added in the writing of this part. → 24-08

What does not move when settled

A column attached to each row of questions and candidate rules, stating what does not move even if that question is settled. Whichever question is settled, the mass gap problem does not move. → 24-08

11

Different quantities under the same name

The same letter or the same word can refer to different quantities in different chapters. This is because the conventions of each field are carried over; the meaning is declared at the start of each chapter, but it is missed when skimming. The heavy one is "gap": the gaps of three operators are called by the same word. The conversion between physical units and lattice units is the following line.

$$\Delta = \frac{m_{\mathrm{lat}}}{a}, \qquad \xi_{\mathrm{phys}} = a\,\xi_{\mathrm{lat}} \quad (10)$$

Example: if the gap in lattice units m_{lat} is held at 0.5 while a goes from 1 to 1/16, Δ increases from 0.5 to 8. If it is made proportional to a, as $m_{\mathrm{lat}} = 0.5a$, then $\Delta = 0.5$ is kept. Computed What the continuum limit asks for is the latter (23-01).

Table 3 Different quantities under the same name

Name	Meaning	Where
β	The inverse of the lattice coupling $2N/g^2$ / the inverse temperature of statistical mechanics $1/k_BT$ / the β function $\mu\,dg/d\mu$	5-01 / 2-08 / 4-04
Δ	Mass gap / Laplacian (on the lattice, Δ_{lat})	4-11 / 1-11
η	Static anomalous dimension η_s (the power of correlations) / dynamical $\eta_{\mathrm{dyn}} = 2 - z$. Always attach the subscript	18-05
b_0	One-loop coefficient. With $\mu\,dg/d\mu = -b_0 g^3$ it is $11N/(48\pi^2)$ (formula (5)) / in the table of 0-03 it is $11C_A/(12\pi)$ (0.5836 for $SU(2)$). The normalisations of the coupling differ	4-06 / 0-03

Name	Meaning	Where
C_P	Vacuum C_P: the Poincaré constant of the distribution of the vacuum at one time (equivalent to the gap of the Hamiltonian) / path C_P: the Poincaré constant of the path measure (the reciprocal of the Langevin gap; what the relaxation-time assumption refers to) / path C_P measured in a graduation: the path C_P with its normalisation named	23-01
Gap	The gap of the Hamiltonian (the mass gap) / the gap of the transfer matrix (the mass m in lattice units) / the gap of Langevin dynamics λ_1	4-11 / 5-03 / 2-10
Mass	The mass gap / the mass of a particle (an isolated eigenvalue) / the coefficient of the mass term of the action (the bare mass)	4-11 / 2-02
Critical	The point where the correlation length becomes infinite / "critical" for stochastic partial differential equations (counting of scales) / the critical dimension of string theory	2-08 / 7-09 / 17-05

Of the three gaps, the problem statement asks only for the first. The second passes to the first if the continuum limit exists and the ratio (10) converges (the physical-unit gap statement, 23-01), and the third is a property of the dynamics that produces samples. The relaxation-time assumption is a statement comparing the third with the second (23-01).

The words of the second road, the band and the ultraviolet (Part 15 and Parts 25–27)

The crossover band (the band)

The interval around the scale $L \sim 1/\Lambda$ at which weak coupling gives way to strong coupling. The subject of Part 26, and the place where the two roads reach the same wall. It is different from the "windows" of Parts 23 and 24 (the window of A_W, the window of $\hbar$, the time window), and the "band statement" of 23-01 § 05, ℓ_band of 23-09, the "two-dimensional strip" of 24-02 and the "twisted strip" of 25-07 also have different meanings (compared in 26-00 Table 0). → 26-00

The twisted T^2

A small torus with the 't Hooft twist $n_{12} = 1$ put in. The container in which the second road shrinks two directions. → 25-01

't Hooft flux

The sector n_{34} and the flux free energy. Uniqueness of the vacuum, the gap of the adjoint field, and the absence of flat configurations in the (1,1) sector. → 25-01

Fractional instanton

A self-dual lump of topological charge ½. Action $4\pi^2/g^2$. Its dilute gas gives the string tension. → 25-02

Container

The region $D \times T^2$ with the frame held fixed. By the container identity (Mack–Petkova), conditioning on the frame makes things independent. → 25-03

The ensemble summed over sectors, the bias θ, the insertion ratio ρ

The ensemble defined by the sum over sectors, the sector bias $\theta(R;T^2_L)$ ((4) of 25-01), and the vortex insertion ratio ρ. Driven by an involution oriented to the sectors. θ is a different quantity from the thresholds θ_B, θ_σ of 23-09 and from the vacuum angle θ (26-00 Table 0). → 25-05

Slab Confinement Theorem

The theorem that, with the lattice spacing and the T^2 held fixed, the area law holds at every β. In the continuum limit the lower bound on the tension goes to 0. → 25-04

Frame decorrelation

How correlations decrease once the frame is fixed. Together with the lateral-widening inequality (strong form, weak form), it becomes the condition for the reduction to a single inequality. → 26-02

The lateral-widening inequality (strong form, weak form)

An inequality that controls the bias when widening sideways. The conclusion of the reduction theorem differs between the strong form and the weak form. → 26-02

The uniform slab tension, the uniform slab gap

The two quantities that remain after the band, with L held fixed, is collected into the density κ_cb. The band is collected into these two. → 26-05

The renormalisation comparison, β-monotonicity

Band = β-monotonicity + renormalisation comparison. A comparison that distinguishes groups; only 26-06 uses the abbreviations. → 26-06

κ_cb (the chessboard density)

The density of frame decorrelation with L held fixed. Different from the κ of Part 23, the κ of the one-sided bridge, κ_BE and others (comparison table in A-01 § 09). → 26-05

Diffusive-relaxation-time assumption

The assumption that the relaxation time grows as a power of the square of the correlation length. The short name is the relaxation-time assumption; the upper side is the upper-bound assumption, and the lower side the lower-bound assumption. The Lean identifiers and the names inside the papers have not been changed. → A-06

A class that closes in one step

A class of effective actions that returns to the same form (a family with the same parameters) after one renormalisation step. Ultraviolet control is the question of whether this class closes across the steps. In 15-03 this is written as the class-closure condition and placed as the first half of the decomposition of the ultraviolet wall. The one-link heat-kernel class is the smallest example that closes with one parameter. → 15-03, 15-04

The telescoping sum of losses

When a class closes with a single parameter, the losses of the steps do not pile up as a product but cancel as a sum of differences of the parameter (a telescoping sum). The one-link heat-kernel class is the smallest example. → 26-11

The properties of the weight (four)

The four properties asked of the weight of one step in multiscale analysis (an arrangement of how the original sources put it). For a group-valued field, the property of closing in one step is missing. The group version of submultiplicativity in sums of fields ((ii) of BBS) is the Background–Fluctuation Decomposition Lemma. → 26-11

Decomposition of the ultraviolet wall (class-closure condition, coupling-flow remainder condition, shape remainder condition, definition and continuity of the effective coupling)

A statement [Assumption] that splits ultraviolet control into four parts: a class that closes with finitely many parameters, the remainder of the flow, the remainder of the shape, and continuity within the domain. The contrast with the decomposition of the band (β-monotonicity × renormalisation comparison, 26-06) is in 15-03 § 06. The questions of this part are in Table 1 of 15-09. → 15-03

Group version of closure under the fluctuation expectation (group version of the exponential moment of fluctuations) and the background dependence of the measure η_k

A statement [Assumption] that restates closure under the fluctuation expectation ((iii) of BBS) and the fourth property, among the four properties of the Balaban–Brydges–Slawny weight, for group-valued fields. For a product of one-link heat kernels it holds by the invariance of Haar measure. The difficulty is the background dependence η of the measure at each step: under a naive estimate Σ η_k diverges, and absorption into the parameters is needed (15-07). → 15-07, 15-01

The invariant curve, the transverse contraction λ, the source σ

The invariant curve: a family of weights indexed by one parameter that the one-step renormalisation map sends to itself. The heat-kernel family is an approximation to it (15-04, 15-06). **The transverse contraction** (λ): the factor by which a deviation of the weight from the invariant curve is multiplied in one step. If $\lambda<1$ the deviation is bounded uniformly in the number of steps (15-03, 15-05). **The source** (σ): the terms outside the class newly fed in at each step (for example, the coefficient of θ^4 of the single coarse plaquette) (15-04, 15-05). → 15-04, 15-05, 15-03

The flux network

A network that finds the commuting configuration of minimal action, with the angle of the single coarse plaquette fixed, as the ratios f of the angles of the plaquettes. In three dimensions with a free boundary it is the current in a resistor network. $\beta'/\beta = \Sigma f^2$, the transverse contraction $\lambda = \Sigma f^4/\Sigma f^2$, and the inequality $1/(64\Sigma f^2) \le \lambda \le \max f^2$. → 15-05

Couplings by loop shape

The coefficients when the weight with the fluctuations integrated out is written as a quadratic form in the angles of classes such as single loops and the composite loop. The coordinates for counting the parameters. For four plaquettes in a row, two are needed at the quadratic level, β and the ratio κ_ab/κ_a, and for the heat kernel a part $2/(3\beta)$ due to the curvature of the group appears. → 15-06

The attracting class

When the one-step map contracts the shape coordinates, the class "one parameter + a fixed shape of $O(1/B)$" that the family approaches. It is determined by the fixed point of the flow of the power of the prefactor. → 15-06

The shift of local couplings

A view (a proposal) that reads the part η by which the measure of the fluctuations depends on the background as a change in local couplings $b_p(W)$ determined by the background. The multi-step induction has not yet been written. → 15-07, 15-08

How the small-field contribution to the coupling is split

The view that splits the one-step remainder for small fields into the local part, the Taylor truncation, the small-field cut and the explicit quadratic term between blocks. The first three have the same form as in the sources for flat fields. → 15-08 § 06

The zero-sum system

A system with only one block, in which the explicit quadratic term between blocks vanishes (four plaquettes in a row, the hierarchical model). → 15-08 § 06

Shape-Remainder Theorem for Four Plaquettes in a Row

In the projected model of four plaquettes in a row, the one-step constant of the shape remainder is given as a number, in the form of an upper bound independent of the number of steps. → 15-06 § 09

The local representative

A representative that agrees with F on im D_c and is bounded uniformly in the volume. Built from the cone of axes whose base point moves with the row (Bundle Network Theorem (local form)). → 15-05

One-Link Heat-Kernel Comparison Lemma

A comparison that bounds the one-link Wilson law by a constant multiple of the heat kernel inside the angle cutoff and by a rare tail outside it. → 15-07 § 05

The group-geometry part (the cubic BCH term)

The part of the derivative in the background that remains only for SU(2) and does not depend on the shape coordinate. Its coefficient is 1/9 in the Gaussian main term of two links. → 15-07 § 07

The budget of background change

The sum over steps of the width of the background change times the coupling. It diverges however the threshold is chosen. → 15-07 § 07

The non-locality of the background map

The fact that a background built from the block constraint moves the fine variables non-locally. A one-loop formula that is local in the fine background looks, in the coarse variables, like a non-local O(1) term. → 15-08 § 06

The tail type

How the weight of (1−Loc)W decays with distance (flat, exponential, power). It is fixed by the definition of the coarse variables (how the fine variables straddle the coarse ones). → 15-08 § 06

Coarse links plus global tags

Coordinates made from the coordinates of coarse links, with the Bianchi constraint promoted to coarse variables, plus finitely many global quantities attached as tags, such as the flux threading the torus and the magnetic flux of the centre Z_2. → 15-08 § 06

The agreement of the two gauges

The fact that, for one step, at one loop and in the background of the critical point, the one-loop effective actions in the tree gauge and in the background-covariant gauge are equal (eq. (5)). → 15-08 § 05

The cost of recentring, and the recentring condition

The loss in the Taylor radius (about a factor of 4) and the shift of the coupling when the conditional maximiser of one link is re-taken at the background. → 15-07 § 05

Proposition on a Class with the Background as a Parameter

The proposal that, in a class that carries the logarithm a of the background holonomy as a parameter of the class, the remainder of the background dependence drops by one order and the budget becomes uniform in the number of steps. → 15-07 § 06

The one-face step

One step in which the coarse variable is a single loop. The update of a is exact as an ordered product in the group. → 15-07 § 07

The corner triple product

$\kappa \, \Sigma \, a_x \cdot (a_y \times a_z)$ of the three faces meeting at a corner of a cube. The first non-commutative coordinate of the multi-loop class. → 15-07 § 07

Row-weighted interpolation

Riesz–Thorin with the rows weighted by their ℓ^1 norms (Stein's analytic family). It brought the factor of the four-dimensional ℓ^4 cone down to 0.823 or less. → 15-05 § 07

The ratio coordinates

The coefficients of the neighbour, the skew neighbour, the cubic and the quartic terms divided by the class coupling B. Their values are fixed at the fixed point of the flow, and they are not independent parameters. → 15-03 § 02, 15-07 § 07

The skew neighbour

The quadratic coupling of a pair of faces shifted by one edge that share no vertex. It is needed in the class in four dimensions; its sign and its being non-zero do not depend on the rule, while its value does. → 15-07 § 07

Normalisation (a)

The coordinate in which the cone constant is divided by the long-wavelength coupling β'_0. The direction-wise normalisation λ is not an amount of contraction. → 15-05 § 07

Blocking by edge sums / by transverse averaging

The rule that takes a coarse edge to be the sum of two fine ones, and the rule that takes it to be the gauge-covariant average of the transverse staples. With the latter, the flow of the covariance heads to a fixed point. → 15-05 § 07

The recursion of W

The one-step recursion of the non-local term. In a row it is "a contracting image + a new term independent of the step + an exponential three-body tail". → 15-08 § 06

Proposal of a one-step inequality with the comparison partner fixed

The one-step inequality that the string tension of the one-step MK image does not exceed that of the genuine coarse theory. On finite boxes the threshold is 1. → 15-04 § 05

Home (of a tool)

The chapter that first places the definition of a tool and the known results about it. The home and label of each tool used by this book's study are in 16-10 Table 1. → 16-10

Where it stops (of a tool)

How far the tool reaches in this book's study and where it stops (the third column of 16-10 Table 1). → 16-10

Estimate (label)

A label used only in Part 24. At the same level as "Reading", it corresponds to a physical estimate ("Physics" among the five labels). The legend is in 0-01 § 03. → 0-01 § 03

Splitting the core in two

Viewing the core of the band as two jobs: "extending the ultraviolet" and "the statement of the last interval to strong coupling". → 16-11 § 01

Extending the ultraviolet

The job of extending the control of the published construction from the threshold to a coupling β_* of order one, while keeping the sum of the per-step errors finite. → 16-11 § 01

Statement of the last interval to strong coupling

A non-perturbative statement over a small number of steps from β_* to the strong-coupling window (distinct from the "Corollary Crossing the Band in Finitely Many Steps" of 26-06). → 16-11 § 01

Lossy chain

Renormalisation comparisons that allow a loss of a factor K at each step, chained over the number of steps. The rate is uniform, but the prefactor grows as $\xi^{\log_2 K}$. → 16-11 § 02

Prefactor

The factor in front of the exponential on the right-hand side of an estimate. In the lossy chain it is $K^{k_0}=\xi^{\log_2 K}$. → 16-11 § 02

Facts on the energy side

Facts about the action of configurations (existence of twisted vacua, the fixed group ± 1, positivity of the action of a box with two twists, upper and lower bounds on S_N). → 16-11 § 03

Shortfall on the entropy side

That this book has no estimate, needed for the band, that counts the fluctuations and the number of arrangements of vortices from below, uniformly in the scale. → 16-11 § 03

Mixing statement for boxes of physical size

That if the side of a box is taken to be the physical length C/Λ, the total variation of the law of gauge-invariant observables inside, when the frame is changed, is smaller than a threshold (uniformly in β). A restatement of the mass gap in a strong form. → 16-11 § 04

Seven test models

The models against which arguments for the band are tested: U(1) at weak coupling, Z_2 at weak coupling, finite non-abelian groups, the limit of the adjoint Higgs, the line and endpoint of the mixed action for SU(2), the bulk transition of SU(N), and G_2. → 16-11 § 04, 26-12 § 02

Three classes of value level

The weak-coupling behaviour of the quantity for a symmetric box with one twist: proportional to β for U(1), constant for a frozen finite group, of the type $(3/2)\ln\beta$ for SU(2). → 16-11 § 04

Mixed action

A lattice action that adds a term in the adjoint representation to the Wilson action (the term in the fundamental representation). → 16-11 § 04, 26-12

Critical endpoint

The point where a line of first-order transitions ends. → 16-11 § 04, 26-12

Frozen phase

In a gauge theory with a finite group, the phase on the side of large β that sticks close to flat connections. → 16-11 § 04, 26-12

Control of the continuum limit of twisted boxes

A comparison across scales, with a finite sum of errors, of quantities fixed by the measure in twisted (framed) boxes. → 16-11 § 06

Coupling of the laws of gauge-invariant observables

A construction that couples the conditional laws of gauge-invariant functions, and a finite-box criterion on top of it. → 16-11 § 06

Sieve

Testing whether an argument for the band would go through equally for other models, and the collection of those models. → 26-12 § 01

Confinement quantity / gap quantity

The two directions of the sieve: a quantity that should be false in a non-confining phase / a quantity that should be true for a model that does not confine but has a gap. → 26-12 § 01

Maximal torus

The commutative subgroup $\{\mathrm{diag}(e^{i\phi}, e^{-i\phi})\}$ of SU(2). It has the same form as U(1). → 26-12

Adjoint Higgs

A model that adds to the SU(2) Wilson action a term coupling a unit vector at each site (in the adjoint representation) with the links. The limit in which the hopping coupling J goes to infinity is formally the U(1) Wilson action. → 26-12 § 02

Bulk transition

A temperature-independent (zero-temperature) transition coming from the lattice action. Not physics of the continuum limit, but a phenomenon on the lattice. → 26-12

Centreless group

A group with trivial centre (G_2 and others). Quantities written with centre twists and vortices are not defined for such a group. → 26-12 § 02

Hysteresis

A disagreement in the value of a quantity between the path that raises the coupling and the path that lowers it (it can be a trace of a first-order transition). → 26-12 § 05

Aspect ratio

For a twisted box, the ratio r = M/N of the side M of the vortex plane to the side N of the transverse plane. It separates comparisons at fixed ratio from the quantification that fixes the transverse side and extends the longitudinal one. → 26-12 § 05

Sign term by term

A test that places a separate coupling β_p on each plaquette and checks whether ΔF increases in each β_p (the sign of $\partial \Delta F / \partial \beta_p$). → 26-12 § 05

Flat connection (flat configuration)

A configuration in which the holonomy of every plaquette equals the identity (or, when there is a twist, the sign of the twist). In 21-07 it is called a "flat configuration". → 16-11 § 04, 26-12

The words of objects, firm theorems and premises (Parts 19–22, Part 28 and the second half of Part 15)

Flux free energy ΔF

$\Delta F = \ln(Z_0/Z_1)$. The logarithm of the ratio of partition functions when one twist n_{34} is inserted. A word of 25-01; the same quantity as the vortex free energy F_v of 26-03. → 19-02

Vortex ratio θ

$\theta_M = (Z_0 - Z_1)/(Z_0 + Z_1) = \tanh(\Delta F/2)$. $|\theta_M| < 1$ follows from the definition. $\theta \geq 0$ does not follow from the definition; it is a theorem when M is even (21-06). → 19-02

Base time

For a link $\ell = (x, \mu)$, the start point x is called the base point of the link; for a link in the time direction, the time of its base point x (the base time) is taken as the time of the link. A different word from the "roots" of the tree of premises. → 19-01

Unnormalised Wilson measure $\tilde{\mu}$, normalised Wilson measure μ

$\mu\tilde{}_{N,\beta}$ is the measure given by the weight $w_\beta(U)$ times the product of Haar measures; $\mu_{N,\beta}$ is the probability measure obtained by dividing it by the partition function. $\mu\tilde{}_{N,\beta} = Z_N(\beta)\ \mu_{N,\beta}$. → 19-01

Bridge between the two Wilson measures

That the measure on the strong-coupling side (Part 20) and the measure on the vortex side (Part 21) are equal, for M = N and no twist, when carried over by a one-to-one correspondence given by definition. An identification shown in Lean. → 19-06

Twist signs n_{12}, n_{34}

Two indices that put in 't Hooft's twisted boundary conditions through the signs $z_p \in \{+1, -1\}$ of the plaquette weights. → 19-02

Twisted vacuum

The configuration V of action 0 built from the unit quaternions i and j. → 19-02

Polyakov loop $P_b(t)$

With the transverse position b and the time t fixed, the real part of the product of the variables of the M links that go once around direction 3. → 19-02

Minimal action of a vortex $S_V(N, m)$, S_N

The number that appears in the exponent of the thick-vortex floor: the minimal action, measured by dividing a plane of height M = 2m into four blocks of side m. $0 \le S_N \le 2N^2$. $S_N > 0$ is the theorem of 21-05. → 19-02

Positive side, reflection pairing

For functions F, G that depend only on the links on one side of the mirror (the positive side), $\langle F, G\rangle = \int F(\Theta\omega)\, G(\omega)\, d\mu$. → 19-03

Vessel of reconstruction

The tuple that takes out of the Osterwalder–Schrader reconstruction the parts that do not depend on the measure (pairing, translation, constants). From it one builds the Hilbert space, the vacuum Ω and the transfer operator T. → 19-03

Mass gap (definition on the vessel)

$\| T - P_\Omega \| \le e^{-m}$. A definition at a fixed lattice spacing, not the definition of the mass gap in the continuum limit. → 19-03

Non-degenerate (two uses)

Non-degeneracy of the vessel means that there are states outside the vacuum (20-04). Non-degeneracy in the statements to be shown means that the coarse field stays in small fields (19-07). → 20-04

Block map B

The map that builds a configuration of the coarse lattice from a configuration of the fine lattice by products along a family of paths, weights, averaging and a return to the group. → 19-04

Path kernel C

The first-order part of the block map B when U is close to the identity. → 19-04

Push-forward measure

$(B{*}\mu)(E) = \mu(B^{-1}(E))$. The measure on the fine lattice carried to the coarse lattice by the block map. → 19-04

Wilson–Dirac operator, measure with fermions

The measure on gauge fields whose weight is the determinant of the Wilson–Dirac operator. A detour outside the main object of the book (pure gauge theory). → 19-05

Cylinder function

A function determined only by the values on a finite set S of links (support S). → 20-03

Dobrushin's condition

The influence that one link receives from the surrounding links is measured by coefficients, and their sum is less than 1. A condition that fixes the infinite-volume limit uniquely. → 20-03

Strong-coupling window

The range of β in which the theorems of Part 20 hold ($18\tanh|\beta| < 1$ and $24(1 - e^{-4|\beta|}) < 1$). The opposite end from the weak coupling that the continuum limit needs. → 20-02

Sign system, sign character

An abstract system, with the lattice forgotten, written with multiplicative functions (sign characters) on a finite group G (on the lattice, the field of signs on the links). → 21-01

Covering inequality

An abstract inequality about sign systems. A part of the thin-vortex floor. → 21-01

Z_2 monotonicity

With non-negative couplings, increasing the couplings term by term does not increase the twisted ratio and does not decrease the flux free energy. → 21-01

Floor

An upper bound on the vortex ratio θ; equivalently a lower bound on the flux free energy ΔF, an estimate of the slab tension from below. → 21-02

Thin-vortex floor

$\theta \le \tanh(N^2|\beta|)^{M^2}$ (all β, N, M and background twists; the SU(2) Wilson measure). Not uniform in N. → 21-02

Thick-vortex floor

An upper bound on θ whose exponent contains the minimal action of a vortex S_N. The thresholds β_0 and $M_0(\beta)$ are not explicit. → 21-03

One-step contraction

The inequality $\boxtimes P \boxtimes (3/5)P$ for the symbol of one step of the free field. → 15-12

Cap h(r)

The open cap $\{U \in SU(2) : |U - 1| < r\}$ around the identity, and its Haar volume $h(r)$. → 22-01

Statements to be shown (the root)

Six statements written only with the objects defined in Part 19: matching, convergence of the full sequence, non-degeneracy, equicontinuity, mass gap (uniform decay), and the band. → 19-07

Matching

The way the sequence of bare couplings β_n is chosen: the adjustment that sets the scale of the lattice spacing, written as a rule using only defined objects. → 19-07

Convergence of the full sequence

For the sequence fixed by matching, the push-forward measures converge without passing to a subsequence. → 19-07

Equicontinuity

Observables are continuous uniformly in the physical position inside the box. → 19-07

Sufficient condition, representation

A sufficient condition is a premise placed in order to derive the root along the present road. Many of them are statements made through effective couplings, activities of large fields, background fields and classes (a representation), and many of the quantities of a representation are not yet defined from the lattice. → 28-00

Labels of premises

The column of the list of premises: theorem, theorem (weak form), paper, form fixed, unproved, form not fixed, depends on design, literature. Separate from the labels of the text (Proved here, Cited, Computer-assisted, Reading, Assumption). → 28-00

Reason marks (a)–(f)

The reasons attached to a premise whose form is not fixed: too light, too strong, the object is undefined, a numerical hint, no form that crosses steps, the order of quantifiers is not fixed. → 28-00

Design choices

Things that the mathematics does not fix uniquely and that we choose and fix: the weights, scale factor and thresholds of the block map, norms, backgrounds and so on. → 28-05

Implication (as used in Part 28)

A conditional theorem with all of its assumptions shown. → 28-01

Skew product

A one-step map of the form in which the coupling direction does not contract, only the irrelevant directions contract, and the two affect each other. → 28-02

Endpoint matching

Choosing the bare value for each number of steps so that the value at the coarse end hits the target. Convergence of the full sequence does not follow from contraction alone, and this becomes part of the statement. → 28-02

Tension-doubling inequality

A premise of the form $4\sigma_{2N} \geq e^{\delta_0}\sigma_N$ ($N \geq N_1$) for some $\delta_0 > 0$ and N_1. → 28-04

Detour: four uses of the word "dimension"

You may skip this. "Dimension" has four meanings. (1) The dimension of spacetime d (for space alone, D; in the chapters on string theory, D is used for the dimension of spacetime to match the literature, 17-05). (2) The physical dimension of a quantity (which power of mass). "Dimensional transmutation" is this one (4-07). (3) The dimension of a linear space ($\mathfrak{su}(N)$ has $N^2 - 1$). (4) The anomalous dimension η_s. "Dimensional regularisation" is the technique of extending (1) to complex numbers (4-05). The four of formula (7) and of the article on four dimensions are both (1).

What this chapter can say

What is said	What is not said
Each entry is a shortened version of the definition in the linked chapter.	Some entries omit the hypotheses of theorems (dimension, group, normalisation). The exact statements are only in the linked chapters.
The numbers in Table 1 were counted on this machine from the drafts.	They move as chapters are revised. They are not an indicator of importance.
The entries of Part 17 are written as physical pictures and conjectures.	We do not say that they hold as theorems for four-dimensional Yang–Mills.
The entries of Part 23 are the names and contents of the assumptions that this book's study set up.	None of them is proved. This book shows nothing about the mass gap of the continuum theory (the mass gap in this book's Lean is a statement about the lattice at strong coupling).

Sources and reproduction

Item	Kind	Source / tools
The meaning of each entry	Summary	The chapters of this book (the links). Bibliographic data for primary sources are in the source table of each chapter and in A-03
Conventions for symbols (normalisation of β and tr, the two ways of writing the gauge field)	Conventions of this book	A-01 Table of notation
Table 1 (number of chapters in which words appear)	Computed	python3: from the drafts of the 105 chapters other than the appendices, script, style, the whole-book contents, the meta line and head were removed, tags stripped, and the occurrences of words counted (`term_counts.py`)
Table 2, Figure 2 (e^{-mt} and $\xi = 1/m$), the example for formula (10)	Computed	python3 (`gap_corr.py`). Formula (4) is the known relation in 4-11
The b_0 and $11N/3$ of formula (5)	Known	Sources for 4-06: D. J. Gross, F. Wilczek, PRL 30 (1973) 1343. H. D. Politzer, PRL 30 (1973) 1346 (both bibliographic data only; from the source table of each chapter)
The entry on the strong-coupling expansion (area law and positive gap)	Known	Source for 5-02: K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract only; from the source table of each chapter)
The Lüscher term of formula (8)	Physics (under the assumption of an effective string)	Source for 17-03: M. Lüscher, NPB 180 (1981) 317 (bibliographic data only; from the source table of each chapter)
The $m/\sqrt{\sigma} = 3.837$ in the example for formula (9)	Computed (a ratio of values from a primary source)	17-12: B. Lucini, M. Teper, JHEP 0106 (2001) 050, arXiv:hep-lat/0103027, Table 1 ($\beta = 2.60$, 24^4, $a\sqrt{\sigma} = 0.13395(62)$, $am = 0.514(8)$). 3.837 = 0.514/0.13395

Next chapter to read: A-03 Guide to the literature — the primary sources, reviews and textbooks of each chapter, with their availability.

Guide to the literature — what to read first, where to get it, and how far this book read it

This chapter is a guide to the literature.

What this chapter shows — The 52 main works cited by the chapters of this book, arranged by lineage (reviews, textbooks, primary sources) and year (in addition, the three items of the construction published in 2026 are in § 07, and 18 works whose bibliographic data for individual chapters' sources were checked against primary sources are in Table 3). A few to read first. Where each can be obtained (arXiv, the publisher, public release by an institute or the authors, books), and whether the chapters of this book read each work in the text, only the abstract, or only the bibliographic data.

Prerequisite chapters — None. It can be consulted from anywhere, and the content of each work is read in the linked chapters. Terms are in A-02 Glossary, symbols in A-01 Table of notation.

Said first — This is not an exhaustive bibliography. The chapters of this book cite several hundred works, and here we select those that matter across chapters. "Extent read" is a value read mechanically from how each chapter's source table is written, not an assessment of the value of the work. Existence in four dimensions is given by the construction published in 2026 (§ 07); the committee's assessment is that it is proved in Lean. The mass gap has not been shown. This book does not claim to solve the problem.

The order of this chapter

1. How to use this guide — three kinds and two statuses. Detour: how to read bibliographic data
2. Map of the literature — seventy years arranged in seven lineages (a moving figure)
3. What to read first: the problem statement and reviews — the text of the problem, and reviews describing where things stand
4. Textbooks and lecture notes — for going deeper into the tools of Parts 1 and 2

5. Primary sources in physics — from the original paper of 1954 to the skeleton of the 1970s (and later mathematical and stochastic-quantisation viewpoints)
6. Lattice and numerics — Wilson's lattice, and measurements of glueballs
7. Constructive field theory and probability — axioms, the four-dimensional ultraviolet, probabilistic constructions in two and three dimensions
8. Strings and holography — models at the level of physics
9. Counting the source tables of this book — which works appear in how many chapters, and how far they were read (computed)
10. Notes on obtaining and reading — differences between versions, unrefereed work, finding old journals. Bibliographic data checked in primary sources for chapters' citations (Table 3)

01

How to use this guide

Works are divided into three kinds. A **primary source** is the paper that first showed a result; the exact hypotheses and statements of theorems are checked here. A **review** is an overall picture of results up to some point, and an entrance for deciding which primary sources to read. **Textbooks and lecture notes** are for acquiring the tools in order.

We attach two statuses. The first is **availability**: arXiv (the public repository for preprints; free), the publisher (only the journal version; often needs a library subscription; those made freely available are marked "Publisher (open)"), PDFs made public by an institute or the authors, and books. Preprints whose journal publication had not been confirmed at the time of this book are marked "not refereed".

The second is **the extent to which this book read it**. The "Sources and reproduction" table of each chapter writes whether a work was read in the text, only the abstract, or only the bibliographic data. We collect this over all chapters and take the maximum extent over chapters.

$$\mathrm{best}(r) = \max_{c \in C(r)} s(c, r), \qquad s \in \{\text{bibliographic data only} < \text{abstract} < \text{text}\} \tag{1}$$

Here r is the work, $C(r)$ is the set of chapters citing it in their source tables, and $s(c, r)$ is the extent read from the source table of chapter c. "Text" includes cases where only the introduction or one section was checked. For example, Osterwalder–Seiler (1978) is cited by 40 chapters, of which 25 write "bibliographic data only" (§ 09).

Claims of this book that rest on works marked "bibliographic data only" go through other works whose text was checked. Where they go through is in each chapter's source table.

Detour: how to read bibliographic data

You may skip this. Bibliographic data for papers in physics and mathematics are usually written in the order "authors, journal volume (year) first page". For example, `Comm. Math. Phys. 102 (1985) 255` means Communications in Mathematical Physics, volume 102, 1985, starting on page 255. Journal names are abbreviated; in this book `CMP` (Comm. Math. Phys.), `PRL` (Phys. Rev. Lett.) and `NPB` (Nucl. Phys. B) appear often.
arXiv numbers come in two forms. Until March 2007 they had the form "field/year-month and serial number", and from April 2007 the form "year-month.serial number". The serial number has 4 digits until December 2014 and 5 digits from January 2015.



The "Year" in the tables of this chapter is the year of the arXiv submission if there is one, otherwise the year of the journal (Shen–Zhu–Zhu: arXiv 2022, journal 2023). The Clay problem statement is given the year of its announcement (2000).

02

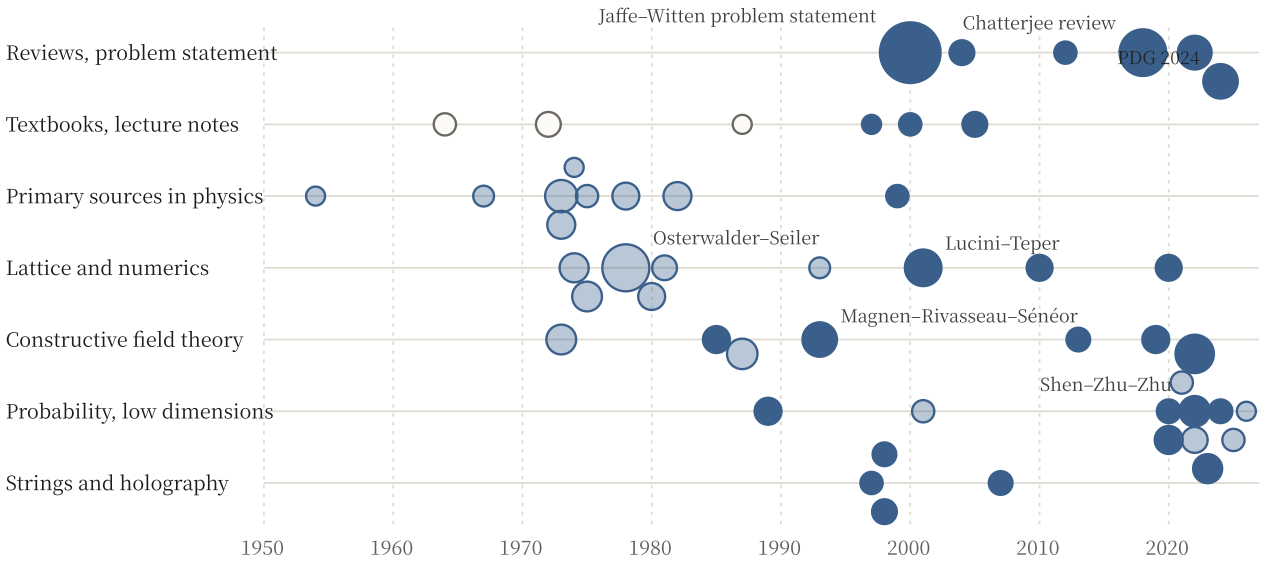
Map of the literature

First we look at the whole on one sheet. The horizontal axis is the year, the vertical the lineage. The size of a dot is the number of chapters citing the work. The fill of a dot is the extent read in this book: solid for "text", light fill for "abstract", hollow for "bibliographic data only".

The primary sources in physics cluster from 1954 to the 1970s. The work aimed at four-dimensional construction (Bałaban, Magnen–Rivasseau–Sénéor) is concentrated in the 1980s and the early 1990s. This line of work did not complete the four-dimensional construction (7-07). For existence in four dimensions, the committee's assessment is that it is proved in Lean in the construction published in 2026 (the table of the published construction in § 07, 6-00). Work from the side of probability theory has been increasing since around 2018.

Moving the year knob leaves dark only the works published up to that year. Moving the part knob makes dark only the works cited by the chapters of that part. Pressing a dot shows its lineage, extent read and the parts citing it below, and lets you jump to its table row (which has links to chapters).

Figure 1 Map of the literature — the year knob brings up "works up to that year", the part knob "works cited by that part". Pressing a dot shows its lineage, extent read and the parts citing it.



Computed The numbers in this figure (positions, sizes and fills of dots, citations by part) were counted on this machine from the source tables of this book (Sources and reproduction). The same content is in the tables by lineage in § 03– § 08, readable even where the figure does not run.

03

What to read first: the problem statement and reviews

If you read only one, read the official problem statement of the Clay Mathematics Institute (Jaffe–Witten). It writes, in the words of those who set it, what the problem asks for and why it is hard in four dimensions. This book reads it word by word in 0-03. It is also the work cited most often among the chapters of this book.

The core of the property the problem statement asks for is the shape of the spectrum of the Hamiltonian H. With the vacuum energy set to 0, it asks that a positive gap Δ open above it.

$$\operatorname{spec}(H) \subset \{0\} \cup [\Delta, \infty), \qquad \Delta > 0 \quad (2)$$

The exact reading of this formula (uniqueness of the vacuum, relation to the decay of correlations) is in 4-11.

The second is Chatterjee's review "Yang–Mills for probabilists". It defines Yang–Mills theory on the lattice in the language of probability theory, and lists what has been shown and what is open in the form of questions (the lattice correlation-length problem and the continuum string-tension problem; Chatterjee's Problems 5.1 and 5.2). Many chapters of Parts 5 and 7 of this book use it as a guide. For a third, Chevyrev

if you follow the probabilistic constructions in two and three dimensions, and the review by Summers for the history of axioms and construction.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2000	A. Jaffe, E. Witten (the official problem statement of the Clay Mathematics Institute)	The official text of the problem, and an exposition of its background	0-01 0-02 and 74 more	The institute's public PDF	Text checked
2004	M. R. Douglas, Report on the Status of the Yang–Mills Millenium Prize Problem (report to the Clay Mathematics Institute, 2004)	A status report a few years after the problem was announced	0-02 4-07 and 4 more	Public PDF of the Clay Mathematics Institute (archive of annual reports)	Text checked
2012	S. J. Summers, arXiv:1203.3991	A review of the history of constructive field theory	4-10 7-03 and 2 more	arXiv	Text checked
2018	S. Chatterjee, arXiv:1803.01950	A review for readers in probability. Lattice definitions, the area law at strong coupling, unsolved questions	0-01 0-02 and 37 more	arXiv	Text checked
2022	I. Chevyrev, J. Math. Phys. 63 (2022) 091101, arXiv:2202.13359	A review of the road approaching Yang–Mills from stochastic quantisation	0-02 0-04 and 14 more	arXiv	Text checked
2024	Particle Data Group (S. Navas et al.), Phys. Rev. D 110 (2024) 030001 (the QCD chapter and others)	The reference for standard physical numbers, such as the running of the coupling constant	0-02 0-03 and 15 more	Publisher (open)	Text checked

04

Textbooks and lecture notes

Parts 1 and 2 of this book prepare, from second-year knowledge, the minimum of tools needed. This table is where to go to learn one step deeper. For groups and representations, Hall; for manifolds and connections, Carroll; for operators and spectra, Reed–Simon; for calculations in quantum field theory, Tong's lecture notes; for axioms and construction, Streater–Wightman and Glimm–Jaffe.

One example. The Baker–Campbell–Hausdorff formula used in 1-03 says that the product of matrix exponentials receives a correction that can be written with commutators alone.

$$e^{X}e^{Y} = \exp\Big(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}\big([X,[X,Y]] + [Y,[Y,X]]\big) + \cdots\Big) \tag{3}$$

This book checked it up to the third-order terms with Hall's notes. Near the identity, the product of the group is determined by the commutators of the Lie algebra alone. This formula is the reason (the group itself is not determined by the Lie algebra alone; 1-03). Many lecture notes have public versions by the authors, and the arXiv numbers are given in the table.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1964	R. F. Streater, A. S. Wightman, PCT, Spin and Statistics and All That, Benjamin (1964)	The Wightman axioms and their consequences	0-03 0-04 and 2 more	Book	Bibliographic data only
1972	M. Reed, B. Simon, Methods of Modern Mathematical Physics I, Academic Press (1972)	Textbook of operator theory and functional analysis (4 volumes)	1-03 1-10 and 4 more	Book	Bibliographic data only
1987	J. Glimm, A. Jaffe, Quantum Physics: A Functional Integral Point of View, 2nd ed., Springer (1987)	Textbook of constructive field theory	4-10 7-01	Book	Bibliographic data only
1997	S. M. Carroll, arXiv:gr-qc/9712019	Lecture notes on general relativity. An introduction to manifolds, connections and curvature	1-05 1-07	arXiv	Text checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2000	B. C. Hall, arXiv:math-ph/0005032 (expanded version Springer GTM 222, 2003)	An introduction to Lie groups, Lie algebras and representations	1-02 1-03 and 2 more	arXiv	Text checked
2005	D. Tong's lecture notes (Gauge Theory, Statistical Field Theory, TASI Lectures on Solitons, arXiv:hep-th/0509216)	Lecture notes on gauge theory, statistical field theory and solitons	3-05 4-04 and 4 more	Author's public version	Text checked

05

Primary sources in physics

Yang–Mills theory appeared in the 1954 paper of Yang and Mills, in the form of building a field from the local symmetry of isospin. In the 1960s the rules of gauge fixing and ghosts (Faddeev–Popov) were set in order, and in 1973 asymptotic freedom was found: the discovery of the sign that makes the coupling weaker at short distances.

In this book's notation (formula (8) of A-01), the one-loop running has the following form.

$$\mu \frac{dg}{d\mu} = -b_0\, g^3 + \cdots, \qquad b_0 = \frac{11N}{48\pi^2} \quad (4)$$

Since $b_0 > 0$, raising the energy μ makes g smaller. Known This is a one-loop perturbative calculation, not a theorem about the four-dimensional theory. Physics This sign is the ground for expecting Yang–Mills theory to make sense at short distances. The account of its content ($11/3 = 4 - 1/3$) is in 4-06.

Many in this lineage are "abstract" or "bibliographic data only" in the chapters of this book. This is because they are in old journals with no public version; formulas and numbers are checked against reviews whose text was checked (PDG, Tong, Chatterjee).

The table also includes three that come after the skeleton of the 1970s. Gribov (1978) showed that gauge fixing is not uniquely determined. Uhlenbeck (1982) is a mathematical paper in geometric analysis showing that singularities of classical Yang–Mills fields can be removed. Baulieu–Zwanziger (1999) is the viewpoint that sees stochastic quantisation as a five-dimensional field theory.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1954	C. N. Yang, R. L. Mills, Phys. Rev. 96 (1954) 191	The original paper on non-abelian gauge fields	3-01 3-02	Publisher	Abstract checked
1967	L. D. Faddeev, V. N. Popov, Phys. Lett. B 25 (1967) 29	The rules of gauge fixing and ghosts	4-03 4-08 and 1 more	Publisher	Abstract checked
1973	D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343	Asymptotic freedom: the sign of the one-loop β function being negative	0-02 0-03 and 12 more	Publisher	Abstract checked
1973	H. D. Politzer, Phys. Rev. Lett. 30 (1973) 1346	Asymptotic freedom (an independent calculation in the same issue as Gross–Wilczek)	0-03 4-06 and 7 more	Publisher	Abstract checked
1974	G. ’t Hooft, Nucl. Phys. B 72 (1974) 461	The expansion in a large number of colours N. That planar diagrams dominate	17-04 18-04	Publisher	Abstract checked
1975	A. A. Belavin, A. M. Polyakov, A. S. Schwartz, Yu. S. Tyupkin, Phys. Lett. B 59 (1975) 85	The instanton solution (a self-dual solution with finite action)	1-12 3-03 and 2 more	Publisher	Abstract checked
1978	V. N. Gribov, Nucl. Phys. B 139 (1978) 1	That gauge fixing is not uniquely determined (Gribov copies)	3-04 4-01 and 6 more	Publisher	Abstract checked
1982	K. Uhlenbeck, Commun. Math. Phys. 83 (1982) 11	Removable singularities of Yang–Mills fields	1-05 1-07 and 7 more	Publisher	Abstract checked
1999	L. Baulieu, D. Zwanziger, Nucl. Phys. B 581 (2000) 604, arXiv:hep-th/9909006	Seeing stochastic quantisation as a five-dimensional field theory	23-00 23-01 and 2 more	arXiv	Text checked

06

Lattice and numerics

In 1974 Wilson put forward the formulation that replaces spacetime by a lattice and puts group elements on the links. With it Yang–Mills theory becomes an integral over finitely many variables, and can be measured by computer. In this book's notation the action has the following form (5-01).

$$S_W(U) = \beta \sum_P \Big(1 - \frac{1}{N} \operatorname{Re}\operatorname{tr} U_P\Big), \qquad \beta = \frac{2N}{g^2} \tag{5}$$

The sum is over all unit squares (plaquettes) P, and U_P is the product of the links around it. Osterwalder–Seiler (1978) gave the mathematical foundation of this formulation (reflection positivity and the transfer matrix). Among the chapters of this book, it is cited most often, together with Jaffe–Witten and Chatterjee.

On the numerical side, starting with the Monte Carlo calculations of Creutz (1980), Lucini–Teper (2001) and Athenodorou–Teper (2020, 2021) have measured glueball masses and their ratios to the string tension. Physics These are values supported by numerical computation, not theorems. The correction coming from fluctuations of the string (Lüscher 1981) has the following form (17-03).

$$V(R) = \sigma R + c_0 - \frac{\pi (d-2)}{24 R} + \cdots \tag{6}$$

In four dimensions ($d = 4$) the last term is $-\pi/(12R)$.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1974	K. G. Wilson, Phys. Rev. D 10 (1974) 2445	Lattice gauge theory and Wilson loops. Confinement at strong coupling	5-01 5-02 and 8 more	Publisher	Abstract checked
1975	S. Elitzur, Phys. Rev. D 12 (1975) 3978	Local gauge symmetry does not break spontaneously (Elitzur's theorem)	3-01 3-07 and 9 more	Publisher	Abstract checked
1978	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440	Reflection positivity, the transfer matrix and strong coupling for lattice gauge theory	0-01 0-02 and 38 more	Publisher	Abstract checked
1980	M. Creutz, Phys. Rev. D 21 (1980) 2308	Monte Carlo calculation of $SU(2)$ lattice gauge theory	4-06 5-02 and 6 more	Publisher	Abstract checked
1981	M. Lüscher, Nucl. Phys. B 180 (1981) 317 (the string fluctuation term)	The universal correction term coming from fluctuations of the string (effective string)	2-02 5-08 and 4 more	Publisher	Abstract checked
1993	R. Sommer, Nucl. Phys. B 411 (1994) 839, arXiv:hep-lat/9310022	Introduction of the length r_0 that sets the lattice scale	5-05 5-08 and 1 more	arXiv	Abstract checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2001	B. Lucini, M. Teper, JHEP 0106 (2001) 050, arXiv:hep-lat/0103027	The string tension of $SU(N)$ (the strength of the force between separated charges) and glueballs, extrapolation to large N	5-02 5-08 and 18 more	arXiv	Text checked
2010	S. Schaefer, R. Sommer, F. Virotta, Nucl. Phys. B 845 (2011) 93, arXiv:1009.5228	Critical slowing down of lattice computations (updates of the computation becoming slower closer to the continuum limit), and the autocorrelation of the topological charge (an integer counting the twisting of the field)	5-07 23-01 and 5 more	arXiv	Text checked
2020	A. Athenodorou, M. Teper, JHEP 11 (2020) 172, arXiv:2007.06422; JHEP 12 (2021) 082, arXiv:2106.00364	Lattice computation of the mass spectrum of glueballs	5-08 5-09 and 5 more	arXiv	Text checked

07

Constructive field theory and probability

The mass gap problem consists of two stages, "existence" and "gap". The works approaching it from the mathematical side fall into three groups: the axioms; the four-dimensional ultraviolet; and probability theory in low dimensions and on the lattice.

Axioms: returning from probability to quantum mechanics

The Osterwalder–Schrader axioms are conditions for rebuilding a Hilbert space and a Hamiltonian from Euclidean (imaginary-time) correlation functions. At the centre is reflection positivity, the condition that the expectation of the product of an observable reflected in time, ΘF, and the original F is not negative (4-10).

$$\langle F, F\rangle := \mathbb{E}\big[\,\Theta F\, F\,\big] \;\geq\; 0 \tag{7}$$

If this holds, a positive inner product can be made, and translation in time appears as e^{-sH} ($H \geq 0$). H in formula (2) comes from here.

The four-dimensional ultraviolet: Bałaban and MRS

The work that came closest to the four-dimensional continuum limit is Bałaban's series (the 1985 paper in three dimensions and the 1987–1989 papers in four dimensions; ultraviolet stability of lattice Yang–Mills by the renormalisation group) and the construction with an infrared cutoff by Magnen–Rivasseau–Sénéor (1993). Neither reaches the stage that this book can read as "complete" in the sense of the problem statement. What is shown and what remains is read in 7-07.

Existence in four dimensions: the published construction (2026)

On 2026-11-10, code written in Lean 4 for a construction of the existence of four-dimensional Yang–Mills theory was published, together with the assessment of the committee that examined it. The assessment's verdict is that the submitted code proves, for every simple compact group, that a theory satisfying the Wightman axioms can be built as the scaling limit of four-dimensional Wilson lattice gauge theory. It also states that the uniqueness of the infinite-volume limit and the mass gap are not shown. This book cites that theorem at the level of "the committee's assessment is that it is proved in Lean". Parts 6–14 of Volume 2 read these three items as primary sources. The way they are cited follows the conventions of 6-00 (for the code, the path relative to the repository root, the declaration name and the line; for the assessment, the section number).

Year	Work	What it contains	Chapters of this book	Availability	Extent read
2026	yang-mills-group/OpenAI-YM-20261028 (https://github.com/yang-mills-group/OpenAI-YM-20261028, Apache-2.0). The Lean 4 source submitted on 2026-10-28. The version this book read is 3456e5e	A construction of the existence of four-dimensional pure Yang–Mills theory (lattice, one renormalisation step, multi-step estimates, infinite volume, reconstruction)	6-00–6-04, Parts 8–14	GitHub (public)	Definitions and statements (part of the proof contents)

Year	Work	What it contains	Chapters of this book	Availability	Extent read
2026	The committee's assessment and checks: docs/assessment-1.pdf (2026-11-10, 35 pages) in yang-mills-group/YM (https://github.com/yang-mills-group/YM), and the checking code in submissions/openai-20261028/. The version this book read is 55ba9fc	The result of the construction compared with the problem statement (Part I); the record of the rebuild, re-checking in the kernel, the audit of axioms, and the matching by Comparator (Part II); five open questions	6-00, 6-02, 16-02 and others	GitHub (public)	Text checked
2026	The committee's third statement (page of the Kavli Institute for Theoretical Physics, 2026-11-09). Signed by Michael R. Douglas, David J. Gross (chair), Arthur Jaffe, Igor R. Klebanov, Nikita Nekrasov	A short statement of the conclusion of the assessment. The committee members' names are not in the text of the assessment; they come from the signatures on this statement	6-00	Public page of the institute	Text checked

The three items of the published construction (table in § 07). They are not included in the count (52 works) of Table 1 and Figure 1.

Two and three dimensions, and probability theory on the lattice

Known In two dimensions the Yang–Mills measure has been constructed (Driver 1989, Lévy 2001 and others). In the 2020s, the road that makes the measure as the equilibrium of a stochastic dynamics (stochastic quantisation) is advancing in two and three dimensions. On the lattice, exponential decay of correlations at strong coupling follows from the cluster expansion of Osterwalder–Seiler (1978), and Shen–Zhu–Zhu showed Poincaré-type inequalities and exponential decay by methods of stochastic analysis in an explicit range of β Known . The reach and where it stops in four dimensions are in 7-10 and 16-01. Unrefereed preprints are marked "not refereed" in the table.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1973	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281	Euclidean axioms and the reconstruction theorem (including the second part of 1975)	0-03 0-04 and 9 more	Publisher	Abstract checked
1985	T. Bałaban, Comm. Math. Phys. 102 (1985) 255 (three dimensions)	Ultraviolet stability of three-dimensional lattice Yang–Mills (controlling short-distance contributions independently of the lattice spacing)	0-02 7-01 and 6 more	Publisher	Text checked
1987	T. Bałaban, Comm. Math. Phys. 109 (1987) 249; 122 (1989) 175; 122 (1989) 355 (the four-dimensional series, 1987–1989)	The series on ultraviolet stability of four-dimensional lattice Yang–Mills by the renormalisation group	0-02 4-04 and 10 more	Publisher	Abstract checked
1993	J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325	Construction of four-dimensional Yang–Mills with an infrared cutoff (an operation that artificially suppresses long-distance contributions)	0-02 0-03 and 15 more	Publisher	Text checked
2013	M. Hairer, Invent. Math. 198 (2014) 269, arXiv:1303.5113	The theory of regularity structures (a framework giving meaning to stochastic partial differential equations that have no solutions in the ordinary sense)	1-11 7-03 and 3 more	arXiv	Text checked
2019	M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163, arXiv:1912.07973	Triviality of the scaling limit (the limit as the lattice spacing goes to 0) of four-dimensional φ^4 and Ising	0-03 1-08 and 6 more	arXiv	Text checked
Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1989	B. K. Driver, Comm. Math. Phys. 123 (1989) 575	Two-dimensional Yang–Mills measure on the plane	0-02 0-04 and 6 more	Publisher	Text checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2001	T. Lévy, Mem. AMS 166 no. 790 (2003), arXiv:math/0101239	Yang–Mills measure on compact surfaces	0-02 7-01 and 2 more	arXiv	Abstract checked
2020	A. Chandra, I. Chevyrev, M. Hairer, H. Shen, Publ. Math. IHÉS 136 (2022) 1, arXiv:2006.04987	Langevin dynamics and the state space of the two-dimensional Yang–Mills measure	3-04 7-09 and 3 more	arXiv	Text checked
2020	S. Chatterjee, Comm. Math. Phys. 385 (2021) 1007, arXiv:2006.16229	A probabilistic mechanism of confinement (centre symmetry)	5-10 17-01 and 7 more	arXiv	Text checked
2021	S. Cao, S. Chatterjee, Comm. Math. Phys. 405 (2024), arXiv:2111.12813	The state space of three-dimensional Euclidean Yang–Mills theory	7-06 7-11 and 2 more	arXiv	Abstract checked
2022	A. Adhikari, S. Cao, Ann. Probab. 53 (2025) 140–174, arXiv:2202.10375	Decay of correlations at weak coupling for lattice gauge theory with finite groups	5-06 5-09 and 10 more	arXiv	Text checked
2022	A. Chandra, I. Chevyrev, M. Hairer, H. Shen, Invent. Math. 237 (2024) 541, arXiv:2201.03487	Stochastic quantisation of three-dimensional Yang–Mills–Higgs	4-05 7-06 and 5 more	arXiv	Abstract checked
2022	H. Shen, R. Zhu, X. Zhu, Comm. Math. Phys. 400 (2023) 805, arXiv:2204.12737	Stochastic analysis of lattice Yang–Mills at strong coupling (Poincaré, exponential decay)	4-11 5-01 and 21 more	arXiv	Text checked
2023	I. Chevyrev, H. Shen, Comm. Pure Appl. Math. (2026) (volume and pages not confirmed), arXiv:2302.12160	The invariant measure and universality of two-dimensional Langevin dynamics	0-02 4-08 and 8 more	arXiv	Text checked
2024	S. Chatterjee, Prob. Math. Phys. 7 (2026) 339, arXiv:2401.10507	The scaling limit (the limit as the lattice spacing goes to 0) of $SU(2)$ lattice Yang–Mills–Higgs	7-08 7-10 and 3 more	arXiv	Text checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2025	S. Cao, R. Nissim, S. Sheffield, arXiv:2505.16585, arXiv:2509.04688	Enlarging the region of the area law for lattice Yang–Mills (2505.16585), and its proof by a dynamical method (2509.04688) (not refereed)	5-02 5-11 and 2 more	arXiv (not refereed)	Abstract checked
2026	I. Chevyrev, T. Klose, A. Mohamed, arXiv:2607.22236 / R. Chhaibi et al., arXiv:2607.24640	New methods of constructing the two-dimensional Yang–Mills measure (not refereed)	7-02 16-01	arXiv (not refereed)	Abstract checked

08

Strings and holography

On the physics side, there is a view that reads confinement and glueball masses of Yang–Mills theory in the geometry of gravity in one more dimension. The starting points are the correspondence proposed by Maldacena (submitted 1997) and Witten (1998), who used it for a model of confinement. Csáki–Ooguri–Oz–Terning derived ratios of glueball masses from this model as solutions of an eigenvalue problem.

Physics These are correspondences and models at the level of physics, not mathematical theorems. Nor have they been shown in a form that applies to pure four-dimensional Yang–Mills theory itself. What the model assumes, and what happens in the limit back to four dimensions, are read in 17-08, 17-09 and 17-10.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1997	J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, arXiv:hep-th/9711200	Proposal of the AdS/CFT correspondence (holography)	17-05 17-08 and 2 more	arXiv	Text checked
1998	C. Csáki, H. Ooguri, Y. Oz, J. Terning, JHEP 9901 (1999) 017, arXiv:hep-th/9806021	Deriving ratios of glueball masses from supergravity (the low-energy theory of gravity) calculations	17-06 17-08 and 4 more	arXiv	Text checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1998	E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505, arXiv:hep-th/9803131	A model, in geometry of one more dimension, of the thermal phase transition and confinement	17-05 17-07 and 3 more	arXiv	Text checked
2007	U. Gürsoy, E. Kiritsis, F. Nitti, JHEP 0802 (2008) 019, arXiv:0707.1349	An improved holographic model of QCD	5-08 17-01 and 3 more	arXiv	Text checked

09

Counting the source tables of this book

The source tables of the 110 chapters of this book (excluding this one) have 1035 rows in total. The arXiv numbers appearing in them are of 206 kinds, and bibliographic data in the form "volume (year) page" of about 602 kinds (the latter may pick up sequences of numbers inside formulas, so it is approximate). Computed

For each work, we counted the number of chapters citing it and the extent read in each chapter. Table 1 is the top 10 in order of the number of citing chapters. "Undetermined" counts chapters whose source-table row had no extent written, or where matching of words could not decide.

Table 1 Works cited often by the chapters of this book (breakdown of the extent in each chapter's source table)

Rank	Reference	Citing chapters	Text checked	Abstract checked	Bibliographic data
1	Jaffe–Witten problem statement	76	61	4	
2	Osterwalder–Seiler	40	0	15	
3	Chatterjee review	39	27	4	
4	Shen–Zhu–Zhu	23	6	12	
5	Lucini–Teper	20	12	0	
6	Magnen–Rivasseau–Sénéor	17	2	7	

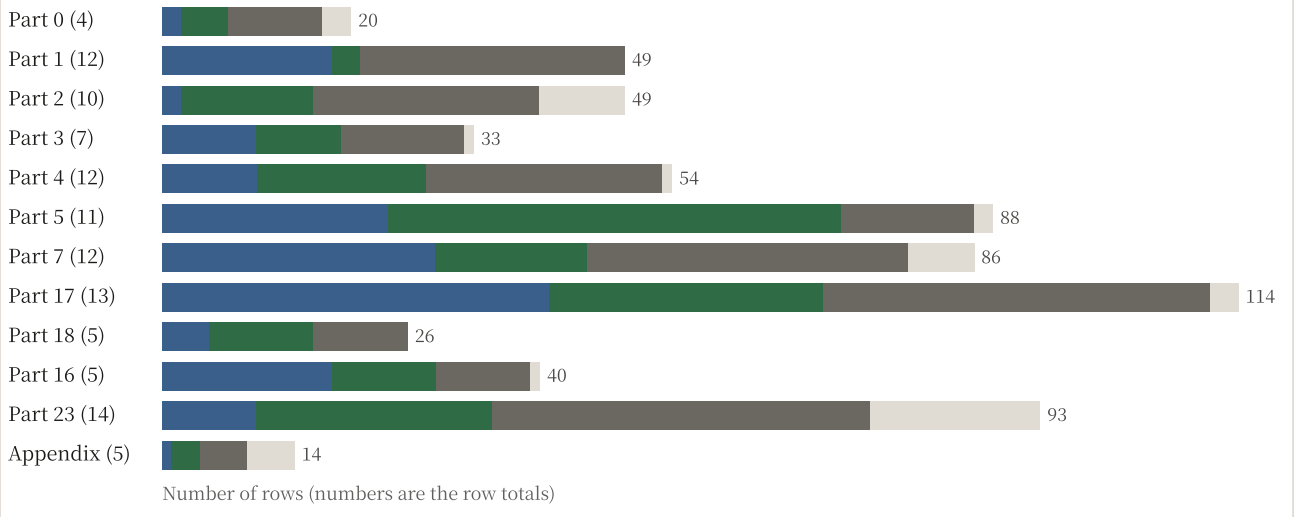
Rank	Reference	Citing chapters	Text checked	Abstract checked	Bibliographic data
7	PDG 2024	17	8	1	
8	Chevyrev review	16	11	1	
9	Gross–Wilczek	14	0	2	
10	Bałaban four-dimensional series	12	0	7	

Osterwalder–Seiler is cited by 40 chapters, but many chapters go through the abstract and reviews whose text was checked (Chatterjee § 4 and others). For Lucini–Teper, 12 of 20 chapters write that they checked as far as the tables of the text.

We also counted by part. Among the source-table rows, for those containing bibliographic data of primary sources, we count the extent in that row. The fraction of text is the following ratio.

$$\text{fraction of text} = \frac{\#\{\text{text}\}}{\#\{\text{text}\} + \#\{\text{abstract}\} + \#\{\text{bibliographic data only}\} + \#\{\text{undetermined}\}}$$

Figure 2 Extent read by part — the button switches between "number of rows" and "fraction".



Computed The numbers in this figure were counted on this machine from the source tables of this book. Table 2 below has the same numbers.

Table 2 Extent of source-table rows containing primary sources, by part

Part	Chapters	Rows	Text checked	Abstract checked	Bibliographic data only	Unde
Part 0	4	20	2	5	10	
Part 1	12	49	18	3	28	

Part	Chapters	Rows	Text checked	Abstract checked	Bibliographic data only	Unde
Part 2	10	49	2	14	24	
Part 3	7	33	10	9	13	
Part 4	12	54	10	18	25	
Part 5	11	88	24	48	14	
Part 7	12	86	29	16	34	
Part 17	13	114	41	29	41	
Part 18	5	26	5	11	10	
Part 16	5	40	18	11	10	
Part 23	14	93	10	25	40	
Appendix	5	14	1	3	5	
Total	110	666	170	192	254	

The extent was decided by the words in the source tables ("text checked", "abstract", "bibliographic data only" and so on), so it is affected by variations in wording. Table 1 reads with priority the words inside the separators (the full-width slash and semicolon) in which the name of the work appears.

10 Notes on obtaining and reading

Differences between versions. The arXiv version and the journal version may differ in page and theorem numbers. The chapters of this book write in their source tables which version was checked.

Publication status. The "journal-ref" in the arXiv catalogue is a field filled in by the authors, and volume and pages are sometimes empty. For example, for Chevyrev–Shen the catalogue gives only "Comm. Pure Appl. Math. (2026)", and volume and pages are not confirmed in this book. For the preprints of 2025 and 2026 (Cao–Nissim–Sheffield, and the two of 2026), publication had not been confirmed at the time of this book. Unrefereed results are not treated with the same weight as refereed results.

Finding old journals. Many papers of the 1970s and 80s exist only in journal versions. Bibliographic data can be checked in INSPIRE (the catalogue of high-energy physics literature) or Crossref. Some old volumes of Comm. Math. Phys. have public versions on Project Euclid (for example Driver 1989, CMP 123). Otherwise, going through a university library is the sure way.

Bibliographic data checked in primary sources for chapters' citations

The tables of § 03– § 08 are the 52 works that matter across chapters. Separately, among the works supporting claims of one or two chapters, those whose bibliographic data or text were rechecked in primary catalogues (Crossref, INSPIRE, DML-CZ, Math-Net.Ru, publishers' pages) are listed in Table 3. "Level of checking" is how far each work was checked. They are not included in the counts of Figure 1, Table 1 or Table 2.

Table 3 Bibliographic data for individual chapters' sources, with the level of checking confirmed

Year	Reference	What it is used for	Chapters of this book	Level of checking
2006	J. Carlson, A. Jaffe, A. Wiles (eds.), *The Millennium Prize Problems*, Clay Mathematics Institute / American Mathematical Society (2006). Jaffe–Witten is pp. 129–152	The book containing the problem statement (Jaffe–Witten). The preface gives the date and place of the announcement (24 May 2000, Collège de France)	0-02	Contents and preface checked in the PDF published by CMI, year of publication on the AMS book page
1907	O. Perron, Math. Ann. 64 (1907) 248–263, DOI 10.1007/BF01449896	The largest eigenvalue of a positive matrix is simple	5-03 17-13	Bibliographic data checked in Crossref. Text: bibliographic data only
1912	R. Jentzsch, "Über Integralgleichungen mit positivem Kern", J. reine angew. Math. 141 (1912) 235–244, DOI 10.1515/crll.1912.141.235	The largest eigenvalue of an integral operator with a positive kernel is simple (uniqueness of the ground state of the transfer matrix)	2-07 5-03 and 3 more	Bibliographic data checked in Crossref. Text not read. The claim checked through a citation in a secondary source (arXiv:1910.01120)
1948	M. G. Krein, M. A. Rutman, Uspekhi Mat. Nauk 3:1(23) (1948) 3–95 (English translation AMS Translation 26, 1950)	Generalisation to operators preserving a cone	5-03 7-01	Bibliographic data checked in the Math-Net.Ru index. Text not read
1974	H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer (1974), the theorem on p. 337	Generalisation of Jentzsch's theorem to positive kernels on compact spaces (continuous positive kernels on a finite box fall under this hypothesis)	5-03 7-01 17-13	Bibliographic data only. The claim checked through citations in arXiv:1910.01120 and arXiv:1501.06011

Year	Reference	What it is used for	Chapters of this book	Level of checking
1958	J. Feldman, Pacific J. Math. 8 (1958) 699–708	Two Gaussian measures are either equivalent or mutually singular	1-09	Text checked (introduction and theorem)
1958	J. Hájek, Czechoslovak Math. J. 8 (1958) 610–618, DOI 10.21136/CMJ.1958.100333	The same dichotomy (independently of Feldman)	1-09	Bibliographic data checked in DML-CZ. In Russian; text not read
2023	M. Schröfl, S. Floerchinger, Ann. Henri Poincaré 26 (2025) 3233–3319, arXiv:2307.15548	Free fields of different masses are mutually singular for $d \geq 4$ (Dirichlet, Neumann, periodic and Robin boundaries), and the relative entropy is finite $\Leftrightarrow d < 4$	1-09	Text checked (§ 4.1, Theorem 3)
2002	H. Boschi-Filho, N. R. F. Braga, JHEP 0305 (2003) 009, arXiv:hep-th/0212207	Deriving glueball masses by placing a Dirichlet condition on a hard wall cutting off AdS	17-09	Abstract checked
2003	M. Kruczenski, D. Mateos, R. C. Myers, D. J. Winters, JHEP 0405 (2004) 041, arXiv:hep-th/0311270	The string tension (2.7) of the D4 model, $\sigma \propto g^2_{\rm YM} N_c M^2_{\rm KK}$	17-09	Text checked (formula (2.7))
2005	H. Boschi-Filho, N. R. F. Braga, C. N. Ferreira, Phys. Rev. D 73 (2006) 106006, arXiv:hep-th/0512295	A linear term appears in the static potential in the hard-wall model	17-09	Text checked ((12)–(16))
2006	A. Karch, E. Katz, D. T. Son, M. A. Stephanov, Phys. Rev. D 74 (2006) 015005, arXiv:hep-ph/0602229	The soft-wall model. The squared meson masses are proportional to c ((15))	17-09	Text checked ((15))
2006	O. Andreev, V. I. Zakharov, Phys. Rev. D 74 (2006) 025023, arXiv:hep-ph/0604204	The string tension of the soft-wall model ((19), $g = R^2/\alpha'$)	17-09	Text checked ((19))

Year	Reference	What it is used for	Chapters of this book	Level of checking
1981	D. Zwanziger, "Covariant quantization of gauge fields without Gribov ambiguity", Nucl. Phys. B 192 (1981) 259–269	Stochastic gauge fixing (fixing the equilibrium distribution of Parisi–Wu stochastic quantisation without imposing a gauge condition)	23-02	Title, volume, pages and abstract checked in INSPIRE. Text not read
1990	C. Parrinello, G. Jona-Lasinio, Phys. Lett. B 251 (1990) 175–180	A modified Faddeev–Popov formula that handles Gribov copies by a weighted average over gauge orbits	23-02 23-13 and 1 more	Title, volume, pages and abstract checked in INSPIRE. Text not read
1990	D. Zwanziger, "Quantization of gauge fields, classical gauge invariance and gluon confinement", Nucl. Phys. B 345 (1990) 461–471	A globally correct gauge fixing (cited by 23-02 alongside Parrinello–Jona-Lasinio)	23-02 23-13 and 1 more	Title, volume, pages and abstract checked in INSPIRE. Text not read
2023	C. Chattopadhyay, J. Ott, T. Schäfer, V. Skokov, Phys. Rev. D 108 (2023) 074004, arXiv:2304.07279	The dynamical exponent of conserving dynamics $z = 3.972(2)$ and the prediction $z = 4 - \eta$	2-10	Text checked. Volume and article number checked in Crossref
2025	R. Masaoka, T. Soejima, H. Watanabe, J. Stat. Phys. 192 (2025) 76, arXiv:2502.09908	$z \geq 2$ for Ising in two or more dimensions (periodic boundary, $\beta \geq \beta_c$)	2-10	Abstract checked

The two papers by Zwanziger are different papers. The 1981 paper is gauge fixing within stochastic quantisation (stochastic gauge fixing), and the 1990 paper, from the same year as Parrinello–Jona-Lasinio, is gauge fixing by a weighted average over orbits. What Part 23 of this book calls "the scheme of Parrinello–Jona-Lasinio and Zwanziger" is the latter pair of 1990, and the source table of 23-02 lists both. For Cao–Nissim–Sheffield in the table of § 07, 5-11 checked arXiv:2509.04688 as far as Definition 1.4 and Remark 1.5.

Chapter authors and overall authors. The PDG review as a whole is written as Navas et al., and when a chapter within it (for example the QCD chapter) is cited, it is written with the names of the chapter's authors. The difference in how 0-03 and 4-06 write it is this distinction, and both point to the same edition (2024).

The meaning of the labels is in A-04, the verification procedure in A-05, and the papers of this site in the list of papers.

What this chapter can say

What is said	What is not said
The bibliographic data of the 52 works listed (those with arXiv numbers checked against the arXiv catalogue)	A list of all works cited by this book (see each chapter's source table)
The chapters citing each work, and the extent read written in each chapter's source table (read by matching words)	Complete correctness of the reading (mix-ups due to variations in wording may remain)
One-line descriptions (summaries within the range of the chapters of this book and the arXiv titles and abstracts)	The exact hypotheses and statements of the theorems of each work (checked in the primary sources and the linked chapters)
The distinction of where to obtain each work. That the preprints of 2025–2026 were not refereed at the time of this book	Whether the journal version can be read for free (this changes with time and subscription). The correctness of unrefereed results
The bibliographic data of the 18 works of Table 3, and the level of checking written in each row (Crossref, INSPIRE, DML-CZ, Math-Net.Ru, publishers' pages)	The content of the text of the works in Table 3 marked "bibliographic data only" or "abstract"
That the mass gap problem is unsolved	A quantitative assessment of how close the works listed, taken together, come to a solution

Sources and reproduction

Item	Kind	Source / tools
Titles, authors, dates of first appearance and journal-ref of works with arXiv numbers	Known	Checked against the arXiv catalogue (API queries). In this chapter only as far as the abstracts. Checking of the texts is as in each chapter's source table
The format of arXiv numbers and when it switched	Known	The arXiv help page "Understanding the arXiv identifier" (text checked)
Where the Clay problem statement is	Known	The Yang–Mills page of the Clay Mathematics Institute (the link to the official problem statement checked). Its content is in 0-03
Where Douglas's report (2004) is	Known	Public PDF in the archive of annual reports of the Clay Mathematics Institute (title, author and date checked)

Item	Kind	Source / tools
Bibliographic data and level of checking of Table 3	Known	Checked against Crossref, INSPIRE (titles, volumes, pages and abstracts of Zwanziger 1981 and 1990 and Parrinello–Jona-Lasinio 1990), DML-CZ, Math-Net.Ru, the PDF published by CMI, and the AMS book page. Rows whose text was checked are the same as in the source tables of the chapters (1-09, 2-10, 17-09)
Bibliographic data of works without arXiv numbers	Known	Taken from the bibliographic data in each chapter's source table. In this chapter, bibliographic data only
Formulas (2)–(7)	Known	Formulas repeated from 4-11, 1-03, A-01, 5-01, 17-03 and 4-10. The sources are in each chapter's source table
Tables 1 and 2, Figures 1 and 2, the counts of § 09, the "Chapters of this book" and "Extent read" columns of the tables	Computed	python3: read the source table of each chapter of this book, collected citing chapters with a regular expression per work, and decided the extent by matching words (`refs.py`, `figdata.py`, `gen.py`; each under 1 second)

Next chapter to read: A-04 Lean and machine checking — what "Lean" means among the labels of this book.

Lean and machine checking — what the label "Lean" guarantees is only the arrow from premises to conclusion

> This chapter is a guide. § 02, § 03, § 04, § 05 and § 08 are established knowledge.

What this chapter shows — What has been confirmed when something is "checked" in Lean (propositions are types, proofs are terms, and what is trusted is the kernel). How to read the goals in the middle of a proof, and how to find `sorry`, `native_decide` and axioms (with the results of checking seven small files on this machine). Four types in which a statement passes but does not say what was meant, how this book decides its labels, and the procedure by which readers can check things again themselves.

Prerequisite chapters — None. You can read this even if you have never used Lean. Detailed usage is in this site's guide to Lean (thirteen pages); this appendix is an entrance to it, and gathers the conventions for reading the labels in this book. The list of Lean theorems in this book is in 23-13, and the legend of the labels in 0-01.

Said first — The Lean label guarantees only that "the conclusion follows logically from the premises"; it does not guarantee that the premises are true, nor that the statement says the mathematics that was intended. The Lean theorems in this book are mainly statements about inequalities of real numbers and finite sums; nothing in this book shows the existence of four-dimensional Yang–Mills theory or the mass gap in Lean. Four-dimensional existence was shown in Lean by a different construction published in 2026, and the committee's assessment is that it is proved (6-00). The mass gap has not been shown.

Order of this chapter

1. The role of this appendix — what the Lean label refers to among the five labels of this book, and the division of work with the guide pages
2. Lean on one sheet — propositions are types, proofs are terms of those types, and only the kernel is trusted; Detour: the name of the correspondence between propositions and types

3. One step at a time — the "goal" in the middle of a proof, and tactics (Figure 1, Table 1)
4. Holes and axioms — `sorry` passes with only a warning; finding it with `#print axioms` (Table 2); Detour: using real numbers brings out the three axioms
5. It passes, but is it what was meant? — the four gaps: definition, name, vacuous truth, assumption
6. What the Lean statements in this book talk about — named real numbers, the "if assumption then conclusion" type, mathematics out of stock
7. How the labels are decided — one label per sentence; the strength of a chain is its weakest step (Figure 2, Table 5)
8. A procedure to check for yourself — the four stages of checking, the record of versions, seven questions (Table 7)

01

The role of this appendix — the label "Lean" and the guide pages

Sentences in this book carry five kinds of label (the legend in 0-01): Lean , Paper , Computed , Known and Physics . Of these, the Lean label looks the strongest and is the most easily misread.

The typical misreading is to read "checked in Lean" as "settled as true". What Lean confirms is that the written conclusion follows logically from the written premises. Whether the premises hold, and whether the statement says the intended mathematics, lie outside Lean. This appendix shows that boundary with small examples that were actually checked.

Written as a formula, a sentence with the Lean label has the following form.

$$\underbrace{P_1,\ P_2,\ \ldots,\ P_k}_{\text{premises: outside Lean}} \quad \underbrace{\Longrightarrow}_{\text{the Lean label}} \quad \underbrace{Q}_{\text{conclusion}} \qquad (1)$$

What the label guarantees is the arrow in the middle. If the premises P_i contain unproved assumptions (for example 23-01 The line of construction and the diffusive-relaxation-time assumption) or measured numbers, the conclusion Q is conditional to the same extent. When this book cites a Lean theorem name, it writes the premises on the same line.

C_P (the Poincaré constant, Part 23), m (the mass) and ξ (the correlation length) are quantities of Part 23, but in this appendix it is enough to read them as plain real numbers.

Example. "If $m > 0$, $CP = 1/m^2$ and $CP \le K\xi^2$, then $1 \le K(m\xi)^2$" is a statement that only multiplies both sides by m^2, and it can be confirmed in Lean (in § 03 this machine actually checks it). But whether $CP \le K\xi^2$ holds in four-dimensional Yang–Mills theory has nothing to do with this arrow. That is an assumption.

The division of work between this appendix and this site's guide to Lean (thirteen pages) is as follows. The guide is an introduction to Lean itself, covering installation, how to assemble concrete proofs, and common pitfalls. This appendix restricts itself to the minimum needed to read this book, and to how this book decides its labels. The correspondence of each theorem's premises with the known results is listed in 23-13.

02

Lean on one sheet — propositions are types, proofs are terms, and the kernel is what is trusted

Lean is both a programming language and a tool for checking proofs. It can do both with one mechanism because it **treats propositions as types, and proofs as values (terms) of those types**.

In a program we write "`3` is a value of the type of natural numbers `Nat`". In Lean, with the same notation, we write "`p` is a proof of the proposition P".

$$\texttt{3} : \texttt{Nat} \qquad\qquad p : P \ \text{(2)}$$

The formula on the right is read "p has type P", and that means "p is a proof of P". Type checking can be done mechanically, so checking proofs becomes mechanical too.

A proposition with premises becomes a function type. A proof of "if P_1 and P_2 then Q" is a function that takes a proof of P_1 and a proof of P_2 and returns a proof of Q.

$$(h_1 : P_1) \to (h_2 : P_2) \to Q \ \text{(3)}$$

This is what the arrow of formula (1) really is. In the declaration of a Lean theorem, the arguments lined up to the left of the colon (brackets like `(hm : 0 < m)`) are the premises, and the right of the colon is the conclusion. **When reading, the most important thing is this row of brackets** ("Make a list of the hypotheses" in the guide).

Example. The smallest example checked on this machine is the following one line.

```
theorem pos_nat (n : Nat) (h : 0 < n) : 0 < n := h
```

The premise `h` is itself the proof of the conclusion. It is a theorem with no content, but the form "a function that takes a premise and returns a conclusion" is clearly

visible.

The other pillar is the **kernel**. The proof a person writes is a script lined up from tactics (commands that direct one step of a proof; § 03), and Lean expands it into a term of the form of formula (2). Finally that term is type-checked by the kernel, a checker kept small. Even if a tactic has an error, a term the kernel does not pass does not become a theorem. Narrowing what is trusted down to the one kernel is where the trust in machine checking comes from ("Kernel and tactics" in the guide, How checking works).

Detour: the name of the correspondence between propositions and types

You may skip this. The correspondence that sees propositions as types and proofs as terms is generally called the "Curry–Howard correspondence". In this view, the logical "implies" corresponds to a function type, "and" to a pair type, and "or" to a type with two kinds of value. Lean's logic is built on this correspondence, as a type theory with dependent types (types that may depend on values, as in "vectors of length n"). For the history and the relation to other proof assistants, see "Where the idea came from" in the guide.

03

One step at a time — goals and tactics

When writing a proof in Lean, "what must be shown now" is always on the screen. This is called the **goal**. The goal is written to the right of the symbol $\vdash$, and the hypotheses that may be used are lined up above it. Each time one tactic is entered, a hypothesis or the goal is rewritten, and when no goals remain the proof is finished.

As an example we use the statement of § 01 written in Lean. There are three premises (excluding the types of the variables). It is the same statement as `hup_is_gap_positivity` in 23-13, with only the names of the variables different.

$$m > 0, \quad C_P = \frac{1}{m^2}, \quad C_P \le K\xi^2 \implies 1 \le K(m\xi)^2 \quad (4)$$

On paper it is three lines: "substitute C_P, multiply both sides by $m^2 > 0$, and tidy up". In Lean it takes four steps, and the second step, `have`, is preparation for multiplying by m^2. The correspondence is: substitution is step 1, multiplication is steps 2 and 3, tidying is step 4. In Lean, what each step rewrote is recorded mechanically. The numbers in Figure 1 (the text of the goals) are copied as they are from the states that Lean 4.33.1 on this machine output in the middle of the proof.

Figure 1 Advancing one tactic with "Next" shows the goal after it. Lines that changed from the previous state are shown in colour

Proof

```
theorem gap_from_bound (CP m K ξ : ℝ) (hm : 0 < m)
    (hCP : CP = 1 / m ^ 2) (hH : CP ≤ K * ξ ^ 2) :
    1 ≤ K * (m * ξ) ^ 2 := by
  rw [hCP] at hH
  have hm2 : 0 < m ^ 2 := by positivity
  rw [div_le_iff₀ hm2] at hH
  calc (1 : ℝ) ≤ K * ξ ^ 2 * m ^ 2 := hH
    _ = K * (m * ξ) ^ 2 := by ring
```

State after step 0

```
CP m K ξ : ℝ
hm : 0 < m
hCP : CP = 1 / m ^ 2
hH : CP ≤ K * ξ ^ 2
⊢ 1 ≤ K * (m * ξ) ^ 2
```

Step 0/4: Right after the declaration. The premises are lined up as hypotheses, and the goal is the conclusion itself.

Computed The numbers in this figure were computed on this machine (checked with Lean 4.33.1 with `trace_state` inserted after each step, and the output copied as it is; the figure omits the `trace_state` lines. The "No goals" of step 4 is not a copy of output; it was confirmed by finishing without errors). On a narrow screen the same content can be read in Table 1.

Table 1 has the same content as Figure 1.

Table 1 Each step of the proof of formula (4), and the state after it Computed

Stage	Tactic entered	What changed
0	(declaration)	Hypotheses `hm : 0 < m`, `hCP : CP = 1 / m ^ 2`, `hH : CP ≤ K * ξ ^ 2`; goal `⊢ 1 ≤ K * (m * ξ) ^ 2`
1	`rw [hCP] at hH`	The `CP` in `hH` is replaced by `1 / m ^ 2`
2	`have hm2 : 0 < m ^ 2 := by positivity`	The hypothesis `hm2 : 0 < m ^ 2` is added (`positivity` is an automatic step that shows something is positive)
3	`rw [div_le_iff₀ hm2] at hH`	`hH` changes to `1 ≤ K * ξ ^ 2 * m ^ 2` (a Mathlib lemma that clears the denominator)
4	`calc … := hH`, `_ = … := by ring`	The goal closes (`ring` is an automatic step that shows ring identities)

How to read it. At step 3, `hH` has almost the same form as the goal; the only difference is the order of the product. The final `ring` confirms that reordering as an identity. You can see that the three lines on paper correspond to Lean's four steps with just one preparatory step added. Detailed examples of choosing steps in Lean are in "The skeleton of the proof" in the guide.

What Lean confirmed in this example is only algebra. Inside Lean, CP is just a named real number, and that it is a Poincaré constant is not checked. We look at this point for the theorems of the whole book in § 06.

04

Holes and axioms — `sorry` passes with only a warning

Lean provides two ways of skipping part of a proof. One is `sorry`, a mark meaning "fill this in later". The other is an axiom, a statement declared true without proof. With either, the theorem passes. So the fact that it "passed" alone does not tell you there are no holes.

The tool for finding holes is `#print axioms`. Given a theorem name, it lists which axioms that theorem ultimately depends on (following the statement and the proof). `sorry` is internally replaced by an axiom called `sorryAx`, so it shows up here. The Lean reference manual writes, as a checking procedure, "confirm that only the three standard axioms `propext`, `Classical.choice` and `Quot.sound` appear". The condition for the Lean label in this book has this form.

$$\mathrm{axioms}(\mathrm{theorem}) \subseteq \{\texttt{propext},\ \texttt{Classical.choice},\ \texttt{Quot.sound}\} \quad (5)$$

The meanings of the three are, in order, "equivalent propositions are equal", "an element can be extracted from a proof of existence (the axiom of choice)" and "equivalent elements are equal in the quotient". A quotient is the set obtained by lumping together the elements regarded as the same under an equivalence relation (the set of equivalence classes), for example the classes of remainders of integers divided by 7. The everyday reasoning of mathematics (including the law of excluded middle) needs nothing more than these three. For details, see 23-13 Lean statements and the limits of machine checking and "Axioms" in the guide.

On this machine we checked seven small files. Table 2 is the result.

Table 2 Results of checking seven small Lean files, eight theorems (Lean 4.33.1) Computed

File (theorem)	Content	Exit code	Warnings, errors	Output of `#print axioms`
`pos_nat`	For natural numbers, "if $0 < n$ then $0 < n$"	0 (success)	None	Does not depend on any axioms

File (theorem)	Content	Exit code	Warnings, errors	Output of `#print axioms`
`pow_mod_decide`	$2^{10} \bmod 7 = 2$ by `decide` (computation inside the kernel)	0	None	Does not depend on any axioms
`pow_mod_native`	The same statement by `native_decide` (computation with compiled code)	0	None	`pow_mod_native._native.native_deci`
`pow_mod_wrong`	The false statement $2^{10} \bmod 7 = 3$ by `decide`	1 (failure)	Error: showed that the proposition is false	—
`gap_from_bound`	Formula (4) (Mathlib's real numbers)	0	None	The standard three
`gap_from_bound'`	The proof of formula (4) replaced by `sorry`	**0 (success)**	Warning: `declaration uses `sorry``	The standard three + `sorryAx`
`vacuous`	$x = 42$ from contradictory assumptions (§ 05)	0	None	The standard three
`mass_gap_positive`	An empty theorem with only a big name (§ 05)	0	None	The standard three

Seconds are wall-clock time including loading `import Mathlib` (a 4-core machine, nice 19). Rows that load Mathlib take about 5 seconds for loading alone. The values are from one run in `results.json` and vary by about 10% from run to run (the exit codes, warnings and axiom rows are the same on re-running).

There are three things to read. First, a file containing `sorry` also passes with exit code 0. All that appears is a warning. If a checking record only says "no errors", the absence of `sorry` is not guaranteed. Second, `native_decide` creates one dedicated axiom just for that call. The Lean reference manual explains that this is so that what each axiom asserted can be audited individually. It differs by version: up to 4.28.0, every call appeared as depending on a common axiom `Lean.ofReduceBool` (with `Lean.trustCompiler` behind it). From 4.29.0, it changed to a dedicated axiom per call (of the form `…ax_1_1` in Table 2). Axiom names change with the version, so this book judges by "is it a subset of the standard three?". Third, a false statement does not pass with `decide`.

Detour: using real numbers brings out the three axioms

You may skip this. `mass_gap_positive` in Table 2 is "if $0 < m$ then $0 < m$", and its proof just returns the premise. Even so, the standard three appear. For `pos_nat` (natural numbers), of the same form, nothing appears. The only difference is the type, so the three come from the side of the real numbers. `#print axioms` follows not only the proof but also the definitions appearing in the statement (including the type $\mathbb{R}$). Mathlib's type of real numbers is built as a quotient of Cauchy sequences, and that definition itself depends on the three. In fact, even the empty theorem "True for an element m of $\mathbb{R}$" brings out the three. So theorems that talk about real numbers almost always bring out the three. That the three appear is not a sign that the proof is heavy.

For how this book uses this, see 23-13 § 02 (the #print axioms at the end of each bundle as the standard of cleanliness).

05

It passes, but is it what was meant? — the four gaps

Even with no holes and only the standard three axioms, something is still unchecked: **whether the statement says what it means to say**. At the start of its checking procedure, the Lean reference manual writes that it is important to distinguish the question "does the theorem have a correct proof?" from the question "what does the statement of the theorem mean?". The latter cannot be checked by machine.

This site's page 8 of the guide divides this gap into four types (Table 3).

Table 3 The four gaps (the guide's classification, with examples from this book)

Type	What happens	How to find it
① Definition	The definition is weaker or stronger than intended (e.g. if "gap" is defined by ≥ 0, it passes even with 0)	Follow every definition appearing in the statement
② Name	The name of the theorem is bigger than the statement	Read the statement, not the name
③ Vacuous truth	The premises cannot hold at the same time, so anything can be concluded	Build one concrete example that satisfies the premises
④ Assumption	What should be proved is placed in an assumption, and when it fails is not written	Make a list of assumptions and look at the label of each

② and ③ actually occur in the last two rows of Table 2. `mass_gap_positive` carries the name mass gap, but the statement is "a positive number is positive". To Lean a name is just a string and guarantees nothing about the content. This is why, when this book cites a theorem name, it writes the statement and the premises on the same line.

An example of ③ is the following statement.

$$(x < 0) \ \wedge \ (0 < x) \implies x = 42 \ \text{(6)}$$

The premises contradict each other, so the conclusion can be anything, and Lean passes it with a single `linarith` (an automatic step that combines linear inequalities). The more premises a theorem has, the harder this danger is to see. The way to check is to give one example that satisfies all the premises (a case where the answer is known, such as a free field). In Part 23 of this book, the role of an example satisfying all the premises is played by checking against the free field (23-16). Negative controls (23-13 Lean statements and the limits of machine checking) go the other way: they show that removing one assumption breaks the conclusion, confirming that the assumption is not decoration.

④ (assumptions) is the most important in this book. In a theorem that draws a conclusion from an unproved assumption such as the diffusive-relaxation-time assumption, the content is on the side of the assumption, and the theorem does not prove a single step of that assumption (23-13 Lean statements and the limits of machine checking).

06

What the Lean statements in this book talk about

A Lean theorem is a statement about typed objects. So looking at the types of a theorem's arguments tells you what the statement actually talks about. 23-13 Lean statements and the limits of machine checking divides the forty-six theorems that this book cites as conclusions into five layers by the types appearing in their statements. Exactly half, 23, are inequalities or identities of real numbers, and the theorems that have the lattice gauge field as a type are the 3 of 5-11.

Written as a formula, a sentence in this book with the Lean label consists of two parts.

$$\text{a sentence of this book} \ = \ \underbrace{(P_1 \wedge \cdots \wedge P_k \Rightarrow Q)}_{\text{checked by Lean}} \ + \ \underbrace{\text{reading: "}C_P \text{ is the Poincaré constant of}}_{\text{attached by the text; not ch}}$$

Just as C_P was a named real number in the example of § 03, in many theorems of this book C_P, ξ and m are real variables. Reading them as physical quantities is done in the text, and whether that reading is right lies outside Lean. The conventions for the reading (the definition of β, the normalisation of tr, the metric of the lattice) are in 23-13 Lean statements and the limits of machine checking and A-01 Table of notation.

Why is there no theorem about the Wilson measure itself? One reason is stock. Counting words in the source of Mathlib, Lean's standard library of mathematics, in the version on this machine (the version is in Table 7 of § 08) gave Table 4.

Table 4 Number of files, out of the 8,311 source files of Mathlib, that contain the word Computed

Word	Files	Word	Files
`Haar` (Haar measure)	70	`plaquette`	0
`Wilson loop`	0	`lattice gauge`	0
`Yang-Mills` (three spellings)	0	`spectral gap`, `SpectralGap`	0
`Poincare inequality`, `PoincareInequality`	0		

There are foundations for the uniform measure on a group (Haar measure), but the words of lattice gauge theory do not appear. The absence of a word is not a proof of the absence of a definition, but at least as named machinery, whoever uses it must build it from scratch. The guide's "Mathematics that mathlib has no stock of" divides the sources of such difficulty into three (quantity, way of stating, stock). The continuum limit and the construction of four-dimensional field theory lie beyond that (7-10).

Example. This site's paper (5-11) is one of the few examples that have the lattice gauge field as a type. Even there the Lean label is split by step: the chain from the action to a certain lower bound is Lean , and the step from that lower bound to the lattice mass gap is Paper (and only a sketch) (0-01).

Table 4b lists the bundles used in the assembly of the first step of Part 23, and Table 4c the bundles cited by Part 24. The single files outside Tables 4b, 4c and 4f whose declarations are cited by Part 23 and other chapters are in Table 4h. All of them are collected in the bundle `principia-ym-lattice-gauge-2026-11-13` (143 modules, about 30,000 lines; every module imports only Mathlib). The rows are the same as Table 1 of 23-13; what a bundle guarantees is only the algebra, arithmetic and one-variable analysis from the list of assumptions to the conclusion. Every bundle follows the convention of writing the assumptions next to the theorem name (23-13 § 04).

Table 4b The bundles of the first step (Part 23). The number of theorems is the number of `#print axioms` lines bundles were rebuilt on another computer, and every line was confirmed to be a subset of the three standa source is the bundle `principia-ym-lattice-gauge-2026-11-13` (the same bundle as Tables 4c and 4f). On 2026-1 built again from this bundle, one at a time, and the kernel re-check and `#print axioms` for the declarations cit were confirmed Lean

Bundle	Theorems	Algebra of what	Representative theorem (assumptic

Bundle	Theorems	Algebra of what	Representative theorem (assumptio
LatticeGaugeOneScale	12	The relaxation-time upper-bound assumption, the relaxation-time lower-bound assumption, and rereading as mass (the equality version)	hup_is_gap_positivity (0<m, C_P= C_P≤Kξ²)
LatticeGaugeHupBridge	18	The one-sided bridge, the lower half of the power, negative control	hup_is_gap_positivity_of_lower (0<m_lat, κ/m_lat²≤C_P, C_P≤Kξ
LatticeGaugeDoubleWellTestbed	7	The skeleton of the finite sums of the One-Dimensional One-Sided Bridge Proposition	local_rayleigh_ratio_le_one (0<m a_k≥m, c_k≥0, Σc_ka_k>0)
LatticeGaugeTwoScalePoincare	8	Splitting of the variance, the convex core, two states, the Glauber band	split_poincare_le (the splitting formula, two local Poincaré inequalities, the mean difference, negativity)
LatticeGaugeDoubleWellKramers	10	From the sandwich of the ratio to the double-well relaxation-time power statement	power_H_of_kink_ratio (0<K,s,δ, 1≤R≤K/ħ, s/ħ≤ln ξ)

Bundle	Theorems	Algebra of what	Representative theorem (assumptic
LatticeGaugeH1Assembly	17	The algebra of the rates of the Six-Node In-Phase Poincaré Proposition, the arithmetic of the margins at the six nodes	H1_assembly (seven named ones)
LatticeGaugeCoarseSideUniform	20	The arithmetic of the crossing row, both sides of Muckenhoupt	T4_paper_numeric (λ_c≥1.3×10
LatticeGaugeIntervalQfreeWindow	150	The Linear-Fractional Lemma, from the cover to the window, recomputation of the rate-limiting cell	win_qfree_window_le_of_interval (the table of interval-arithmetic o
LatticeGaugeRingUnconditionedWindow	20	The assembly of the window of the chain outside the ring	unc_window_lt_one_of_table (the of interval-arithmetic output)
LatticeGaugeRingInnerChain	22	The ratio at the corner at the wall, the S_W of the ring	sw_inner_le (q_in≤13/10, q_out≤0.998888530)
LatticeGaugePartitionFramework	14	The sector sum, the Gibbs stage, the closing condition	comp_close (X>0, b,A≥0, ℓ>0, Bar componentwise localisation, Bocl
LatticeGaugeRemVProxy	23	The assembly of the Scalar-Strand Proposition	path5_conditional (the remainder proxy ratio, the rarity and the ma named)
LatticeGaugeDetourKills, DetourKills2	17, 16	The arithmetic of the lines that killed the detour plans	Lemmas by type

Bundle	Theorems	Algebra of what	Representative theorem (assumptio
LatticeGaugeBoxReduction	23	The Two-Projection Lemma, even and odd boxes, averaging over shifts, controls	box_shift_average (the two-family inequality for each shift, C_box≥ sum of overlaps≤$\mathcal{E}$)
LatticeGaugeBoxGap	18	The Gap-Inside-the-Box Proposition, c_kg, composition of H3 and H4	H3_H4_compose (product of boxe of the box, D=ħ/S_0, n,ħ,S_0>0, c_alt<
LatticeGaugeCoreMixing, CoreMaxCorr, CoreThreeStage	15, 20, 18	The three stages of the core	three_stage (the upper bound of ϵ stage)
LatticeGaugeCoreActionGap	15	The vanishing of the local term, drawing the line at the ends	cloc_term_vanishes (the core acti barrier assumption, the reading c ΔS_*<2S_0)
LatticeGaugeGlobalLyapunov	21	The algebra of the Lyapunov criterion, and a positive control	gauss1d_bbcg, gauss1d_equality (assumptions)

Table 4c The bundles cited by Part 24. The source is the bundle `principia-ym-lattice-gauge-2026-11-13`. The l paper, EuclideanPath.GapBridge, is in `principia-gap-bridge-2026-10-09.tar.gz` Lean

Bundle (namespace)	Theorems	Algebra of what	Representative theorem (assumptions)
LatticeGaugeLocalH (LatticeGauge.LocalH)	31	Properties of the supremum defining the local Poincaré constant, the algebra of the charges of subregions, monotonicity of the window factor, the local bridge	local_le_global (bounded above, non-empty, subset), wr_strictAntiOn (no assumptions), gap_lower_of_local_window (κ/(m^2+w^2)≤C_P, w^2≤2m/(cℓ), C_P≤K$ℓ^2$)
LatticeGaugeLocalHReference (LatticeGauge.LocalHReference)	11	The bridge in ratio form, the algebra of the threshold K_*	ratio_bridge, kstar_le_kappa (the assumptions of each are in the classification table of 24-04)

Bundle (namespace)	Theorems	Algebra of what	Representative theorem (assumptions)
LatticeGaugeLocalHDoubleWell (LatticeGauge.LocalHDoubleWell)	14	The one-site ratio in 0+1, the three regimes	one_slice_ratio (0<ε, 0<m, 0<ω)
LatticeGaugeGaugeReference (LatticeGauge.GaugeReference)	16	The gauge quotient reference, and that the reference vanishes when gauge fixing is removed	kstar_gauge_measured (measured values), kstar_vanishes_without_gauge_fixing
LatticeGaugeWilsonLoopReference (LatticeGauge.WilsonLoopReference)	14	That the family of loops spans the quotient, that colour singlets have no first-order term, the threshold of the second chaos	no_linear_singlet, chaos2_bracket
LatticeGaugeConservationLength (ConsLen)	14	The sandwich of the Gaussian gap of the strip	neumann_gap_lower / upper (1≤M)
LatticeGaugeConservationExponent (ConsExp)	16	The interval arithmetic of the slope of the strip, the relation for ℓ_c, the decomposition of the rate	lc_eq_iff (0<E, 0<λ, 0<ℓ), lamQ_log_split (0<κ, positivity)
LatticeGaugeHierarchicalLink (HierLink)	14	The arithmetic of coordinates and gaps on the hierarchical lattice	lam_le (no assumptions), tree_no_decay (0<β)

Bundle (namespace)	Theorems	Algebra of what	Representative theorem (assumptions)
LatticeGaugeFourDTrial (LatticeGauge.FourDTrial)	8	The finite sums of $\tau_int \le C_P$, the one line in which the upper-bound assumption breaks, and a negative control	not_hup_of_fast_tau ($\tau_n \le C_n$, $c\xi_n^p \le \tau_n$, $p>2$, $\xi_n \to \infty$), z_two_compatible_with_hup (no assumptions)
LatticeGaugeHupBridge (LatticeGauge.HupBridge)	18	The one-sided bridge, the rotor ratio	hup_is_gap_positivity_of_lower ($0<m$, $\kappa/m^2 \le C_P$, $C_P \le K\xi^2$), rotor_ratio_unbounded ($0<r$)
LatticeGaugeCoreMaxCorr (LatticeGauge.CoreMaxCorr)	20	The algebra of the maximal correlation of the core	Not cited by name in the text of Part 24 (the core bundle of 23-11; background to the reduction to the box in 24-03)
AlternatingMaxCorrBloch	33	The Bloch quadratic formula for the alternating maximal correlation	Not cited by name in the text of Part 24 (background to the alternating maximal correlation of the reduction to the box)
The bundle of the paper, EuclideanPath.GapBridge (`principia-gap-bridge-2026-10-09.tar.gz`)	17	The paper's version of the one-sided bridge (optimal over the family local in time, the rotor)	local_rayleigh_le (the assumptions of each are in the classification table of 24-04)

Table 4h Single files outside Tables 4b, 4c and 4f that are named by Part 23 and other chapters (40). The numb of theorem and lemma declarations in each file (excluding private ones), 725 in all. The source is the bundle `2026-11-13` (the same bundle as the files of Tables 4b, 4c and 4f). On 2026-11-13 each file was built one at a ti and `#print axioms` for the declarations cited by the chapters (all the three standard axioms) were

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugePoincareNormalisation (LatticeGauge.PoincareNormalisation)	9	Normalisation of the Dirichlet form and invariance of the exponent	`exponent_invariant_of_slowly_vary`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeAutocorrelationBound (LatticeGauge.AutocorrelationBound)	10	Comparison of the integrated autocorrelation time of a reversible chain with the Poincaré constant	`rayleigh_le_tau`, `tau_int_le`
LatticeGaugeBlockKernel (LatticeGauge)	26	A one-parameter family of one-step block kernels, and the condition for a fixed step to be bounded	`fixed_step_bounded_iff`
LatticeGaugeCoarseGraining (LatticeGauge)	24	The finite part of one coarse-graining step (quaternion identities, upper and lower bounds on one-link quantities)	`log_F0_ge`, `log_F0_le`
LatticeGaugeFiberAverage (LatticeGauge)	37	Counting for one packing step, and that blocking alone does not lower it	`no_blocking`
LatticeGaugeFlowedPoincare (LatticeGauge.FlowedPoincare)	17	Families of flowed functions and the chain rule for the Dirichlet form, the price of one smearing step	`comp_class_eq_of_equivariant_bije`
LatticeGaugeGapVersusCost (GapVersusCost)	13	That the gap bounds the target but does not bound the cost of local updates	`cost_ge`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeGaugeFixing (LatticeGauge.GaugeFixing)	10	The cost of gauge-fixed long-wavelength test functions	`jump_energy_ge`, `prop_at_gap_scal`
LatticeGaugeHardyLadder (LatticeGauge.HardyLadder)	19	A weighted Hardy inequality on a finite chain, and the ladder of scales	`conditional_scale_bound`, `const_defect_scale_unbounded`
LatticeGaugeMarginalAccumulation (LatticeGauge.Marginal)	22	Accumulation of one-step deficits along the renormalisation ladder, and marginal dimension	`dev_eq_of_marginal`, `dev_le_of_contracting`
LatticeGaugeMultiStep (LatticeGauge)	19	Slope observables of multi-step blocking, the functional equation for the correlation length, anomalous exponents	`S4_two_sided`, `kappaDisj_lt_one`, `xi_eq_exp`
LatticeGaugeOverlapExponent (LatticeGauge.Overlap)	20	Zero-momentum overlap, operator dimensions, what the renewal series does not see	`eta_ge_two_of_bounded_susc`, `fisherEta_eq_zero_iff_canonical`, `overlap_le_susc_mul_gap`, `renewal_chi_mul_massSq`, `renewal_residue_pos`
LatticeGaugeProfileFreedom (LatticeGauge.ProfileFreedom)	9	The freedom in the shape of translations in the variational lower bound	`profileQuotient_isGreatest`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeRunning (LatticeGauge)	19	What one blocking step does to the coupling constant (the strong-coupling end and the weak-coupling end)	`discStep_semigroup`, `step_lowers_coupling`
LatticeGaugeSlicePositivity (SlicePositivity)	17	Smearing of time slices, positivity of the transfer matrix, monotonicity of the effective mass	`meff_antitone`
LatticeGaugeSmearedSquare (LatticeGauge.SmearedSquare)	10	The variational quotient of the squared smeared field, and what smearing gains	`sqQuotient_between`
LatticeGaugeSmoothGaugeFixing (LatticeGauge.SmoothGaugeFixing)	11	Orbit averages for smooth gauge fixing, and that they do not act on the hinge	`orbitAvg_continuous`, `orbitAvg_gauge_invariant`
LatticeGaugeSpectralHinge (LatticeGauge.Hinge)	22	Decomposition of the hinge between the Poincaré constant and the square of the correlation length (the Gaussian optimum, the drain exponent)	`c2_ge_of_nonneg_weights`, `drainExp_eq_zero_iff`, `drain_expo`
LatticeGaugeStepSequence (LatticeGauge)	19	The sequence of steps (decomposition of the packing integral and the one-step Hessian lower bound)	`blocking_shortfall`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeSubspacePoincare (LatticeGauge.SubspacePoincare)	16	Poincaré constants on a restricted space of functions	`autocov_sum_le_on_invariant`
LatticeGaugeSusceptibilityRatio (SusceptibilityRatio)	8	What the ratio of the susceptibility to the equal-time correlation can and cannot tell	`gap_ge_of_overlap`, `light_admixture_invisible`
LatticeGaugeTubeDefect	14	The sign of the tube deficit in the model of surfaces with positive weights	`linear_deficit_breaks_inv`
LatticeGaugeTubeEntropy (LatticeGaugeTube)	18	Counting surfaces (the entropy of the tube) and the lower half at one scale	`mass_sq_ge_of_tube`, `linear_regime_breaks_onescale`
LatticeGaugeTwoLink (LatticeGauge.Character)	28	The finite part of the two-link integral	`intact_frac_le`
LatticeGaugeVariationalBound (VariationalBound)	9	That the variational effective mass is an upper bound on the gap	`meff_ge_of_corr`
LatticeGaugeWulffCorrection	13	The tension of a tilted wall and the Wulff subtraction (the deficit of a closed tube)	`casimir_only_no_bound`, `mass_bound_localized`
LatticeGaugeH1TaylorStage (LatticeGauge.H1TaylorStage)	19	Three-point interpolation in the β direction and the cubic remainder, endpoints of the linear fraction on a small piece	`cube_le_exp_split`, `lebesgue_le`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeH1Tilt (LatticeGauge.H1Tilt)	22	Upper bound on the chord in the β direction and the Hölder loss, the number of subintervals	`chord_beta_upper`, `holder_r117_exceeds`, `nodes_reach`
LatticeGaugeRingMirror (LatticeGauge.RingMirror)	20	Algebra of the mirror image at the seam of the ring	`chain_action_reflect`
LatticeGaugeAdjointSandwich (LatticeGauge.AdjointSandwich)	19	Arithmetic that sandwiches the remaining assumption (MC-sup) between the reading from below and path-by-path upper bounds	`mbar_bound_false`
LatticeGaugeDriftDual (LatticeGauge.DriftDual)	20	Algebra of the weak dual measured by the dual of the conditional drift, and arithmetic on the measured values	`tilt_g1`
LatticeGaugeHbarZero (HbarZero)	23	Algebra of the form that fixes the range of ħ by re-choosing the window	`p5_assembly`
LatticeGaugeLocalizedBochner (LatticeGauge.LocalizedBochner)	29	Composition that localises the negative curvature of the Bochner step to the defect set, and the cost of the tilt	`bochner_localized`, `judge_i_insid` `negw_le_localized`, `tilt_cost_le`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeBTDressed (BTDressed)	17	The form of one-dimensional sequences counted in the dressed direction (the margin does not depend on the direction, scale invariance)	`threshold_gap_invariant`
LatticeGaugeBartaRefine (LatticeGauge.BartaRefine)	19	How to split the tilt between the Barta and Bochner defects, the ceiling lemma, negative controls	`ceiling_same_set`, `ceiling_split`
LatticeGaugeEigenGradient (LatticeGauge.EigenGradient)	16	A form that uses only the gradient of the eigenfunction, one step with the Agmon weight, negative controls	`agmon_no_gain_on_negative_curvatu`
LatticeGaugeExtendedChain (LatticeGauge.ExtendedChain)	19	Composition of the LSI constant of the extended chain, and the condition for that constant to be bounded	`S_ext_01_iff`
LatticeGaugePlateauLSI (LatticeGauge.PlateauLSI)	26	Decomposition of the LSI on the plateau at the top (collective coordinate and transverse directions), transverse eigenvalues	`transverse_fails_k17`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeWallCut (LatticeGauge.WallCut)	21	Composition of the multi-step decomposition cut at the position of the wall	`chain_rule_cells`
LatticeGaugeKinkPosition (LatticeGauge.KinkPosition)	16	Bistability at the negative end of the core, and conditioning on the binary of the bump	`pn_vs_bistab`, `y_side_k9_3em4`

Table 4d The bundle of Parts 25 and 26 (Y7, 11 files). The number of theorems is the number of theorem and lemma declarations in the files (32). Rechecked on another computer on 2026-10-15: all use the three standard axioms, `sorry` 0. `Lean`

File (namespace)	Theorems	Content	Chapters that cite it
InsertionIdentity (Insertion)	5	The identity for the sum of insertion ratios, mean_one, mean_rho_sigma, sector_bias_le	25-05
BoundedInsertion (Bounded)	5	single, single_mul, multi, averaged, averaged_bound	25-05
SecondMoment	2	sqrt_identity, pz_bound	25-05
MultiCell	3	pushforward, expand_identity, bound ("odd under insertion" is an assumption)	25-05
FlipLemma	2	flip_le, cut_formula	25-06
FluxLemma	1	flip_le_one_sub_abs	25-06, 26-05
Containers	2	factorize, container_bound	25-03
FluxDuality	3	split, twisted_le, bias_mem (positivity is assumed for $\theta \leq 1$)	26-04
Doubling	2	iterate, decay	26-02, 27-01
ReflPos	2	refl_pos, refl_cs	26-04, 26-05, 27-02
ColumnChessboard	5	signed_sum, unsigned_sum, prod_expand, binomial_bound, exp_bound	25-02 (paper on lattice fractional instantons)

Only two files have a file name different from the namespace, InsertionIdentity (Insertion) and BoundedInsertion (Bounded); the text cites them in the form "file (namespace.theorem)". The bundle is `principia-ym-insertion-identities-2026-10-15.tar.gz` (package `yminsertion`, lib `Y7`, sha256 `58d60c433cb23cb9cfab1f2299732385f79a362b96a1d49ac109334b4832bdf8`, record of axioms axioms-ym-insertion-identities-2026-10-15.txt).

Table 4e The bundle of slabs and containers (lean-y6, 30 files + Lemma5Core). The number of theorems is the number of lines in the `#print axioms` record (150: of these, 137 are declarations written in this book's study

for Parts 25 and 26, and 13 are declarations added on the verification branch). On 2026-10-15 it passed `lake build` on another computer: all use the three standard axioms, `sorry` 0. The bundle is `principia-ym-slab-containers-2026-10-15.tar.gz` (package `ymslabcontainers`, lib `Y6`, sha256 `4892aca93897af4940828fd14d5fc45e e72bdcfca0914a2dcd20dd2d77229341`, record of axioms axioms-ym-slab-containers-2026-10-15.txt). `Lemma5Core.lean` is outside the lean_lib and is not checked by `lake build` alone (it is checked separately with `lake env lean`; its record of axioms is `axioms-Lemma5Core.txt`). `Lean`

Main files	Content	Chapters that cite it
SlabConfine, SlabAverage	The Slab Confinement Theorem and the slab average	25-04
FrameLemma, CommutingFlips	The frame lemma of the container identity, commuting flips	25-03, 25-04
GapFromDecay, LevyArea, TwistLocality and others	Gap from decay, Lévy area, locality of the twist	25-03–26-05

Table 4f The bundles of Part 15 (ultraviolet control) (72 single files; the number of theorems is the number of lemma declarations in each file, 704 in all; the last twenty-five files were added on 2026-10-17, 2026-10-22 and 2 have been re-verified on this book's side; the four from DominantMatrixContraction to ConeCalculus bel systematic chapter 15-10, the seven after them were added on 2026-10-22, and the last seven on 2026-10-29). R file at a time on the main machine (a different computer) on 2026-10-15: all rc=0, `sorry` 0, `native_decide` standard axioms. The assumptions of each theorem and "what is not included" are in 15-09 Table 3. An inde verification (same environment, one file at a time, the 62-file inventory) also gave rc=0 and the three standard (evening of 2026-10-15). `Lean`

File (namespace)	Theorems	Main theorems	Chap
LatticeGauge.ClassResidualTelescoping	6	`telescoping_residual`, `telescoping_residual_multi`, `inv_sq_sum_le`, `uniform_deviation`	15-03
LatticeGauge.HierarchicalShapeFlow	11	`quarticStep_fixed`, `shapeStep_contract`, `flow_coefficient`, `affine_contraction`, …	15-04
LatticeGauge.FluxNetwork	11	`ratio_le_max`, `ratio_ge_of_unit_flux`, `energy_split`, `thomson_lower`, …	15-05 Lemr Step Back Indep Fluct
LatticeGauge.AllFacesBlocking	6	`fine_energy_lower`, `block_energy_lower`, `smooth_ratio`, `equal_split_ratio`	15-05
LatticeGauge.RatioTwoCriterion	9	`ratio_two_iff`, `one_loop_D`, `gamma_family_one_loop_lower`, `fund_adj_one_loop_root`, …	15-06
LatticeGauge.FundAdjointRefit	8	`refit_solution`, `tree_quadratic_split`, `tree_value_at_saddle`, `image_rho`, …	15-06

File (namespace)	Theorems	Main theorems	Chap
LatticeGauge.PrefactorFlowFixedPoint	12	`step_k2`, `step_k2_fixed_iff`, `step_k2_contract`, `fixed_point_general`, …	15-06
LatticeGauge.HeatKernelFourPlaquette	8	`conv_assoc`, `convPow_add`, `four_plaquette_of_semigroup`, `four_plaquette_convPow`	15-06
QuaternionReImSplit	9	`re_mul_split`, `commutator_re`, `J_imI`, `two_link_split`	15-07
LatticeGauge.ShapeRemainderUniform	9	`uv_uniform`, `stays_in_box`, `pow_sub_pow_norm_le`, `keff_half_le`, …	15-06 Rema Theo Plaqu Row
LatticeGauge.HierarchicalTubeChain	8	`tube_exit`, `cell_descent`, `chain_compose`, `tube_weight_mono`, …	15-04
LatticeGauge.ShapeResidualGeometric	4	`forced_uniform`, `geom_partial_le`, `residual_sum_uniform`, `harmonic_part_unbounded`	15-07
LatticeGauge.LargeFieldSource	4	`weighted_tail_le_exp_moment`, `union_tail_le_sum`, `coarse_source_commute`, `coarse_source_commute_comp`	15-08
LatticeGauge.TreeGaugeFixing	29	`exists_gauge_tree_trivial`, `gauge_tree_trivial_unique`, `configEquiv`, `fiber_card_const`, …	15-08
LatticeGauge.WilsonWindowComparison	4	`one_sub_cos_window`, `wilson_weight_window`, `window_constant_chain`, `window_split_bound`	15-07
LatticeGauge.QuarticRowColumnBound	7	`weighted_quartic`, `row_quartic`, `schur_quartic`, `representative_bound`, …	15-05 Netw (loca
LatticeGauge.BCHCubicResidual	6	`nT_sum_diff`, `shape_split`, `score_generator_split`, `bch3_coefficient`, …	15-07
LatticeGauge.SmallFieldLocSplit	3	`loc_split_unique`, `geom_tail_le`, `taylor_tail_bound`	15-08
LatticeGauge.BackgroundBudget	4	`budget_term_sq`, `budget_unbounded_of_threshold`, `inv_sq_partial_le`, `inv_sq_budget_bounded`	15-07

File (namespace)	Theorems	Main theorems	Chap
LatticeGaugeInterBlockLocAbsorption	7	`loc_split`, `nonlocal_vanishes_on_constant`, `rowSum_add`, `two_site_split`, …	15-08
LatticeGaugeBlockCouplingCancellation	28	`locM_idem`, `locLin_idem`, `split_unique`, `bil_split`, …	15-08
LatticeGaugeSmallFieldRemainderAssembly	16	`one_step_remainder`, `gamma_ge_one`, `cut_sum_le`, `taylor_step_le`, …	15-08
LatticeGaugeToeplitzTailSplit	10	`split`, `split_unique`, `qform_split`, `cform_split`, …	15-08
LatticeGaugeWeightedTailBound	5	`weighted_geom_le`, `weighted_tail_le`, `weighted_sum_uniform_linear`	15-08
LatticeGaugeTailTypeClassification	6	`flat_weighted_sum`, `flat_weighted_unbounded`, `exp_weighted_le`, `power_sum_uniform`, …	15-08 § 07
LatticeGaugeClassParameterBudget	4	`ghat3_forms`, `ghat3_zero_iff`, `inv_sq_partial_le`, `class_parameter_budget_bounded`	15-07
LatticeGaugeStokesSecondOrder	7	`log_quadratic_part`, `log_quadratic_part_five`, `commSum_eq_zero_of_comm`, `pure_quaternion_commutator`	15-07
LatticeGauge.CubeFaceCoupling	9	`total_sq`, `opp_eliminate`, `general_to_diag_adj`, `diag_adj_unique`, …	15-07
LatticeGaugeStapleSumLowerBound	5	`sum_norm_ge_of_inner_ge`, `card_cos_ge_quadratic`, `first_order_ratio`, `quadratic_margin`, …	15-07
LatticeGauge.QuarticSplit	4	`titu_two`, `quartic_two`, `quartic_triangle`, `class_split`	15-05
LatticeGauge.GhostDeterminantMatching	8	`det_gauge_plus_ghost_split`, `log_tree_eq_cov_sub_two_ghost_blocks`, `coeff_tree_eq_coeff_cov`, `coeff_drop_ghost_shift`, …	15-08
LatticeGauge.BondMovingSandwich	6	`bond_moving_finite`, `iterate_le_of_le_of_monotone`, `comparison_fails_at_fixed_point`, `shrink_shape_shift`, …	15-04

File (namespace)	Theorems	Main theorems	Chap
LatticeGauge.InfluenceRecursion	6	`tp2_comp`, `cdf_upper`, `cdf_lower`, `k_upper`, …	The s row c Poinc unifo assur Part 2 text i 15-00
LatticeGauge.CornerCouplingAffineFlow	5	`fixpt_is_fixed`, `iterate_dev`, `abs_iterate_dev`, `ratio_gt_one_below_fixpt`, …	15-07 § 02
LatticeGauge.HypercubeFaceCoupling	5	`sigma_is_relation`, `relation_is_multiple_of_sigma`, `opposite_in_cell_reduces`, `diag_adj_not_enough`, …	15-07
LatticeGauge.SkewCouplingSpectrum	5	`coeffs_of_spectrum`, `skew_zero_iff`, `one_step_skew`, `increment_pow`, …	15-07
LatticeGauge.HypercubeCochainCount	6	`faceCount_four`, `euler_cube`, `hypercube_cochain_ranks`, `edge_face_dof_agree`, …	15-03 § 07
LatticeGaugeNonlocalImageRecursion	12	`affine_iter`, `fixed_point`, `affine_bound_sum`, `affine_bound_uniform`, …	15-08 § 03
LatticeGaugeThreeBodyTailRecursion	8	`weighted_geom_le`, `three_body_sum_le`, `affine_uniform`, `pair_tail_uniform`, …	15-08
CoarseFaceHodgeSplit	7	`range_sup_tags`, `finrank_ker_eq`, `disjoint_range_tags`, `decomp_unique`, …	15-08 § 07
LatticeGauge.QuarticConeNorm	5	`ratio_sandwich`, `ledger`, `transfer`, `cone_b_to_a`, …	15-05
LatticeGauge.BlockCovariance	7	`cone_chain`, `cone_chain_uniform`, `growth_ledger`, `inverse_decay`, …	15-05
LatticeGauge.AveragedBlockFlow	7	`affine_iter`, `affine_dist`, `tail_sum`, `tail_bound`, …	15-05
LatticeGaugeHierarchicalUpperTube	8	`tube_exit_top`, `chain_top`, `steps_numeric_128`, `steps_numeric_64`, …	15-04
LatticeGauge.TensionDomination	6	`iterate_bound_var`, `iterate_bound`, `lower_bound_four`, `area_law_transfer`, …	15-04
LatticeGauge.BlockStepBetaShift	6	`beta_shift_of_inv_sq`, `su2_coeff`, `log_steps`, `shift_bounds`, …	15-01 § 02

File (namespace)	Theorems	Main theorems	Chap
LatticeGaugeTwoBlockRowSumLoc	14	`sRow_eq_sComp_add`, `nRow_eq_nComp`, `row_sum_cancels`, `row_split_unique`, …	15-08
LatticeGauge.MultiStepCone	5	`step_pair`, `contraction`, `cone_quartic`, `block_three`, `cone_transfer`	15-08 boun re-fix
LatticeGauge.MultiStepPerturb	3	`three_product_perturb`, `three_product_perturb_grouped`, `block_three_affine`	15-08 boun re-fix
LatticeGauge.QuarticCone	2	`four_amgm`, `quartic_sum_le`	15-08
LatticeGauge.ColorAverage	10	`colorAverage`, `colorAverage_of_design`, `design_identity`, `isQuarticDesign`, …	15-08 boun re-fix
LatticeGauge.CubicCone	7	`three_amgm`, `cubic_sum_le`, `triple_amgm`, `cubic_cone_pullback`, `block_three_uniform`, …	15-08
SmallSetRowSum	20	`rowSum_total`, `row_cancel`, `total_split`, `piece_loc_zero`, `theta_ledger`, …	15-08
ThetaSquareOneStep	6	`one_step_affine`, `theta_ledger_half`, `sup_monotone_in_mu`, …	15-08
DominantMatrixContraction (DominantMatrix)	21	`liftWeight_step`, `weighted_step`, `weighted_iter`, `weight_change`, `norm_pow_le_const_mul`, `sum_norm_pow_le`, `aggregate_step`, `two_by_two_weight`, `signedEx_sq`, …	15-10
SignLossBound (SignLoss)	16	`stein_le_inv_sq_mul_test`, `stein_form_le`, `schur_four`, `loss_le_of_coherent`, `comp_coherent`, `row_l1_comp_le`, …	15-10
ConstrainedExtension (ConstrainedExtension)	20	`isExtension_iff_mem_coset`, `finrank_freedom`, `exists_extension_with_conditions`, `finrank_freedom_with_conditions`, `eq_of_eqOn_iSup`, `comp_isExtension`, `comp_depends_of_not_le`, …	15-10
ConeCalculus (ConeCalculus)	20	`cone_pullback`, `contracts_comp`, `block_decay`, `lp_schur`, `lp_schur_sharp`, `delta_dilution`, `weighted_rpow_midpoint`, `weighted_rpow_shift`, `midpoint_below_one`, `affine_varying`, `affine_block`, …	15-10

File (namespace)	Theorems	Main theorems	Chap
ColorAverageN (LatticeGauge.ColorAverageN)	13	`colorAverage_of_design`, `design_identity`, `colorAverage`, `oneColour_of_colorAverage`, `design_identity_two`, `cube_moment`, …	15-08 boun re-fix § 07
OneStepWindow (LatticeGauge.OneStepWindow)	20	`holder_dual_ratio`, `exists_lpRatio_ge_row`, `lamMax_mono`, `steinWeighted_ge`, `stein_row_weight_saturation`, `window_ordConnected_of_logConvex`, …	15-10
LargeSetWeight (LatticeGauge.LargeSetWeight)	16	`summable_shell_iff`, `bbs_weight_vs_decay`, `bsv_weight_vs_decay`, `card_shell`, `manhattanShell_three`, …	15-08
LatticeGaugeAcceptanceSandwich (LatticeGauge.AcceptanceSandwich)	15	`half_metro_le_fermi`, `fermi_le_metro`, `barker_le_of_crooks`, `overlap_sandwich`, `mixture_sandwich`, `jarzynski_of_crooks`, …	25-06 § 03
LatticeGaugeAbelianTwistAction (LatticeGauge.AbelianTwistAction)	13	`abelianAction_lt_pi_sq`, `totalAction_eq`, `plaqTerm_mixed`, …	27-02 small
LatticeGaugeNestedBlocks (LatticeGauge.NestedBlocks)	7	`nested_supermultiplicative`, `doubling_iterate`, `doubling_bound`, …	26-05 § 03
CenterFluxPolymerCertificate (PolymerCert)	6	`tree_k`, `kp_bound`, `kp_lt_one`, `rate_rational`, `exp_le_inv_one_sub`, `final_real`	26-06
ClassFixedPoint (LatticeGauge.ClassFixedPoint)	24	`exists_unique_fixed`, `exists_unique_fixed_in_ball`, `norm_iterate_sub_le`, `exists_unique_fixed_of_iterate`, `dist_fixed_fixed_le`, `tracking`, `exists_unique_entire_orbit`, `entire_tracking`, …	15-03
LocalCancellationOneRow (LatticeGauge.LocalCancellationOneRow)	8	`pairs_sum_to_total`, `loc_remainder0_eq_zero`, `pairAB_rank_one`, `pairAB_local_plus_gradient`, `mirror`, …	15-08
RankOneRemainder (LatticeGauge.RankOneRemainder)	12	`pair_split`, `edge_sum_split`, `chain_remainder_eq`, `chain_remainder_zero_of_match`, `deflation_split`, …	15-08
BlockDeflation (LatticeGauge.BlockDeflation)	9	`blockDeflate_split`, `blockDeflate_left_zero`, `blockDeflate_right_zero`, `twoStep_zero`, `twoStep_split`, …	15-08

File (namespace)	Theorems	Main theorems	Chap
ConeCalculusSOS (LatticeGauge.ConeCalculusSOS)	27	`cert_sound`, `certValue_anti`, `floor_of_coeff`, `classSum_pushforward`, `classSum_pullback`, `shift_rigid`, `tensor_ratio_product`, `schur_weighted_bound`, …	15-10
LatticeGaugeH1HbarInterval (LatticeGauge.H1HbarInterval)	15	`cell_lower`, `cell_upper`, `rho_lower`, `pieces_min`, `margin_on_interval`, `margin_I1`, `margin_I2`, …	23-15 unifo doub
StrongCouplingOneSidedTerm (StrongCouplingOneSided)	9	`area_le_strip`, `strip_coeff_lower`, `tube_coeff_neg`, `h_main_term`, `h_neg`, `h_neg_on_interval`, …	27-02 langu coup

The labels in the text of Part 15 stay the five grades (paper proofs are [Proved here]); Lean is attached in the text of each chapter as the theorem name and its assumptions on the same line (15-09 §01). The source is the bundle `principia-ym-lattice-gauge-2026-11-13` (the tar.gz under `principia/lean/`, and the axiom log `axioms-ym-lattice-gauge-2026-11-13.txt`). Of these, 23 files are also in the bundles published in October 2026 (flux-network-blocking, hierarchical-chain, one-row-renormalisation, coarse-chain-poincare, h1b-margin-interval, boundary-mixed-derivative), as the same files.

07

How the labels are decided — one label per sentence

This book attaches labels sentence by sentence, not chapter by chapter or paper by paper, because even within one paper the label differs from step to step (the example in § 06). Which label is attached is decided by the kind of guarantee the sentence has. Figure 2 puts that decision in the form of questions asked from the top down.

There is a reason for the order of the questions. The Lean label guarantees the arrow from premises to conclusion; the Known label refers to a proof in the literature, the Paper label to a proof on this book's side, the Computed label to a check over a finite range, and the Physics label to acceptance at the level of physics. The strong guarantees are asked first, and the first label that applies is attached. A sentence that matches none gets no label and is written as an "assumption" or a "question" (the questions of 23-17 carry no label).

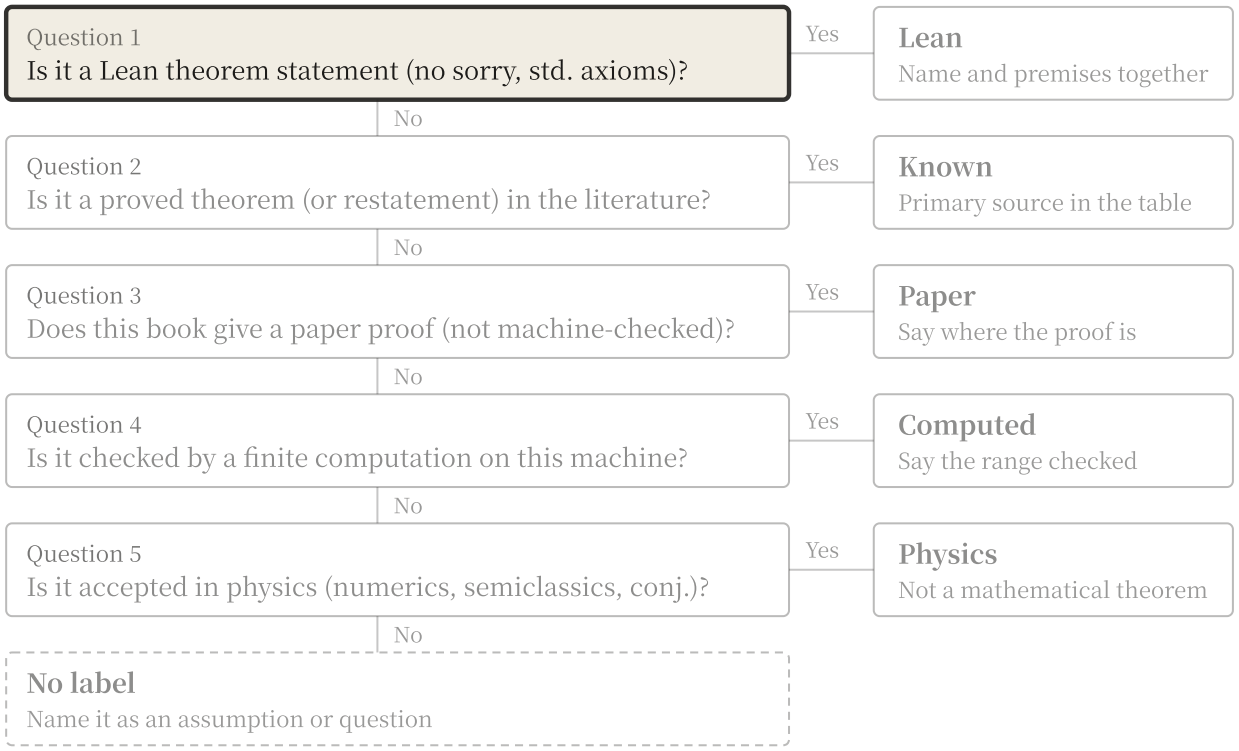
When the sentences form a chain, read it as follows.

$$\text{for the chain } A_0 \Rightarrow A_1 \Rightarrow \cdots \Rightarrow A_n\text{, the certainty of the conclusion } = \text{ the label of the weal}$$

If just one step is `Paper` , then even if all the other steps are `Lean` , the conclusion is at the level of Paper. If the starting point A_0 is an assumption, the conclusion is also conditional on it. The premises P_i of formula (1) correspond to this starting point.

Figure 2 Answering the questions from the top with "Yes / No" shows the path taken and the label attached. The

example buttons pick one sentence of this book and show its path



`Question 1: Is it a Lean theorem statement (no sorry, std. axioms)?`

This figure puts the conventions for labels in 0-01 in the form of questions. It contains no numbers.

Table 5 Examples of sentences in this book, and the path they take in Figure 2

Sentence	Path taken (answers to questions 1–5)	Label	Chapter
Formula (4): $m > 0$, $C_P = 1/m^2$, $C_P \le K\xi^2 \Rightarrow 1 \le K(m\xi)^2$ (the same statement as `hup_is_gap_positivity` in 23-13)	Yes	Lean (with premises stated alongside)	§ 03, 23-13
The one-loop coefficient b_0 (including $11N/3$)	No → Yes	Known	4-06
The step from the one-link lower bound to the lattice mass gap	No → No → Yes	Paper (sketch)	5-11
The behaviour of C_P/L^2 in small boxes ($L = 3, 4, 5$) for the two-dimensional Ising model	No → No → No → Yes	Computed	2-10
The ratio of the glueball mass to the string tension from lattice numerical simulations	No ×4 → Yes	Physics	5-08

Sentence	Path taken (answers to questions 1–5)	Label	Chapter
Relaxation-time assumption (at the level of powers, $\ln C_P / \ln \xi \to 2$; the band statement is named the band form of the diffusive-relaxation-time assumption, and its upper half is the upper-bound assumption: $C_P \le K\xi^2$)	No ×5	No label (assumption)	23-01

How to read it. The Lean premise $CP \le K\xi^2$ in the first row of the table is the upper half of the band statement of the assumption in the last row (the band form of the relaxation-time assumption), that is, the upper-bound assumption (A-06 Index of names). In the first row it is a premise; in the last row, the claim of an assumption. The labels differ because the roles in the sentence differ.

08

A procedure to check for yourself

The Lean label presupposes that readers can check again on their own machines. The Lean reference manual describes checking as a staircase of four stages (Table 6). The lower stages are easier; the higher stages widen the range of doubt.

Table 6 The four stages of checking (from the structure of "Validating a Lean Proof" in the Lean reference manual)

Stage	What to do	What it guards against
1	See that the editor's check mark appears, or that `lake build` finishes with no warnings or errors	Unfinished proofs, `sorry` inside that theorem, honest mistakes in tactics
2	Write `#print axioms theorem-name` after the theorem, and confirm formula (5)	`sorry` inside theorems it depends on, extra axioms, `native_decide`
3	Use `lean4checker` to pass the proofs saved by the build through the kernel again	Bugs in the core of Lean that handles the kernel's state, metaprograms or tactics that deliberately bypass checking
4	Write only the statements in a trusted place, assemble the proofs in isolation with `lake comparator`, pass them again through both Lean's kernel and an external checker written independently of it, and also confirm that the statements match	Malicious proofs, implementation bugs present in only some of the checkers used

The manual itself positions the isolated-comparison step for high-risk settings, such as markets for proofs, competitions with large prizes, and AI whose intentions are not aligned. About the build-check step, the manual warns that the check mark appears even if a dependency contains `sorry`. This is the same situation as `gap_from_bound'` in Table 2 having exit code 0, and it is why the axiom-check step (a checking step of the Lean reference manual) is needed. The Lean labels in this book are attached to theorems that have been through at least the axiom-check step.

The concrete procedure has the following form. How to install and match versions is in "Installing" in the guide and "Checking a distributed proof locally".

```
lake build                      # Stage 1: check the whole project
lake env lean File.lean         # check a single file only
#print axioms gap_from_bound    # Stage 2: write this inside the file
```

This site's Lean record (the list of theorems and the record of checking) is at the Lean record. Even the same proof may stop passing when the versions of Lean and Mathlib differ, because lemma names change. For this reason this book records the versions used for checking.

Table 7 The environment in which the examples of this appendix were checked Computed

Item	Version
Lean	4.33.1 (commit `819816b2`)
Mathlib	rev `0df444a3`
Machine	x86_64, 4 cores, Linux. Checking was run at low priority (nice 19)

Finally, we gather into seven the things to ask when you receive a sentence with the Lean label. This is the checklist of "The procedure for reading a statement" in the guide, adapted to the way this book is read.

1. Is the theorem name given, so that you can check it again yourself (build-check step)?
2. Is there no `sorry`, and do the axioms fit within the three of formula (5) (axiom-check step)?
3. Have you followed every definition that appears in the statement (① of § 05)?
4. Did you read the statement rather than the name (② of § 05)?
5. Is one example satisfying all the premises given (③ of § 05)?
6. Are the list of premises and the label of each premise written (④ of § 05, formula (8))?
7. Where in the text is the reading of the real variables as physical quantities declared (§ 06, formula (7))?

The seventh is the question specific to this book. Many of the Lean theorems in this book connect to the mass gap problem at this point, and that connection lies outside the Lean label.

> 09

> **Declarations of Parts 19–22 and 28, the second half of Part 15, and the sections "What this book's study could see" of Volume 2 — list, and the checking procedure**

The Lean declarations cited by Parts 19–22 and 28, the second half of Part 15, and the sections "What this book's study could see" of Volume 2 (Parts 9, 10, 12 and 14) are listed chapter by chapter. The detailed form of the scope and of what is not claimed is in the scope chapter of each part (20-07, 21-08, 15-14), in "What this chapter can say" of 14-02, and in the correspondence table of definitions (19-06).

The checking procedure. Every declaration has been checked again in the core (kernel) of Lean 4 (kernel recheck), confirming that there is no `sorry` and that the axioms used are only the three standard ones. In addition, with eyes other than those that wrote the proofs, we have compared whether the Lean definitions and statements point to the same things as the definitions and statements of the text, and whether the assumption fields can be satisfied by the actual objects (cross-checking by a reader other than the writer). What machine checking guarantees goes only as far as the written statement following from the written assumptions (the four gaps of § 05).

The bundle. The declarations of Table 4g, and all the files on which they depend through `import`, are collected in the bundle `principia-ym-three-volumes-2026-11-13`, a single lake package. It uses Lean 4 v4.33.1 and Mathlib v4.33.1, has 400 modules and about 115,000 lines, and the declarations cited by the chapters number 677 (of which 455 are theorems). A lexical search finds 0 occurrences of `sorry`, `admit`, `native_decide` and custom `axiom`. How to check it (`lake build`, and `#print axioms` for each declaration), and the pairs of modules that cannot be `import`ed together because they have declarations of the same name, are written in the README of the bundle. The earlier bundle of the same day, `principia-ym-premise-first-2026-11-13` (317 modules), is entirely contained in this bundle (the 317 module files are the same).

Table 4g Lean declarations cited by Parts 19–22 and 28, the second half of Part 15, and the sections "What this book's study could see" of Volume 2 (Parts 9, 10, 12 and 14) (bundle `principia-ym-three-volumes-2026-11-13`, 400 modules)

Chapter	Theorem or definition	Scope	Lean declaration
19-06	Strong-coupling side ↔ vortex side (measure): $\tilde{\mu}_{N,\beta}$ (`TorusRP.wilsonMeasure N β`) equals the measure on the vortex side, a Haar product multiplied by a weight, transported by a one-to-one correspondence given by definition. No constant factor	Only for $M = N$ and no twist. There is no bridge for the twisted measure (background $n_{12} = 1$)	`WilsonBridge.wilsonMeasure_e`
19-06	Same (expectations, total mass): the expectations of measurable observables and the total mass $Z_N(\beta)$ agree on the two sides	As above	`WilsonBridge.integral_wilson` `WilsonBridge.wilsonMeasure_u`
19-06	Same (correspondence with aligned time): the measures also agree under the correspondence that puts the time of the strong-coupling side (index 0) onto index 3 of the vortex side, and the reflections in planes through lattice sites on the two sides become the same map	As above	`WilsonBridgeT.wilsonMeasure_` `WilsonBridgeT.refl_toBallT`
19-06	Same (vortex ratio): the vortex ratio with no background twist ($n_{12} = 0$) is written as an expectation of the probability measure on the strong-coupling side. It is not the book's θ_M ($n_{12} = 1$)	As above	`WilsonBridgeV.vortex_ratio_e`
19-06	Strong-coupling side ↔ fermion side: $\tilde{\mu}_{2h,\beta}$ is $e^{6(2h)^4\beta}$ times the pure-gauge measure `WilsonGauge.gaugeMeasure` transported by a relabelling of indices	Spatial side $N = 2h$, $2h > 1$	`WilsonGauge.wilsonMeasure_ec`

Chapter	Theorem or definition	Scope	Lean declaration
20-01	Reflection positivity in planes through lattice sites	$(\mathbb{Z}/2M)^4$, all β, bounded measurable f depending only on the closed positive half	`TorusRP.wilson_rp`
20-01	Reflection positivity in mid-planes	$(\mathbb{Z}/2M)^4$, $M \geq 2$, $\beta \geq 0$, gauge-invariant bounded measurable F	`MidRP.wilson_rp_mid`
20-01	Reflection positivity in planes through lattice sites, in four directions	$(\mathbb{Z}/2M)^4$, all β, for each direction ρ	`PeierlsPerm.wilson_rp_dir`
20-01	Symmetry of the reflection pairing	$(\mathbb{Z}/2M)^4$, all β, measurable f, g	`TorusRP.wilson_refl_symm`
20-02	Decay of correlations at strong coupling in finite volume	Narrow window, for each finite N, no N in the constants, distance in time	`TorusRP.wilson_strong_coupli`
20-02	Decay of covariances in the Dobrushin distance	$\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$, finite N, no N in the constants	`LinkSpec.wilson_cov_dist`
20-03	Theorem on the limit without subsequences	$\beta \not= 0$, $q(\beta) < 1$	`InfVol.wilson_infvol_unique`
20-03	Theorem on the limit independent of the size	$\beta \not= 0$, $q(\beta) < 1$, all sizes $m + 1$, any $s(k) \to \infty$	`AllN.wilson_infvol_allN`
20-03	Translation invariance of the limit	$\beta \not= 0$, $q(\beta) < 1$, unit translations in four directions	`TransI.wilson_infvol_transla`
20-03	Clustering of limit points	$\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$, at every limit point, four directions	`WilsonLimit.cluster_limit`
20-04	Mass gap with non-degeneracy (Dobrushin window)	$\beta \not= 0$, $q(\beta) < 1$, the statement chooses one limit point (the same estimate at every limit point: `WilsonLimit.wilson_massGap_dob`), $\|T - P_\Omega\| \leq q(\beta)^{1/2}$, $\mathrm{Var}(F_0) \geq \frac{1}{4} e^{-12\|\beta\|}$	`WilsonLimit.wilson_massGap_d`
20-04	Mass gap with non-degeneracy (all limit points)	$\beta \not= 0$, narrow window, all limit points, rate $-\log 24(1 - e^{-4\|\beta\|})$ (not included in the combined statement)	`WilsonLimit.wilson_massGap_r`
20-04	Mass gap in the gauge-invariant sector	$\beta \not= 0$, $q(\beta) < 1$, the statement chooses one limit point, witness $F_p = \mathrm{Re}\, U_p$	`GaugeZ.wilson_massGap_gauge_`
20-04	$T \geq 0$ (on the whole of W)	$\beta \geq 0$, $\lambda > 0$, $e^{\lambda} q(\beta) < 1$, all limit points	`FullT.wilson_Tpos_full`
20-04	The image of T lies in the gauge-invariant sector	$\lambda > 0$, $e^{\lambda} q(\beta) < 1$, all limit points	`Bind.T_range_gauge_all`

Chapter	Theorem or definition	Scope	Lean declaration
20-04	The combined statement	$0 < \beta$, $18 \tanh \beta < 1$. Convergence, translation, reflection positivity, gap and non-degeneracy in the two sectors, $T[F_0] = 0$, $T \geq 0$, clustering	`InfMain2.wilson_infvol_main2`
20-05	Reflection positivity of the Wilson–fermion weight	$m > -1$, all β and j, time planes through lattice sites	`WilsonGauge.site_reflection_`
20-05	The partition function is positive	All β, m, j	`WilsonGauge.partition_pos_wi`
20-05	Global gauge invariance of the determinant	Unconditional, constant g	`GaugeDet.physDirac_det_conj`
20-05	Invariance under the generators of the lattice symmetries	All N, j, m, β: global gauge, translations in four directions, spatial 90° rotations, reflections of spatial axes, time reflection	`CubicSym.wfProb_lattice_gene`
20-05	Invariance under the hypercubic generators	Isotropic lattice $N = 2h$; the above with 90° (space, time) rotations added	`CubicSym.wfProb_hypercubic_g`
20-05	Invariance of the determinant under time reflection	Unconditional (all N, h, m, ω), reversal $t \mapsto 2h - 1 - t$	`CubicSym.physDirac_det_trefl`
21-01	The three results of the sign systems (covering inequality, monotonicity for $\mathbb{Z}_2$, comparison with $\mathbb{Z}$ 2)	Abstract groups and sign characters. Implications with assumption fields	`CoveringInequality.covering_` `dominated_covering_ineq`, `in`
21-02	Thin-vortex floor (ratio form)	All real β, $N, M \geq 1$, any background twist: `twisted_vortex_bound` is the signed form $(Z_0 - Z_1)/(Z_0 + Z_1) \leq t^{M^2}$, `_abs` is the absolute-value form $\lvert Z_0 - Z_1\rvert/(Z_0 + Z_1) \leq t^{M^2}$	`TwistedWilson.twisted_vortex` `twisted_vortex_bound_abs`
21-02	Thin-vortex floor (exponential form)	$\beta \neq 0$: $\sigma_0 = -\ln t > 0$, $\lvert\theta_M\rvert \leq e^{-\sigma_0 M^2}$	`TwistedWilson.thin_vortex_fl`
21-04	Block estimate	$mk \leq M$, all real β: $\lvert\theta_M\rvert \leq \tanh(K_{\text{eff}})^{k^2}$	`TwistBlocks.block_bound_abs`
21-03	Thick-vortex floor	All $N \geq 1$, any $\varepsilon > 0$, for each sufficiently large β a sufficiently large M: $\lvert\theta_{M+1}\rvert \leq \exp(-(M+1)^2 e^{-\beta(S_N + \varepsilon)})$	`TwistBlocks.thick_vortex_flo`

Chapter	Theorem or definition	Scope	Lean declaration
21-05	Positivity of the minimal vortex action	All $N \geq 1$: $\delta \leq S_{\mathrm{v}}(N, m)$ with $\delta > 0$ uniform in m, and $S_N > 0$	`VortexPos.Svortex_ge_unif`,
21-05	Certificates of upper bounds Computer-assisted	$S_{\mathrm{v}}(2,8) \leq 6.67401$, $S_{\mathrm{v}}(3,8) \leq 9.09321$, $S_{\mathrm{v}}(4,4) \leq 12.02057$, hence $S_N < 2N^2$ $(N = 2,3,4)$	`VortexCert.Svortex_two_eight` `Svortex_three_eight_le`, `Svo` `SN_two_lt`, `SN_three_lt`, `SN_` `VortexCert.Svortex_le_of_cer`
21-05	Thick-vortex floor with explicit exponents	$N = 2,3,4$, for each sufficiently large β a sufficiently large M: exponents 6.675, 9.094, 12.021	`TwistBlocks.thick_vortex_flo`
21-06	Sign of the vortex ratio	M even, all real β: $Z_1 \leq Z_0$, $\theta_M \geq 0$	`HalfSplit.Z1_le_Z0`, `TwistBl`
21-06	Signed thick-vortex floor	Within the range of the thick-vortex floor, with $M+1$ even: $0 \leq \theta_{M+1} \leq \exp(-(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	`TwistBlocks.thick_vortex_flo`
21-06	Estimate of Polyakov loop pairs by θ	M even, all real β, all b, b': $\lvert\langle P_b(0) P_{b'}(M/2)\rangle_0\rvert \leq 2\sqrt{\theta}/(1+\theta)$	`HalfSplit.polyakov_pair_thet`
21-06	Polyakov loop pairs at general time separation	$M \equiv 0 \bmod 4$, all real β, $0 \leq T \leq M$: $\lvert\langle P_b(0) P_{b'}(T)\rangle_0\rvert \leq 2t^{M \min(T, M-T)}$	`PolyakovT.polyakov_pair_all`
21-06	Same (other parities)	M even ≥ 4 with $\beta \geq 0$ / N, M even $(M \geq 4)$ with all β	`PolyakovT.polyakov_pair_ever`
21-06	Pairs at distance $M/2$ and the thin-vortex floor	M even (including $M = 2$), all real β: $\lvert\langle P_b(0) P_{b'}(M/2)\rangle_0\rvert \leq 2t^{M \cdot M/2}$	`PolyakovT.half_pair_tanh`
21-06	Polyakov loop pairs combined with the thick-vortex floor	For each sufficiently large β a sufficiently large M, $M+1$ even: pair at distance $(M+1)/2 \leq 2\exp(-\frac{1}{2}(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	`HalfSplit.polyakov_pair_floo`
15-12	Upper side	All steps k, all $q \in \mathbb{R}^4$ with $\omega(q) > 0$	`OSUpQpos.os_up_Qpos`, `OSUpQp`
15-12	Lower side	As above (a family closed under a different name)	`HsSplit.lower_side_c`
15-12	Bracketing of the propagator	As above; constants uniform in k and q	`LatticeRec.lattice_bracket`, `OneStepLower.bracket_Qpos_fr` `NearAliasB.bracket_Qpos_spli`
15-12	One-step contraction	All q	`OneStepBridge.Mfam_P_le_row`
15-12	Contraction uniform in the step	All steps k, all q	`TorusLam.stage_tau_17`
15-12	Convergence of Γ_k	All steps; constants uniform in the step and the momentum	`AliasPoinc.coupling_converge`

Chapter	Theorem or definition	Scope	Lean declaration
15-13	Locality, gauge covariance	$r0 > 0$, $\lvert g\rvert = 1$	`BoxWalk.boxMap_local`, `BoxWa`
15-13	Linearisation	$0 < r0 \le 1$, $\varepsilon M \le 1/32$	`BlockLinearKernel.blockMap_e`
15-13	Covariance of the pushforward	Any measure (meaningful when second moments exist), one colour	`PushCov.lattice_pushCov`
15-13	Covariance of plaquettes	All steps m, $N \ge 2$. The field obtained by pushing the free field forward with the linearised blocking	`LatticeFace.lattice_face_T1h`
22-01	Chessboard Estimate	$N = 2m$ ($m \ge 1$), all real β, functions on a unit cube	`FoldWilson.wilson_chessboard`
22-01	Rarity uniform in the volume	$\beta \ge 81$, $m \ge 2$, any set X of plaquettes. $q1$ does not depend on N or X	`FoldCap.rarity_explicit`
22-01	Rarity at rate $1/4$	$0 < y \le 2$, at or above some $\beta0$, $m \ge 1$	`FoldQuarter.rarity_uniform4`
28-01	Subsequential limits, and the properties that pass to the limit (implications with assumption fields)	The assumption fields are as in the table of 28-01	`WeakUltravioletExistence.exi` `limit_consistent`, `BlockConc` `ChainBlockMap.chain_end_to_e` `continuous_blockMap`; `BoxWal` `chain_end_to_end_box_wilson2`
28-02	Convergence of skew-product orbits with matched endpoints (implications with assumption fields)	The one-step map is an abstract skew product and is not defined from the lattice (28-02 § 08)	The 13 declarations of `SkewOr` `Admissible`, `two_point_bound` `orbit_compare`, `coarse_step`, `coarse_tendsto`, `coarse_rate` `observable_tendsto` and the r `SkewOrbitProbe.theta_exists_` `one_step_after` and others; `WeakUltravioletExistence.ter` `weak_uv_observable_full`
28-03	From decay to the mass gap; reduction of the band (implications with assumption fields)	The assumption fields are as in the table of 28-03. Inside the strong-coupling window all fields can be discharged	`OSGap.massGap_of_measures_su` `OSGap.Datum.hasMassGap_of_de` `BandReduction.band_reductior`

Chapter	Theorem or definition	Scope	Lean declaration
20-06	Area law at strong coupling (finite volume): the expectation of the Wilson loop on the boundary of a rectangle in directions 1 and 2 with a corner at the origin is exponentially small in the area of the rectangle (the formula is (3) of 20-06)	Torus of equal size in the four directions, β at least 0, inside the strong-coupling window (the 4096 window), both sides of the rectangle at least 1 and smaller than the side of the lattice. The rate and the prefactor depend neither on the size of the lattice nor on the size of the rectangle. The form whose exponent is exactly the area, when twice a side is at most the side of the lattice, is area_law_wilson_area	`AreaLaw.area_law_wilson`, `ar`
20-06	Same (rectangles with any corner and orientation)	As above. From the invariance of the Wilson measure under translations and permutations of directions	`AreaLaw.area_law_wilson_gene` `wilson_cfgShift_invariant`,
20-06	Same (version with tightened constants, and version with the exponent taken over the larger of the two directions)	β at least 0, inside the tightened window (the 441 window). Only the constants of the prefactor and the rate differ. Nothing is said outside the window, about general loops, or about lower bounds on Wilson loops	`AreaLaw.area_law_wilson_shar` `area_law_wilson_general_shar` `area_law_wilson_max_sharp`
20-06	Area law at strong coupling (infinite volume): in the limiting measure, the expectation of the Wilson loop on the boundary of a rectangle with a corner at the origin is exponentially small in the area	β positive, the intersection of the window of the limit without subsequences of 20-03 with the 4096 window, all side lengths. Also at every limit point (β at least 0, the 4096 window). There is no infinite-volume version with tightened constants or with any corner and orientation	`AreaLaw.area_law_infvol`, `ar` `area_law_limit_sub`, `zRectWo`
20-06	Parts of the proof of the area law: positive mixtures, the domination lemma, terms that vanish by sign flips and mod 2 duality, connectedness, lower bound on the area and counting of minimal sets of plaquettes, definition of the rectangular path	As in the statement of each declaration (20-06 § 05– § 08). The family of spanning sets of plaquettes is placed concretely inside the definition and is not an assumption	`AreaLaw.rectWord`, `rectWordG` `rectWord_one_one`, `abs_wordE` `exp_eq_mix`, `mixVar_mem`, `pro` `integral_mul_prod_one_add`, `sum_superset_le`, `q_eq`, `flip` `hol_eq_wordHol`, `integral_word_mul_prod_eq_ze` `not_mem_span_iff_exists_sep` `rect_closed_cases`, `minSpan_` `card_ge_of_spans_rect`, `word` `card_rootP_le`, `card_nbQ_le`

Chapter	Theorem or definition	Scope	Lean declaration
20-06	Upper bound on the vortex free energy (isotropic torus): the difference between the partition function with one central twist inserted and the ordinary partition function is exponentially small in the area of the twisted plane	Torus of equal size in the four directions, β at least 0, the 4096 window (the 441 window for the tightened version). The rate per unit area does not depend on the size of the lattice. A prefactor (the transverse area) remains. Nothing is said about the sign of the difference or outside the window	`VortexStrong.vortex_floor`, `vortex_floor_rel`, `vortex_fl`
20-06	Same (torus with transverse size N and longitudinal size M)	β at least 0, the 441 window (the 9216 window for the untightened version vortex_floor_MN), all backgrounds. The difference is exponentially small in the area of the longitudinal plane, and the rate per unit area depends neither on the transverse nor on the longitudinal size. A prefactor (the transverse area) remains	`VortexStrongMN.vortex_floor_`
20-06	Parts of the proof of the vortex upper bound: good sets and bad sets, independence of where the twist is placed, lower bound on the size of bad sets and their connectedness	As in the statement of each declaration (20-06 § 07, § 08)	`VortexStrong.Good`, `ZM_gauge` `good_twist_iff`, `card_bad_ge` `VortexStrongMN.card_bad_ge`,
21-07	Continuity of the vortex ratio, and its value at coupling 0: the twisted partition function and the vortex ratio are continuous in the coupling; at coupling 0 the partition function is 1 and the vortex ratio is 0	Transverse N and longitudinal M fixed at 1 or more, all real couplings	`ThetaTuning.ZW_continuous`, `thetaM_zero`
21-07	Weak form of tuning: if the target value is at least 0 and at most the vortex ratio at some coupling, then between 0 and that coupling there is a solution, and a largest solution among them	With the size of the lattice fixed. The largest is the largest within that interval	`ThetaTuning.tuning_exists`,

Chapter	Theorem or definition	Scope	Lean declaration
21-07	A box with one twist on each of two orthogonal planes has no flat configuration (from the discrete Stokes theorem and the algebra of quaternions)	All N, M (at least 1, including degenerate lattices), no assumptions. In a background with only one of the twists, flat configurations exist	`TwistFlat.no_flat_twist11`, `no_twisted_quadruple`, `Theta`
21-07	The minimal defect is positive and is attained at some configuration	All N, M (at least 1), no assumptions. No explicit value is stated	`TwistFlat.defect11`, `defect1` `ThetaLimit.Smin`
21-07	Uniform lower bound on the defect: in a symmetric box, the defect is at least 1/2 independently of the transverse size	All N, M (at least 1) and all configurations, no assumptions. In a background with only one twist there is no positive lower bound. The true value of the minimal defect is not stated	`TwistFloor.defect11_ge_two`, `defect11_uniform_half`, `Smin` `QuadFloor.quad_floor_two`, `FlatStokes.approx_torus_comm`
21-07	With the lattice fixed, the vortex ratio approaches 1 as the coupling grows. Its values lie above −1 and at most 1	With the size of the lattice fixed. The constant of the lower bound depends on the size of the lattice and has no explicit value. Nothing is said about the speed being uniform in the size of the lattice	`ThetaLimit.tendsto_theta_one` `Z10_ge`, `volV_pos`, `ratio_ter` `theta_gt_neg_one`, `theta_le_`
21-07	Slope of the flux free energy: the log ratio of the two twisted partition functions, divided by the coupling, approaches the minimal defect	The limit of large coupling with the size of the lattice fixed. In a symmetric box the limit is at least 1/2. No uniform estimate at finite coupling is stated	`ThetaLimit.tendsto_logratio_` `TwistFloor.logratio_limit_ha`
21-07	Tuning the vortex ratio: for every target value at least 0 and less than 1 there is a coupling that makes the vortex ratio equal to it, and the set of solutions has a largest element	With the size of the lattice fixed. If the target is positive, the largest solution is positive too. Monotonicity, uniqueness of solutions, and sequences moving the transverse size and the coupling together are not stated	`ThetaLimit.tuning_all`, `tuni` `exists_theta_gt`, `sanity_hal` `envelope_max_sol`
14-02	Block map of the side-3, centred design: a coarse link is made as the uniform average over a family of paths through the centre of the block (definition)	1944 path indices, weights summing to 1. Not the design of the main line of this book's study (side 2)	`Block3.paths`, `wts`

Chapter	Theorem or definition	Scope	Lean declaration
14-02	Same: the map commutes with reflections in planes through lattice sites without a gauge transformation, and flipping the signs of the links crossing a section flips only one coarse link	One step and all steps, with values in the sphere	`Block3.blockMap_refl_cfg`, `d` `down_flipSet`, `down_local`
14-02	Same: reflection positivity, in planes through lattice sites, of the pushed-forward Wilson measure	The SU(2) Wilson measure (normalised or not) pushed forward by a composite of several steps of the side-3 map, all real β, bounded measurable functions depending only on coarse links strictly between the two planes through the coarse lattice sites at the ends (gauge invariance not imposed). The version with the plane through the middle of the links, infinite volume and the mass gap are not stated	`Block3.rp_push`, `rp_push_pro`
14-02	Same: the map commutes with coarse translations and with reflections in boundary planes	One step and all steps. Within the lattice symmetries	`Block3.blockMap_trans`, `down`
14-02	Same: existence of a tuned sequence. The expectation of the coarse plaquette average of the pushed-forward measure is 0 at coupling 0 and approaches 1 as the coupling grows, and at each step there is a non-negative coupling tuned to a target value in between	On the side-3 design, retract threshold at most 1, side of the coarse lattice at the tuned step at least 2, target value at least 0 and less than 1 (the assumptions are only the values of the design and arithmetic). Monotonicity, divergence and uniqueness of the coupling are not stated	`Block3.Y3_zero`, `tendsto_Y3_`
10-01	In the book's current design (side 2), the expectation of the coarse plaquette average of the pushed-forward measure is continuous in the coupling, and for a target between its values at two couplings there is a solution	For each step and each coarse plaquette. That a tuned sequence exists at all steps is not stated by these two	`TunedCoupling.continuous_Yc`

Chapter	Theorem or definition	Scope	Lean declaration
9-05	In the book's current design (side 2), the pushed-forward Wilson measure is invariant under coarse translations	No assumption on the sequence of couplings. For each step, when the fine-side step is taken sufficiently large, for bounded continuous functions	`TunedCoupling.hsym_translati`

What this chapter can say

What is said	What is not said
The output of Table 1, Table 2 and Figure 1 is exactly the result of checking seven small files with Lean 4.33.1 and Mathlib (the version in Table 7) on this machine.	We do not say that other versions give the same output. In particular the names of the axioms for `native_decide` change with the version.
The Lean label guarantees the arrow from premises to conclusion (formula (1)). `sorry` passes with only a warning, and is found by `#print axioms` (Table 2).	This appendix does not confirm the soundness of Lean's logic itself, or that the kernel has no bugs (the kernel-recheck step and the isolated-comparison step in the sense of the reference manual were not carried out for the examples of this appendix).
Of the four gaps, name and vacuous truth were shown with examples that actually pass (the last two rows of Table 2).	This appendix does not confirm that the individual theorems of this book have no gaps. Read the premises and readings in each row of 23-13.
The words of lattice gauge theory (Wilson loop, plaquette, Yang–Mills and so on) do not appear in the source of Mathlib on this machine.	The absence of a word is not a proof that the same content is not defined under another name.
How the labels are decided (Figure 2, Table 5) is a restatement of this book's conventions.	A label shows the kind of guarantee; it is not a ranking of the importance or plausibility of sentences.
The Lean theorems of this book show neither the existence of four-dimensional Yang–Mills theory nor the mass gap (23-16). Four-dimensional existence was shown by a construction published in 2026, and the committee's assessment is that it is proved in Lean (6-00). The mass gap has not been shown.	

Sources and reproduction

Item	Kind	Source / tools
Table 1, Figure 1 (the states of the goal)	Computed	Lean 4.33.1 + Mathlib: `e1_gap.lean` (with `trace_state` after each step) checked with `run.sh`, and the output copied

Item	Kind	Source / tools
Table 2 (exit codes, warnings, axioms and times of the seven files)	Computed	python3: `table.py` checks the seven files `e1` – `e7` in turn with `run.sh` and keeps them in `results.json`. The seconds column is from that one run
The observation in the detour "Using real numbers brings out the three axioms"	Computed	Lean 4.33.1 + Mathlib: `e8_real_probe.lean` (an empty theorem, and `#print axioms` on `Real`, `CauSeq.Completion.Cauchy`, `Real.instZero` and so on)
Word counts in the source of Mathlib	Computed	`mathlib_grep.sh` (a case-insensitive search of 8,311 files)
Table 7 (versions)	Computed	`lean --version` and the Mathlib rev in `lake-manifest.json` (`versions.txt`)
The four steps of checking; distinguish "is there a correct proof?" from "what does the statement mean?"; the mark of the build-check step appears even with `sorry` in a dependency; the three standard axioms; the common axiom `Lean.ofReduceBool` up to 4.28.0 (with `Lean.trustCompiler` behind it) and the dedicated axiom per call from 4.29.0	Known	Lean Reference Manual, “Validating a Lean Proof” (lean-lang.org/doc/reference/latest/ValidatingProofs/). Text read
`native_decide` creates a dedicated axiom for each call	Known	Lean Reference Manual, “Axioms” (lean-lang.org/doc/reference/latest/Axioms/). Text read. Agrees with the output of Table 2
Lean 4	Known	L. de Moura, S. Ullrich, “The Lean 4 Theorem Prover and Programming Language”, CADE-28, LNCS 12699 (2021), doi:10.1007/978-3-030-79876-5_37. Bibliographic data only
Lean (the first system description)	Known	L. de Moura, S. Kong, J. Avigad, F. van Doorn, J. von Raumer, “The Lean Theorem Prover (System Description)”, CADE-25, LNCS 9195 (2015), doi:10.1007/978-3-319-21401-6_26. Bibliographic data only
Mathlib	Known	The mathlib Community, “The Lean Mathematical Library”, CPP 2020, doi:10.1145/3372885.3373824, arXiv:1910.09336. Bibliographic data only
The classification of the four gaps, the procedure for reading a statement, the three-way classification of stock	Article on this site	Guide 08, 09, 10

Item	Kind	Source / tools
The layers of types of the theorems in this book (46 theorems; 23 only real numbers; 3 gauge fields)	Computed (another chapter)	23-13 Lean statements and the limits of machine checking
The legend of the labels, and how the labels of the paper in 5-11 are split	Conventions of this book	0-01, 5-11
The name "Curry–Howard correspondence"	Known	As a general name. The history is left to Guide 13; years and attributions are not stated here

Next chapter to read: A-05 How this book was made — how sentences and numbers were checked before labels were attached.

How this book was made — labels and source tables record how far a sentence was checked

This chapter is a guide (how this book was made and checked).

What this chapter shows — The seven stages a sentence passes through before it appears in the text, and what each stage catches and does not catch (Figure 1).
The meaning of "text", "abstract" and "bibliographic data only" in the source tables, and that about half of the verdicts in the refutation check were corrections (with tables counted on this machine).
The convention for the label "Computed", the stage that aligns symbols and names across chapters, the mechanical checks of assembly, and what this procedure does not guarantee.

Prerequisite chapters — None. It is easier to read after looking at the legend of the labels (0-01). The procedure of machine checking with Lean is not dealt with in this appendix; it is in A-04.

Said first — What this appendix explains is a procedure for reducing errors, not a guarantee of correctness. This book has not been through expert peer review. Also, no step of this book shows the existence or the mass gap of four-dimensional Yang–Mills theory. Four-dimensional existence was shown by a construction published in 2026, and the committee's assessment is that it is proved in Lean (6-00). The mass gap has not been shown.

The order of this chapter

1. The role of this appendix — labels and source tables are a graduation for measuring how far one may believe; the procedure that marked that graduation
2. The road a sentence travels — seven stages, and what each catches and does not catch (Figure 1, Table 1)
3. Cross-checking against primary sources — three levels: text, abstract, bibliographic data only (Table 2); Detour: writing in the form of hearsay
4. Looking for refutations — another reader checks "is this too strong?"; the breakdown of verdicts and the types of correction (Tables 3 and 4)

5. Making computations reproducible — the label "Computed" covers a finite range; write the conventions; a scale that finishes in minutes
6. Attaching labels — a record of how far a sentence passed; the composition of labels by part (Figure 2, Table 5)
7. Aligning across chapters — five things: symbols, conventions, names, duplication, degree of certainty
8. Assembling and checking mechanically — pre-rendering of formulas, links, forbidden words (Table 6)
9. What this procedure does not guarantee — not peer review; texts not obtained; finite computations; problems that remain open

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01

The role of this appendix — so that readers can measure for themselves how far to believe

Sentences in textbooks tend to be written on the premise that readers will believe them as they are. This book does not do that. Each sentence carries a label (the legend in 0-01), the source tables write "whether the text was read", "only the abstract" or "only the bibliographic data", and computations are kept in a form that can be reproduced at hand. Labels and source tables are a graduation with which readers can measure for themselves "how far this sentence may be believed". Precisely, a label is a record of how far a sentence was checked, not a number expressing a degree of correctness.

A graduation cannot be read unless the procedure that marked it is known. For example, even for a sentence labelled "Known", the weight differs according to whether it was checked word for word against the text of a primary source or copied from the description in a review. This appendix explains that procedure from the reader's side. Stage by stage, it writes what was checked and what was not.

The flow in one line:

research → cross-check against primary sources → look for refutations → reproduce com

Each of the seven stages is placed to catch what the previous stage overlooks. But no stage guarantees "correctness". What is not guaranteed is collected in § 09.

Example. The sentence of Part 5 that "the Wilson action has reflection positivity" carries the label "Known", and the source table lists the paper of Osterwalder–Seiler (5-03). The label "Known" shows that it is a known result checked in a primary source. The depth to which it was checked (text, abstract, bibliographic data only) is written in

the source table, and the source table for this example says "abstract checked, text not checked". Whether it passed the refutation check (§ 04) is not distinguished by the label. If the source table says "bibliographic data only", the cross-checking stopped at the level of bibliographic data.

The procedure of machine checking with Lean is not dealt with in this appendix. It is in A-04. How labels are decided (one label per sentence; the strength of a chain is its weakest stage) also has its master copy in A-04 § 07.

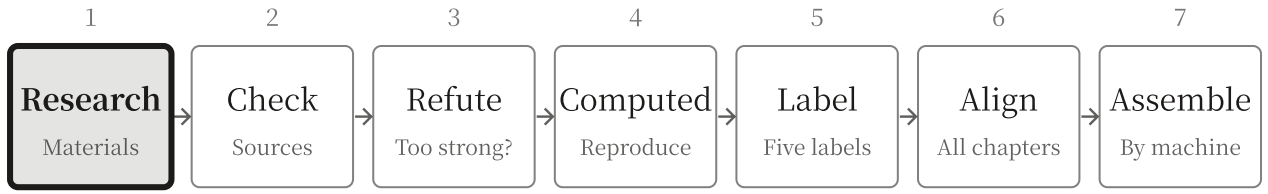
02

The road a sentence travels — seven stages

Figure 1 lines up the stages a sentence passes through before it appears in the text. Moving the knob shows below what that stage catches and what it does not.

The order of the steps has reasons. If the refutation step (step 3) were placed before the cross-checking step (step 2), the refuting side would not know what to compare against. The label step (step 5) comes after the cross-checking step and the reproduction step because a label is a record of "which step it passed up to". The consistency step (step 6) and the assembly step (step 7) are steps for errors that cannot be caught by looking at one chapter at a time, and are run each time a part is complete (and finally for the whole book).

Figure 1 Choosing a stage with the knob shows below what that stage catches and does not catch. The frame of the chosen stage becomes thicker



What the previous stage overlooks, the next stage catches

```
Step 1/7 Research. Catches: Literature on each topic, known results, candidate numbers. Misses: Whether the candidates are correct (here candidates are only collected).
```

Table 1 has the same content as Figure 1 (the figure is schematic and contains no numbers).

Table 1 What the seven stages catch and do not catch

Stage	Name of the stage	What it catches	What it does not catch
1	Research (survey step)	Literature on each topic, known results, candidate numbers	Whether the candidates are correct (here candidates are only collected)

Stage	Name of the stage	What it catches	What it does not catch
2	Cross-check against primary sources (cross-checking step)	Errors in bibliographic data, over-paraphrased citations, misattribution	Errors in the primary sources themselves, the content of texts that could not be obtained
3	Look for refutations (refutation step)	Wording stronger than the source, promotion of the degree of certainty, omitted conditions	Cases where the cross-checking reader and the refuting reader share the same misunderstanding
4	Reproduce computations (reproduction step)	Miscopied numbers, mixed-up conventions, counterexamples within a finite range	Whether it holds outside the range computed
5	Attach labels (label step)	Mixing of "proved", "conjecture" and "numerically supported"	Errors in the judgement of the label itself
6	Align across chapters (consistency step)	Discrepancies in symbols, names and conventions; the same name pointing to different sentences; duplication	All chapters being wrong in the same direction
7	Assemble and check mechanically (assembly step)	Formulas that cannot be rendered, broken links, forbidden words, unfilled placeholders	Errors of content (the machine does not read meaning)

03

Cross-checking against primary sources — three levels: text, abstract, bibliographic data

What this book calls a "primary source" is the paper that first stated a result, or the problem statement itself. Reviews and textbooks are used as guides to the primary sources, not as substitutes for them. Years and attributions (who showed what, when) are not written from memory but checked in the sources.

Cross-checking has three levels of depth, and the source table writes which level was reached.

Table 2 Depth of cross-checking, and how it is written in the source tables

Depth	What was read	How it is written in the source table	What can be written in the text
Text checked	The text of the paper (a public PDF or the like) was read, and the relevant passage checked word for word	Authors, year, journal or arXiv number, section	Exactly the strength stated by the source

Depth	What was read	How it is written in the source table	What can be written in the text
Abstract checked	Only the abstract (the abstract in a literature database) was read	As above + "checked in the abstract", "abstract only" and so on (the wording differs slightly by chapter)	As far as is written in the abstract
Bibliographic data only	Only that the authors, title, journal and year exist was checked	"Bibliographic data only"	As far as the form in which another source reports that "... did ..."

Word-for-word checking is the work of confirming by search that the original text enclosed in quotation marks exists literally in the source. Even when a translated sentence is placed in the text, the cross-checking is done on the original.

Example. The problem statement of the Clay Institute (Jaffe–Witten) was cross-checked in the text of the public PDF. The requirement of the problem, in the original, is as follows.

> “Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$.”

0-03 reads this sentence word by word. Because it was cross-checked in the text, 0-03 can use each of the words "existence", "non-trivial" and "mass gap" at the strength of the problem statement.

On the other hand, for papers of which only the abstract could be read, claims in the abstract are written as "checked in the abstract", and claims not in the abstract are not brought into the text. The same goes for section numbers: if the abstract does not mention sections, which section something is written in is not known until the text is read.

Example. The abstract of the review by Wilson–Kogut (Phys. Rep. 12 (1974) 75) is a single sentence and does not mention sections. So this book does not write that some topic "is discussed in § 10 of Wilson–Kogut", and writes " § 10 not checked" in the source table.

Detour: writing in the form of hearsay

You may skip this. That the upper limit on the spacetime dimension in which supersymmetry can be realised with fields of spin at most 2 (supergravity, including the graviton) is 11 is widely known as a consequence of Nahm's 1978 classification. Yet the number 11 does not appear in the abstract of that paper. While the text cannot be obtained, this book writes "known as a

consequence of Nahm's classification", and does not attribute the conclusion directly to Nahm's paper (17-05). It is not that the fact is doubtful. It is to show honestly, in the form of the sentence, how far this book checked. Precisely, the form of hearsay is not a mark that "it may be wrong" but a mark that "the cross-checking did not reach the text, and the abstract does not contain the number".

04

Looking for refutations — another reader checks "is this too strong?"

Material that has been cross-checked is handed to another reader. That reader's job is not confirmation but refutation. Going back to the primary sources once more, they look for "is this sentence stronger than what the source says?", "it was written as settled, but does it not need a condition?", "does this bibliographic entry exist?". There are three verdicts.

- **Confirmed** (confirmed): against the source, the written strength is fine as it is.
- **Corrected** (corrected): it needs to be weakened, given a condition, or have its bibliographic data fixed. The wording of the correction is attached.
- **Unverifiable** (unverifiable): the primary source could not be reached, and neither can be said. It is lowered to the form of hearsay.

At the stage of writing chapters, the rule is to reflect every correction. Material that received a correction is not cited at its strength before the correction.

Table 3 counts the verdicts for the six materials used mainly by Parts 7 to 23 (and partly by chapters of Parts 0, 4 and 5). Many chapters of Parts 1 to 5 have not been through the materials of this refutation check.

Table 3 Verdicts of the refutation check by material (counting the materials as of 2026-10-01) Computed

Subject of the material	Confirmed	Corrected	Unverifiable	Fraction corrected
Dictionary of the physical and the mathematical vocabulary	25	20	0	44%
Extra dimensions and where gauge fields come from	16	12	1	41%
The physical view (confinement, lattice numerics, holography)	25	17	0	40%
Arrangement of this book's study	7	14	0	67%
Where things stand, dimension by dimension	25	8	0	24%
Where constructive field theory stands, and candidates for new concepts	16	29	0	64%

Subject of the material	Confirmed	Corrected	Unverifiable	Fraction corrected
Total	114	100	1	47%

How to read it. About half of the verdicts are corrections. The highest fraction of corrections is for "arrangement of this book's study": in this material corrections were many, 14 of 21, and those that weakened wording stronger than the record, or added conditions that had been dropped, stood out. Conversely, materials close to a list of theorems, such as where things stand dimension by dimension, have fewer corrections.

Table 4 sorts the content of the corrections into four types by words. The sorting is mechanical, by words appearing in the wording of the correction, and one correction may fall into several types.

Table 4 Types of the 100 corrections (mechanical sorting by words; with overlaps) Computed

Type	Cue words	Count	Typical
Strength	too strong, promoted, showed, restricted	39	Lowering "showed" to "proposed"; "settled" to "conditional"
Level of the source	abstract, text, not obtained, not checked	34	Rewriting a claim not in the abstract as "text not obtained"
Numbers and signs	number, coefficient, sign, digit	15	Adding the convention for the coefficient (which β)
Bibliographic data	bibliographic data, volume, page, author, journal	12	Corrections to the list of authors, such as a missing co-author
None of the above	—	30	Supplements, suggestions for rewording

Example. What § 6.5 of the Clay problem statement says is that "with many regularisations, such as dimensional regularisation and Pauli–Villars, reflection positivity is lost, and there are few ways to recover it"; it does not reject continuum regularisations across the board. This book cites it at this strength, and does not write "continuum regularisations do not preserve reflection positivity". What the refutation stage catches is summaries that have become one step stronger than the source in this way (regularisation and reflection positivity are in 4-10 and 5-03).

05

Making computations reproducible — the content of the label "Computed"

Of the numbers in the text, those that can be recomputed on this machine are recomputed and given the label "Computed". All computations are short programs (Python), kept chapter by chapter. Their names are written in the "tools" column of the source tables, so running the same programs gives the same tables.

What the label "Computed" means is that it was **checked on this machine, over the range written**. As a formula, having checked a property $P(n)$ for finitely many n is the distinction

$$P(n_1) \wedge P(n_2) \wedge \cdots \wedge P(n_k) \qquad \text{and not} \qquad \forall n\, P(n)$$

The left is a computation, the right a theorem. A sentence with the computed label is not cited as a theorem.

Computations have three conventions. The first is not to make numbers inside figures. The numbers in moving figures are limited to tables computed in advance and embedded, or simple formulas that can be computed on the spot (rotations, partial sums of series and so on). The second is to write the conventions. If the definition of the lattice coupling β or the normalisation of the trace tr differs, the same quantity is off by several times, so the conventions are stated in the premises of a computation (conversion between conventions is in 5-01 and A-01). The third is to keep to a scale that finishes within a few minutes on an ordinary computer. A computation that readers cannot reproduce loses the meaning of the label.

Example. Tables 3, 4 and 5 of this appendix are also computations. We wrote one program that reads the materials and chapter pages and counts, and copied its output. Over the whole book, the places where computations are kept, chapter by chapter, number over 100, and there are several hundred programs.

06

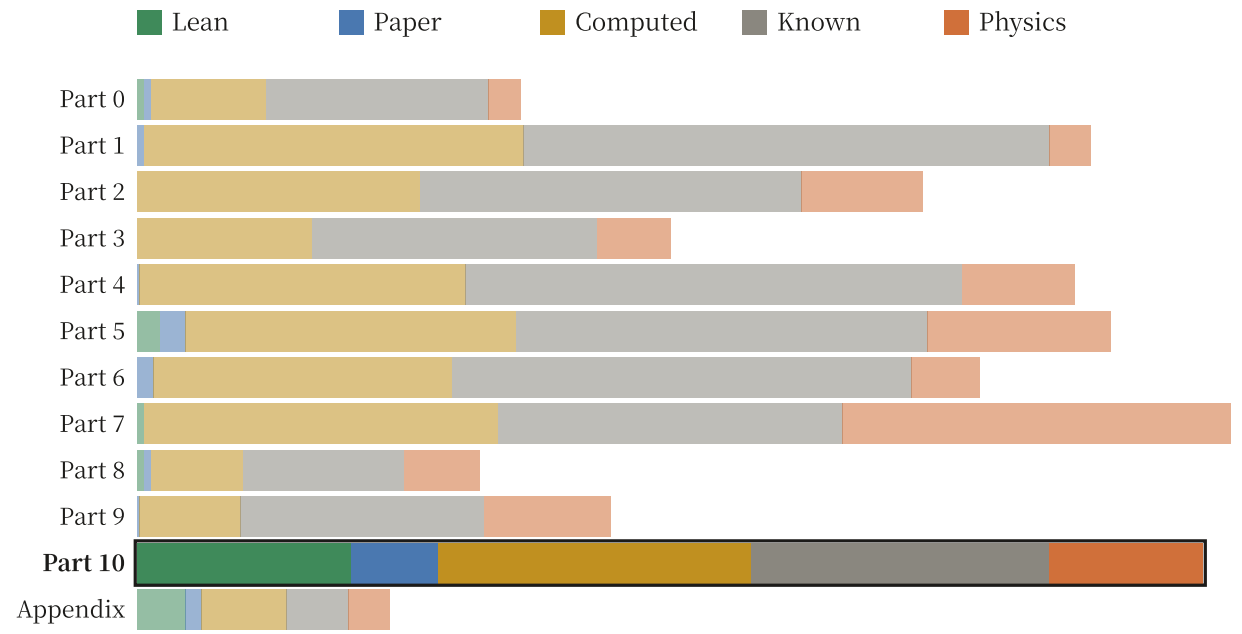
Attaching labels — a record of how far a sentence passed

A label is a record that tells readers how far a sentence passed. A sentence that stopped at cross-checking gets "Known", a number reproduced at hand gets "Computed", something accepted in physics but without proof gets "Physics", and so on. The meaning of the five labels, and what the labels do not guarantee, are in the table of 0-01 § 03.

There are two rules for attaching labels. One sentence gets one label. And for a claim connected through several stages, the label of the weakest stage is attached to the whole (A-04 § 07). For example, if numbers are put as assumptions into an inequality checked in Lean and a conclusion is drawn, the label of the conclusion is "Computed" or "Physics". Labels are not pushed towards the strong side.

Figure 2 counts, by part, the labels appearing on the chapter pages of this book. The composition of labels differs clearly from part to part.

Figure 2 Number of labels by part. "View by fraction" scales each part to 100%. Choosing a part with the knob shows its breakdown below



`Part 10 (total 464) : Lean 93 (20%) , Paper 38 (8%) , Computed 136 (29%) , Known 130 (28%) , Physics 67 (14%)`

Computed The numbers in this figure were computed on this machine (counting the chapter pages of 110 chapters as of 2026-10-01, excluding this appendix itself. Labels placed in legend tables are also counted. The numbers change as chapters are added).

Table 5 Number of labels, number of figures, and occurrences of "bibliographic data only" by part (as of 20 Computed)

Part	Chapters	Lean	Paper	Computed	Known	Physics	Figures	Bibliographic
Part 0 Entrance	4	3	3	50	97	14	8	
Part 1 Mathematical preparation	12	0	3	165	229	18	40	
Part 2 Physical preparation	10	0	0	123	166	53	28	
Part 3 Classical theory	7	0	0	76	124	32	13	
Part 4 Quantum theory	12	0	1	142	216	49	23	
Part 5 Lattice	11	10	11	144	179	80	22	

Part	Chapters	Lean	Paper	Computed	Known	Physics	Figures	Bibliographic
Part 7 Constructive field theory	12	0	7	130	200	30	24	
Part 17 From the side of physics	13	3	0	154	150	169	28	
Part 18 A dictionary of two languages	5	3	3	40	70	33	7	
Part 16 Where things stand, and open tasks	5	0	1	44	106	55	10	
Part 23 The record of this book's study	14	93	38	136	130	67	25	
Appendices (A-01, A-02, A-03, A-04, A-06)	5	21	7	37	27	18	10	
Total	**110**	**133**	**74**	**1241**	**1694**	**618**	**238**	

How to read it. Most labels are "Known" and "Computed". This is because much of this book is an exposition of known mathematics and physics, with numbers reproduced at hand added to it. "Physics" is most frequent in Part 17, because its centre is material accepted in physics but not theorems, such as mechanisms of confinement and the string picture. The "Lean" labels are concentrated in Part 23. However, many of those Lean theorems are statements about inequalities of real numbers or finite sums, and do not show the existence or the mass gap of four-dimensional Yang–Mills theory (23-16, A-04 § 06).

07

Aligning across chapters — symbols, names, conventions

Checking chapters one at a time cannot catch discrepancies between chapters. For example, if one chapter writes $\beta = 2N/g^2$ and another $\beta = 1/g^2$, each chapter may be correct on its own, but readers take them to be the same β and get lost when the numbers do not match. So each time a part is complete, a stage of reading across parts is run.

This stage looks at five things.

1. **Symbols**: whether the same symbol is used for the same quantity. The standard is the table of notation A-01.
2. **Conventions**: when the coupling β, the normalisation of the trace, or the lattice metric differs between chapters, whether a conversion is written (5-01).
3. **Names**: whether one name points to different sentences in different chapters. For example, the diffusive-relaxation-time assumption of Part 23 can be read in eight ways, according to which length it is measured in, which normalisation it is measured in, and whether it is stated as a power or a band. A name is assigned to each reading and fixed (calling the power level the diffusive-relaxation-time assumption and the band statement the band form of the diffusive-relaxation-time assumption; A-06), and other chapters link there and use the same name. The list of assumptions is in A-06.
4. **Duplication**: when the same explanation is in two chapters, make one the master copy, and reduce the other to a one-line summary and a link.
5. **Degree of certainty**: whether one chapter writes the same correspondence as "settled" and another as "conditional". In the dictionary part, correspondence as a definition (which can be settled) and actually holding in four-dimensional Yang–Mills theory (unsettled) are written in two separate columns (18-01).

When a discrepancy is found, it is aligned to one side or the other. Discrepancies that cannot be aligned (for example when conventions genuinely differ between papers) are written in the text as discrepancies.

Example. CP in Part 23 is a number that serves as a measure of the time it takes for fluctuations to be smoothed out (defined in 1-10 and 23-01). With the same symbol CP, it can point to different numbers according to which fluctuations are meant (of the field at one time, or of the whole path including the time direction). For the free field one is 4 times the square of the other, so mixing them up doubles the power. This is an example distinguished at this stage, and the whole book writes the two as different quantities (23-01).

08

Assembling and checking mechanically — what can be left to the machine

The last stage is done by machine. The tool that assembles the chapter pages does the following mechanically for all chapters, and reports what gets caught.

Table 6 Mechanical checks at the assembly stage

Check	What it looks at	What it does not look at

Check	What it looks at	What it does not look at
Rendering of formulas	Renders formulas written in TeX to static HTML with KaTeX, and reports formulas that cannot be rendered	Mathematical correctness of the formulas
Links and headings	Whether the files linked between chapters and the heading `id`s exist	Whether the link target fits the flow of the argument
Forbidden words	Whether real names, paths, internal numbers or unfilled-placeholder marks remain	The strength of wording (this is looked at by people and by the refutation stage)
Templates	Whether a class name not to be used for figure frames (`fig`) remains	The content of figures
PDF	Makes a PDF for printing for each part (moving figures go in as still images of their initial state)	Whether it is easy to read in print

Formulas are rendered at the assembly stage rather than in the reader's browser so that no JavaScript is needed on the reader's side, and so that they do not break when printed or made into PDF. Only moving figures use JavaScript, and even when JavaScript is disabled, the same content can be read in the table below the figure.

Mechanical checks do not read meaning. Passing this stage guarantees that the page is not broken, not its content. Judgements of the strength of wording, the choice of labels and the depth of sources are not left to the machine; people and the writers read them at the stages of cross-checking, refutation and consistency.

09

What this procedure does not guarantee

Errors can remain even after the seven stages. We write the limits of this book from the side of the procedure.

It has not been through expert peer review. The reader at the refutation step is a different reader from the writer, but not a referee of an academic journal. Much of the research, drafting and refutation for this book was done by the AI writers of this site; people decide the policy and read before publication. Readers of the same kind can share the same misunderstanding. What the refutation step cannot catch is exactly such errors (the "Look for refutations" row of Table 1).

The content of texts that could not be obtained has not been checked. For works marked "bibliographic data only" or "checked in the abstract" in the source tables, this book does not know what is written in the text. The count of "bibliographic data only" in Table 5 is a measure of the size of that range.

Computations cover finite ranges. Numbers with the label "Computed" are what the computations at hand gave over the range written, and say nothing outside the range or about the continuum limit (§ 05).

The mass gap problem remains open. No step of this book shows the existence or the mass gap of four-dimensional Yang–Mills theory. Four-dimensional existence was shown by a construction published in 2026, and the committee's assessment is that it is proved in Lean (Part 6). A list of what this book does not claim is in 23-16.

If you notice an error, please let us know through the board or the contact page of this site. Corrections are recorded with dates in the "Revised" field at the end of each chapter.

What this chapter can say

What is said	What is not said
That the sentences of this book appear in the text after passing through the seven stages (research, cross-checking, refutation, computation, labels, consistency, assembly). What each stage catches (Table 1)	That sentences that passed the seven stages are correct. No stage guarantees correctness
That "text", "checked in the abstract" and "bibliographic data only" in the source tables are records of the depth of cross-checking (Table 2)	What is written in the texts of works marked "bibliographic data only" or "abstract"
That in the refutation check of the materials as of 2026-10-01, of 215 verdicts, 100 were corrections and 1 was unverifiable Computed (Table 3)	That the 114 judged confirmed contain no errors. If the refuting reader and the cross-checking reader share the same misunderstanding, this stage cannot catch it
The counts from sorting the wording of corrections by words Computed (Table 4)	That the sorting classifies the content of corrections correctly (it is a mechanical classification by words, with overlaps)
The numbers of labels, figures and "bibliographic data only" in the chapter pages of 110 chapters as of 2026-10-01 (excluding this appendix itself) Computed (Table 5, Figure 2)	That the number of labels represents the weight of claims. Labels placed in legends are also counted, and the numbers change as chapters are added
What the assembly stage looks at mechanically (Table 6)	That the content of pages that passed the mechanical checks is correct
That this book has not been through expert peer review, and that much of the research and refutation was done by AI writers	—

What is said	What is not said
—	The existence and mass gap of four-dimensional Yang–Mills theory. No step of this book shows them (existence was shown by the published construction, and the committee assessed it so; the mass gap is open; 23-16)

Sources and reproduction

Item	Kind	Source / tools
Tables 3 and 4 (verdicts of the refutation check and types of correction)	Computed	`tally.py` (counts the verdicts of this book's materials and sorts the wording of corrections by words; output `tally.json`)
Table 5 and Figure 2 (numbers of labels, figures and "bibliographic data only" by part)	Computed	`tally.py` (counts `class="label …"`, figure frames and the words "bibliographic data only" on the chapter pages, excluding the inside of figures' JavaScript)
Quotation of the Clay problem statement (§ 03)	Known	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official problem statement of the Clay Mathematics Institute, public PDF). Checked in the text
What the problem statement says about regularisation and reflection positivity (the example in § 04)	Known	Same, § 6.5. Checked in the text (details in 4-10)
Reflection positivity of the Wilson action (the example in § 01)	Known	K. Osterwalder, E. Seiler, “Gauge Field Theories on the Lattice”, Ann. Phys. 110 (1978) 440–471. Abstract checked, text not checked (how the text is handled is in the source table of 5-03)
An example where the abstract does not mention sections (§ 03)	Known	K. G. Wilson, J. Kogut, Phys. Rep. 12 (1974) 75–199. Abstract only (the abstract is a single sentence; § 10 not checked)
The upper limit 11 on the dimension for supersymmetry (detour)	Known	W. Nahm, Nucl. Phys. B135 (1978) 149. Abstract only. "11 is the upper limit" is not in the abstract and not checked. The upper limit carries the condition of fields of spin at most 2 (17-05)
Pre-rendering of formulas	Tool	KaTeX (a JavaScript typesetting library that renders TeX formulas to static HTML. Used only at the assembly stage; it does not run on the reader's side)
Figure 1, Tables 1, 2 and 6 (description of the procedure)	—	A summary of this book's writing conventions and assembly procedure

Next chapter to read: A-06 Index of names.

Index of names

This chapter is guide material (an index).

What this chapter shows — What the names in this book (assumptions, conditions, propositions, lemmas, theorems, statements, routes, table rows, questions) and its symbols refer to, and in which chapter each was introduced. The names and symbols given in Volume 2 (the volume that reads the published construction) are gathered in § 02; the rest are listed in § 03 in the order of the parts of their introducing chapters.

Prerequisite chapters — None. The list of symbols is A-01, and explanations of words are in A-02.

Said first — This book does not use names made only of letters or numbers; it uses names that show their content. Standard labels from the literature are added at the first occurrence of a name, in the form "(author label)". "What it refers to" is a one-line guide; the exact statement is in the introducing chapter.

Order of this chapter

1. How to use this index
2. Names and symbols of Volume 2 — conventions for names and symbols when reading the published construction (Tables 1 and 2)
3. Index of names — in the order of the parts of the introducing chapters (Table 3)
4. Eight readings of the diffusive-relaxation-time assumption — why the normalisation is named
5. Bookkeeping and exponent quantities — the one-step slope, the shortfall, the exponents
6. Assumptions of the second road and the band — name, content, status

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01

How to use this index

1. Look things up by name. The quickest way is to search for part of a name ("relaxation time", "widening", and so on) with the browser's in-page search.

2. The "introducing chapter" in Table 3 is the chapter where the statement of that name was first placed. Later chapters may use the same object in another form, but there is one name for it across the whole book.
3. Long names are sometimes called by a short form from their second occurrence on. The short form is added in the name column of Table 3 as "(short: ...)".
4. "Kind" is one of: assumption (a premise that has not been shown), condition (a condition imposed within a statement), proposition or theorem (something shown), statement (a sentence to be shown, or a sentence in the form of a question), table row (a row of a table referred to by name), question, and so on. What has been shown and what has not is read from the labels in the introducing chapter.

02

Names and symbols of Volume 2

Volume 2 is the volume that reads the construction of four-dimensional Yang–Mills existence published on 2026-11-10 (the Lean code and the committee's assessment report). Objects of the published construction are called by names that carry their meaning, and the identifiers of the code are added as sources (6-00). There are three labels, Source (we read the source of the published construction), Inferred (this book's inference) and Assessment (from the committee's assessment report); the labels of the whole book (Proved here, Cited, Computer-assisted, Reading, Assumption, Lean) have the same meaning in all volumes. Theorems of the published construction are written with Source , and their level (the committee's assessment is that they are proved in Lean) with Assessment ("Labels" in 6-00).

Name	What it refers to	Introducing chapter
Published theorem	The statement in the form of the published question (its only assumption is that the Lie algebra of the group is simple)	6-02
Construction theorem	A stronger statement, for each penalty coefficient, that contains the published theorem as a projection	6-02
Committee's combined target	A statement in which the committee bundles independent checks on a single family, on top of the construction theorem	6-02
Headline theorems, outside the dependencies	The 17 headline theorems listed in the verification record, and the declarations not reached from any of them	6-00
Family certificate	The bundle of conditions that the construction theorem imposes on a family (renormalisation relations at all steps, local cross-sections, label templates, calibration of insertion coefficients)	6-02
Run	A construction satisfying the conditions, chosen once for each penalty coefficient. All later objects are built on this single run	6-03
Template	A finite set of offsets of lattice sites, used to make lattice representatives of local fields	8-03

Name	What it refers to	Introducing chapter
Probability kernel, predictor	One step of renormalisation: a probability kernel that draws a coarse link with Gaussian-type fluctuations around a central value (the predictor) made from the fine configuration	9-01
Representation	The fixed form in which the density of each step is written (vacuum term, main action, regular activity, gas of forced activities). Not unique	9-06
Regular activity, forced activity	The local correction terms in the representation, and the gas of corrections that exactly covers the large-field flags	9-06
Cut action, flag	The action in which the plaquette terms are cut off linearly above a step-dependent threshold, and the mark of a large field exceeding the threshold	9-07
Step coupling	The coefficient multiplying the term of the squared normalised curvature in the representation at step m. Not determined by the measure alone	10-01
Rereading	After one step, reading the coupling of the representation again from the log ratio of the densities at two configurations	10-03
Shooting method	Choosing, by nested intervals and the intermediate value theorem, the bare coupling that makes the coupling at step 0 equal to the reference value	10-04
Calendar	The bundle of conditions that chooses, in a fixed order, the constants used in the one-step estimate (sides, smoothing exponents, entry coupling and so on). Conditions uniform in the step are discharged here	10-05, 11-08
Induction hypothesis, one-step certificate	That the density at every step has a representation of the same form and carries the ledger; and making the representation at step m from the representation at step $m+1$ together with the pushforward identity	11-00
Seed, seed type, region	Each large fine plaquette, each edge deviating strongly from the prediction, and each flag (seeds); the triple of these three sets (seed type); and the condition that divides configurations by seed type (region)	11-03
Price, mark	The factor per seed, and the part of its allocation left for the exit (half of the guaranteed lower bound)	11-05
Strong estimate, weak ledger	The exit estimate of one step, $e^{-\mathrm{rate}\cdot P^2}$, and the estimate carried by the induction hypothesis, $e^{-P^{1/5}}$	11-07
Polymers with and without flags, hard and soft clusters	The way of sorting polymers, after the integration of one step, into the exit gas and the regular activity	11-06
Early representation	In the small-field step, the way of writing the expansion of the corrections into the form of the representation first	11-09
Convergence of ingredients, anchor, forgetting the top end, the step-removing step	The parts that give convergence of the cutoff limit at fixed step (uniform convergence of the ingredients, an anchor of uniqueness, vanishing of the influence of the fine-side end, and one step that passes from step 0 to the limit)	11-10

Name	What it refers to	Introducing chapter
Slice, thread, inner function	In reflection positivity: the configuration of links in one layer (slice), the set of links joining layers (thread), and a bounded gauge-invariant function depending only on the inside of a band (inner function)	12-01
Chessboard root, centred tiling	The expectation of the product of a function copied to all boxes, raised to the power one over the number of boxes (a seminorm); and the tiling that puts the reference box in the middle of the observation points	12-02
Column	An arrangement placing a block and its mirror image alternately around the time circle (the chessboard transcribed in the time direction)	12-04
Zero-energy space, phase measure	The subspace $\ker H$ fixed at all times, and the probability measure that splits a mixed theory into phases	12-07
Order bundle	The input bundling the choice of kernel, the step constants of fine and coarse steps, the number of convolutions of the block weights, and so on (`Teller` in the source)	11-04
Extraction sequence	The pair of a subsequence of cutoffs and a subsequence of volumes (`Extraction.Order` in the source). Used in the two-stage Arzelà–Ascoli extraction	12-03
Minimal defect of the twisted box	The lower bound $D_{\min}(N, M)$, over the whole twisted box, of the loss from the maximum of the action to the actual action. A different quantity from the minimal vortex action	21-07

Table 1 The main names given in Volume 2 (and one name from a new chapter of Volume 3). Explanations of the words are also in A-02.

Symbol	What it refers to	Introducing chapter
s	The one-loop slope $s = 11C_g/(24\pi^2)$ ($2b_0$ in the assessment report, `slopeConstant` in the source). Each time the lattice is made one step coarser, the coupling decreases by about $s \log b$	10-01
κ	The one-step slope of Volume 3, $\kappa = 11/(3\pi^2)$ (SU(2)). The coefficient in the one-step bookkeeping $\beta \mapsto \beta - \kappa(\ln b - \delta(b))$ for a step coarsening by a factor b. For SU(2), $C_g = 8$ and $s = 11 \cdot 8/(24\pi^2) = 11/(3\pi^2)$, so κ and s are the same number, and both are the coefficient of the logarithm in the decrease of the coupling over one step Proved here (this book's $\beta = 4/g^2$ and the published construction's $\beta = 1/g^2$ are the same number for SU(2), 8-01)	23-02, 10-01
C_g	The Casimir constant ($2N^2$ in the defining representation of SU(N))	8-01
$\bar\beta$, $\hat\beta_m$	The reference value of the coupling at step 0, and the one-loop running $\hat\beta_m = \bar\beta + s \log D_m$	10-01
c_{pen}	The penalty coefficient of the probability kernel (`kernel.penalty` in the source text)	9-01
cE	The constant of the energy per seed	11-04
c_{sup}	The covering exponent	11-01

Symbol	What it refers to	Introducing chapter
σ	The chordal distance (given a different letter so as not to be confused with the slope s)	11-01
$\kappa 0$	The tuning target value (the value to which the vortex ratio or the expectation of the coarse plaquette average is tuned, $0 \le \kappa 0 < 1$)	19-07, 14-03
$\mu_{N,p}^{\mathrm{W}}$, μ_{β}^{W}	The Wilson lattice measures of the published construction	8-01
μ	The phase measure (`phaseMeasure` in the source; ν in the assessment report). This book's infinite-volume limiting measure at strong coupling stays ν_β	12-07
σ_A	The spectral measure of the state A	12-08
τ	The Euclidean time of the heat estimate	6-04

Table 2 Symbols of Volume 2. The full list of symbols is in A-01.

03

Index of names

The entries are listed in the order of the parts of their introducing chapters, and within a part in the order in which they appear in the text. A row whose "What it refers to" is empty is one for which the introducing chapter does not yet have a one-line statement of its content.

Part 0 Entrance — how to read this book

Name	Kind	What it refers to	Introducing chapter
local-field footnote · weak-existence footnote (footnotes of Jaffe–Witten) (short: local-field footnote · weak-existence footnote)	Name from the literature	Two footnotes of the problem statement (Jaffe–Witten): asking for a correspondence between local fields and quantum operators / not accepting as a solution a weak existence obtained from compactness alone	0-03

Part 2 Physical preparation

Name	Kind	What it refers to	Introducing chapter
Maxwell equations with sources	Name	The two equations by which charges and currents produce fields (Gauss's law and the Ampère–Maxwell law)	2-03
source-free Maxwell equations	Name	The two equations constraining the form of the fields (absence of magnetic charge and Faraday's law of induction). They hold automatically from the potential representation	2-03
potential representation	Name	The formulas writing E and B in terms of potentials	2-03
dynamical exponent (symbol kept)	Name	The exponent z when the relaxation time grows as a power of the correlation length, C_P ≍ ξ^z (together with η_dyn = 2 − z)	2-10

Name	Kind	What it refers to	Introducing chapter
gauge-transformation formula	Formula	A relabelling of the potentials that leaves the electric and magnetic fields unchanged (a transformation changing ϕ and A simultaneously by a single function χ)	2-03
Euler–Lagrange equation	Formula	The equation of motion obtained from the condition that the action is stationary (obtained by a variation with fixed ends and one integration by parts)	2-01

Part 4 Quantisation

Name	Kind	What it refers to	Introducing chapter
states-and-symmetry axiom (Wightman axioms) (short: states-and-symmetry axiom)	Name from the literature	States are elements of a separable Hilbert space, on which the Poincaré group acts as a strongly continuous unitary representation	4-09
spectral condition (Wightman axioms) (short: spectral condition)	Name from the literature	The joint spectrum of $P^\mu = (H, \boldsymbol{P})$ lies in the forward light cone V_+ (energy is never negative)	4-09
uniqueness of the vacuum (Wightman axioms) (short: uniqueness of the vacuum)	Name from the literature	There is only one unit vector Ω (the vacuum), up to phase, invariant under the spacetime transformations U	4-09
field axiom (Wightman axioms) (short: field axiom)	Name from the literature	Fields are operator-valued tempered distributions (for each test function $f \in \mathcal{S}(\mathbb{R}^4)$, $\phi(f)$ is defined on a dense domain containing the vacuum)	4-09
covariance (Wightman axioms) (short: covariance)	Name from the literature	Moving a field by the unitary operator of a Poincaré transformation gives the field with the test function moved by the same transformation (there is no special coordinate system)	4-09
local commutativity (Wightman axioms) (short: local commutativity)	Name from the literature	If the supports of f, g are spacelike separated, then $[\phi(f), \phi(g)] = 0$ (no influence faster than light)	4-09
cyclicity (Wightman axioms) (short: cyclicity)	Name from the literature	Linear combinations of products of fields applied to the vacuum, $\phi(f_1)\cdots\phi(f_n)\Omega$, are dense in $\mathcal{H}$ (there are no states unrelated to the fields)	4-09
regularity (Osterwalder–Schrader axiom E0) (short: regularity)	Name from the literature	Schwinger functions are tempered distributions on test functions vanishing where points coincide (used together with the linear-growth condition)	4-10
Euclidean covariance (Osterwalder–Schrader axiom E1) (short: Euclidean covariance)	Name from the literature	Schwinger functions are unchanged by Euclidean rotations and translations	4-10

Name	Kind	What it refers to	Introducing chapter
reflection positivity (Osterwalder–Schrader axiom E2) (short: reflection positivity)	Name from the literature	The quadratic form of correlations between past and future, with a time reversal in between, is never negative (the condition for the theory to have a Hilbert space)	4-10
permutation symmetry (Osterwalder–Schrader axiom E3) (short: permutation symmetry)	Name from the literature	Schwinger functions are unchanged by permutations of the points	4-10
clustering (Osterwalder–Schrader axiom E4) (short: clustering)	Name from the literature	The correlation of two groups of points pulled apart splits, at large distance, into the product of the separate correlations (corresponds to uniqueness of the vacuum)	4-10
Counterterm Uniqueness Theorem (Theorem B of Costello) (short: Counterterm Uniqueness Theorem)	Name from the literature	Once the renormalisation scheme is fixed, local counterterms can be chosen uniquely and the limit of the effective action exists	4-05
Classification Theorem for Families of Effective Actions (Theorem C of Costello) (short: Classification Theorem for Families of Effective Actions)	Name from the literature	Local actions correspond one-to-one to families of effective actions (with the scheme fixed)	4-05

Part 5 Lattice gauge theory

Name	Kind	What it refers to	Introducing chapter
lattice area-law problem (Chatterjee's Problem 4.1) (short: lattice area-law problem)	Name from the literature	The problem of showing the area law for Wilson loops at all β in four-dimensional lattice gauge theory with a non-abelian compact group	5-02, 16-07
Theorem from Uniform Decay to the Area Law (Theorem 2.4 of Chatterjee 2021) (short: Theorem from Uniform Decay to the Area Law)	Name from the literature	If a lattice gauge theory has uniform exponential decay under arbitrary boundary conditions, then centre symmetry is unbroken, and Wilson loops in representations on which the centre acts non-trivially obey the area law	5-10
definition of unbroken centre symmetry (Definition 2.1 of Chatterjee 2021) (short: definition of unbroken centre symmetry)	Name from the literature	The theory on a slab is invariant under centre transformations under every boundary condition (centre symmetry is not broken)	5-10
definition of strong decay (Definition 2.3 of Chatterjee 2021) (short: definition of strong decay)	Name from the literature	Correlations decay exponentially, uniformly over every cube and every boundary condition	5-10

Name	Kind	What it refers to	Introducing chapter
Strong-Coupling Uniqueness Theorem · Exponential-Decay Corollary (labels of Shen–Zhu–Zhu) (short: Strong-Coupling Uniqueness Theorem · Exponential-Decay Corollary)	Name from the literature	Under the strong-coupling condition, the invariant measure of stochastic quantisation (the Langevin equation) is unique and has an infinite-volume limit, and correlations decay exponentially (a set of assumption, theorem and corollary)	5-04

Part 7 Constructive field theory

Name	Kind	What it refers to	Introducing chapter
six steps of construction	Table row	The six steps of the work of building a quantum field theory (regularisation, finite volume, ultraviolet limit, infinite volume, axioms, uniform gap)	7-01
regularisation step	Table row	Cutting off short distances, by a lattice or otherwise	7-01
finite-volume step	Table row	Building the finite-volume measure μ_{a,L}	7-01
ultraviolet-limit step	Table row	The ultraviolet limit $a \to 0$ and uniform estimates	7-01
infinite-volume step	Table row	The infinite-volume limit $L \to \infty$ and clustering	7-01
axiom step	Table row	The limit satisfies the OS axioms (reflection positivity)	7-01
uniform-gap step	Table row	A mass gap uniform in the volume	7-01
level of perturbation theory	Table row	Computation as a power series in the coupling (perturbation theory). The premise of the levels of control	7-07
level of uniform estimates	Table row	The level at which estimates uniform in the lattice spacing are obtained (Bałaban's main four-dimensional theorem reaches this far, with assumptions)	7-07
level of subsequential limits	Table row	The level at which, from uniform estimates, a limit can be taken along a subsequence of a sequence of finer lattice spacings	7-07
level of uniqueness	Table row	The level at which the limit is determined uniquely, independently of the choice of subsequence	7-07
level of symmetry and reflection positivity	Table row	The level at which the limiting theory has Euclidean symmetry and reflection positivity	7-07
level of non-triviality	Table row	The level at which the limit is not a free (Gaussian) field. The problem statement further asks that the short-distance behaviour agree with the predictions of asymptotic freedom	7-07

Name	Kind	What it refers to	Introducing chapter
level of infinite volume	Table row	The level at which the limiting theory with infinite volume exists	7-07
level of a uniform gap	Table row	The level at which there is a mass gap uniform in the volume	7-07
obstacle of the infrared parameter	Table row	The obstacle that at long distances the coupling is strong and there is no small parameter to expand in	7-11
obstacle invisible to perturbation theory	Table row	The obstacle that there are contributions (of the form e^{−c/α}) that appear at no order of the power series in the coupling	7-11
obstacle of non-triviality	Table row	The obstacle that it must be shown separately that the limit does not collapse to a free field (non-triviality)	7-11
obstacle of the curved state space	Table row	The obstacle that the space of fields (the space of gauge orbits) is curved, and the signed sum of gauge fixing becomes 0 (0/0 remains)	7-11
obstacle of positive regularisation	Table row	The obstacle that, apart from the lattice, there are few regularisations that preserve reflection positivity	7-11
obstacle of infinitely many degrees of freedom	Table row	The structure of infinitely many degrees of freedom: the obstacle that as the volume grows, the overlap of different vacua tends to 0	7-11
Marginal-Triviality Theorem and others (labels of Aizenman–Duminil-Copin) (short: Marginal-Triviality Theorem and others)	Name from the literature	A set of results of Aizenman–Duminil-Copin: the scaling limits of four-dimensional ϕ^4 and Ising are Gaussian (Theorem 1.2), an improved tree-diagram upper bound (Theorem 1.3), an error estimate for the characteristic function (Proposition 1.4)	7-08

Part 15 Record of the attempts at ultraviolet control

Name	Kind	What it refers to	Introducing chapter
ultraviolet control	Condition	Keeping, at every step from the lattice spacing a to the scale L_0 where the coupling is small, the effective density in the same step-independent form, with the three conditions of small-field analyticity, large-field rarity and the coupling staying small	15-00
small-field analyticity condition	Condition	The localised correction E_k(X,·) of each step is analytic in a complex neighbourhood of radius r of the background-field variables, with $\lvert E_k\rvert \le C\, e^{-\kappa d}$ (constants independent of the step k and the lattice spacing a)	15-00

Name	Kind	What it refers to	Introducing chapter
large-field rarity condition	Condition	The large-field factor satisfies $\|R_k(\Omega^c)\| \le e^{-c\, p(g_k)^2\, \|\Omega^c\|}$ (constants independent of the step and the lattice spacing)	15-00
condition that the coupling stays small (short: the coupling stays small)	Condition	The sequence of effective couplings $g_0,\dots,g_K$ all lies in $(0,\gamma]$ (γ independent of the step and the lattice spacing)	15-00
three ultraviolet conditions (twisted-box version)	Condition	Ultraviolet control stated up to the last step of the twisted M^4 lattice, with constants depending only on M	15-02
class-closure condition	Condition	The number of parameters is finite independently of the step, and the main part of the one-step map returns into the family. The class is taken to be the one-parameter class into which the one-step map pulls (a fixed form of coupling $B + O(1/B)$)	15-03
coupling-flow remainder condition	Condition	The β component of Φ satisfies $\beta_{k+1} = \beta_k - 8\, b_0 \ln L + O(1/\beta_k)$ (the net one-step bracket of the coupling written in the variable β)	15-03
shape remainder condition	Condition	$\| r_{\theta_k}[\rho_k] \|_{G_{k+1}} \le (1+\eta_k)\, 2^{\|B\|}\, \| \rho_k \|_{G_k}$ with $\sum \eta_k < \infty$, and the deviation from the class ρ_k stays uniformly bounded in the step	15-03
definition and continuity of the effective coupling (short: continuity of the coupling)	Condition	On $D_k = \{g_0 : g_0,\dots,g_k \in (0,\gamma_1]\}$, the map $g_0 \mapsto g_{k+1}(g_0)$ is continuous	15-03
net one-step bracket of the coupling	Condition	$1/g_{k+1}^2 = 1/g_k^2 - b + r_k$, $\|r_k\| \le C\, g_k^2$ (the change of the coupling at each step deviates from the one-loop value b only by $O(g_k^2)$)	15-03
small-field contribution to the coupling	Condition	The part of the remainder of the net one-step bracket of the coupling conditioned on small fields, $r^{\text{small}} = b(\rho_S) - \beta_{\text{pred}}$ (local part + Taylor truncation + small-field cut + explicit quadratic inter-block term)	15-03
large-field contribution to the coupling	Condition	The large-field part of the remainder of the net one-step bracket of the coupling, $r^{\text{large}} = b(\rho) - b(\rho_S)$	15-08
Coupling-Smallness Induction Lemma	Lemma	Under the net one-step bracket of the coupling and the definition and continuity of the effective coupling, for every $\gamma_0 \in (0,\gamma_1)$ there is a bare coupling g_0 with $g_0 < g_1 < \dots < g_n = \gamma_0$ (monotonicity and intermediate values; the condition that the coupling stays small is split into these two)	15-03

Name	Kind	What it refers to	Introducing chapter
Background–Fluctuation Decomposition Lemma	Lemma	In SU, $\| U_{\partial p} - 1 \| \le \| W_{\partial p} - 1 \| + \sum \| A_{l_i} \|$, and its squared version (decomposition into background × fluctuation, U = W e^A)	15-07
Splitting of the Weight	Corollary	$G_c(X, W e^A) \le G_c(X, W)^2 \exp(48c \sum \| A_l \|^2)$	15-07
Lemma on One Step with Background-Independent Fluctuations	Lemma	If the law of the fluctuations ν does not depend on the background and satisfies the background-independence condition for fluctuations, the scale-contraction condition and the exponential-moment condition for fluctuations, then $\| \theta_\nu K \|_{G_{h+}} \le 2^{|B|} \| K \|_{G_h}$ (one step only; no loss in exchanging the expectation and the seminorm)	15-07
background-independence condition for fluctuations	Condition	ν does not depend on the background W	15-07
scale-contraction condition	Condition	$h_+^2 \le h^2/2$ (the scale of the next step is at most half)	15-07
exponential-moment condition for fluctuations	Condition	$M_\nu(8qc/h^2, X) \le 2^{|B|}$ (exponential moment of the fluctuations)	15-07
One-Link Heat-Kernel Comparison Lemma	Lemma	Bounds the single-link Wilson law μ^b of SU(2) by $K(b,\theta_0)$ times the heat kernel $p_{1/b'}$ inside the angle bound θ_0, and by a rare tail outside it	15-07
One-Link One-Step Fluctuation Lemma	Lemma	The chain of the Lemma on One Step with Background-Independent Fluctuations also goes through for a single Wilson link, and the loss splits into K (on the side of the closure under the fluctuation expectation) and 1+η (the movement of the coupling)	15-07
recentring condition	Condition	A condition that assumes the angle of the background plaquette is at most $\theta_0 < \pi/2$, together with the geometry of recentring (the Taylor radius of the map to the direction of the sum of staples)	15-07
group version of the exponential moment of fluctuations	Condition	For all small-field backgrounds W and $0<t\le t_0$, $E_{\mu_{k,W}} \exp((48tc/h_k^2) \sum \| A_l \|^2) \le 2^{|B|}$ (for the true fluctuation measure)	15-07
background-regularity condition	Condition	For W in the image of the background map, $G_k(X,W)^{2t} \le G_{k+1}(X,W)$ (fine curvature is bounded by coarse curvature)	15-07
group version of exchanging the expectation and the seminorm	Condition	$\| E \theta_W F \|_{T_W(h_{k+1})} \le (1+\eta_k) E \| F \|_{T_{We^A}(h_k)}$, $\sum \eta_k < \infty$ (the group version of exchanging the expectation and the seminorm)	15-07

Name	Kind	What it refers to	Introducing chapter
product factorisation ((i) of BBS) (short: product factorisation)	Name from the literature	The first of the four properties of the weight (regulator) of BBS Appendix A.6.2: on disjoint sets the weight splits into a product	15-01
submultiplicativity in sums of fields ((ii) of BBS) (short: submultiplicativity in sums of fields)	Name from the literature	The second of the four properties of the weight: for sums of fields, $\tilde{G}(X,\phi+\phi') \leq \tilde{G}^2(X,\phi)\tilde{G}^2(X,\phi')$	15-01
closure under the fluctuation expectation ((iii) of BBS) (short: closure under the fluctuation expectation)	Name from the literature	The third of the four properties of the weight: taking the expectation over fluctuations, the weight closes with the weight of the next step	15-01
monotonicity to the next step ((iv) of BBS) (short: monotonicity to the next step)	Name from the literature	The fourth of the four properties of the weight: the weight of the current step is bounded by the weight of the next step	15-01
One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS) (short: One-Step Bound on the Activity Norm)	Name from the literature	If the weight satisfies submultiplicativity in sums of fields, closure under the fluctuation expectation and monotonicity to the next step, then $\| E_+\theta K \|_{\tilde{G}_+} \leq 2^{\|B\|} \| K \|_{\tilde{G}}$ (for flat fields the activity norm stays within a factor $2^{\|B\|}$ over one step)	15-01
inequality exchanging the expectation and the seminorm ((A.6.5) and others of BBS) (short: inequality exchanging the expectation and the seminorm)	Name from the literature	Formulas of BBS Appendix A.6: the inequality exchanging the fluctuation expectation and the seminorm (together with the definition of the norm and how the properties of the weight are applied)	15-02
Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989) (short: Conditional Ultraviolet Stability Theorem)	Name from the literature	If one assumes that the effective coupling stays small, then the form of the effective density and the estimates of small-field analyticity and large-field rarity hold at all steps (four dimensions, finite volume)	15-02
basic inequality (unpublished) (Theorem 2 of Bałaban 1989) (short: basic inequality (unpublished))	Name from the literature	The inequality for removing the assumption of the Conditional Ultraviolet Stability Theorem (that the coupling stays small). Unpublished	15-02
Orbit-Direction Flatness Proposition	Proposition	At a minimum point of a gauge-invariant smooth function, $\mathrm{Hess}\, V(x_0)(\xi,\xi) = 0$ for tangent vectors $\xi \neq 0$ of the orbit	15-01
Proposition on a Class with the Background as a Parameter	Proposition	In the class $C_2 = (B,q,a)$ that has the logarithm a of the background holonomy as a parameter, the background-dependent remainder drops to $\theta^0\beta^{-3/2}$, $\theta^1\beta^{-1}$, and the budget is uniform in the number of steps	15-07

Name	Kind	What it refers to	Introducing chapter
Shape-Remainder Theorem for Four Plaquettes in a Row (short: Four-in-a-Row Theorem)	Theorem	In the projected model of four plaquettes in a row (SU(2), diagonal definition, ψ^8 projection, $B \geq 32$, O(1) box), $r_n \leq \lambda^n r_0 + C/((5-\lambda)B_n)$ with $C = 0.0108$ (independent of the number of steps)	15-06
One-Step Contraction of the ℓ^4 Cone	Corollary	The four-dimensional ℓ^4 cone contracts by a factor of at most 0.823 in one step (normalisation $\beta'_0 = 1$, row-weighted interpolation)	15-05
Mixing-Limit Lemma	Lemma	In the volume limit, $\sup \lambda = 4 \sup \mu_{loc}$ (the limit of mixing between localised directions and the coarsest direction)	15-05
Floor Lemma for the Stein Route	Lemma	The row-weighted Stein route cannot go below $\Sigma\, s\, \nu_s^2/16 \geq 0.452 > 1/4$, for any local representative and any weights	15-05
Conditional-Mean Covariance Lemma	Lemma	One step at tree level is a Gaussian conditional mean, and the inverse of the coarse action (the covariance of the blocked flux) iterates linearly	15-05
Heat-Kernel Gaussian Upper-Bound Lemma	Lemma	For the SU(2) heat kernel p_t and non-decreasing $g \geq 0$, $E\, g(\theta) \leq e^{t/2}\, E\, g(\lvert Z\rvert)$, $Z \sim N(0, tI_3)$	15-08
Heat-Kernel Fluctuation Rarity Lemma	Lemma	Under the edgewise independent heat-kernel condition through the background-independence condition for fluctuations, the probability that a link exceeds the threshold h and the exponential moment are bounded in a closed form independent of the background (with heat-kernel fluctuations, the group version of the exponential moment of fluctuations holds)	15-08
Large-Field Remainder Bound Lemma	Lemma	Splitting the remainder as $r = r^{small} + r^{large}$, the Lipschitz constant of the read-out b with respect to total variation gives $\lvert r^{large}\rvert \leq (4/3)(b_S^2/E_S)\, P_L$	15-08
edgewise independent heat-kernel condition	Condition	The condition that the law of the fluctuations is an edgewise independent heat kernel (a premise of the Heat-Kernel Fluctuation Rarity Lemma and others)	15-08
single-step condition	Condition	The condition restricting the claim to a single step of renormalisation (multistep induction is not included)	15-08
analyticity condition at complex backgrounds	Condition	Analyticity at complex backgrounds and an upper bound independent of β (a Cauchy-type estimate of the Taylor coefficients of the small-field effective action)	15-08

Name	Kind	What it refers to	Introducing chapter
definition of the inter-block term	Condition	The definition of the explicit quadratic inter-block term W^gauge (algebra)	15-08
local cancellation	Condition	Local cancellation (the Loc identity)	15-08
cubic-remainder estimate	Condition	The cubic estimate (the remainder is $O(g^3)$, uniformly in complex backgrounds and in the volume)	15-08
Gaussian comparison condition for the partition function	Condition	e^{−CM⁴} ≤ Z_n/Z_n^G ≤ e^{CM⁴} (the ratio to the sum of Gaussian integrals around the isolated flat connections)	15-02
observable-source condition	Condition	The statement holds with the same constants even when a source of a smooth observable (strength ≤ s_0) with support larger than the scale L_0/M is inserted	15-02
ultraviolet statement for a small twisted four-dimensional torus (short: twisted-torus ultraviolet statement)	Statement	On a small twisted M^4 lattice, with a suitable choice of bare coupling, the three ultraviolet conditions (twisted-box version), the Gaussian comparison condition for the partition function and the observable-source condition hold (proposal, unproved)	15-02
row of the remainder including large fields	Table row	One of the four rows of 15-08 that do not close: the one-step remainder of the coupling including large fields (showing the net one-step bracket of the coupling including large fields)	15-08
row of background-dependent absorption and many steps	Table row	The core of the four rows that do not close: the row that absorbs the background-dependent loss η into a parameter of the class and passes it through multistep induction	15-08
row of the boundary of gauge re-fixing	Table row	One of the four rows that do not close: the handling of the boundary when the gauge fixing is re-fixed at each step	15-08
covariance-flow fixed-point condition	Condition	The flow of the covariance of the blocked field has a scale-invariant fixed point	15-03
cone-contraction condition at the fixed point	Condition	At the fixed point of the covariance flow, the contraction rate of the cone is less than 1 (sup μ_∞ < 1)	15-03
ultraviolet questions	Question	The thirty-one questions left by the chapters of Part 15 (Table 1 of 15-09). All touch on parts of the row of ultraviolet control	15-09
questions of the systematic attempts	Question	The sixteen questions left by 15-10 (the systematic attempts) (Table 15 of 15-10)	15-10

Name	Kind	What it refers to	Introducing chapter
contraction-map window for the fixed point of the class	Name	When the fixed point of the class-closure condition is stated in the form of a contraction map, the range of β in which the contraction map opens (from β of about 28–54, with $\tau = 0.695$, $a = 4.335$, $c_0 = 0.02$–0.04)	15-03
question "transverse contraction on a one-parameter class" (short: transverse contraction on a one-parameter class)	Question	Whether the one-step map contracts transversally on the one-parameter class it pulls into (whether λ and the sources stay finite, whether the constants are independent of the step, whether a weight on the sign of the centre is needed)	15-09
question "minimal model of the remainder including large fields" (short: minimal model of the remainder including large fields)	Question	Whether, in the minimal model (hierarchical SU(2)) in which the net one-step bracket of the coupling can be shown including large fields, the constrained absorption of one plaquette can be restated at the true critical point	15-09
question "smallest adjoint eigenvalue on the twisted lattice" (short: smallest adjoint eigenvalue on the twisted lattice)	Question	Whether, on the twisted M^4 lattice, the smallest eigenvalue of the adjoint fluctuation operator around the isolated flat connections is at least c/M^2	15-09
question "extending the tube of the hierarchical model" (short: extending the tube of the hierarchical model)	Question	Whether the tube and the attracting curve of the hierarchical model can be extended to $\beta > 64$ (the seam between the asymptotic estimates and interval arithmetic)	15-09
question "interval arithmetic for curvature outside the layer" (short: interval arithmetic for curvature outside the layer)	Question	Whether, by confirming the lower bound on the curvature outside the layer and the weight of the layer with interval arithmetic, the flow coefficient on the curve $c^* = 1/60$ can be separated from 1/48	15-09
question "lateral constancy of the coarse-plaquette source" (short: lateral constancy of the coarse-plaquette source)	Question	Whether, along a sequence of coarse-plaquette sides L, source $\times L^4$ approaches a constant value even when the lateral extent is widened	15-09
question "local representative of the tree-level network" (short: local representative of the tree-level network)	Question	Whether, in the tree-level network with the coarse plaquette fixed, a local representative can be made and an upper bound uniform in the volume obtained	15-09
question "one-step correction for four plaquettes in a row" (short: one-step correction for four plaquettes in a row)	Question	Whether the O(1/B) one-step correction for four plaquettes in a row can be derived on paper, giving the constant C of the one-step inequality of the shape remainder condition	15-09

Name	Kind	What it refers to	Introducing chapter
question "uniqueness of the class for other loop shapes" (short: uniqueness of the class for other loop shapes)	Question	Whether the class pulled into is determined uniquely even when loop shapes other than four in a row (2×2, L-shaped, mixed quartic) are included	15-09
question "splitting the density of the joint law" (short: splitting the density of the joint law)	Question	Whether, for four in a row, the joint law of two plaquettes can be split into a product of local couplings × a density, giving the lowest order in θ of the density and the isotropy of the quadratic form	15-09
question "minimal multistep statement" (short: minimal multistep statement)	Question	Whether a minimal multistep statement can be written in a hierarchical model or a two-dimensional model (the background-regularity condition and contraction of many-body norms)	15-09
question "Lipschitz constant of the read-out" (short: Lipschitz constant of the read-out)	Question	Deriving on paper the Lipschitz constant of the read-out used in the large-field remainder estimate, and measuring the difference in total variation of the densities with and without large fields	15-09
question "shift of the coupling when a source is added" (short: shift of the coupling when a source is added)	Question	Whether, when a source s Re W is added to four plaquettes in a row, the shift of the coupling is linear in s	15-09
question "parity of critical points" (short: parity of critical points)	Question	Whether the parity of critical points holds for four in a row (paired with the question of splitting the density of the joint law)	15-09
question "one-loop Hessian and ghosts" (short: one-loop Hessian and ghosts)	Question	Whether the one-loop Hessian of a block with factor 2 can be split and the F^2 coefficient of the ghosts measured (together with formalising the direct-product decomposition of the tree gauge)	15-09
question "smallest system that is not zero-sum" (short: smallest system that is not zero-sum)	Question	Whether, in the smallest system that is not zero-sum (two sets of four in a row side by side, sharing the link between them), the breaking of the local cancellation of the second-order contribution to the marginal coupling can be turned into a number	15-09
question "odd BCH orders on a 2×2 slab" (short: odd BCH orders on a 2×2 slab)	Question	Deriving, for the central plaquette of a 2×2 slab, the cyclically symmetrised BCH coefficients of odd order, and comparing them with the two-link values	15-09
question "class with the holonomy as a parameter" (short: class with the holonomy as a parameter)	Question	Whether, in a class having the logarithm of the background holonomy as a parameter, the background-dependent part moves exactly into the flow of that parameter	15-09

Name	Kind	What it refers to	Introducing chapter
question "radius of recentring" (short: radius of recentring)	Question	Whether, in the recentring of the One-Link One-Step Fluctuation Lemma, the Taylor radius can be made explicit and the loss coefficient made a single number	15-09
question "direct upper bound for the four-dimensional ℓ^{4}" (short: direct upper bound for the four-dimensional ℓ^{4})	Question	Bounding the contraction of the four-dimensional ℓ^4 cone directly (a finite network for the near part, Riesz–Thorin only for the tail), and seeing whether sup $\mu < 1$	15-09
question "two definitions of Loc" (short: two definitions of Loc)	Question	Whether, when the localisation Loc is defined in two ways, by row sums (zero momentum) and by matrix components, the quadratic inter-block terms agree (and which cut the original source uses)	15-09
question "three-dimensional non-local weights" (short: three-dimensional non-local weights)	Question	Whether, in three dimensions without boundary, the one-loop non-local weights in the coordinates of coarse links + global labels decay exponentially	15-09
question "second step of the multi-loop class" (short: second step of the multi-loop class)	Question	The second step of the multi-loop class: whether these are coordinates in which the triple product of angles κ is preserved at each step, and whether the same coordinates suffice in four dimensions with 2^4	15-09
question "gauge agreement on non-abelian backgrounds" (short: gauge agreement on non-abelian backgrounds)	Question	Measuring, on non-abelian backgrounds (curvature including commutators), the agreement of the two gauges and the coefficient of the ghosts	15-09
question "normalisation factors by direction" (short: normalisation factors by direction)	Question	Whether the normalisation factor by direction, sup $\lambda \leq 3.29$, can be brought below 1 with symbols weighted by momentum band	15-09
question "class with the coupling as a parameter" (short: class with the coupling as a parameter)	Question	Whether, giving the class the coupling as a parameter, the movement of the coupling of a single Wilson link moves into the update of the parameter	15-09
question "fixed point of the four-dimensional class ratio" (short: fixed point of the four-dimensional class ratio)	Question	Whether the value at the fixed point of the ratio coordinate of the four-dimensional class can be derived directly from a semi-infinite problem near the corner	15-09
question "upper bound on the fixed point of the lateral average" (short: upper bound on the fixed point of the lateral average)	Question	Whether, at the fixed point of lateral-average blocking, an upper bound on sup μ_∞ below 1 can be obtained by a route other than Stein	15-09

Name	Kind	What it refers to	Introducing chapter
question "quartic width of the three-body tree" (short: quartic width of the three-body tree)	Question	Measuring, with four-dimensional coarse links + global labels, the rate of dependence of the three-body tree term on the quartic width	15-09
question "one-loop coefficient of the long-wavelength network" (short: one-loop coefficient of the long-wavelength network)	Question	Whether, with lateral-average blocking or a large factor, the dependence on the factor of the one-loop coefficient of the long-wavelength network approaches the one-loop value	15-09
question "checking the whole range of the recentred tube" (short: checking the whole range of the recentred tube)	Question	Whether, passing the recentred tube through the whole of $\beta \in [15.26, 128]$ in intervals of width 0.02, the curve entering the tube and the enclosure of the flow coefficient c* can be made unconditional	15-09
question "alignment of rows in the worst direction" (short: alignment of rows in the worst direction)	Question	Whether, by estimating the lower bound of the alignment of rows in the worst direction with the kernel in the limit of steps, the loss from discarding signs can be bounded uniformly at all steps	15-10
question "periodic shift of the Stein maximiser" (short: periodic shift of the Stein maximiser)	Question	Where the amount by which the maximiser of the Stein upper bound on a large torus shifts away from period 2 comes from (upper bound 10.2 against a loss of 6.4)	15-10
question "cancellation on the side of the true operator" (short: cancellation on the side of the true operator)	Question	Once a quantity measuring the cancellation on the side of the true operator is fixed (for example, the ratio of $\|T^{k}\|$ to $\||T|^{k}\|$ divided by the minimum within the similarity class of signs), what can be said to cause the regularity of the loss that grows only slowly with the step	15-10
question "finite types and aggregation of returns" (short: finite types and aggregation of returns)	Question	Whether, aggregating the finite types and the remaining returns into two-by-two, $bc < (1-a)(1-d)$ holds for the recorded matrix	15-10
question "rereading with Schur weights" (short: rereading with Schur weights)	Question	What becomes of the one-step contraction constant when the record of the W coordinates (measured in ℓ^2) is reread with the weights of Schur's test	15-10
question "supremum of constrained sum rules" (short: supremum of constrained sum rules)	Question	Whether, for all $k \le \operatorname{ord}^{V}(F)$, a local representative with $\operatorname{ord}(K) \ge k$ can be taken with finite range	15-10
question "canonical choice of degrees of freedom" (short: canonical choice of degrees of freedom)	Question	Whether a representative minimising a Stein-type norm exists, is unique, and is local	15-10

Name	Kind	What it refers to	Introducing chapter
question "consistency of composing representatives across steps" (short: consistency of composing representatives across steps)	Question	Whether the output of the representative at step k lies in the constrained subspace of step $k+1$ (if not, the composition of representatives depends on the choice)	15-10
question "decay of the Moore–Penrose kernel" (short: decay of the Moore–Penrose kernel)	Question	How fast the kernel of the Moore–Penrose representative (the minimal-norm representative) decays	15-10
question "zero modes of the symmetric representative" (short: zero modes of the symmetric representative)	Question	Why the zero modes of the symmetric representative on a finite torus appear to approach $U/4$ faster than exponentially	15-10
question "band of triple valuations" (short: band of triple valuations)	Question	Whether, with triple valuations adding a third valuation (the exponent of the transseries), the difference between the bands of U(1) and SU(2) can be distinguished	15-10
question "ties in the four-dimensional series" (short: ties in the four-dimensional series)	Question	When the ties of the errors are found for the genuine four-dimensional series (higher orders at strong and weak coupling), how they overlap with the crossover band	15-10
question "skeleton of the space of two couplings" (short: skeleton of the space of two couplings)	Question	Whether, in the space of the two couplings, fundamental and adjoint, the skeleton of the tie set becomes a non-trivial polyhedral complex	15-10
question "motion of the lower edge of the window" (short: motion of the lower edge of the window)	Question	How the lower edge of the window moves with the number of steps along a sequence of blocking maps (whether the true value goes below 1 for one step with $3.5 \le p \le 5$)	15-10
question "divergence of the number of bundles" (short: divergence of the number of bundles)	Question	How the number of bundles diverges as p $\downarrow$ 2 (of the c/(p−2) type, or with a critical exponent fixed by the geometry of the lattice)	15-10
question "Schur threshold and Stein window" (short: Schur threshold and Stein window)	Question	Where the gap between the exponent of the Schur threshold and the lower edge of the Stein window comes from, and how far p^* can be lowered with weighted Schur (positive test functions)	15-10

Part 16 Where things stand, and the questions that remain

Name	Kind	What it refers to	Introducing chapter
seven stages a proof must contain	Table row	The seven stages a proof of the mass gap must pass through (vessel, ultraviolet, small-T^2 gap, small-T^2 confinement, band, R^4, general groups)	16-09

Name	Kind	What it refers to	Introducing chapter
vessel stage	Table row	Lattice, reflection positivity, block observables, compactness. The foundational stage that derives a gap of the transfer matrix from decay on a dense set	16-09
ultraviolet-control stage	Table row	The stage of multiscale analysis uniform in a (locality of the effective action, rarity of large fields, flow of the coupling), from the lattice spacing a to the scale L	16-09
small-T^2 gap stage	Table row	The stage that removes zero modes with twisted boundary conditions and shows, uniformly in a, a gap of the transfer matrix $\geq c/L$ on a small T^2	16-09
small-T^2 confinement stage	Table row	The stage that reduces confinement on a small T^2, by container estimates and insertion estimates, to "insertion of one vortex" and "independence between regions"	16-09
band stage	Table row	That the gap holds with the same constant even when T^2 is widened. The stage that crosses the intermediate scale $L \sim 1/\Lambda$ non-perturbatively	16-09
R^4 stage	Table row	The stage of the infinite-volume limit, the OS axioms, recovery of rotation invariance and non-triviality	16-09
general-group stage	Table row	The stage that extends to SU(N) and to groups without centre (such as G_2)	16-09
seam of the premise of existence	Table row	The seam where physics posits that the theory exists and computes quantities, while mathematics must build a limit (not a subsequence) satisfying the axioms	16-05
seam of "there is one scale"	Table row	The seam where physics regards both the mass and the string tension as constant multiples of a single scale Λ, while mathematics must show this holds uniformly in β	16-05
seam of the different shores	Table row	The seam where physics computes on shores where computation works (small coupling, small β, large N), while mathematics must control all scales at finite N	16-05
seam of finite and infinite	Table row	The seam where physics measures in finite boxes and extrapolates, while mathematics must show a gap uniform in the volume	16-05
seam of spacetime and space	Table row	The seam where physical intuition lives on spacetime (Minkowski), while mathematics returns from the Euclidean measure by reflection positivity	16-05
seam of the word "particle"	Table row	The seam between the physics statement "the gap is the mass of the lightest glueball" and the mathematical spectral gap (an isolated one-particle state is an extension problem)	16-05
seam of extra dimensions	Table row	The seam where physics produces the gap from the geometry of extra dimensions, while mathematics needs the limit shrinking the extra dimensions to 0 and the exchange of limits	16-05

Name	Kind	What it refers to	Introducing chapter
type in which a premise becomes a goal	Table row	The type of seam in which a premise of physics becomes a goal of mathematics (premise of existence; there is one scale)	16-05
type of limits and uniformity	Table row	The type of seam in which physics computes at particular values of parameters (shores), while mathematics needs limits to the values of the problem statement and uniform estimates (different shores; finite and infinite; extra dimensions)	16-05
type in which the same word points to different things	Table row	The type of seam in which the same word points to different things in physics and mathematics (spacetime and space; the word "particle")	16-05
task of the T^4 construction	Task	On T^4, the expectations of gauge-invariant quantities converge as $a \to 0$ along the whole sequence	16-07
task toward infinite volume	Task	The limit $L \to \infty$ exists along the whole sequence	16-07
task of a uniform gap	Task	$\inf_L \Delta_L > 0$	16-07
task of the remaining axioms	Task	Recovery of rotation symmetry, and the regularity needed for reconstruction	16-07
task of non-triviality	Task	The limit is not Gaussian	16-07
task of local fields and short distances	Task	Local fields, and agreement with asymptotic freedom at short distances	16-07
task of strong decay	Task	Uniform exponential decay under arbitrary boundary conditions	16-07
task of one scale	Task	$c\sigma \le m^2 \le C\sigma$ for $\beta \ge \beta_0$	16-07
task of the three-dimensional measure	Task	Construction of the measure on T^3	16-07
rule of existence	Table row	The rule that the existence of the theory on R^4 (the requirement of existence) follows from the task of the T^4 construction and the task toward infinite volume (a definition)	16-07
rule of the axioms	Table row	The rule that the requirement of the axioms follows, by OS reconstruction, from the existence of the theory and the task of the remaining axioms (a theorem)	16-07
rule of the gap	Table row	The rule that the requirement of the gap follows from the existence of the theory and the task of a uniform gap (a theorem: a uniform gap survives in the limit)	16-07
rule of non-triviality	Table row	The rule that the requirement of non-triviality follows from the task of non-triviality and the task of local fields and short distances (a definition)	16-07
rule of the problem statement	Table row	The rule that the whole problem statement for the group G follows from the three requirements of the axioms, non-triviality and the gap (a definition)	16-07

Name	Kind	What it refers to	Introducing chapter
rule from decay to the area law	Table row	The rule that the lattice area law follows from the task of strong decay (the Theorem from Uniform Decay to the Area Law of Chatterjee 2021)	16-07
rule of the string tension	Table row	The rule that the requirement of the string tension follows from the continuum string-tension problem (a definition)	16-07
rule of the problem setters' route	Table row	The route indicated by the problem setters (Jaffe–Witten), in which the task of a uniform gap opens the task toward infinite volume	16-07
rule from the string tension to the gap	Table row	The physical expectation that the requirement of the gap follows from the existence of the theory, the requirement of the string tension and the task of one scale (there is one scale)	16-07
rule from the gap to the string tension	Table row	The physical expectation that the requirement of the string tension follows from the existence of the theory, the requirement of the gap and the task of one scale (the reverse of the rule from the string tension to the gap)	16-07
requirement of existence	Name	That the theory on R^4 exists (one of the requirements of the problem statement)	16-07
requirement of the axioms	Name	That the theory satisfies axioms at least as strong as the Wightman and Osterwalder–Schrader axioms (one of the requirements of the problem statement)	16-07
requirement of non-triviality	Name	That the theory is non-trivial, has local fields, and agrees with asymptotic freedom at short distances (one of the requirements of the problem statement)	16-07
requirement of the gap	Name	That there is a mass gap $\Delta > 0$, and $m < \infty$ (one of the requirements of the problem statement)	16-07
requirement of the string tension	Name	That the continuum string tension d is not 0 (confinement; outside the requirements of the problem statement)	16-07
requirement of the problem statement	Name	The whole problem statement for the group G: the four requirements of existence, axioms, non-triviality and the gap taken together	16-07
two-way split of the core	Name	Viewing the core of the band as two jobs: "the ultraviolet extension" and "the statement of the last interval to strong coupling"	16-11 § 01
ultraviolet extension	Name	The job of extending the control of the published construction from the threshold to a coupling β_* of order one, while keeping the sum of the errors over the steps finite	16-11 § 01
statement of the last interval to strong coupling	Statement	A non-perturbative statement over a small number of steps, from β_* to the strong-coupling window (different from the "Corollary Crossing the Band in Finitely Many Steps" of 26-06)	16-11 § 01

Name	Kind	What it refers to	Introducing chapter
lossy chain	Name	Renormalisation comparisons allowing a loss of a factor K at each step, joined over the number of steps. The rate is uniform, but the prefactor grows as $\xi^{\log_2 K}$	16-11 § 02
facts on the energy side	Name	Facts about the action of configurations (existence of twisted vacua, stabiliser group ± 1, positivity of the action of the box with two twists, upper and lower bounds on S_N)	16-11 § 03
shortfall on the entropy side	Name	That this book lacks the estimate the band needs, which would count vortex fluctuations and the number of placements from below, uniformly in the scale	16-11 § 03
mixing statement for boxes of physical size	Statement	If the side of the box is taken to be the physical length C/Λ, the total variation of the law of the gauge-invariant observables inside, when the frame is changed, is smaller than a threshold (uniformly in β). A restatement of the mass gap in a strong form	16-11 § 04
seven test models	Name	The models against which arguments for the band are tested: U(1) at weak coupling, Z_2 at weak coupling, finite non-abelian groups, the adjoint Higgs limit, the line and endpoint of the SU(2) mixed action, the bulk transition of SU(N), and G_2	16-11 § 04, 26-12 § 02
three classes of value levels	Name	The weak-coupling behaviour of the quantity for a symmetric box with one twist: proportional to β for U(1), constant for frozen finite groups, of the $(3/2) \ln \beta$ type for SU(2)	16-11 § 04
control of the continuum limit of twisted boxes	Task	A comparison of scales with a finite sum of errors, for quantities determined by the measure, in twisted (framed) boxes	16-11 § 06
coupling of the laws of gauge-invariant observables	Task	A way of coupling the conditional laws of gauge-invariant functions, and a finite-box criterion on top of it	16-11 § 06
method for computing lower bounds on the entropy side	Task	Insertions adapted to the configuration, with relative entropy tending to 0	16-11 § 06
ordering of actions in the twisted sector	Task	An inequality saying that ΔF increases in the plaquette couplings, or an ordering of actions replacing it	16-11 § 06
substitute for the vortex free energy for groups without centre	Task	A quantity that plays the role of the vortex free energy for groups without centre, such as G_2	16-11 § 06
termwise sign inequality	Question	$\partial \Delta F / \partial \beta_p \geq 0$ for the plaquette-wise couplings β_p	16-11 § 04, 26-12 § 05

Part 18 A dictionary of two languages

Name	Kind	What it refers to	Introducing chapter
entry "path integral (quantum mechanics)"	Dictionary entry	Path integral (quantum mechanics) ⟷ Feynman–Kac formula	18-02
entry "functional integral"	Dictionary entry	Functional integral ⟷ Euclidean probability measure	18-02
entry "space of states"	Dictionary entry	Space of states ⟷ reconstructed Hilbert space	18-02
entry "Wick rotation"	Dictionary entry	Wick rotation ⟷ OS reconstruction theorem	18-02
entry "Green functions"	Dictionary entry	Green functions ⟷ Wightman and Schwinger functions	18-02
entry "Källén–Lehmann representation"	Dictionary entry	Källén–Lehmann representation ⟷ spectral measure	18-02
entry "vacuum and uniqueness"	Dictionary entry	Vacuum and uniqueness ⟷ Ω, clustering	18-02
entry "Hamiltonian"	Dictionary entry	Hamiltonian ⟷ logarithm of the transfer matrix	18-02
entry "one-particle state"	Dictionary entry	One-particle state ⟷ isolated eigenvalue of the mass operator	18-02
entry "S-matrix"	Dictionary entry	S-matrix ⟷ Haag–Ruelle scattering theory	18-02
entry "positive norm"	Dictionary entry	Positive norm ⟷ reflection positivity	18-02
entry "unitarity of the S-matrix"	Dictionary entry	Unitarity of the S-matrix ⟷ positive norm + asymptotic completeness	18-02
entry "spacetime and lattice"	Dictionary entry	Spacetime and lattice ⟷ $a\mathbb{Z}^4 \to \mathbb{R}^4$	18-02
entry "regularisation"	Dictionary entry	Regularisation ⟷ family of measures with lattice spacing a	18-02
entry "renormalisability"	Dictionary entry	Renormalisability ⟷ BPHZ, Epstein–Glaser	18-02
entry "field φ(x)"	Dictionary entry	Field $\phi(x)$ ⟷ operator-valued distribution	18-02
entry "gauge fixing"	Dictionary entry	Gauge fixing ⟷ section into the orbit space	18-02
entry "physical states"	Dictionary entry	Physical states ⟷ reconstruction from gauge-invariant quantities	18-02
entry "local symmetry does not break"	Dictionary entry	Local symmetry does not break ⟷ Elitzur's theorem	18-02
entry "mass gap"	Dictionary entry	Mass gap ⟷ spectral gap, uniform decay	18-03

Name	Kind	What it refers to	Introducing chapter
entry "correlation length ξ"	Dictionary entry	Correlation length ξ ⟷ inverse of the gap of the transfer matrix	18-03
entry "gap in physical units"	Dictionary entry	Gap in physical units ⟷ uniform lower bound in lattice units	18-03
entry "V(R)"	Dictionary entry	$V(R)$ ⟷ decay rate of Wilson loops	18-03
entry "string tension σ"	Dictionary entry	String tension σ ⟷ coefficient of the area law, d of Chatterjee's Problem 5.2 (the continuum string-tension problem)	18-03
entry "confinement"	Dictionary entry	Confinement ⟷ one-directional theorems	18-03
entry "centre symmetry"	Dictionary entry	Centre symmetry ⟷ centre invariance on slabs	18-03
entry "Λ, one scale"	Dictionary entry	Λ, one scale (all masses are constant multiples of a single scale) ⟷ no corresponding term on the mathematical side	18-03
entry "m/√σ"	Dictionary entry	$m/\sqrt{\sigma}$ ⟷ the limit combining Chatterjee's Problems 5.1 and 5.2 (the lattice correlation-length problem and the continuum string-tension problem)	18-03
entry "glueball"	Dictionary entry	Glueball ⟷ isolated eigenvalue of the mass operator	18-03
entry "Langevin relaxation"	Dictionary entry	Langevin relaxation ⟷ generator L	18-03
entry "z=2"	Dictionary entry	$z = 2 \longleftrightarrow \mathrm{gap}(L) \asymp \mathrm{gap}(H_T)^2$ (the Langevin gap is the square of the gap of the transfer matrix)	18-03
entry "exponential autocorrelation time"	Dictionary entry	Exponential autocorrelation time ⟷ Poincaré constant C_P	18-03
entry "thermalisation time"	Dictionary entry	Thermalisation time ⟷ mixing time	18-03
entry "dynamical exponent z"	Dictionary entry	Dynamical exponent $z \longleftrightarrow C_P \asymp \xi^z$	18-03
entry "z=2-η_s"	Dictionary entry	$z = 2 - \eta_s$ ⟷ variational lower bound $C_P \geq \chi$	18-03
entry "no slowing down away from criticality"	Dictionary entry	No slowing down away from criticality ⟷ uniform mixing ⟺ uniform log-Sobolev	18-03
entry "Lüscher term"	Dictionary entry	Lüscher term ⟷ Casimir energy of the strip	18-03

Name	Kind	What it refers to	Introducing chapter
entry "roughening, shape of the tube"	Dictionary entry	Roughening, shape of the tube ⟷ Wulff construction	18-03
entry "topological freezing"	Dictionary entry	Topological freezing ⟷ bottleneck of the state space	18-03
entry "renormalisation group"	Dictionary entry	Renormalisation group ⟷ sequence of steps, UV stability	18-04
entry "β function"	Dictionary entry	β function ⟷ recursion for the effective coupling	18-04
entry "Polchinski's equation"	Dictionary entry	Polchinski's equation ⟷ Polchinski equation of measure decomposition (heat flow)	18-04
entry "Δβ of MCRG"	Dictionary entry	Δβ of MCRG ⟷ one-step map	18-04
entry "change of scheme for Λ"	Dictionary entry	Change of scheme for Λ ⟷ reparametrisation of the coupling	18-04
entry "Migdal–Kadanoff"	Dictionary entry	Migdal–Kadanoff ⟷ recursion that closes on hierarchical lattices	18-04
entry "continuum limit"	Dictionary entry	Continuum limit ⟷ scaling limit	18-04
entry "a(β), setting the scale"	Dictionary entry	a(β), setting the scale ⟷ ε(β) (the scale function of Chatterjee's Problem 5.2)	18-04
entry "scaling"	Dictionary entry	Scaling ⟷ limits of dimensionless ratios	18-04
entry "asymptotic scaling"	Dictionary entry	Asymptotic scaling ⟷ asymptotic form of ε(β)	18-04
entry "universality"	Dictionary entry	Universality ⟷ uniqueness of the limit	18-04
entry "Symanzik's $O(a^2)$"	Dictionary entry	Symanzik's $O(a^2)$ ⟷ the shortfall of a step δ(b)	18-04
entry "improved action"	Dictionary entry	Improved action ⟷ loss of reflection positivity (contributions with negative weights)	18-04
entry "marginal coupling"	Dictionary entry	Marginal coupling ⟷ Sobolev critical exponent	18-04
entry "large N, planar diagrams"	Dictionary entry	Large N, planar diagrams ⟷ planar maps, sums over strings	18-04
entry "master field"	Dictionary entry	Master field ⟷ limit of non-commutative distributions	18-04
entry "radial direction of holography"	Dictionary entry	Radial direction of holography ⟷ Fefferman–Graham expansion	18-04

Name	Kind	What it refers to	Introducing chapter
entry "KK tower"	Dictionary entry	KK tower ⟷ λ_1 of the internal Laplacian	18-04
entry "topological charge"	Dictionary entry	Topological charge ⟷ second Chern number	18-04
entry "instanton"	Dictionary entry	Instanton ⟷ anti-self-dual connection	18-04
entry "Wilson flow"	Dictionary entry	Wilson flow ⟷ Yang–Mills heat flow	18-04
entry "Gribov region"	Dictionary entry	Gribov region ⟷ region where the FP operator is positive	18-04
entry "trace anomaly"	Dictionary entry	Trace anomaly ⟷ no corresponding term on the mathematical side	18-04
Theorem from Centre Symmetry to Confinement (Theorem 2.2 of Chatterjee 2021) (short: Theorem from Centre Symmetry to Confinement)	Name from the literature	If centre symmetry is not broken, the static pair potential V(R) tends to infinity as $R \to \infty$ (confinement)	18-01
definition column	Table row	The first column of the dictionary's grade: whether the correspondence as a definition is settled, conditional or unsettled	18-01
existence column	Table row	The second column of the dictionary's grade: whether the object exists in four dimensions (constructed, conditional, none)	18-01
measurement column	Table row	The third column of the dictionary's grade: in which direction finite measurements can reach	18-01
static exponent (symbol kept)	Name	The static correlation exponent η_s: the exponent determined from the scaling dimension Δ of an operator by $\eta_s = 2\Delta - (d - 2)$ (distinct from the dynamical exponent)	18-05

Part 19 Objects, and the statements to be shown

Name	Kind	What it refers to	Introducing chapter
uniform-decay statement	Statement	One of the statements to be shown: along a tuned sequence of couplings, correlations decay exponentially, uniformly in the lattice spacing and the box (its consequence is the mass gap)	19-07
statements to be shown	Statement	The set of statements to be shown toward the continuum limit and the mass gap, written only with the objects defined in Part 19 (tuning, convergence of the full sequence, non-degeneracy, equicontinuity, uniform decay, the band)	19-07

Name	Kind	What it refers to	Introducing chapter
base time	Statement	The time of the starting point x of the link (x,μ). For a link in the time direction, this is taken as the time of that link	19-01

Part 21 Firm theorems — the vortex floor and Polyakov loops

Name	Kind	What it refers to	Introducing chapter
upper bound on the flipping cost	Statement	For some S, for every $\varepsilon > 0$ there is an m such that $\kappa_c(N,M,m,\lfloor M/m \rfloor) \le S + \varepsilon$ for all $M \ge m$ (an upper bound on the cost of flipping the class of the window)	21-04

Part 23 The first road, first half — from the direction of time

Name	Kind	What it refers to	Introducing chapter
diffusive-relaxation-time assumption (short: relaxation-time assumption)	Name	For the family of Wilson measures, $\lim_{\beta\to\infty} \ln C_P(\beta)/\ln \xi(\beta) = 2$ (the relaxation time grows with the same power as the square of the correlation length; a statement at the level of powers)	23-01
relaxation-time upper-bound assumption (short: upper-bound assumption)	Name	$a^2 C_P \le C$, that is, $C_P \lesssim \xi_{RG}^2$ (ξ_{RG} is the bookkeeping length; the upper half of the band statement, a Poincaré inequality for all test functions f)	23-01
relaxation-time lower-bound assumption (short: lower-bound assumption)	Name	$a^2 C_P \ge c$, that is, $C_P \gtrsim \xi_{RG}^2$ (ξ_{RG} is the bookkeeping length; the lower half of the band statement, for which one test function suffices)	23-01
band form of the diffusive-relaxation-time assumption (short: band form of the relaxation-time assumption)	Name	For some $0<c\le C$ and β_0, $c\,\xi^2 \le C_P[E] \le C\,\xi^2$ (a band statement agreeing up to a constant factor; it names the graduation E)	23-01
double-well relaxation-time power statement (short: double-well power statement)	Name	For the double-well path measure, $\lim_{\hbar\to 0} \ln C_P(\hbar)/\ln \xi(\hbar) = 2$ (a statement of the same form as the diffusive-relaxation-time assumption, stated for the family of measures of the first step)	23-03
relaxation-time band at the bookkeeping length	Assumption	The band statement $a^2 C_P \asymp 1$ ($a = 1/\xi_{RG}$) that follows from the band assumption (angle) together with the one-scale chain and the bookkeeping-length chain	23-01

Name	Kind	What it refers to	Introducing chapter
eight statements of construction	Statement	The eight component statements into which the requirements of the problem statement are split along the road entering from the lattice (infinite-volume uniqueness, reflection-positivity preservation, the Wilson-loop limit, the physical-unit gap, rotation recovery, OS reconstruction, non-triviality and so on)	23-01
infinite-volume uniqueness statement	Statement	At each β the infinite-volume limiting measure is determined uniquely	23-01
reflection-positivity preservation statement	Statement	A positive self-adjoint transfer operator, preserved under weak limits	23-01
Wilson-loop limit statement	Statement	Renormalised Wilson loops have a non-trivial limit as $a \to 0$	23-01
physical-unit gap statement	Statement	The gap in physical units $\Delta(\beta)$ converges as $\beta \to \infty$ to $\Delta_\infty \in (0,\infty)$	23-01
rotation-recovery statement	Statement	Rotation and translation invariance return in the limit	23-01
OS reconstruction statement	Statement	A Hilbert space and a Hamiltonian are built from the OS axioms	23-01
non-triviality statement	Statement	The limit is not Gaussian and matches asymptotic freedom at short distances	23-01
relaxation chain	Assumption	$\lambda_1 \asymp m^2$ (the Langevin gap and the square of the mass agree)	23-01
one-scale chain	Assumption	$m^2 \asymp \sigma$ (the square of the mass and the string tension agree)	23-01
bookkeeping-length chain	Assumption	$\sigma \asymp a^2$ (the string tension and the bookkeeping length agree)	23-01
tube transfer-operator inequality (short: tube inequality)	Assumption	The tube transfer operator satisfies $T \le \exp(-(\sigma\ell + c/\ell))$ (a tube of circumference ℓ is heavier by at least c/ℓ in addition to the tension)	23-02
localised tube inequality	Assumption	T for short loops, a Casimir floor for large loops, and the junction between them. The three give $m^2 \ge 4c\sigma$	23-02
assumption of projection onto a one-dimensional chain of scales (short: projection assumption)	Assumption	The state space can be projected onto a one-dimensional chain in the scale $s = \ln \ell$ (four parts, from the scale decomposition of the variance to the single-line property)	23-02
summability-of-shortfalls assumption	Assumption	The sum of the one-step shortfalls $\delta(b_k)$ is finite	23-02

Name	Kind	What it refers to	Introducing chapter
time-step comparison assumption ((D) in the paper "one-sided bridge") (short: time-step comparison assumption)	Assumption	On a lattice with discrete time, $E\|dF\|^2_G \le c_D\langle\phi,(1-\hat{T})\phi\rangle$ (a comparison of the increment between neighbouring times with the gradient, with c_D uniform in β)	23-01
ground-state assumption ((A1) in the paper) (short: ground-state assumption)	Assumption	The spectrum is discrete and the ground state is simple and positive	23-01
stationary path-measure assumption ((A2) in the paper) (short: stationary path-measure assumption)	Assumption	A stationary path measure with $T = \infty$ (infinitely long in the time direction) exists	23-01
metric comparison assumption ((A3) in the paper) (short: metric comparison assumption)	Assumption	Comparison of the Langevin metric, $G \le \lambda_G M^{-1}$	23-01
vacuum C_P	Name	The Poincaré constant C_P of the vacuum measure at a single time (one of the three C_P)	23-01
path C_P	Name	The Poincaré constant C_P of the distribution of all fields in spacetime (the path measure = the Wilson measure). The relaxation-time assumption is a statement about this	23-01
path C_P measured in a graduation	Name	The C_P for the same measure as the path C_P, with the graduation of the Dirichlet form named (measured in angles or in fields). Multiplying the graduation by c multiplies C_P by 1/c	23-01
One-Sided Bridge Proposition (Theorem 3.1 of the paper) (short: one-sided bridge)	Proposition	With continuous time, a finite spatial lattice and $T=\infty$, under the ground-state assumption, the stationary path-measure assumption and the metric comparison assumption, $C_P \ge \kappa\xi^2 = \kappa\hbar^2/\Delta^2$ (best possible within the family of test functions local in time)	23-01
Discrete-Time One-Sided Bridge Proposition	Proposition	On a time lattice with step 1, under the time-step comparison assumption, $C_P \ge 2/(c_D \Delta^2)$	23-01
One-Dimensional One-Sided Bridge Proposition	Proposition	For any one-dimensional well with continuous time and $T\to\infty$, the ceiling of the family of test functions local in time is ξ^2, hence $C_P \ge \xi^2$ ($R \ge 1$)	23-04
Free-Particle-on-a-Circle Counterexample	Proposition	For a free particle on a circle, the gap $\Delta = \hbar^2/2 > 0$, yet the C_P of the path measure is ∞ (a negation of the converse direction)	23-01

Name	Kind	What it refers to	Introducing chapter
Double-Well Relaxation-Time Upper-Bound Proposition (short: First-Step Upper-Bound Proposition)	Proposition	For the double-well path measure, under the four double-well conditions and the premise rows, $C_P \le K\xi^2 S_0/\hbar$, so with $1 \le R \le K'/\hbar$ the power statement follows	23-03
Six-Node In-Phase Poincaré Proposition	Proposition	For the phase measure $\mu^{(+)}$ with $V=(x^2-1)^2/8$, $\varepsilon=1/2$ and ring $N=12n$, at each of the six nodes in $\hbar$, $\mathrm{Var}\, f \le C\, E(f)$ with $C \le 1.6\times 10^{10}$ (658 with numerical input), independent of n	23-06
Box-Reduction Proposition	Proposition	If $c_{alt} < 1$, then $C_P \le (1+1/n)\, C_{box}/(1-c_{alt})$ (two families, even and odd, of overlapping boxes; Poincaré inequalities only)	23-05
Gap-Inside-the-Box Proposition	Proposition	Splitting the inside of the box by kink configurations, $C_{box} \le C_{loc} + (1+\chi)C_{kg} \le K_4(C\xi)^2/D$	23-05
Three-Range Core-Constant Proposition	Proposition	Bounds the Poincaré constant of the core by splitting according to the core length k into three ranges: convexity for $k\le 7$, two blocks for $8\le k\le 14$, and box reduction applied to the core for $k\ge 15$ (the Box-Reduction Inequality for Long Cores)	23-11
Two-Projection Lemma	Lemma	If the maximal correlation of two σ-algebras is $c < 1$, then $\mathrm{Var}\, f \le (1-c)^{-1}(E\,\mathrm{Var}(f\|G_1) + E\,\mathrm{Var}(f\|G_2))$	23-05
Short-Core Convexity Lemma	Lemma	If $\kappa(k) > 0$ ($k \le 7$), then $C_P(\mu_k^{\{\alpha,\beta\}}) \le \varepsilon/\kappa(k)$ for all end values and $\hbar$	23-11
End-Reduction Lemma	Lemma	For a Markov chain, the maximal correlation of two separated blocks equals the maximal correlation of the two facing end points	23-11
Two-Block Lemma	Lemma	Bounds C_P for cores with $8 \le k \le 14$ by splitting them into two blocks	23-11
Closed Formula for the Alternating Maximal Correlation	Proposition	For a stationary Gaussian chain with correlation t within pieces and correlation s across seams, the maximal correlation between the even pieces and the odd pieces is $\phi(\|s\|,\|t\|)$ (a closed formula)	23-11
Box-Reduction Inequality for Long Cores	Proposition	The inequality for the Poincaré constant of the core obtained by applying the box reduction of the Box-Reduction Proposition to cores of length $k \ge 15$	23-11
Tilted Second-Order Taylor Lemma	Lemma	Tilts the measure at the centre of the box with given end values, and bounds the whole box by the second-order Taylor expansion of the cumulant generating function with a Hölder remainder	23-06

Name	Kind	What it refers to	Introducing chapter
Gap Non-Monotonicity Proposition	Proposition	FKG gives an ordering of likelihood ratios, but the single-site gap is not monotone in the neighbouring values	23-06
Linear-Fractional Lemma	Lemma	Setting $h = g - g^*$ and $\beta = \beta_0 + \tau$ makes the expectation a linear-fractional function of τ	23-08
Tilted-Enclosure Lemma	Lemma	Keeps the enclosure of the fixed point of the step map tilted in τ, in the form $m_k - \tau\mu_k - \tau^2\nu_k$	23-08
Moving-Centre Lemma	Lemma	Moves the centres of the two one-sided parts of the variance bound with τ, as $c_\tau = c + \tau\kappa$	23-08
Scalar-Strand Proposition	Proposition	At 8 nodes with $\hbar \in [3\cdot10^{-4}, 10^{-3}]$, bounds the Poincaré constant of the core using Bakry–Émery for the convex bottom and paying for the band at the top with the Gibbs step (the condition is $\ln k < \ell_{band}$)	23-09
Product Bound for the Window Log-Sobolev Constant	Proposition	An assembly bounding the log-Sobolev constant of the site window from above by a product of factors, $A_W \le c_W C_{1D}^W S_W T_W$	23-07
Window Log-Sobolev Threshold Proposition	Proposition	The log-Sobolev constant of the site window is within the required value: $A_W < A_{max}(\hbar)$	23-09
marginal log-Sobolev input for the window	Assumption	The marginal log-Sobolev inequality for the window, $\mathrm{Ent}(E[g^2\|\sigma_W]) \le A_W E(g) + D_W \int g^2$	23-09
Coarse-Chain Dobrushin Theorem (Theorem CC of the paper) (short: Coarse-Chain Dobrushin Theorem)	Name from the paper	For a core with both ends inside the wells, the Poincaré constant of the core is bounded uniformly by the Dobrushin criterion for the coarse chain (for $\hbar \in [10^{-5},1]$, $\rho_{col} \le 0.400$ by interval arithmetic over 239 intervals)	23-05
Small-ħ Coarse-Chain Proposition (Proposition PIN of the paper) (short: Small-ħ Coarse-Chain Proposition)	Name from the paper	For small $\hbar$ ($\hbar \le 10^{-5}$), the Dobrushin coefficient of the coarse chain stays within $\rho_{col} \le 0.874$ (the small-ħ side of the Coarse-Chain Dobrushin Theorem)	23-05
Coarse-Chain Brascamp–Lieb Proposition (Proposition BLc of the paper) (short: Coarse-Chain Brascamp–Lieb Proposition)	Name from the paper	The coarse-chain version of the Brascamp–Lieb inequality (a part of the Coarse-Chain Dobrushin Theorem)	23-05
Closed-Core KKT-Point Proposition	Proposition	When the end values are below the box ($a,b \le -x_c$), the only KKT point on the closed core is the bottom corner	23-15

Name	Kind	What it refers to	Introducing chapter
four double-well conditions	Condition	The four conditions into which the upper bound $C_P \le K\xi^2 S_0/\hbar$ of the first step (the double well) is split (in-phase local Poincaré, saddle tube, box reduction, gap inside the box)	23-03
in-phase local Poincaré condition	Condition	The Poincaré constant C_loc of the kink-free phase measure $\mu^{(+)}$ is O(1) uniformly in ħ and the length of the path (inside the box, the segment local Poincaré assumption)	23-05
saddle-tube condition	Condition	In the tube around the family of kink–antikink pairs, the transport constant and the Poincaré constant of the tube are uniform in the dimension and ħ (inside the box, the saddle-tube Poincaré assumption and the core Poincaré uniformity assumption)	23-05
box-reduction condition	Condition	$C_P(\mu) \le C_{box}(L)/c_M$, with c_M depending only on the length in units of ξ (reducing the overall relaxation time to the relaxation time of boxes of length L)	23-05
gap-inside-the-box condition	Condition	For the conditional measure on a box of length Cξ, $1/C_P \ge cD/(C\xi)^2$, with c uniform in ħ and the boundary values	23-05
segment local Poincaré assumption	Condition	The local Poincaré inequality for segments with end values (length $\le C\xi$) is uniform also in the end values	23-05
saddle-tube Poincaré assumption	Condition	The Poincaré constant of the tube is uniform in tube lengths $\le C\xi$	23-05
core Poincaré uniformity assumption	Condition	The Poincaré constant of the core within the tube (the interval that stays near the top) is uniform in the core length k and the end values	23-05
alternating maximal-correlation assumption for boxes	Assumption	The alternating maximal correlation between boxes, $c_{alt} < 1$, is uniform in ħ and the boundary values	23-05
alternating maximal-correlation assumption for the core	Assumption	An upper bound $\bar{c} < 1$ on the alternating maximal correlation of the core ($k \ge 15$)	23-11
core action-barrier assumption	Assumption	C_P of the core and the window $\le c\,\hbar^{-p}\, e^{(\Delta S/\hbar)}$, with barrier $\Delta S < 2S_0$	23-11
kink-gas gap assumption	Assumption	The box gap of the reduced kink gas is of the Glauber type of the one-dimensional Ising model ($C_{kg} \le (C\xi)^2/(c_{kg}(1-\eta_{KG})D)$)	23-05
cross-term uniformity assumption	Assumption	The cross term satisfies $E_Y(E[f\|Y]) \le (1+\chi)E(f)$, with χ uniform in ħ and the boundary values	23-05

Name	Kind	What it refers to	Introducing chapter
kink diffusion-coefficient row	Table row	The mobility of a single kink, $D = \hbar/S_0$ (at the level of the process)	23-05
bridge volume-locality input	Assumption	Volume locality on the bridge	23-09
seam limit assumption	Assumption	The limit of stacking steps without end lies, at each $\hbar$, in the enclosure of the fixed point	23-07
random-end-value assumption	Assumption	The assumption that the seam estimate holds also when the end values of the outer chain are given by a distribution rather than a single point	23-07
ratio conditioned on the plateau	Assumption	The ratio measured at the seam of the ring (the plateau at the top), conditioned on the length of the plateau	23-07
row of the contraction of the chain outside the window	Table row	$q_{\mathrm{free}} \le 0.998854904$ (contraction of the chain outside the window of the bridge core, over the whole window of $\hbar$)	23-07
small assumptions of the sector input	Assumption	The list of eleven small assumptions needed as input to the sector framework (the bridge volume-locality input, the core log-Sobolev uniformity assumption, the longitudinal term, regularity and others; seven of them do not yet have a one-line statement of content)	23-07
margin row	Table row	The first row of the two-scale assembly: the coarse-side margin $\hbar\rho - 2\hbar\kappa \ge 0.067$ (for neighbouring and end values in $[0, 2.5]^2$)	23-06
nearest-neighbour chain row	Table row	A row of the two-scale assembly: the marginal measure of the retained sites is a nearest-neighbour chain	23-06
two-site gap row	Table row	A row of the two-scale assembly: the gap of the marginal measure of the two sites (x_4, x_8) is $\lambda_c \ge 1.3 \times 10^{-4}$ (end values in $[0, 4]^2$, $\hbar \in (0, 0.2]$)	23-06
crossing row	Table row	A row of the two-scale assembly: the strength of the crossing between the two scales, $v/\hbar \le 5888$	23-06
variance-decomposition row	Table row	The last row of the two-scale assembly: the formula that splits the variance into the retained sites and the rest, and assembles the other rows into a single Poincaré constant	23-06
lattice target level	Condition	The first level of the goal: at each β the lattice correlation length $\xi(\beta)$ is finite, and it diverges as $\beta \to \infty$	23-15
physical-unit target level	Condition	The lattice mass in physical units, m_{lat}/a, tends to a positive finite value	23-15

Name	Kind	What it refers to	Introducing chapter
continuum target level	Condition	The third level of the goal: a non-trivial continuum theory on R^4 with a positive gap	23-15
remaining item between nodes	Condition	A row remaining for closing the in-phase local Poincaré condition: ħ between the six nodes	23-06
small-ħ remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: the side $ħ < 0.005$ (computer assistance that subdivides into intervals would need infinitely many intervals)	23-06
large-end-value remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: the case where neighbouring and end values exceed 2.5	23-06
sharp-constant remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: sharpening the constant $C \le 1.6\times 10^{10}$	23-06
segment-version remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: the version for segments with end values (the segment local Poincaré assumption)	23-06
bibliographic remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: checking the bibliographic data of the cited theorems	23-06
sector sum	Step	The first step of the sector framework: even when the configuration space is split into a product of boxes, the Dirichlet form, the variance and the entropy split into sums over sectors	23-09
Gibbs decomposition	Step	The second step of the sector framework: by the Gibbs variational inequality, the weight on rare sets is paid for by the entropy of the marginal of the coarse variables	23-09
componentwise localisation	Step	The third step of the sector framework: for each component of the gradient, the marginal log-Sobolev constant of the site window is applied, so that the core length k does not enter the sum	23-09
volume locality	Step	The fourth step of the sector framework: the rarity exponent ℓ_min of the defect set does not depend on the core length k	23-09
sector composition	Step	The fifth step of the sector framework: combines the three inequalities of Barta, componentwise localisation and Bochner, and bounds the gap of the core independently of k under the closing condition	23-09

Name	Kind	What it refers to	Introducing chapter
quantitative uniform Laplace method	Condition	Local Poincaré inside the wells of the first step, and the reduction of the margin as $\hbar \to 0$	23-15
step aligning the assumptions	Step	The first step of the proof of the First-Step Upper-Bound Proposition: aligning the premise rows	23-05
box-reduction step	Step	A step of the proof of the First-Step Upper-Bound Proposition: by the Box-Reduction Proposition, the whole is reduced to boxes of length $C\xi$	23-05
step inside the box	Step	A step of the proof of the First-Step Upper-Bound Proposition: the gap inside the box is bounded by splitting according to kink configurations	23-05
composition step	Step	A step of the proof of the First-Step Upper-Bound Proposition: combining the box reduction and the gap inside the box gives $C_P \le K\xi^2 S_0/\hbar$	23-05
step bracketing the ratio	Step	A step of the proof of the First-Step Upper-Bound Proposition: the ratio $R = C_P/\xi^2$ is bracketed as $1 \le R \le K'/\hbar$	23-05
step reading the power	Step	The last step of the proof of the First-Step Upper-Bound Proposition: the power statement $\ln C_P/\ln \xi \to 2$ is read from the bracketed ratio	23-05
short-core step	Step	The first of the three ranges of the core constant: cores of length $k \le 7$ are bounded by convexity	23-11
medium-core step	Step	The second of the three ranges of the core constant: cores of length $8 \le k \le 14$ are bounded by splitting into two blocks	23-11
long-core step	Step	The third of the three ranges of the core constant: box reduction is applied to cores of length $k \ge 15$	23-11
seven routes of the saddle tube	Route	The seven routes set up to close the core row of the saddle-tube condition	23-07
route of the chain outside the window	Route	Contraction of the chain outside the window ($q_free < 1$ for the bridge core, computer-assisted)	23-07
route of the window-constant numerics	Route	The remaining numerics (putting the site-window constants into intervals)	23-07
route of the sector framework	Route	The sector framework (five steps and the closing condition)	23-07
route of the sector input	Route	The small assumptions (the list of sector inputs)	23-07

Name	Kind	What it refers to	Introducing chapter
route of the window log-Sobolev inequality	Route	The route that attacks the one remaining proposition (the Window Log-Sobolev Threshold Proposition) with three strands	23-07
route of the seam of the ring	Route	The seam of the ring (outer chain and inner chain)	23-07
route of the range of ħ	Route	The range of ħ (outside the window and ħ→0, outside the scope)	23-07
componentwise strand	Route	The first strand of the route of the window log-Sobolev inequality: the componentwise form of the gradient (stops at 1.4–5.5 times the required value)	23-09
scalar strand	Route	The second strand of the route of the window log-Sobolev inequality: the scalar form (the Scalar-Strand Proposition)	23-09
Feynman–Kac strand	Route	The third strand of the route of the window log-Sobolev inequality: the Feynman–Kac form (Monte Carlo readings)	23-09
road of the core on the box side	Route	The road that supports the fork (the core Poincaré uniformity assumption) with the core on the box side: the Three-Range Core-Constant Proposition, and the local term vanishing as ħ→0	23-15
road of the core on the sector side	Route	The road that supports the fork (the core Poincaré uniformity assumption) with the core on the sector side: a uniform lower bound on the gap of the core in the sector framework, and the seam limit assumption	23-15
road of blocks and mixing	Route	A third branch that supports the fork (the core Poincaré uniformity assumption) with log-Sobolev inequalities for blocks ($\ell_B \geq 12$) and strong mixing (the fourteen rows of Table C1 of 23-15)	23-15
summit-curvature type	Type	A type of breakage of detour plans: there is a single source — the curvature of the single site at the summit	23-10
type of the ceiling on structural factors	Type	A type of breakage of detour plans: structural factors that enter as ratios have a ceiling	23-10
double-payment type	Type	A type of breakage of detour plans: the same place was being paid for twice	23-10
currency-mix-up type	Type	A type of breakage of detour plans: a mix-up of currencies — exit speed and rarity	23-10
type of the address of 1/ħ	Type	A type of breakage of detour plans: the address of 1/ħ (a mistake about where the factor 1/ħ enters)	23-10

Name	Kind	What it refers to	Introducing chapter
type of the misaligned fluctuation direction	Type	A type of breakage of detour plans: a mismatch between the direction of fluctuation and the soft direction	23-10
transfer type	Type	A type of breakage of detour plans: transfer — the weight only moves to another column	23-10
seven types of breakage	Table row	The types classifying into seven the reasons why detour plans broke. Used as questions to put to new candidates	23-10
detour plan "localising negative curvature" (short: localising negative curvature)	Detour plan	Localising Bochner's negative curvature on the defect set	23-10
detour plan "optimising the Gibbs tilt" (short: optimising the Gibbs tilt)	Detour plan	Setting the Gibbs tilt to $t^{*}=\ell-5.10$	23-10
detour plan "raising the threshold" (short: raising the threshold)	Detour plan	Raising the threshold to $1/\sqrt{3}$ and making the localisation remainder 0	23-10
detour plan "layering defects by depth" (short: layering defects by depth)	Detour plan	Dividing the defect depth b into layers	23-10
detour plan "narrowing the defect set" (short: narrowing the defect set)	Detour plan	Narrowing the defect set to the set where the row falls below κ	23-10
detour plan "defects as long clusters" (short: defects as long clusters)	Detour plan	Defining defects by clusters of length 8 or more	23-10
detour plan "paying for the summit band" (short: paying for the summit band)	Detour plan	Charging the summit band to the defects	23-10
detour plan "scalar payment" (short: scalar payment)	Detour plan	Paying for the sector defects in scalar form	23-10
detour plan "non-linear Barta weight" (short: non-linear Barta weight)	Detour plan	Making the Barta weight non-linear	23-10
detour plan "Agmon-type payment" (short: Agmon-type payment)	Detour plan	Paying only with the gradient of the true eigenfunction (Agmon type)	23-10
detour plan "re-examining downstream needs" (short: re-examining downstream needs)	Detour plan	Re-examining what is needed downstream	23-10

Name	Kind	What it refers to	Introducing chapter
detour plan "coarse and propagation steps at once" (short: coarse and propagation steps at once)	Detour plan	Doing the coarse step and the propagation step at once	23-10
detour plan "replacing the functional inequality" (short: replacing the functional inequality)	Detour plan	Poincaré / modified LSI / Φ-entropy instead of log-Sobolev	23-10
detour plan "log-Sobolev with clusters fixed" (short: log-Sobolev with clusters fixed)	Detour plan	Log-Sobolev for the conditional measure with the cluster state fixed	23-10
detour plan "perturbing only the summit band" (short: perturbing only the summit band)	Detour plan	Bakry–Émery outside the window, Holley–Stroock only for the summit band	23-10
detour plan "known uniform log-Sobolev" (short: known uniform log-Sobolev)	Detour plan	Known theorems on uniform log-Sobolev in one dimension	23-10
detour plan "finite-volume criteria" (short: finite-volume criteria)	Detour plan	Finite-volume criteria (Martinelli–Olivieri, Lu–Yau)	23-10
detour plan "Glauber log-Sobolev" (short: Glauber log-Sobolev)	Detour plan	Known log-Sobolev for Glauber dynamics	23-10
detour plan "two-step cluster expansion" (short: two-step cluster expansion)	Detour plan	Two-step cluster expansion, block spins	23-10
detour plan "extension to the classical limit" (short: extension to the classical limit)	Detour plan	Taking $\hbar$ below the window (the classical limit)	23-10
detour plan "one- or two-site windows" (short: one- or two-site windows)	Detour plan	Making the window one or two sites	23-10
detour plan "localising at the cluster core" (short: localising at the cluster core)	Detour plan	Localising at the core of the cluster rather than at the window	23-10
detour plan "Hölder and hypercontractive steps" (short: Hölder and hypercontractive steps)	Detour plan	Replacing the Gibbs step with Hölder and hypercontractivity	23-10

Name	Kind	What it refers to	Introducing chapter
detour plan "paying with a height function" (short: paying with a height function)	Detour plan	Paying for the negative curvature with a height function	23-10
list of detour plans	Table row	The twenty-four detour plans that moved either side, or the form, of the one remaining proposition of the route of the window log-Sobolev inequality	23-10
road of strong-coupling curvature	Table row	Widening the range of β where the curvature method works (the strong-coupling side)	23-02
road of one-step bookkeeping	Table row	Bookkeeping the continuum limit as a sequence of single steps	23-02
road of the sum of shortfalls	Table row	Whether the sum of the shortfalls δ(b) is finite	23-02
road of the strong-coupling series	Table row	Whether the dynamical exponent η can be read from the strong-coupling series	23-02
road of the dimension of complete families	Table row	In which dimensions complete families of edges and plaquettes exist	23-02
road of the dimensional threshold	Table row	Why four dimensions is the threshold	23-02
road of reflection positivity	Table row	Using reflection positivity for estimates from below	23-02
road of an area law in the bookkeeping	Table row	Whether the form of an area law appears in the bookkeeping	23-02
road of multi-link integrals	Table row	Whether the closed forms of k-link integrals are analytic	23-02
road of comparison with the defect of a submission	Table row	Whether the defect of a submission claiming a construction is the same quantity as δ	23-02
road of multiscale curvature	Table row	Stacking the curvature method scale by scale	23-02
road of volume and boundary	Table row	Whether the estimate of C_P is uniform in the volume and the boundary conditions	23-02
road of lower-bound test functions	Table row	Whether there is a test function giving the lower-bound assumption	23-02
road of gauge-fixed long wavelengths	Table row	Whether gauge-fixed long-wavelength modes can be made single-valued	23-02
road of spectroscopy	Table row	Bounding the gap with smoothed operators and spectroscopy	23-02
road of the lower half at one scale	Table row	Making the lower half of one scale, $m^2 \gtrsim \sigma$	23-02
road of a one-dimensional chain of scales	Table row	Bracketing the upper-bound assumption with a one-dimensional chain of scales	23-02

Name	Kind	What it refers to	Introducing chapter
seventeen roads tried on the four-dimensional lattice	Table row	The seventeen roads tried on the four-dimensional lattice (each a pair of one question and one set of tools)	23-02
one-sided test-function wall	Wall	Test functions bound C_P only from below	23-02
spectroscopic upper-bound wall	Wall	Spectroscopy ends with an upper bound on the gap	23-02
local-operator dimension wall	Wall	Local gauge-invariant operators have dimension 4 or more, so η_s=2	23-02
gauge-fixing single-valuedness wall	Wall	Gauge fixing: if sharp, it is multivalued; if blunt, the long wavelengths disappear	23-02
measurement-cost wall	Wall	The cost of measuring the numerator of smoothed operators is $\geq cL^{d+2}$	23-02
positive-weight kernel wall	Wall	A kernel with $\delta(2)=0$ has negative neighbouring weights	23-02
renewal-series wall	Wall	The strong-coupling series has renewal form and forces $\eta=0$	23-02
single-step integral wall	Wall	The one-link integral of a complete family works for one step only	23-02
limit-property wall	Wall	The Wilson-loop limit statement and the rotation-recovery statement do not follow from compositions of one-step maps	23-02
upper-bound tool wall	Wall	There are no tools of the type of the upper-bound assumption (they stop at the cross term)	23-02
tube-inequality wall	Wall	$m^2 \gtrsim \sigma$ has moved to the tube inequality, but there is no pattern of proof for it	23-02
projection-assumption wall	Wall	The projection assumption is not weaker than the relaxation-time assumption	23-02
checking-tool wall	Wall	There are few tools for machine checking	23-02
wall of the assumption itself	Wall	The relaxation-time assumption and the three chains have not been proved	23-02
equivalence-scaffold wall	Wall	The scaffold of equivalences for the physical-unit gap statement (power form, $\eta \geq 0$, additive step errors, δ of the one-loop kernel)	23-02
periodic-boundary conjecture wall	Wall	The breaking of the upper-bound assumption with periodic boundary conditions depends on a conjecture	23-02
fitting wall	Wall	That the violation of $m/\sqrt{\sigma}$ is $O(a^2)$ is a fit	23-02
box-size wall	Wall	The boxes are small	23-02
gauge dependence of the shortfall wall	Wall	Differences in c_2 depending on the choice of gauge	23-02

Name	Kind	What it refers to	Introducing chapter
tube-model sign wall	Wall	The sign of the 1/P coefficient of the tube model is undecided	23-02
primary-source wall	Wall	Missing primary sources	23-02
wall of principle	Type	One of the five types of walls: a wall that comes from the direction of an inequality and does not depend on the details of the tools	23-02
wall of structure	Type	One of the five types of walls: a wall where the requirements are incompatible within the chosen family, bookkeeping or series (outside the family it is undecided)	23-02
wall of tools	Type	One of the five types of walls: a wall where tools of the needed kind do not yet exist	23-02
wall of assumptions	Type	One of the five types of walls: a wall where the assumption relied on has itself not been proved, or depends on a conjecture	23-02
wall of measurement and literature	Type	One of the five types of walls: a wall coming from the sizes that can be measured and from missing primary sources	23-02
question "complete families of k=3 cells" (short: complete families of k=3 cells)	Question	Whether complete families of $k = 3$ cells exist in $d = 9$	23-17
question "largest dimension with a complete family" (short: largest dimension with a complete family)	Question	The largest dimension $d_{\max}(k)$ in which complete families exist	23-17
question "constant of the Hermite family" (short: constant of the Hermite family)	Question	The question of finding the constant in 1 − λ_k for the Hermite family	23-17
question "SU(N) version of the ratio on forests" (short: SU(N) version of the ratio on forests)	Question	The $SU(N)$ version, and a Lean statement, of the fact that the ratio of functions on forests saturates at $1/\lambda_1(G)$	23-17
question "C_P(β) on the two-dimensional lattice" (short: C_P(β) on the two-dimensional lattice)	Question	$C_P(\beta)$ in the link variables of two-dimensional lattice YM	23-17
question "curvature after one-link integration" (short: curvature after one-link integration)	Question	The best constant $c^* \in [21, 27]$ for the curvature after one-link integration	23-17

Name	Kind	What it refers to	Introducing chapter
question "the N^{−3} coefficient of the column model" (short: the N^{−3} coefficient of the column model)	Question	Deriving analytically the N^{-3} coefficient $C \approx 320$ of the column model (finite-size expansion of the XX chain)	23-17
question "sign of 1/P in the column model" (short: sign of 1/P in the column model)	Question	In the column model, the sign of $1/P$ at finite u	23-17
question "gauge dependence of the shortfall" (short: gauge dependence of the shortfall)	Question	How much the shortfall c_2 changes with the choice of gauge of the block transformation	23-17
question "C_P and ξ² of the O(3) σ model" (short: C_P and ξ² of the O(3) σ model)	Question	C_P and ξ^2 of the two-dimensional $O(3)$ σ model	23-17
question "C_P/ξ² of three-dimensional Z_2 gauge theory" (short: C_P/ξ² of three-dimensional Z_2 gauge theory)	Question	C_P/ξ^2 of three-dimensional $\mathbb{Z}_2$ gauge theory	23-17
question "volume monotonicity of C_P(V)" (short: volume monotonicity of C_P(V))	Question	Volume monotonicity of $C_P(V)$ and boundary conditions (computing all eigenvalues of the generator on small lattices)	23-17
question "dynamical exponent of four-dimensional SU(2)" (short: dynamical exponent of four-dimensional SU(2))	Question	Collecting primary sources on the dynamical exponent of local algorithms for four-dimensional $SU(2)$	23-17
question "interval arithmetic for the maximal correlation of the core" (short: interval arithmetic for the maximal correlation of the core)	Question	The question of bounding from above, by interval arithmetic, the two-point maximal correlation of medium cores ($8 \le k \le 14$)	23-17
question "power of the prefactor for the core and window" (short: power of the prefactor for the core and window)	Question	Reading the power of the prefactor $\hbar^{-p}$ of C_P for the core and the window from the capacity of degenerate points	23-17
question "comparison for the alternating maximal correlation of boxes" (short: comparison for the alternating maximal correlation of boxes)	Question	A comparison theorem showing the alternating maximal correlation between boxes $c_{\mathrm{alt}} < 1$ while avoiding examples that do not reduce to two-point quantities	23-17

Name	Kind	What it refers to	Introducing chapter
question "length set by a conservation-breaking term" (short: length set by a conservation-breaking term)	Question	Asks, when a small conservation-breaking term is added to a conservative dynamics, at what length the gap departs from the behaviour as the square of the side	23-17
row of common premises	Table row	Uniqueness of the $T=\infty$ measure, the Markov property, extension from local functions, the limiting step, regularity (the common premises)	23-11
row of interval-arithmetic premises	Table row	Premises of the interval arithmetic: correct rounding of the IEEE 754 arithmetic operations, errors of sums, errors of the self-written exponential function	23-07
summit-curvature shortfall wall	Table row	A row of Table 4 of 23-07: the curvature of the single site at the summit is short by 1/4 (a wall common to the three strands)	23-07
core lower-bound row	Table row	A row of Table 4 of 23-07: a uniform lower bound on the gap of the core	23-07
sector-framework row	Table row	A row of Table 4 of 23-07: the sector framework (five steps and the closing condition)	23-07
row of small assumptions	Table row	A row of Table 4 of 23-07: the small assumptions of the route of the sector input	23-07
window log-Sobolev threshold row	Table row	A row of Table 4 of 23-07: the one remaining proposition (the Window Log-Sobolev Threshold Proposition)	23-07
assumption lifting the proxy ratio	Table row	The lifting assumption of the Scalar-Strand Proposition (lifting the proxy ratio linearly beyond the measured range)	23-09
box-product inequality row	Table row	The box-product inequality (of Efron–Stein type)	23-05
row of the growth of the correlation length	Table row	The reading $\ln \xi \geq s/\hbar$ ($\xi \propto \hbar^{1/2} e^{S_0/\hbar}$)	23-05
kink-action row	Table row	A row of Table 3 of 23-04: the action S_0 of a single kink (paired with the zero-mode row)	23-04
zero-mode row	Table row	A row of Table 3 of 23-04: the translational zero mode of the kink (paired with the kink-action row)	23-04
next-eigenvalue row	Table row	A row of Table 3 of 23-04: the next eigenvalue 3/4 of the fluctuation operator around the kink	23-04

Name	Kind	What it refers to	Introducing chapter
Box-Dynamics Proposition (Propositions 3.4 and 3.5 of Martinelli's lectures) (short: Box-Dynamics Proposition)	Name from the literature	Box dynamics (the pattern that, for discrete spins, bounds the overall gap by the box dynamics and the gap inside the box)	23-05
Variance-Splitting Lemma (Lemma 2.4 of Menz–Schlichting) (short: Variance-Splitting Lemma)	Name from the literature	The formula for splitting the variance (the skeleton of the Poincaré version of Eyring–Kramers)	23-05
Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux) (short: Poincaré Version of Helffer's Criterion)	Name from the literature	The Poincaré version of Helffer's criterion (Proposition 3.1 of Bobkov–Ledoux 2000 is cited with the same number)	23-06
Integrated Bochner Formula (Proposition 1.3 of Ledoux) (short: Integrated Bochner Formula)	Name from the literature	The identity obtained by integrating Bochner's formula (used in sector composition to bound the Dirichlet form of the gradient by the curvature)	23-09
Holley–Stroock Perturbation Lemma (Proposition 6.1 of Ledoux) (short: Holley–Stroock Perturbation Lemma)	Name from the literature	A lemma that perturbs the Poincaré constant by the oscillation of the difference from a convex action (of Holley–Stroock type)	23-06
Single-Interface Diffusion Theorem (Theorem 2.1 of Bertini–Brassesco–Buttà) (short: Single-Interface Diffusion Theorem)	Name from the literature	The diffusion coefficient of a single interface	23-04
Two-Block Gibbs-Sampler Rate Theorem (Theorem 5 of Roberts–Sahu) (short: Two-Block Gibbs-Sampler Rate Theorem)	Name from the literature	The rate of convergence of the two-block Gibbs sampler	23-11
lattice correlation-length problem (Chatterjee's Problem 5.1) (short: lattice correlation-length problem)	Name from the literature	Chatterjee's Problem 5.1: at each β the lattice correlation length $\xi(\beta)$ is finite, and $\xi(\beta) \to \infty$ as $\beta \to \infty$	23-01, 16-07
continuum string-tension problem (Chatterjee's Problem 5.2) (short: continuum string-tension problem)	Name from the literature	Chatterjee's Problem 5.2: in the continuum limit of Wilson loops with a chosen lattice spacing, the string tension d is not 0	23-01, 16-07
angle graduation	Name	The Dirichlet form measured in the angles of the links	23-01
field graduation	Name	The Dirichlet form measured in the field $A = \sqrt{\beta}\,\vartheta$	23-01

Name	Kind	What it refers to	Introducing chapter
exponent of accumulation of the shortfall	Name	The exponent θ of the speed at which the shortfall accumulates (aξ ∝ a^θ)	23-01
one-step slope (symbol kept)	Name	The one-step slope $\kappa = 4Nb_0 = 11/(3\pi^2)$	23-02
one-step shortfall (symbol kept)	Name	The one-step shortfall. It vanishes as c_2/b^2	23-02
bookkeeping length	Name	The length ξ_RG = b^{k(β)} determined by the number k(β) of coarsening steps of the lattice. The lattice spacing is set to a := 1/ξ_RG	23-01
band exponent	Name	The exponent p when a^2C_P behaves as a power a^p of a (p = 0 is the band statement)	23-01
core log-Sobolev uniformity assumption	Condition	The log-Sobolev constant of the core is bounded uniformly	23-09
upper-side target row	Table row	A row of Table 3 of 23-04: the upper-side target $C_P \le K\xi^2 S_0/\hbar$ (the form of the conclusion of the First-Step Upper-Bound Proposition)	23-04
row of the numerical kink diffusion coefficient	Table row	A row of Table 3 of 23-04: the numerical kink diffusion coefficient $D/(\hbar/S_0) = 1.00–1.14$	23-04
row of the semiclassical leading term of the correlation length	Table row	A row of Table 3 of 23-04: the semiclassical leading term of the correlation length ξ (ξ ∝ ħ^{1/2} e^{S_0/ħ})	23-04
row of the lattice ln ξ	Table row	A row of Table 3 of 23-04: numerical values of the lattice ln ξ and the lattice kink action S_0^lat	23-04
row of the growth of the lower bound on R	Table row	A row of Table 3 of 23-04: the lower bound on the ratio $R = C_P/\xi^2$ grows from 1.06 to 6.5	23-04
dilute-gas accounting row	Table row	A row of Table 3 of 23-04: the accounting $C_P \approx \xi^2 S_0/\hbar$ obtained by counting kinks as a dilute gas	23-04
band-ratio row	Table row	A row of Table 3 of 23-04: the ratio to the component orthogonal to the magnetisation approaches 1	23-04
Glauber-eigenvalue row	Table row	A row of Table 3 of 23-04: the Glauber eigenvalue appears in the spectrum of the full state	23-04
row of the curvature of a convex core	Table row	A row of Table 3 of 23-04: for a convex core, $C_P \le 2/(3a^2-1)$	23-04
assumption that the maximum over searched starting points suffices	Assumption	The assumption that the maximum over the starting points searched by Monte Carlo suffices (it is not an upper bound over the whole fibre)	23-09

Name	Kind	What it refers to	Introducing chapter
assumption of the Barta row	Table row	The assumption of the inequality of the Barta row used in sector composition (lower bound $\kappa = 3/1000$ outside the defects, defect depth $b = 493/1000$)	23-09
eigenfunction-regularity assumption	Table row	The assumption, used in sector composition, that the Bochner boundary term vanishes, and of the regularity of the eigenfunction	23-09

Part 24 The first road, second half

Name	Kind	What it refers to	Introducing chapter
local diffusive-relaxation-time assumption (short: local relaxation-time assumption)	Name	The local Poincaré constant C_P^loc(c), with support restricted to a window c times the yardstick ℓ, has the same power as ℓ^2 (the upper side is C_P^loc(c) $\leq$ K(c)ℓ^2). Stated in the ratio form K_rel	24-03
sector diffusive-relaxation-time assumption (short: sector relaxation-time assumption)	Name	A statement of the diffusive-relaxation-time assumption for the measure conditioned on each topological sector (value of the charge) (the sector Poincaré constant C_P^sec(n))	24-03
local-excitation moment assumption	Assumption	The moment ratio of the spectral measure of local observables supported in a spatial box satisfies $M_3/M_1 \leq m_\infty^2 + \gamma/(c\ell)^2$	24-04
alternating maximal-correlation assumption for four-dimensional boxes	Assumption	The alternating maximal correlation between families of closed four-dimensional boxes is less than 1, uniformly in β	24-03
gap assumption for closed four-dimensional boxes	Assumption	For closed four-dimensional boxes, the relaxation time of a box satisfies C_box(cξ_RG) $\leq$ K'ξ_RG^2 (ξ_RG is the bookkeeping length)	24-03
infinite-volume ground-state transformation identity	Assumption	The assumption that the ground-state transformation identity of the lattice with continuous time (the relation between the gap and the vacuum measure) holds also in infinite volume	24-04
Window-Factor Monotonicity Proposition	Proposition	$g(x) = \phi(x)/x^2$ ($\phi(x) = 1 - (1-e^{-x})/x$) is strictly decreasing for $x > 0$ (the window factor decreases)	24-04
In-Window Supremum Proposition	Proposition	Under the assumptions of the one-sided bridge and the canonical graduation, writes in closed form, via the root of a transcendental equation, the supremum over the family local in time with weights supported in the window [0,T]	24-04

Name	Kind	What it refers to	Introducing chapter
Gap and Vacuum Measure Proposition	Proposition	On a lattice with continuous time, the gap is exactly the reciprocal of the Poincaré constant of the vacuum measure at a single time (ground-state transformation)	24-05
Electric–Magnetic Splitting Frustration Proposition	Proposition	With the splitting into electric and magnetic parts, the Kogut–Susskind Hamiltonian is not frustration-free	24-05
Local Poincaré Constant Properties Proposition	Proposition	The local Poincaré constant C_P^loc(c) is determined only by the marginal law of the window; D_c is not dense; it is monotone in c and returns to C_P as c→∞	24-03
Sector Poincaré Constant Proposition	Proposition	The sector Poincaré constant C_P^sec(n) is finite if the sector is connected; the variance splits into within-sector and between-sector parts, and can be bracketed with C_trans	24-03
Convexity Ceiling on Groups Lemma (short: Convexity Ceiling Lemma)	Lemma	For measures on closed manifolds, the lower bound of the Bakry–Émery curvature has a ceiling	24-06
No Concentration at the Identity Corollary	Corollary	On SU(2), a convex measure cannot concentrate at the identity	24-06
One-Link Wilson Weight Proposition	Proposition	The range of convexity of the one-link Wilson weight (at weak coupling, the non-convex region is pushed into a region of small probability)	24-06
rule of local convexity	Rule	One of the candidate rules for a class of measures that pay with rarity: the measure is locally convex on the good set	24-06
rule of rarity of bad sets	Rule	One of the candidate rules for a class of measures that pay with rarity: the non-convex bad set is rare	24-06
rule of return drift	Rule	One of the candidate rules for a class of measures that pay with rarity: there is a drift returning from the bad set to the good set	24-06
rule of paying for entropy	Rule	One of the candidate rules for a class of measures that pay with rarity: the rarity of the bad set is paid for with an entropy inequality	24-06
rule of closing under coarse-graining	Rule	One of the candidate rules for a class of measures that pay with rarity: the estimates close on the coarse-grained variables	24-06

Name	Kind	What it refers to	Introducing chapter
source: dimensionless coupling	Question	One of the sources of difficulty: the coupling is dimensionless, all scales act with the same weight, and infinitely many steps must be controlled uniformly in the number of steps	24-07
source: no expansion linking scales	Question	One of the sources of difficulty: there is no expansion linking the ultraviolet and the infrared	24-07
source: no global gauge fixing	Question	One of the sources of difficulty: because of non-abelian gauge invariance, there is no continuous global gauge fixing	24-07
three sources of difficulty	Question	The three sources of the difficulty of four-dimensional Yang–Mills (dimensionless coupling, no expansion linking scales, no global gauge fixing)	24-07
rule of axiomatising scale	Table row	A candidate rule in Table 2 of 24-08: treating scale as an axiom (axiomatising scale)	24-08
rule of the loop algebra	Table row	A candidate rule in Table 2 of 24-08: rules of the algebra of Wilson loops	24-08
rule of conserved quantities and sectors	Table row	A candidate rule in Table 2 of 24-08: rules for conserved quantities and the sectors divided by them	24-08
rule of the rarity class	Table row	A candidate rule in Table 2 of 24-08: the class of measures that pay with rarity	24-08
rule of an asymptotic-freedom axiom	Table row	A candidate rule in Table 2 of 24-08: a rule that sets asymptotic freedom as an axiom	24-08
curvature rule	Table row	A candidate rule in Table 2 of 24-08: a rule on the Bakry–Émery curvature on the space of gauge orbits	24-08
question "comparing local excitations of the free field" (short: comparing local excitations of the free field)	Question	The question of comparing energies of local excitations in the free field	24-08
question "monotonicity of the window factor" (short: monotonicity of the window factor)	Question	A question about the monotonicity of the window factor $\phi(x)/x^2$	24-08
question "sign of the Polyakov loop" (short: sign of the Polyakov loop)	Question	Asks whether, with the sign of the Polyakov loop in boxes of small period, a static ratio captures the slowness of relaxation	24-08
question "U(1) gap on two-dimensional layers" (short: U(1) gap on two-dimensional layers)	Question	Asks about the gap of U(1) on two-dimensional layers when the strip is lengthened	24-08

Name	Kind	What it refers to	Introducing chapter
question "data on topological-charge autocorrelation" (short: data on topological-charge autocorrelation)	Question	The question of collecting primary sources on the autocorrelation of the topological charge of SU(2)	24-08
question "many-family version of the Two-Projection Lemma" (short: many-family version of the Two-Projection Lemma)	Question	Asks for a many-family version of the Two-Projection Lemma (families of 2^d colours of d-dimensional boxes) and its condition on the maximal correlation	24-08
question "Hessian of the one-link weight" (short: Hessian of the one-link weight)	Question	Asks about the Hessian of the one-link Wilson weight convolved once with the heat kernel, and how the parameters of the class are mapped	24-08
question "time-step comparison for the transfer operator" (short: time-step comparison for the transfer operator)	Question	Asks whether the time-step comparison assumption holds for the Wilson transfer operator	24-08
question "Lean for τ_int and C_P" (short: Lean for τ_int and C_P)	Question	The question of writing τ_int ≤ C_P and the ground-state transformation identity in Lean	24-08
question "rereading the modes of CP^{N−1}" (short: rereading the modes of CP^{N−1})	Question	The question of rereading the local modes and topological modes of the CP^{N−1} model	24-08
question "loop families of two-dimensional SU(2)" (short: loop families of two-dimensional SU(2))	Question	The question of finding λ_W and Γ for loop families of two-dimensional SU(2)	24-08
question "values of large four-dimensional loops" (short: values of large four-dimensional loops)	Question	The question of finding the values for families of large loops in four-dimensional boxes	24-08
question "reduction to kink positions" (short: reduction to kink positions)	Question	The question of reducing the double-well problem to the dynamics of kink positions	24-08
question "the single point β=7 of the band" (short: the single point β=7 of the band)	Question	The question of measuring the single point β = 7 of the band	24-08
question "mass in the band between the threshold and 1" (short: mass in the band between the threshold and 1)	Question	The question of building another argument that sees the mass in the band between the threshold and 1	24-08

Name	Kind	What it refers to	Introducing chapter
question "window bridge on a time-step lattice" (short: window bridge on a time-step lattice)	Question	The question of making a version of the In-Window Supremum Proposition (the window bridge) on a lattice with discrete time	24-08
question "frustration-free sector splitting" (short: frustration-free sector splitting)	Question	Asks whether a sector splitting of the Kogut–Susskind Hamiltonian can be frustration-free	24-08
question "extending the CP^{N−1} test bed" (short: extending the CP^{N−1} test bed)	Question	The question of extending the CP^{N−1} test bed by adding seeds and values of N	24-08
question "conservation length of subregion charge" (short: conservation length of subregion charge)	Question	Asks at what length conservation breaks when measured with the charge of a four-dimensional subregion	24-08
type of a nearly conserved global quantity	Type	The type in which a global quantity not reflected in the gap slows relaxation at a length different from the correlation length (three conditions: the variance of the quantity is large, its Dirichlet form is small, and it is not reflected in the gap)	24-01
Infrared Asymptotic-Freedom Proposition (Proposition 6.1.7 of BBS) (short: Infrared Asymptotic-Freedom Proposition)	Name from the literature	The one-step estimate of the large-field weight norm; infrared asymptotic freedom	24-06
Covariance-Compression Lemma (Lemma 2.2 of Hairer–Stuart–Voss–Wiberg) (short: Covariance-Compression Lemma)	Name from the literature	An expression for the operator norm of a covariance compressed to a subspace	24-03

Part 25 The second road — from the direction of space

Name	Kind	What it refers to	Introducing chapter
five moves of the proof	Name	The five steps of the proof of the Slab Confinement Theorem	25-04
Moving-Surface Lemma	Lemma	Flipping the signs of all plaquettes containing a link leaves the partition function unchanged, the surface Π_q can be moved, and the partition function is determined only by the parity of the number of surfaces	25-01
definition of the sector and flux free energies	Name	The definitions $\Delta F = \ln(Z(0)/Z(1))$ and $\theta = \tanh(\Delta F/2)$	25-01

Name	Kind	What it refers to	Introducing chapter
Parity Representation of the Bias	Proposition	$\theta = E[(-1)^{\wedge}\lvert Q\rvert]$ (the expectation of the parity of the number of surfaces in the ensemble summed over twists)	25-01
Zero-Action Configuration Lemma	Lemma	$S^{\wedge}z=0 \iff U_p = z_p \cdot 1$ for all plaquettes	25-01
Lemma on the Sign of the Holonomy Commutator	Lemma	If all plaquettes take central values, the commutator of the holonomies around the two directions is the product of the signs of that plane	25-01
Determination-by-Holonomy Lemma	Lemma	A configuration with central values is determined, up to gauge, by the four holonomies at the base point	25-01
Lemma on the Configuration that Absorbs the Twist	Lemma	U^{0} has action 0 in the sector (1,0) ($\Omega_1\Omega_2\Omega_1^{-1}\Omega_2^{-1} = -1$)	25-01
Rigidity Lemma for Anticommuting Pairs	Lemma	A pair with $ABA^{-1}B^{-1} = -1$ is conjugate to $i\sigma_1$, $i\sigma_2$, and the only elements commuting with both are ± 1	25-01
Uniqueness of the Twisted Vacuum	Proposition	The configuration of action 0 in the sector (1,0) is U^{0} alone, up to gauge and centre transformations in the plane directions	25-01
Twisted Adjoint Gap Lemma	Proposition	The smallest eigenvalue of the adjoint Laplacian around U^{0} is $4\sin^2(\pi/2N) > 0$ (there are no zero modes)	25-01
Positivity of the Minimal Action in the Twisted Sector	Proposition	There is no configuration of action 0 in the sector (1,1), and $0 < S_{min} \leq 2N^2$	25-01
Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3) (short: Confinement Lower Bound by Fractional Instantons)	Theorem	For fixed $N \geq 2$, $\ell \geq 2$, $\beta \geq 1$, $-\ln \theta \geq c\, \beta^{\{-a\}}\, e^{\{-\beta(S_{*}+\varepsilon(\beta))\}}\, M^2$ (bounding the flux free energy from below by the action of fractional instantons)	25-02
Route to Existence by the Nahm Transform	Proposition	A route that, assuming equivariance of the Nahm transform, gives the existence of self-dual solutions of charge 1/2 on twisted $T^2 \times R^2$	25-02
Container Identity	Lemma	With the frame fixed, the containers become independent, and $\lvert\langle W\rangle\rvert \leq E_{\Phi}[\prod \theta_j]$, $\theta_j = \tanh(\lvert\Delta F_j\rvert/2)$	25-03
$\sigma_{avg}(\ell)$	Name	The estimate of the string tension of two-dimensional SU(2) obtained from the Container Identity (frame average) when tiling with containers of side ℓ	25-03

Name	Kind	What it refers to	Introducing chapter
Inner and Outer Container Proposition	Proposition	If containers are placed inside and outside the loop, the Wilson loop in a fixed sector is bounded by the product of the container biases and an outer term	25-03
Nesting Inequality in the Plane Directions	Proposition	The bias of a large container is bounded by the average of the product of the biases of the smaller containers into which it is split by the inner frame	25-03
assumption on the form of the thick-container free energy ((23) of Kovács–Tomboulis) (short: assumption on the form of the thick-container free energy)	Name from the literature	The free energy of a thick container has the form $\lvert V\rvert\, e^{-cd^2}$, uniformly in the boundary values	25-03
Slab Confinement Theorem	Theorem	With N and the lattice spacing fixed, for all β, $\lvert\langle \tfrac{1}{2}\mathrm{tr}\, U_C\rangle\rvert \le \tanh(\beta N^2)^A$ and $\sigma \ge -\ln\tanh(\beta N^2)$	25-04
finite configuration-space condition	Condition	One of the three conditions of the insertion identities: the configuration space X is a finite set	25-05
positive-weight condition	Condition	One of the three conditions of the insertion identities: the weight $w : X \to (0,\infty)$ is positive	25-05
sector-flipping bijection condition	Condition	One of the three conditions of the insertion identities: there is a bijection $T : X \to X$ that flips the sector sign $\sigma : X \to \{\pm 1\}$ ($\sigma(Tx) = -\sigma(x)$)	25-05
kernel moving only inside the container	Condition	A condition of driven insertion: the kernel updates only the variables inside the container and does not change the frame	25-06
Mean Insertion-Ratio Identity	Lemma	$E[\rho] = 1$, $E[\rho\sigma] = -E[\sigma]$	25-05
Bias Estimate by Insertion-Ratio Fluctuation	Theorem	$4E[\sigma]^2 \le \mathrm{Var}(\vec{\rho})$	25-05
Bounded Insertion Identity	Lemma	For an involution τ and an odd H, $E[H] = E[H\tanh(\Delta S/2)]$	25-05
Insertion Fluctuation Relation Lemma	Lemma	$P(\Delta S{=}s) = e^{s}\, P(\Delta S{=}{-}s)$, $E[t] \ge 0$, and $1-E[t]$ is twice the acceptance rate	25-05
Position-Averaged Insertion Estimate	Theorem	Bounds $\lvert E[H]\rvert$ by the product of the tanh factors averaged over positions	25-05
Square-Root Insertion Identity	Lemma	$E[H\sqrt{\rho}] = 0$	25-05
Paley–Zygmund-Type Bias Estimate	Theorem	$E[H]^2 \le 1 - E[Y]^2/E[Y^2]$, $E[Y^2] \le 1$	25-05

Name	Kind	What it refers to	Introducing chapter
Many-Cell Second-Moment Estimate	Theorem	Under "odd under insertion", $4^{\lvert C\rvert}\,\mathbb{E}[H]^2 \le \mathbb{E}[\prod(1-r_c)^2]$	25-05
assumption that composed insertions are odd	Assumption	For all $S \subseteq C$, $\mathbb{E}[H \prod_{c\in S} r_c] = (-1)^{\lvert S\rvert}\,\mathbb{E}[H]$	25-05
Crooks Relation	Lemma	Under the path involution ι, the ratio of weights is e^{-W}, and $W\circ\iota = -W$	25-06
Driven Insertion Identity	Theorem	$\mathbb{E}[H] = \mathbb{E}_{\Omega}[H \tanh(W/2)]$	25-06
Jarzynski Equality	Corollary	$\mathbb{E}[e^{-W} \mid \Phi, \sigma=+] = Z_-/Z_+$	25-06
Driven Insertion Acceptance-Rate Corollary	Corollary	$1 - \mathbb{E}[\tanh(W/2)] = 2\times$ the Barker-type acceptance rate	25-06
Many-Cell Driven Estimate	Theorem	Bounds by the product of the position averages of $\lvert\tanh(W/2)\rvert$ of driven insertions over many cells	25-06
Cut-Measure Formula	Proposition	$Z_{\pm}/Z_{\mathrm{cut}} = \mathbb{E}_{\mathrm{cut}}[e^{\pm\beta X/2}]$, $\theta = \lvert E\sinh\rvert/E\cosh$	25-06
Flip-Probability Lemma	Theorem	For a stationary kernel, the probability of a flip in one step is $\le 1-\theta^2$	25-06
Flip-Probability Corollary for Positive Kernels	Corollary	For a reversible positive kernel, $(1-\theta^2)/2$	25-06
Flow Lemma	Theorem	For the one-step flip probability a of a stationary kernel, $\theta \le 1-a$	25-06
Simultaneous-Flip Estimate	Theorem	$\mathbb{E}\prod\theta_t \le P(\text{no container flips})$	25-06
Transport Identity	Proposition	$\Delta F = \mathbb{E}_+[S\circ T - S - \ln J_T] - \mathrm{KL}$ (the transport identity and the Gibbs variational principle)	25-07
Isometric-Map Loss Proposition	Proposition	The loss of a map isometric in the fluctuations is $\tfrac{1}{2}\Sigma(\lambda-1-\ln\lambda)$, which is not diluted by the degrees of freedom	25-07
Fluctuation-Shaping Proposition	Proposition	The optimal value for maps of determinant 1 is $(n/2)(r^{1/n}-1)$, with loss $\le (\Delta F^2/n)e^{2\lvert\Delta F\rvert/n}$	25-07
First-Order Vanishing of the Spin Term	Proposition	The spin term does not appear in the first-order trace (at one-loop level)	25-07
Isometric versus Shaping Comparison Proposition	Proposition	(a) The isometric comparison gives $\beta' \ge \beta+0.023$; (b) the shaping comparison gives $\beta-\beta' = 0.258$	25-07
Relative-Entropy Obstruction to the Area Law	Proposition	If the remaining $\mathrm{KL} \ge c'A$, the area law does not follow from the variational argument	25-07
Additivity of Relative Entropy over Containers	Proposition	For a map that is a product over containers, the remaining KL is the sum over containers	25-07

Part 26 The crossover band

Name	Kind	What it refers to	Introducing chapter
Hidden-Convexity Proposition	Proposition	On the box [−c,0]^l the action has exactly one local minimum, and it is global	26-07
jump step	Name	The first step of the proof of the Conditional Reduction Theorem: from frame decorrelation and the base point at small T^2, jump to the first bias by the lateral-widening inequality	26-02
doubling step	Name	The second step of the proof of the Conditional Reduction Theorem: doubling the container and the transverse T^2 at each step, show that the bias satisfies $t_{k+1} \le K\, t_k^4$	26-02
iteration step	Name	The third step of the proof of the Conditional Reduction Theorem: iterating $t_{k+1} \le K\, t_k^4$ gives that the bias becomes small with an area exponent	26-02
Tension Lower Bound from the Naive Form	Proposition	The lower bound on the tension $\ln(1/\theta_0)/L^2$ obtained by container estimates from the naive form of the band $\theta \le \theta_0$ goes to 0	26-01
Area Law from the Area Exponent	Proposition	If $\theta \le e^{-cL^2}$, then $\lvert\langle W\rangle\rvert \le C_D\, e^{-cA}$	26-01
Tension Lower Bound from Tomboulis–Yaffe	Corollary	Under Tomboulis–Yaffe, if $F_\nu \le e^{-\rho\, L_\mu\, L_\nu}$, then the tension is $\ge \rho$	26-01
naive form of the band	Name	The naive form of the band: the sector bias stays away from 1 uniformly in the scale ($\sup \theta(L,L) \le \theta_0 < 1$). From this, container estimates give no positive lower bound on the tension	26-01
correct form of the band	Name	The correct form of the band: the sector bias becomes small with an area exponent ($\ln(1/\theta(L,L)) \ge cL^2$)	26-01
exclusion of a deconfined phase	Condition	One of the intermediate milestones toward the band: the vortex free energy $F_\nu(L,L)$ tends to 0 (excluding a non-confining phase)	26-01
almost area law	Condition	One of the intermediate milestones toward the band: $\ln(1/\theta) \ge cL^2/\ln L$ (the area exponent up to a logarithmic loss)	26-01
base point at small T^2	Assumption	One of the assumptions of the reduction theorem: at some physical scale L_0, the sector bias on a small T^2 satisfies $\theta(L_0;T^2_{L_0}) \le 1-\varepsilon_0$	26-02

Name	Kind	What it refers to	Introducing chapter
frame decorrelation	Assumption	One of the assumptions of the reduction theorem: multiplying the container by s raises the bias to the power s^2, up to a constant ($\theta(sR;T^2_L) \le C_D\, \theta(R;T^2_L)^{s^2}$)	26-02
lateral-widening inequality (strong form)	Assumption	One of the assumptions of the reduction theorem: with the container fixed, widening the transverse T^2 by a factor L'/L increases the bias by at most a factor $C(L'/L)^2$	26-02
lateral-widening inequality (weak form)	Assumption	The lateral-widening inequality weakened to a form that may lose, at each step doubling the transverse size, a fraction ε_k of $\ln(1/\theta)$ (with finite sum) and a constant	26-02
Conditional Reduction Theorem	Theorem	Under the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form), $\ln(1/\theta(R_k,R_k)) \ge \gamma R_k^2$ (the bias becomes small with an area exponent)	26-02
Reduction Theorem in the Weak Form	Theorem	Under frame decorrelation, the weak form and a condition on the size of the base point, $u_k \ge \gamma' R_k^2$	26-02
Slab-Tension Monotonicity Proposition	Proposition	If the lateral-widening inequality (strong form) holds, the effective tension σ_eff does not decrease in L (the reason the strong form is stronger than confinement)	26-02
Reduction Read in the Chessboard Density	Corollary	Read in the chessboard density κ_cb, the same reduction goes through with nesting constant 1 (under the chessboard-density base point and the lateral comparison)	26-02
chessboard-density base point	Assumption	The assumption of the base point at small T^2 read in the chessboard density: $\kappa_cb(L_0;L_0) \le 1-\varepsilon_0$	26-02
lateral comparison of the chessboard density	Assumption	The assumption of the lateral-widening inequality (strong form) read in the chessboard density: $\kappa_cb(R;L') \le C(L'/L)^2\, \kappa_cb(R;L)$	26-02
base-point translation inequality	Condition	The single line linking the bias at the base point and the chessboard density: $1-\kappa_cb(L_0;L_0) \ge \eta(1-\theta(L_0;L_0))$	26-02
Flux-Equalisation Lemma	Lemma	$\int_\Omega f^2 \ge \Phi^2/\lvert\Omega\rvert$, with equality for constants (Cauchy–Schwarz)	26-03

Name	Kind	What it refers to	Introducing chapter
Contrapositive of the Reduction	Proposition	If the conclusion of the reduction theorem is false, one of the three assumptions is false (contrapositive)	26-03
Electric-Flux Decomposition Lemma	Lemma	$Z(1,0) = Y(0)+Y(1)$, $Z(1,1) = Y(0)-Y(1)$, $\theta = Y(1)/Y(0) \geq 0$	26-04
positivity assumption for the twisted partition function	Assumption	The twisted partition function $Z(1,1)$ is at least 0 (positivity of the Boltzmann weight)	26-04
tanh Comparison Lemma	Lemma	If $F' \leq KF$ ($K \geq 1$), then $\tanh(F'/2) \leq K \tanh(F/2)$	26-04
Reflection-Positivity Theorem for z (short: reflection positivity of z)	Theorem	The z ensemble with the twist fixed is positive under reflections through lattice sites in directions 3 and 4 (the row "reflection positivity of z" in the ledger of conditions of 27-02 also refers to this theorem)	26-04, 27-02
Lemma on the Non-Positivity of Lateral Reflections	Lemma	The weight of a twisted plaquette has negative coefficients at half-integer j, so lateral reflections are not of positive type	26-04
Lemma that Structural Inputs Alone Do Not Suffice	Lemma	The set of structural inputs ☒ also holds for Z_2 and U(1), so the lateral-widening inequality does not follow from it alone	26-04
set of structural inputs	Name	The set consisting of positivity of the transfer matrix, the centre and duality identities, reflection positivity, the lattice symmetries and finite lemmas	26-04
thin-vortex floor	Name	The strong form reaching down to the floor of slab confinement: $\theta \leq \tanh(N^2\|\beta\|)^{M^2}$	26-04
Chessboard Estimate (Fröhlich–Israel–Lieb–Simon) (short: Chessboard Estimate)	Name from the literature	For a reflection-positive measure, $\|\langle\prod f_t\rangle\| \leq \prod\langle\prod f_t^{(s)}\rangle^{1/\|T\|}$	26-05
Chessboard Estimate with the Lateral Size Fixed	Corollary	$E\prod_{t\in S}\theta_t \leq \kappa_{cb}^{\|S\|}$, $\|\langle W\rangle\| \leq \kappa_{cb}^{n}$	26-05
Nested Supermultiplicativity Lemma	Lemma	$\kappa_{cb}(sR;L) \leq \kappa_{cb}(R;L)^{s^2}$ (constant 1, all L)	26-05
$\kappa_{cvx}(k)$	Name	The convexity constant at the top of a chain of length k ($\kappa(k)$ of the Short-Core Convexity Lemma of 23-11)	26-07
β-monotonicity	Condition	With the lattice geometry fixed, $\partial_\beta \Delta F \geq 0$	26-06
renormalisation comparison	Condition	$\Delta F(2N,2M,\beta) \leq K\cdot\Delta F(N,M,\beta')$, $\beta' < \beta$	26-06

Name	Kind	What it refers to	Introducing chapter
rate form of the renormalisation comparison	Name	$\sigma_N(\beta)$, $s(N,\beta) = N^2\sigma_N(\beta)$, and the rate form $s(N,\beta') \le s(2N,\beta)$	26-06
Strong-Coupling-End Expansion Proposition	Proposition	For $u \le u_0 \approx 10^{-80}$, $\Delta F = 2N^2 e^{-\sigma_N M^2}(1+\varepsilon)$	26-06
Comparison at the Strong-Coupling End	Corollary	At the strong-coupling end, choosing $\sigma_N(\beta') = 4\sigma_{2N}(\beta)$ gives $\Delta F(2N,2M,\beta)/\Delta F(N,M,\beta') = 4(1+O(u))$	26-06
assumption of two-sided area laws inside the band	Assumption	An assumption combining, on the band I, the two-sided area law at fixed β (with errors of the order of the perimeter) and the uniform positivity and continuity of the rate	26-06
two-sided area law at fixed β	Assumption	The assumption that at each β in the band I the vortex free energy is bounded above and below by the area rate $\sigma_N(\beta)$ (with errors of the order of the perimeter)	26-06
uniform positivity and continuity of the rate	Assumption	The assumption that the rate $\sigma_N(\beta)$ is uniformly positive on the band and uniformly continuous in β (for nearby β, $\sigma_N(\beta') \le \sigma_{2N}(\beta) + \sigma_*$)	26-06
Lemma Reducing the Comparison inside the Band	Lemma	Under the assumption of two-sided area laws inside the band, the renormalisation comparison inside the band holds with a constant K	26-06
Corollary Crossing the Band in Finitely Many Steps	Corollary	The band can be crossed in $\lceil(\beta_b - \beta_a)/\delta\rceil$ steps, with a loss independent of N and M	26-06
criterion "distinguish from the U(1) transition"	Condition	Show that SU(2) has no transition, distinguishing it from the transition of U(1) inside the band	26-06
tension-doubling inequality	Condition	$4\sigma_{2N}(\beta) \ge e^{\delta_0}\sigma_N(\beta)$ (for all $N \ge N_1$): the ratio of the tensions when the box is doubled exceeds 1 uniformly	26-06
softness in the twist direction	Name	One of the four faces of the middle band: the variance of the force along the generator of the twist within a band of width w (adapted stiffness)	26-06
mixing of box magnetic flux	Name	One of the four faces of the middle band: the total variation distance of the law of the inner cube when the central magnetic flux on the boundary of the box is changed	26-06

Name	Kind	What it refers to	Introducing chapter
twist cost of a column	Name	One of the four faces of the middle band: the twist-conditioned free energy and bias of the column $\mathrm{T}^2_\mathrm{N} \times \ell \times \ell$ with the outer frame fixed	26-06
pressure margin	Name	One of the four faces of the middle band: the difference in pressure $p_{\infty} - p_{G,L}$ between systems with and without the twist pattern inserted (the exponent of the chessboard density κ_{cb})	26-06
interpolation-parameter commonality inequality ((3.5) of Ito–Seiler) (short: interpolation-parameter commonality inequality)	Name from the literature	The inequality that reduces to A(α) > A⁺(α) the claim that the Migdal–Kadanoff interpolation parameter t* can be taken in common for two sectors	26-06
candidate "reduction by compactness of the band" (short: reduction by compactness of the band)	Name	One of the candidates for the band: since the band is a bounded closed interval of β, reduce the comparison inside the band to the two-sided area law at fixed β at each β, and cross in finitely many steps	26-10
candidate "log-concavity of the Weyl measure" (short: log-concavity of the Weyl measure)	Name	One of the candidates for the band: using the log-concavity of the density obtained by projecting the uniform measure on the sphere onto one coordinate, and a height–azimuth decomposition, bring it to a conditional ferromagnetic model	26-10
candidate "local step function of a column" (short: local step function of a column)	Name	One of the candidates for the band: replace the global renormalisation comparison by a comparison of the twist free energies of columns with the outer frame fixed (the local step function of a column)	26-10
candidate "confinement as sign cancellation" (short: confinement as sign cancellation)	Name	One of the candidates for the band: write θ = E[tanh(X/2)] in the measure summed over twists, and view the bias as becoming small through cancellation of the sign of X	26-10
candidate "centre-only influence coefficients" (short: centre-only influence coefficients)	Name	One of the candidates for the band: the influence coefficient $\delta_{\mathcal{C}}(L_0, \beta)$ when the outside is conditioned only on its SO(3) part and only the central flux on the boundary is flipped	26-10

Name	Kind	What it refers to	Introducing chapter
candidate "flux diffusion by monopole flow" (short: flux diffusion by monopole flow)	Name	One of the candidates for the band: since the sign flux of each plane changes with the parity of the closed $\mathbb{Z}_2$ monopole curves crossing between them, view the flux as being mixed by the winding density of monopoles	26-10
candidate "determinant in self-dual coordinates" (short: determinant in self-dual coordinates)	Name	One of the candidates for the band: in each sector, look at the difference between sectors of the Jacobian when the measure is viewed as a pushforward to its anti-self-dual part	26-10
candidate "quantitative 1-form symmetry of the slab" (short: quantitative 1-form symmetry of the slab)	Name	One of the candidates for the band: read slab confinement as the lattice version of "a discrete 1-form symmetry is not broken in two dimensions", and make its quantitative version uniform in N	26-10
stopped road "isometric comparison" (short: isometric comparison)	Table row	A road stopped in the band chapters: an isometric comparison that does not change the shape of the fluctuations (the one-loop spin term becomes 0, and only the anti-asymptotic-freedom direction comes out)	26-08
stopped road "structural inputs only" (short: structural inputs only)	Table row	A road stopped in the band chapters: showing it with structural inputs only — reflection positivity, duality, the transfer matrix and the centre identities (these also hold for $\mathbb{Z}_2$ and U(1), so they cannot distinguish groups)	26-08
stopped road "deterministic squeeze" (short: deterministic squeeze)	Table row	A road stopped in the band chapters: obtaining the area law by a squeeze with deterministic maps (the remaining relative entropy grows with the area)	26-08
stopped road "glassy squeeze" (short: glassy squeeze)	Table row	A road stopped in the band chapters: a squeeze with the $\mathbb{Z}_2$ glass in the upper half of the band (for $\beta \gtrsim 2$ the glass is on the ordered side and the squeeze is loose)	26-08
stopped road "flip certificate" (short: flip certificate)	Table row	A road stopped in the band chapters: using container flip certificates and simultaneous flips to test the band (the flip probability saturates at about 1/2, and only the weak form comes out)	26-08
stopped road "supremum estimate over frames" (short: supremum estimate over frames)	Table row	A road stopped in the band chapters: estimates using the supremum over frames (they get super-exponentially worse in L, and for uniformity only the frame-averaged form can be used)	26-08

Name	Kind	What it refers to	Introducing chapter
stopped road "Dobrushin-type criterion" (short: Dobrushin-type criterion)	Table row	A road stopped in the band chapters: using the criteria of Dobrushin, Dobrushin–Shlosman and van den Berg–Maes with frames (continuous fields follow the boundary and the influence coefficients are nearly 1)	26-08
stopped road "β-monotonicity from easy parts" (short: β-monotonicity from easy parts)	Table row	A road stopped in the band chapters: treating β-monotonicity as an easy part and splitting the band into β-monotonicity and the renormalisation comparison (β-monotonicity turned out to be as heavy as monotonicity of the tension)	26-08
stopped road "one-directional lateral widening" (short: one-directional lateral widening)	Table row	A road stopped in the band chapters: restricting lateral widening to one direction (doubtful given the behaviour of the high-temperature spatial string tension; restricted to square shapes only)	26-08
stopped road "transport by Wilson flow" (short: transport by Wilson flow)	Table row	A road stopped in the band chapters: using Wilson flow to transport vortex insertions (numerically negative; a form recast as forward transport remains open)	26-08
stopped road "conformal frame" (short: conformal frame)	Table row	A road stopped in the band chapters: turning scale into position with the conformal frame $H^2 \times S^2$ (with hyperbolic nesting the number of containers does not grow, and the crux remains in the same place)	26-08
stopped road "strong-coupling expansion only" (short: strong-coupling expansion only)	Table row	A road stopped in the band chapters: covering the band with the strong-coupling expansion alone (the flat-surface expansion stops at roughening $\beta_R \approx 1.9$)	26-08
approach of the holonomy to Haar	Table row	An idea that turned out to be only a restatement: the law of the holonomy of large loops approaches the Haar measure (the decay of the odd part is the Wilson area law itself)	26-08
anomaly of the centre and the θ term	Table row	An idea that turned out to be only a restatement: consistency of the mixed anomaly of the centre and the θ term (a kinematic constraint; at θ = 0 it forces nothing)	26-08

Name	Kind	What it refers to	Introducing chapter
success in the hierarchical model	Table row	An idea that turned out to be only a restatement: taking success in the hierarchical model (the model where Migdal–Kadanoff decimation is exact) as evidence for the band (U(N) also confines there, so it is not a test)	26-08
criterion "stands on the lattice alone" (short: stands on the lattice alone)	Condition	A criterion a tool for crossing the band must meet: it holds on the lattice alone and does not rely on rotation invariance	26-08
criterion "distinguishes phases" (short: distinguishes phases)	Condition	A criterion a tool for crossing the band must meet: it distinguishes phases (it breaks down in a massless phase, or the limiting values differ)	26-08
criterion "does not rely on positivity" (short: does not rely on positivity)	Condition	A criterion a tool for crossing the band must meet: it does not rely on positivity or FKG	26-08
criterion "uses non-commutativity essentially" (short: uses non-commutativity essentially)	Condition	A criterion a tool for crossing the band must meet: it uses non-commutativity essentially (the same argument does not go through for U(1) or $\mathbb{Z}_2$)	26-08
criterion "compatible with asymptotic freedom" (short: compatible with asymptotic freedom)	Condition	A criterion a tool for crossing the band must meet: it is compatible with asymptotic freedom	26-08
criterion "handles topological sectors" (short: handles topological sectors)	Condition	A criterion a tool for crossing the band must meet: it can handle topological sectors (twists)	26-08
criterion "not a rewording of one loop" (short: not a rewording of one loop)	Condition	A criterion a tool for crossing the band must meet: it is a non-perturbative estimate, not a rewording of one loop	26-08
criterion "dynamical" (short: dynamical)	Condition	A criterion a tool for crossing the band must meet: it is a dynamical argument, not a kinematic constraint	26-08
criterion "passes through non-convex regions" (short: passes through non-convex regions)	Condition	A criterion a tool for crossing the band must meet: it can pass through non-convex (negatively curved) regions. Tools based only on convexity, log-concavity or curvature lower bounds cannot be used in the band	26-08
criterion "crosses the smallest band" (short: crosses the smallest band)	Condition	A criterion a tool for crossing the band must meet: it can first cross the smallest band of two-dimensional asymptotically free models (such as the O(N) model at large N) with the ultraviolet cutoff removed	26-08

Name	Kind	What it refers to	Introducing chapter
criterion "not broken by small loops" (short: not broken by small loops)	Condition	A criterion a tool for crossing the band must meet: it does not break down for small loops (short scales), or it is in a form that claims only large loops	26-08
criterion "non-isometric linearisation" (short: non-isometric linearisation)	Condition	A criterion a tool for crossing the band must meet: the linearisation of the comparison map is not isometric in the fluctuations	26-08

Part 27 Closing — conditional statements

Name	Kind	What it refers to	Introducing chapter
Reverse Estimate at a Fixed Plane (Theorem B of Paper 3) (short: Reverse Estimate at a Fixed Plane)	Theorem	The reverse estimate at fixed M (an upper estimate on the tension) of the paper on lattice fractional instantons	27-01
Doubling-Iteration Theorem	Theorem	Under θ(2R_k;T²_{R_k}) ≤ K₁θ^4 and lateral widening (s=2), θ(R_k;T²_{R_k}) ≤ K^{−1/3}exp(−cR_k²)	27-01
uniform acceptance of insertion	Table row	One of the assumptions of the conditional statement: under small-field frames, the average of tanh(W/2) for vortex insertions driven by dynamics is at most 1 − δ uniformly in the lattice spacing	27-01
frame decorrelation (form with L fixed)	Table row	One of the assumptions of the conditional statement: correlations between container frames decay, and E∏θ can be split into the product of the individual θ with a constant loss (form with L fixed)	27-01
decay of block observables (form with L fixed)	Table row	One of the assumptions of the conditional statement: correlations of block observables decay exponentially at a rate $m \geq c/L$ uniform in a (the form assumed by the theory on a small T^2)	27-01
lateral-widening inequality	Table row	One of the assumptions of the conditional statement: with the container fixed, widening the transverse T^2 by a factor s multiplies θ by at most Cs^2 (strong form), or a form with a finite total loss (weak form)	27-01
uniform slab tension	Table row	One of the assumptions of the conditional statement: the string tension $\sigma_{\text{eff}}(L)$ of the slab $T^2_L \times \mathbb{R}^2$ is positive uniformly in L	27-01
uniform slab gap	Table row	One of the assumptions of the conditional statement: correlations of block observables decay exponentially, uniformly in the lattice spacing and the slab size L (a premise of the arrow of the R^4 gap)	27-01

Name	Kind	What it refers to	Introducing chapter
limit to R^4	Table row	One of the assumptions of the conditional statement: the infinite-volume limit, the OS axioms, recovery of rotation invariance, uniqueness of the limit, non-triviality (outside the ledger)	27-01
tying the two roads together	Table row	One of the assumptions of the conditional statement: the road of the gap and the road of confinement give both conclusions for the same theory on R^4 (outside the ledger)	27-01
theory on a small T^2	Name	The theory on $R^2 \times$ a small twisted T^2 with the lattice spacing taken to 0	27-01
theory on R^4	Name	The theory on R^4 (where the T^2 of the theory on a small T^2 has been widened)	27-01
arrow of the small-T^2 gap	Condition	The arrow that gives, from ultraviolet control and the decay of block observables (form with L fixed), the existence of the theory on a small T^2 and a gap $m \geq c/L$	27-01
arrow of the R^4 gap	Condition	The arrow that gives the R^4 gap from the arrow of the small-T^2 gap, the uniform slab gap and the limit to R^4	27-01
arrow of slab confinement	Condition	The arrow of slab confinement with the lattice spacing and N fixed	27-01
arrow of the fractional-instanton rate	Condition	The arrow that gives the rate of the flux free energy from the Confinement Lower Bound by Fractional Instantons	27-01
arrow of the small-T^2 area law	Condition	The arrow that gives the area law of the theory on a small T^2 from ultraviolet control and uniform acceptance of insertion	27-01
arrow of the reduction of the band	Condition	From the base point, frame decorrelation and lateral widening, $\theta(R;T^2_R) \leq e\text{^}\{-cR^2\}$	27-01
arrow from the container estimate to the area law	Condition	From the reduced form and frame decorrelation, an area law uniform in L (the container estimate)	27-01
chessboard arrow	Condition	The arrow of the Chessboard Estimate with T^2 fixed	27-01
arrow from the two uniformities to the area law	Condition	The arrow that gives the area law from the uniform slab tension and the uniform slab gap	27-01
arrow of the R^4 area law	Condition	The arrow that gives the R^4 area law from the arrow from the container estimate to the area law, or the arrow from the two uniformities to the area law, together with the limit to R^4	27-01
arrow tying the two roads together	Condition	Both the mass gap and confinement for the same theory on R^4	27-01

Name	Kind	What it refers to	Introducing chapter
arrow of the road from the direction of time	Condition	From the upper-bound assumption and the one-sided bridge, m_phys $\geq$ (κ/K)^{1/2} at each β	27-01
small-T^2 gap	Table row	With the twist, zero modes disappear and the gap of the transfer matrix $\geq$ c/L uniformly in a	27-02
band comparison (in the language of the gap)	Table row	The lowest mass of the neutral sector of the box measured in units of the box size, x(L) = L m_0(L), satisfies x(2L) $\geq$ x(L) + δ over the interval of the band	27-02
uniform decay of block observables	Table row	Correlations of block observables with vacuum expectation 0 decay at a rate independent of a and L	27-02
positivity of the 't Hooft tension	Table row	$\ln(1/\theta(L,L)) \geq cL^2$ (c independent of L)	27-02
uniform gap of the double well	Table row	A gap, uniform in k and ħ, of the path measure of the 0+1-dimensional double well	27-02, A-08
uniformity of the relaxation-time upper bound and the one-sided bridge	Table row	The upper side of the diffusive-relaxation-time assumption and the one-sided bridge hold uniformly as $\beta \to \infty$	27-02, A-08
ledger of conditions	Table row	The ledger listing, one row each, the assumptions set by the conditional statement toward the mass gap and confinement, with what is to be shown and its status (Table 2 of 27-02)	27-02
lower end of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: the lower end, which pushes confinement and the comparison up to $\beta \approx 0.3$–1	27-02
upper end of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: the semiclassical upper end at fixed N and large β	27-02
middle of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: the middle, where one inequality using non-commutativity is needed (one of the four faces, uniformly in the scale)	27-02
strong-coupling end of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: the strong-coupling end $\beta \lesssim 10^{-79}$ (the record has a proof, with crude constants)	27-02

Name	Kind	What it refers to	Introducing chapter
one-loop end of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: at one-loop level, the comparison that shapes the fluctuations gives $\beta - \beta' = 0.258$, and the comparison that does not change their shape gives $\beta' \geq \beta + 0.023$	27-02
band in the language of the coupling	Table row	The row that states the band in the language of the coupling: written as a mixed derivative of a box with uniform background holonomies placed at both ends in time (a Schrödinger-functional box)	27-02
ultraviolet control in arbitrary volume	Condition	Ultraviolet control extended to arbitrary volume with extensive constants, with continuity in the sources made uniform in the lattice spacing and the volume	27-02
limiting measure on R^4 and its positivity	Table row	A gap outside the ledger: within the limit to R^4, the limiting measure, translations, preservation of reflection positivity, the Hilbert space and $H \geq 0$	27-02
recovery of rotation invariance	Table row	A gap outside the ledger: recovery of rotation invariance in the limit	27-02
local fields and OS regularity	Table row	A gap outside the ledger: local fields, OS regularity and the linear-growth condition (Wightman fields)	27-02
uniqueness and non-triviality of the limit	Table row	A gap outside the ledger: uniqueness of the limit, non-triviality, clustering	27-02
all compact simple groups (short: general groups)	Table row	A gap outside the ledger: extending to all compact simple groups	27-02
verification and peer review	Table row	A gap outside the ledger: writing out the whole, Lean, expert peer review, publication	27-02
boundedness of the insertion free energy	Condition	One of the premises of uniform acceptance of insertion: the insertion free energy ΔF is bounded uniformly in the lattice spacing	27-02
uniform lower bound on the path overlap	Condition	One of the premises of uniform acceptance of insertion: the overlap of the forward and reverse paths is bounded from below uniformly in the lattice spacing	27-02
ultraviolet control around a vortex background	Condition	One of the premises of the base point at small T^2: ultraviolet control around a background of constant curvature	27-02
insertion acceptance in a symmetric container	Condition	One of the premises of the base point at small T^2: uniform acceptance of insertion in a symmetric container of size L_0	27-02

Name	Kind	What it refers to	Introducing chapter
Lemma on the Upper Bound for the Minimal Action of a Symmetric Box	Lemma	For all N, the minimal action of the symmetric box satisfies S_min(N,N) $\leq 2N^4(1-\cos(\pi/N^2)) < \pi^2$	27-02
twelve tasks	Table row	The list dividing the continuum tasks into twelve (T^4 construction, infinite volume, uniform gap, remaining axioms, non-triviality, local fields and short distances, and so on; Table 1 of 16-07)	27-03
linear-growth condition (Osterwalder–Schrader axiom E0′) (short: linear-growth condition)	Name from the literature	A condition controlling the growth of Schwinger functions in n (added in 1975 to go back from the OS axioms to the Wightman axioms)	27-02
question "reverse direction of the fractional-instanton lower bound" (short: reverse direction of the fractional-instanton lower bound)	Table row	The reverse direction of the Confinement Lower Bound by Fractional Instantons: the question of showing, at fixed N, a dilute-gas upper bound uniform in M (an upper estimate on the tension)	27-03
question "agreement of the minimal actions" (short: agreement of the minimal actions)	Table row	The question of showing that the minimal action of a column with the frame frozen to the vacuum agrees with the minimal action of a free fractional instanton	27-03
question "existence of fractional instantons" (short: existence of fractional instantons)	Table row	The question of showing, on twisted $T^2 \times R^2$, the existence of a solution of topological charge 1/2 attaining the lower bound $4\pi^2/g^2$ of the action	27-03
question "SU(N) versions of the three facts" (short: SU(N) versions of the three facts)	Table row	The question of showing SU(N) versions of the uniqueness of the vacuum that absorbs the twist, the gap of the adjoint field, and the absence of flat configurations in the (1,1) sector	27-03
question "bias of the outer container" (short: bias of the outer container)	Table row	The second term when moving to a fixed sector: the question of showing that the bias of the outer container is typically small (0 as $M \to \infty$)	27-03
question "estimating the container free energy" (short: estimating the container free energy)	Table row	The question of estimating the container free energy uniformly over good frames and uniformly in the lattice spacing	27-03
question "cost of inserting a thick vortex" (short: cost of inserting a thick vortex)	Table row	The question of bounding the cost of insertion independently of N with the thick-vortex map (an improvement of the Slab Confinement Theorem toward the continuum limit)	27-03

Name	Kind	What it refers to	Introducing chapter
question "gap from insertion to the area law" (short: gap from insertion to the area law)	Table row	The question of showing the three things missing between the insertion estimates and the area law (rarity of simultaneous rejections, uniformity of the lower bound on the acceptance rate, an upper bound on the variance of the insertion ratio)	27-03
question "U(1) negative control" (short: U(1) negative control)	Table row	A negative-control question: applying the insertion estimates to four-dimensional U(1) to see where they stop working	27-03
question "oddness of composed insertions" (short: oddness of composed insertions)	Table row	The question of showing that the map composing insertions in separate cells satisfies the identity "odd under insertion"	27-03
question "literature check for slab confinement" (short: literature check for slab confinement)	Table row	The question of checking the Slab Confinement Theorem against arguments of Tomboulis–Yaffe type and the area law of high-temperature spatial Wilson loops (Borgs 1985)	27-03
question "kernels with flip probability near 1" (short: kernels with flip probability near 1)	Table row	The question of building a kernel with flip probability near 1 (for one-step kernels, the flip probability saturates at about 1/2)	27-03
question "quasi-static limit" (short: quasi-static limit)	Table row	The question of showing that, as the number of driving steps goes to infinity, the work W approaches the conditional free energy of the container	27-03
question "error of the shaping map" (short: error of the shaping map)	Table row	The question of bounding the error of the map that shapes the fluctuations (the relative entropy near the vortex surface and higher loops) at intermediate coupling and for large fields	27-03
question "ultraviolet cost of a fixed map" (short: ultraviolet cost of a fixed map)	Table row	Asks whether the extra ultraviolet cost of a fixed map grows in the same way in four-dimensional gauge theory	27-03
question "extending the measurements and their errors" (short: extending the measurements and their errors)	Table row	The question asking for measurements extended beyond the measured sizes of T^2, and for error estimates of driven insertion and replica exchange	27-03
question "non-perturbative proof of the lateral-widening inequality" (short: non-perturbative proof of the lateral-widening inequality)	Table row	The question of showing the lateral-widening inequality non-perturbatively, through $L \sim 1/\Lambda$ between the dilute region and the confining region	27-03
question "uniformity of frame decorrelation" (short: uniformity of frame decorrelation)	Table row	The question of showing frame decorrelation uniformly in L (with L fixed, it becomes a comparison of the chessboard density across scales)	27-03

Name	Kind	What it refers to	Introducing chapter
question "proof of the base point at small T^{2}" (short: proof of the base point at small T^{2})	Table row	The question of showing the base point at small T^{2} under ultraviolet control and uniform acceptance of insertion	27-03
question "excluding a deconfined phase" (short: excluding a deconfined phase)	Table row	The question of showing the exclusion of a non-confining phase, $\Delta F(L,L) \to 0$ (at any speed)	27-03
question "almost area law" (short: almost-area-law question)	Table row	The question of showing the almost area law $\ln(1/\theta(L,L)) \geq cL^{2}/\ln L$	27-03
question "β-monotonicity" (short: β-monotonicity question)	Table row	The question of showing $\partial_{\beta} \Delta F \geq 0$ with the lattice geometry fixed (β-monotonicity)	27-03
question "Z_{2} and U(1) on twisted slabs" (short: Z_{2} and U(1) on twisted slabs)	Table row	The question of checking that, on twisted slabs, β-monotonicity for $\mathbb{Z}_{2}$ and U(1) does not follow from Ginibre's inequality	27-03
question "criterion at the lower end" (short: criterion at the lower end)	Table row	The question of pushing up the lower end with a Dobrushin–Shlosman-type finite-size criterion based on the mixing of box magnetic flux, and with a computer-assisted count of the growth constant of surfaces	27-03
question "bookkeeping at the strong-coupling end" (short: bookkeeping at the strong-coupling end)	Table row	The question of writing out the bookkeeping of the Strong-Coupling-End Expansion Proposition and enlarging the constant $u_{0} \approx 10^{-80}$	27-03
question "four faces of the middle" (short: four faces of the middle)	Table row	The question of showing any one of the four faces of the middle, uniformly in the scale, in a form that breaks down for U(1)	27-03
question "two-sided area laws inside the band" (short: two-sided area laws inside the band)	Table row	The question of showing, at each β inside the band, the two-sided area law at fixed β (if it can be said, the comparison inside the band shrinks to finitely many steps)	27-03
question "what remains of the road of blocks and mixing" (short: what remains of the road of blocks and mixing)	Table row	The question of clearing up what remains of the road of blocks and mixing for the first step (the second and third stages of strong mixing, the check against a Varadhan-type limit and Witten's Morse theory)	27-03
question "convexity threshold and mixing length" (short: convexity threshold and mixing length)	Table row	Asks why, at the first step, the convexity threshold (the first conjugate point at the top ≈ 7.84) and the mixing length (about 9) are numerically close	27-03
question "touchstone of the smallest band" (short: touchstone of the smallest band)	Table row	The question of first testing a tool claimed to cross the band on two-dimensional asymptotically free models with the ultraviolet cutoff removed	27-03

Name	Kind	What it refers to	Introducing chapter
question "comparison principle for large loops" (short: comparison principle for large loops)	Table row	The question of showing the asymptotic form of the comparison principle for large loops only, σ_tw(L) ≤ σ_∞	27-03
question "testing the three new candidates" (short: testing the three new candidates)	Table row	The question of putting the three new candidates (reduction by compactness of the band, log-concavity of the Weyl measure, local step function of a column) through the tests of the selection criteria	27-03

Part 28 The tree of premises

Name	Kind	What it refers to	Introducing chapter
premise of the box continuum limit	Step	One of the groups of the tree of premises: the premises for the chain of block maps on a box of fixed size to give subsequential limits and the properties that pass to the limit (Table 2 of 28-04)	28-04
premise of ultraviolet stability	Step	One of the groups of the tree of premises: the premises for deriving the representation-level rows of the premise of the box continuum limit from one-step estimates (Table 3 of 28-04)	28-04
free-field premise	Step	One of the groups of the tree of premises: the absolute conditions on the free field. Given by reference to the free-field rows of the premise of ultraviolet stability (bracketing of the propagator and others), without separate rows in the table	28-04
premise of full-sequence convergence	Step	One of the groups of the tree of premises: the premises for the full sequence, along a tuned sequence of couplings, to converge without taking a subsequence (Table 4 of 28-04)	28-04
observable premise	Step	One of the groups of the tree of premises: the premises for handling fine observables (flowed observables and others) (Table 4 of 28-04)	28-04
infinite-volume premise	Step	One of the groups of the tree of premises: the premises for taking the infinite-volume limit (Table 4 of 28-04)	28-04
rotation premise	Step	One of the groups of the tree of premises: the premises for recovering rotation invariance in the limit (Table 4 of 28-04)	28-04
non-triviality premise	Step	One of the groups of the tree of premises: the premises that the limit is not a free field (Table 4 of 28-04)	28-04
fermion premise	Step	One of the groups of the tree of premises: the premises for theories containing fermions (outside the scope of the list of 28-04)	28-04

Name	Kind	What it refers to	Introducing chapter
implication of subsequential limits	Route	A conditional implication that, for the pushed-forward measures, if the premise fields hold then subsequential limits exist, and the limit has reflection positivity, symmetry and a lower bound of non-degeneracy	28-01
implication of full-sequence convergence	Route	A conditional implication that gives convergence of the full sequence, without taking a subsequence, as a statement about orbits of an abstract skew-product map	28-02
implication of the gap and the band	Route	A conditional implication that derives the mass gap from the uniform decay of block observables and reduces the band	28-03
continuity, composition and push-forward consistency of the block map	Table row	The block map B is continuous, the map from step to step is the composition of single steps, and the pushed-forward measures are consistent ($(B_m)_*\nu_{n,m+1} = \nu_{n,m}$)	28-04
gauge covariance and locality	Table row	The block map is covariant under gauge transformations ($B(U^g) = B(U)^{g\vert_{\text{coarse points}}}$), and the value of a coarse link depends only on fine links within a finite distance	28-04
commuting with the lattice symmetries	Table row	The block map commutes with the lattice translations, reflections and the hypercubic group	28-04
reflection positivity of the fine lattice	Table row	Reflection positivity of the Wilson measure on the fine lattice: for bounded measurable f depending only on links on the positive side, $\int f(\Theta U)\, f(U)\, d\mu_W \geq 0$	28-04
reflection positivity at the coarse level	Table row	Reflection positivity of the pushed-forward measure: for F in the positive-time class, the integral of the product of $F\circ\Theta$ and F with respect to the coarse-level measure is at least 0	28-04
symmetries of the measure	Table row	The pushed-forward measure $\nu_{n,m}$ is invariant under the lattice symmetries, and the symmetry breaking of observables tends to 0	28-04
family of observables and small-field set	Table row	The family of observables D is countable, dense and uniformly bounded, and the small-field set is closed and contains the two-step events	28-04
recursion of the activity	Table row	The recursion of the activity z_k, $z_{k+1} \leq \rho z_k + A w_k + B z_k^2$ (together with the existence of the limiting expectation and the large-field weight)	28-04
rarity of supply	Table row	The supply w_k is bounded as $w_k \leq \Phi(u_k)$ from exponential moments of the plaquette variables uniform in the step	28-04

Name	Kind	What it refers to	Introducing chapter
single-scale rarity on the fine lattice	Table row	For $\beta \geq 81$, the probability that all plaquettes of a set X have $\mathrm{Re}\, U_p < 0$ is $\leq q_1^{\lfloor \lvert X \rvert / 6 \rfloor}$ ($q_1 < 1$ independent of the volume)	28-04
moving invariant interval and margin	Table row	The effective coupling stays in a moving invariant interval, and the remainders can be paid for by the margin (inequalities of real functions such as ρW(s) + AΦ(s) + BW(s)^2 ≤ W(s'))	28-04
rarity of two-step events: plaquette part	Table row	For the pushed-forward measure, the probability that the plaquette variable β_m(1 − Re U_p) exceeds the threshold y_h(m) is at most P_1(m) (for large n)	28-04
rarity of two-step events: background-invariant part	Table row	For the pushed-forward measure, the probability that the invariant f of the background field exceeds the threshold f_h(m) is at most S(m) (for large n)	28-04
equicontinuity of observables	Table row	Differences of observable values are bounded uniformly as |s_n(d) − s_n(d')| ≤ τ_n + Ω(dist(d,d')) (τ_n → 0, Ω(r) → 0)	28-04
bracketing of the propagator	Table row	For all steps k and momenta, $c_0\, P/\omega \preceq G_k \preceq P/\omega$, $c_0 = 3/100$	28-04
positivity with background	Table row	The smallest eigenvalue of the constrained Hessian around the background field is at least $\lambda_0(1 - f_1/f_\star)$	28-04
contraction of irrelevant terms	Table row	$\mathcal{M}^{17}(\omega G_k) \preceq \tau\, \omega G_{k+17}$, $\tau < 1/170$, uniformly in k	28-04
coefficient of the one-step decrease of the coupling	Table row	The free-field coefficients b_k converge at rate $(3/5)^k$, and the sum of the transients is finite	28-04
value of the one-loop coefficient	Table row	The limit of the coefficient of the one-step decrease of the coupling is the one-loop value $b_\infty = 2b_0 \ln 2$ (a value from the literature; not rederived for this map)	28-04
contraction of the background map	Table row	The sum of the displacements per graduation, $\sum_j \Delta_j < \infty$ (the free-field part is of the type $\|\psi_l\|^2 \leq 0.608^l$)	28-04
geometry of many-body sums (correlation norm)	Table row	The correlation norm of the plaquette fields, $\Lambda \leq 640/9$	28-04
constrained inverse identity	Table row	The identity for the constrained Green function, C = G − GT*(TGT*)$^{-1}$TG, and its upper and lower estimates	28-04
Constrained Decay Theorem	Table row	The one-step constrained Green function at step k decays with respect to exponential weights, with constants uniform in the step	28-04

Name	Kind	What it refers to	Introducing chapter
zero-order transport of the background	Table row	For the step-by-step plaquette ratio θ_n of the background, sup 4ⁿθ_n < ∞ and Σ4ⁿθ_n^{3/2} < ∞ (the present form is known to be false)	28-04
one-step background-field lemma	Table row	For small fields, the minimiser of the constrained action is unique, analytic, exponentially local, and commutes with coarse gauge transformations	28-04
multistep background field	Table row	Unique and regular in nested small-field regions. The inverse of the linearisation is at most 1 in operator norm, independently of the number of steps	28-04
row sums of the lift uniform in the step	Table row	$\sup \lvert Eg\rvert \le R \sup \lvert g\rvert$, with R uniform in k	28-04
single-scale expansion tools	Table row	The cluster expansion on product white-noise measures, and its convergence	28-04
forest interpolation formula	Table row	The forest interpolation formula used for the cluster expansion (from the literature; its body is not in Lean)	28-04
small-field one-step renormalisation theorem	Table row	Coarse action = $\beta_{k+1} S_W$ (background) + local expansion + many-body terms. The many-body terms are exponentially small in the distance	28-04
gauge fixing step by step	Table row	Landau gauge at each step. The determinant is block lower-triangular. An estimate of the Haar Jacobian	28-04
large-field estimate	Table row	For each step t and each hole γ, $\lvert z_t(\gamma)\rvert \le \exp(-\gamma E \beta S^\flat(\gamma))$, $S^\flat(\gamma) \ge s_* \lvert\gamma\rvert$, with $s_* > 0$ uniform in the step and the volume	28-04
large-field gas	Table row	$\sum_{X \ni x} \lvert z(X)\rvert\, e^{2\lvert X\rvert} \le 1$, and the partition function of the gas equals the corresponding ratio	28-04
class and localisation	Table row	The only local gauge-invariant terms of dimension at most 4 are constants and $\sum \operatorname{tr} F^2$	28-04
induction uniform in the step	Table row	$r_{k+1} \le \tau_k r_k + a_k w_k + s_k$, $w_{k+1} \le b_k r_k + \sigma_k w_k + s'_k$; if the products of the coefficients are less than $1/4$, the iterates stay in the ball	28-04
contraction of the one-step map and matching at the endpoint	Table row	With estimates of the remainder of the one-step map and Lipschitz estimates of the coordinates, the map contracts and the coupling at the endpoint can be matched	28-04
approximation of flowed observables by block variables	Table row	Observables at flow time t are approximated by Lipschitz functions of the block variables at step m, with an error $\varepsilon_m(t) \to 0$ uniform in the lattice	28-04

Name	Kind	What it refers to	Introducing chapter
ultraviolet estimate uniform in the box size	Table row	The premise of the box continuum limit and the premise of ultraviolet stability hold with constants independent of the box size L	28-04
one-step ultraviolet estimate for the bridge	Table row	The one-step image from a rotated fine lattice to the standard coarse lattice lies in the ball of the chain	28-04
independence of the infinite-volume limit from the box shape	Table row	The limits along sequences of tori of different shapes agree	28-04
the limit is not a free field	Table row	The limiting theory is not a free field (which quantity to state it in is undecided)	28-04
non-negativity of θ	Table row	If M is even, $0 \le \theta_{N,M}(\beta)$ for all real β	28-04
single-scale criterion	Table row	At a physical size L_*, $\theta(L_*; T^2_{L_*}) < (s^2)^{-1/(s^2-1)}$ (0.63 for $s=2$, 0.7598 for $s=3$)	28-04
question of the cost of a twist-inserting map	Table row	For a gauge-covariant smooth map T, $\mathbb{E}_+[S\circ T - S - \ln J_T] \le 2\,\mathrm{artanh}(0.7598)$ uniformly in the lattice spacing	28-04
monotonicity of the vortex free energy and renormalisation comparison (short: monotonicity and renormalisation comparison)	Table row	The free energy of twist insertion is monotone in β. $s(N,\beta') \le s(2N,\beta)$	28-04
gap of a small twisted box and uniform acceptance of insertion (short: small-box gap and acceptance)	Table row	A gap $\ge c/L$ uniform in the lattice spacing. The acceptance rate of driven vortex insertion is uniform in the lattice spacing	28-04
from the area law to the gap	Table row	From the band (area exponent) to the uniform decay of block observables	28-04
from the area exponent of vortices to the area law of Wilson loops (short: from the vortex exponent to the area law)	Table row	The step that derives the area law of Wilson loops from the vortex free energy being small with an area exponent (from the literature; the typical-frame version has not been rederived)	28-04
tying the thresholds together	Table row	Tying the scale factor and threshold of the plaquette variables of two-step events to the effective coupling and threshold on the coupling side (for example β_m = 4u_m, y_h(m) = y_s(u_m))	28-04
choice of the small-field set and extension of the background invariant (short: choice of the small-field set)	Table row	The two-step events are contained in the small-field set (this holds if the background invariant is extended by a value larger than the threshold, and is false if it is extended by 0)	28-04

Name	Kind	What it refers to	Introducing chapter
constants of the quadratic type of the background	Table row	The constants of the three quadratic types in the background estimates, which make up the Lipschitz constant of the background map	28-04
geometry of many-body sums (reach constant)	Table row	The row sum R of the correlations of the plaquette fields is finite (the reach constant)	28-04
seam to the vessel of the mass gap	Table row	The implication of the vessel (28-03) assumes weak convergence of the full sequence of measures, and that the subspace of observables is closed under time translations. When used along subsequences, it takes the form "at every limit point"	28-04

Appendices

Name	Kind	What it refers to	Introducing chapter
upper side of the chains · lower side of the chains	Assumption	Collecting the upper halves of the three chains gives the upper side, and the lower halves the lower side	A-06
scale decomposition of the variance	Assumption	The variance can be split scale by scale	A-06
uniform Poincaré at each scale	Assumption	The conditional Poincaré constant of each step is uniform	A-06
upper bound on cross-scale conductance	Assumption	The conductance across steps is bounded	A-06
property that the state space is a single line (short: single-line property)	Assumption	The state space is a single line of scales	A-06
build-check step (a checking step of the Lean reference manual) (short: build-check step)	Name from the literature	The first of the four checking steps of the Lean reference manual: see that the editor's check mark, or lake build, finishes with neither warnings nor errors	A-04
axiom-check step (a checking step of the Lean reference manual) (short: axiom-check step)	Name from the literature	The second of the four checking steps: write #print axioms after the theorem and confirm that the axioms it depends on are only the three standard ones	A-04
kernel-recheck step (a checking step of the Lean reference manual) (short: kernel-recheck step)	Name from the literature	The third of the four checking steps: with lean4checker, pass the proofs saved by the build through the kernel again	A-04
isolated-comparison step (a checking step of the Lean reference manual) (short: isolated-comparison step)	Name from the literature	The fourth of the four checking steps: write only the statements in a trusted place, build the proofs in isolation, pass them again through an independent external checker, and confirm that the statements agree	A-04

Name	Kind	What it refers to	Introducing chapter
substitution move, multiplication move, simplification move	Table row	The names of the moves when the three lines on paper (substitute, multiply both sides, simplify) are matched to the four Lean moves	A-04
survey step	Table row	The first of the seven steps a sentence passes through before it appears in the text: collecting literature on each topic, known results and candidate numbers	A-05
cross-checking step	Table row	The second of the seven steps: cross-checking against primary sources, catching errors in bibliographic data, over-paraphrasing and misattribution	A-05
refutation step	Table row	The third of the seven steps: another reader looks for refutations, catching wording stronger than the source, promotion of the degree of certainty and omitted conditions	A-05
reproduction step	Table row	The fourth of the seven steps: reproducing computations, catching miscopied numbers, mixed-up conventions and counterexamples within a finite range	A-05
label step	Table row	The fifth of the seven steps: attaching labels, catching mixtures of "proved", "conjecture" and "numerically supported"	A-05
consistency step	Table row	The sixth of the seven steps: aligning symbols, names and conventions across chapters, catching discrepancies and duplication	A-05
assembly step	Table row	The seventh of the seven steps: assembling and checking mechanically, catching formulas that cannot be rendered, broken links and forbidden words	A-05
gap of definition	Table row	One of the gaps where a Lean proof passes but does not say what was meant: a definition weaker or stronger than intended (for example, defining the gap with ≥ 0 lets 0 pass)	A-04
gap of naming	Table row	One of the gaps where a proof passes but does not say what was meant: the name of the theorem is bigger than the statement	A-04
gap of vacuous truth	Table row	One of the gaps where a proof passes but does not say what was meant: the premises cannot hold simultaneously, so anything can be concluded	A-04
gap of assumptions	Table row	One of the gaps where a proof passes but does not say what was meant: what should be proved is placed among the assumptions, and when it fails is not written	A-04

Name	Kind	What it refers to	Introducing chapter
bracketing of the tension rate (Corollary C of Paper 3) (short: bracketing of the tension rate)	Name from the paper	A corollary of the paper on lattice fractional instantons: bracketing the limit superior and limit inferior of the tension rate	A-07
paper on vortex-insertion identities	Name	The paper, which came out of this book's study, collecting the vortex-insertion identities (Table 4 of A-07)	A-07
paper on the reduction of the band	Name	The paper, which came out of this book's study, collecting the reduction of the band and the numerics for SU(2) and U(1) (Table 5 of A-07)	A-07
paper on lattice fractional instantons	Name	The paper, which came out of this book's study, collecting confinement by lattice fractional instantons (Table 6 of A-07)	A-07

Table 3 Index of names (902 entries). Look things up by name, not by table serial numbers or ranges of labels.

04

Eight readings of the diffusive-relaxation-time assumption (master copy) — why the normalisation is named

The statement "C_P agrees with the square of a length" involves three choices: whether to say it by powers or by a band; whether the length of comparison is the correlation length ξ or the bookkeeping length ξ_{RG}; and whether C_P is measured with the angle $\mathcal{E}_\vartheta$ or the field $\mathcal{E}_A = \mathcal{E}_\vartheta/\beta$. With $2 \times 2 \times 2$ there are eight readings. The following table is the master copy.

Table 4 The eight readings (this table is the master copy; Table 5 of 23-01 is a summary. The two right-hand columns are accounts that presuppose the relaxation-time assumption. The exponent is the power of β multiplying C_P/ξ^2 or $a^2 C_P$)

Level, length, normalisation	Statement	Name in this book	Free approximation	One-loop logarithms Physics
Power, ξ, $\mathcal{E}_\vartheta$	$\ln C_P / \ln \xi \to 2$	Relaxation-time assumption ($\eta_{\mathrm{dyn}} = 0$)	Holds	Does not move
Power, ξ, $\mathcal{E}_A$				
Power, bookkeeping length, $\mathcal{E}_\vartheta$	$\ln C_P / \ln \xi_{\mathrm{RG}} \to 2$	Relaxation-time assumption with the one-scale chain and the bookkeeping-length chain (power)	Holds (the two-loop difference does not show in the power)	Does not move
Power, bookkeeping length, $\mathcal{E}_A$				
Band, ξ, $\mathcal{E}_\vartheta$	$C_P^{(\vartheta)} \asymp \xi^2$	Band form of the relaxation-time assumption [$\mathcal{E}_\vartheta$]	The lower-bound assumption [$\mathcal{E}_\vartheta$] breaks (ratio $g^2/4$, exponent -1)	$-9/22$: the lower-bound assumption [$\mathcal{E}_\vartheta$] breaks

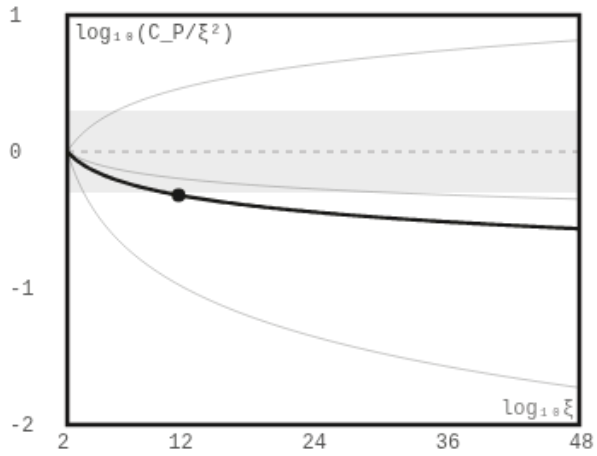
Level, length, normalisation	Statement	Name in this book	Free approximation	One-loop logarithms Physics
Band, ξ, $\mathcal{E}_A$	$C_P^{(A)} \asymp \xi^2$	Band form of the relaxation-time assumption [$\mathcal{E}_A$]	Holds (equality)	$+13/22$: the upper-bound assumption [$\mathcal{E}_A$] breaks
Band, bookkeeping length, $\mathcal{E}_\vartheta$	$a^2 C_P^{(\vartheta)} \asymp 1$	Band form of the relaxation-time assumption [$\mathcal{E}_\vartheta$] with the one-scale chain and the bookkeeping-length chain	Decided by the order of the bookkeeping (23-01 § 07)	$-303/242$
Band, bookkeeping length, $\mathcal{E}_A$	$a^2 C_P^{(A)} \asymp 1$	Band form of the relaxation-time assumption [$\mathcal{E}_A$] with the one-scale chain and the bookkeeping-length chain		$-61/242$

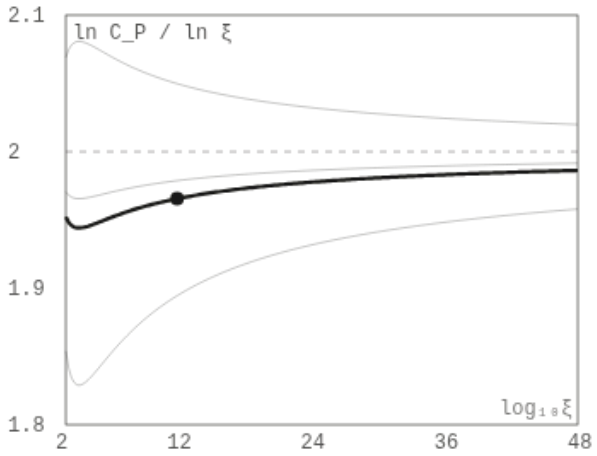
The derivation is in 23-01 The line of construction and the diffusive-relaxation-time assumption and 4-06 § 05; here only the sum is shown (this passage may be skipped). The fractions in the right-hand column are sums of three logarithms: the -1 of the normalisation ($C_P^{(\vartheta)} = C_P^{(A)}/\beta$), the $+13/22$ of the renormalisation of stochastic time (23-01 The line of construction and the diffusive-relaxation-time assumption), and, entering only the rows with the bookkeeping length, the difference between one-loop bookkeeping and two-loop running, $-2 \cdot \frac{51}{121} = -\frac{102}{121}$ ($51/121 = b_1/(2b_0^2)$, independent of N for pure SU(N); 4-06 § 05). All are accounts under the renormalisation group of one-loop perturbation theory, and carry the Physics label.

$$-1 + \frac{13}{22} = -\frac{9}{22}, \qquad \frac{13}{22} - \frac{102}{121} = -\frac{61}{242}, \qquad -\frac{9}{22} - \frac{102}{121} = -\frac{303}{242} \quad (3)$$

We confirmed this sum on this machine in exact rationals Computed . The following figure draws, under this account, how the ratio of the band readings C_P/ξ^2 and the exponent of the power readings $\ln C_P / \ln \xi$ move. β and ξ are tied by the one-loop formula $\xi \approx e^{3\pi^2\beta/11}$ (up to a constant factor, 4-07).

Figure 1 Choosing a reading draws, on the left, the ratio of the band reading C_P/ξ^2 (1 at $\xi = 10^2$; logarithmic vertical axis) and, on the right, the exponent of the power reading $\ln C_P/\ln\xi$. The three buttons choose level, length and normalisation, and the knob chooses $\log_{10}\xi$. The panel with the thicker frame decides the truth of the chosen level. The pale shading on the left is a guide for ratios 1/2–2 (one example of the constants of a band)





Band, ξ, angle graduation → band form of the relaxation-time assumption [angle graduation]: s = −9/22. ξ = 10^12: β = 10.27, ratio 0.4805, power exponent 1.9655. Band: the ratio keeps moving away from 1 (it does not close with a uniform constant)

Computed The numbers in this figure were computed on this machine (the exponent of (3) and the one-loop formula are evaluated on the spot; Table 5 is the Python values).

When JavaScript is disabled, it can be read in Table 5.

Table 5 Ratio and exponent under the one-loop account (SU(2); $\beta = \frac{11}{3\pi^2}\ln\xi$; the ratio is $(\beta/\beta(10^2))^s$ and the expone

Computed Physics

Length, normalisation	s	Ratio $\xi = 10^3$	Ratio 10^{12}	Ratio 10^{48}	Exponent 10^3	Exponent 10^{12}
ξ, $\mathcal{E}_\vartheta$	−9/22	0.8472	0.4805	0.2725	1.9442	1.9655
ξ, $\mathcal{E}_A$	+13/22	1.2707	2.8828	6.5400	2.0806	2.0498
Bookkeeping length, $\mathcal{E}_\vartheta$	−303/242	0.6019	0.1061	0.0187	1.8292	1.8945
Bookkeeping length, $\mathcal{E}_A$	−61/242	0.9028	0.6366	0.4488	1.9656	1.9788
(For reference) free approximation, $\mathcal{E}_\vartheta$	−1	0.6667	0.1667	0.0417	1.8636	1.9157

There are two ways to read it. In the ratio columns all four rows keep moving away from 1, and no band reading closes with a uniform constant. In the exponent columns all four rows approach 2, and the power reading does not move. But the approach is slow: even at $\xi = 10^{48}$ the exponent is still 0.008–0.042 away from 2. The last row is the same formula as Table 9 of A-01; the exponents agree, and the ratios agree once the values of A-01 are divided by the value at $\xi = 10^2$. This table does not measure C_P of the Yang–Mills measure; it is an example laying out the consequences of the account. The difference between the two kinds of columns in this table is the reason this book placed the relaxation-time assumption at the level of powers.

05

Bookkeeping and exponent quantities — κ, δ, θ, the bookkeeping length, p, η_{dyn}

This book's study in Part 23 kept the books on a step that coarsens the lattice by a factor b with the following map (23-02 Roads tried on the four-dimensional lattice, and walls).

$$\beta \longmapsto \beta - \kappa\big(\ln b - \delta(b)\big), \qquad a^2 C_P \propto a^p, \qquad p = \eta_{\mathrm{dyn}} + (2 - \eta_{\mathrm{dyn}})\,\theta \ \text{(8)}$$

The two numbers in the left formula and the two exponents in the right formula are the quantities of this section. The quantities are not assumptions, but sentences such as "$\sum \delta < \infty$" and "$\eta_{\mathrm{dyn}} = 0$" are assumptions, and the phrasing of Part 23, "the physical-unit gap statement $\Longleftrightarrow \sum \delta < \infty$ and $\eta = 0$", is the algebra of $p = 0$ presupposing the power form and the model of the one-loop kernel.

Table 6 Quantities of bookkeeping and exponents

Symbol	Kind	One-line meaning	Master	Status
κ	Quantity	The slope per step. $\kappa = 4Nb_0 = 11/(3\pi^2) = 0.3715110$ $(N = 2)$	23-02 Roads tried on the four-dimensional lattice, and walls, 4-06 § 02	Known A rewriting of the one-loop coefficient. Agreement to 5 digits in the window of the model is Computed (agreement within the model)
$\delta(b)$	Quantity	The shortfall per step: the amount by which the window falls short of the continuum $\ln b$. Vanishes as c_2/b^2	23-02 Roads tried on the four-dimensional lattice, and walls	A number measured in the model of the one-loop kernel (not a non-perturbative quantity) Computed
$\sum_k \delta(b_k) < \infty$	Assumption	The shortfalls are summable	Same as above, 18-05 § 06	Assumption. For fixed b, $\delta(b) = 0$ is necessary and sufficient Lean fixed_step_bounded_iff
θ	Quantity	The exponent of accumulation of the shortfall. $a\xi \propto a^{\theta}$. For fixed b, $\delta(b)/\ln b$	23-01 The line of construction and the diffusive-relaxation-time assumption	Definition. If $\sum \delta < \infty$ then $\theta = 0$
ξ_{RG}, a	Quantity	The bookkeeping length b $k(\beta)$, and $a := 1/\xi_{\mathrm{RG}}$. At one loop $\xi_{\mathrm{RG}} \approx e^{3\pi^2\beta/11}$	23-01 The line of construction and the diffusive-relaxation-time assumption, 23-02	The definition of Part 23. The one-loop part is Known (the same formula as $1/(a\Lambda)$)
p	Quantity	The power in $a^2 C_P \propto a^p$. The right-hand side of (8)	23-01 The line of construction and the diffusive-relaxation-time assumption	$p = 0 \iff \eta_{\mathrm{dyn}} = 0 \wedge \theta = 0$ (under $0 \le \eta \le 2$, $\theta \ge 0$) Lean drainExp_eq_zero_iff
z, η_{dyn}	Quantity	The dynamical exponent $C_P \asymp \xi^z$ and $\eta_{\mathrm{dyn}} = 2 - z$. The relaxation-time assumption is $\eta_{\mathrm{dyn}} = 0$	2-10 § 06, 18-05 § 02	Definition. The premise $\eta_{\mathrm{dyn}} \ge 0$ includes $z \le 2$

Symbol	Kind	One-line meaning	Master	Status
η_s	Quantity	The power of static correlations. Different from η_{dyn}. They are equal only within $z = 2 - \eta_s$	Same as above, A-01 § 08	For the two-dimensional Ising model, $\eta_s = 1/4$ and $\eta_{\mathrm{dyn}} = -0.1665$ Physics

The symbol η needs care. In the study of Part 23, η originally referred to two quantities. This book fixes the η of the equivalence between the power form and the physical-unit gap statement as η_{dyn}, and uses η_s only for the lower bound by test functions. An η without qualification is η_{dyn}.

Example. Fixing $b = 2$ gives $\delta(2) = 0.1335$ (weight w) and $\theta = \delta(2)/\ln 2 \approx 0.19$ (23-02 Roads tried on the four-dimensional lattice, and walls). Even with $\eta_{\mathrm{dyn}} = 0$, $p = 2\theta \approx 0.39 > 0$, and one falls on the side $a^2 CP \to 0$. If $b_k = k + 2$ is made larger at each step, then $\sum c_2/(k+2)^2 < \infty$ and $\theta = 0$ can be restored.

06

The assumptions of the second road and the band — names, content, status, what is needed to remove them, U(1)/$\mathbb{Z}_2$

We collect in one table the assumptions set by Parts 25–27 (the road from the direction of space, the crossover band, closing). In this section alone, the status column is written with the five grade labels of Part 23 (23-00 § 01: Proved here, Cited, Computer-assisted, Reading, Assumption). Rows at the stage of a line of proof are graded Assumption, and the existence of the line is written in the same column. The stages are the seven stages a proof must contain of 16-09, and the ledger rows are the rows of 27-02 Table 2; both are referred to by name. The last column is the behaviour of the same assumption when placed in $U(1)$ and in four-dimensional $\mathbb{Z}_2$ gauge theory; it is there to show which rows non-abelianness enters.

Table 7 The assumptions of the second road and the band (the conditional statement is in 27-01; the current state of each row and the rule by which it disappears are in 27-02)

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$
Ultraviolet control (the ultraviolet stage among the stages a proof must contain; the ledger row has the same name)	Multiscale analysis uniform in a, from the lattice spacing a to the scale L: the small-field analyticity condition (locality and analyticity of the effective action), the large-field rarity condition, and the condition that the coupling stays small (the effective coupling stays in $(0, \gamma]$)	Assumption (Bałaban's inequality (0.31) contains a step that the original source says is unpublished)	Completing the write-up of Bałaban's steps, or checking the original source	An assumption relying on asymptotic freedom; it does not apply to U(1) or $\mathbb{Z}_2$
Uniform acceptance of insertion (small-T^2 confinement stage)	Under small-field frames, the average of $\tanh(W/2)$ for vortex insertions driven by dynamics is at most $1-\delta$ uniformly in a	Assumption (the numbers showing that the work drops as the procedure is lengthened are Reading)	Estimating the distribution of the work at one-loop (determinant) level	—
Base point at small T^2 (small-T^2 confinement stage)	At a small physical L_0, $\theta(L_0; T^2_{L_0}) \le 1 - \varepsilon_0$, $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$	Assumption (line: under ultraviolet control and uniform acceptance of insertion. The form of the tension is Cited formula (2.37) of Tanizaki–Ünsal)	A proof under ultraviolet control and uniform acceptance of insertion	For U(1), the dependence of ε_0 on L_0 goes the other way
Frame decorrelation (of the same family as "uniform block decay" on a small T^2; small-T^2 confinement stage and band stage)	Correlations between container frames decay, and the Wilson-loop estimate $\mathbb{E}\prod\theta$ can be split into the product of the individual θ	Assumption . If T^2 is fixed, under reflection positivity of z the chessboard replaces it by a single density κ_{cb} Proved here	An estimate of κ_{cb}. Making it uniform in L requires the uniform slab gap	—

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$
Reflection positivity of z (small-T^2 confinement stage and band stage)	The ensemble of z summed over twists is positive under reflections through lattice sites in directions 3 and 4	Proved here (no longer an assumption. The algebra of the kernel is in Lean `ReflPos.refl_pos`, `refl_cs`. The chapter giving the paper proof is 26-04)	—	Holds in the same way for $\mathbb{Z}_2$ and U(1) (a structural fact)
Contraction through good blocks (small-T^2 confinement stage and band stage)	Passing through a good block, the influence of the frame contracts to e^{-mw} in the gauge-invariant distance	Assumption	Bałaban-type analysis	—
Lateral-widening inequality, strong form (band stage; a form within the ledger row "positivity of the 't Hooft tension")	With the container fixed, widening the transverse T^2 by a factor s multiplies the sector bias θ by at most Cs^2	Assumption. The numbers are consistent Reading. It does not follow from the structural inputs alone (a negative lemma, 26-04)	Input from non-abelian dynamics (self-dual clusters, asymptotic freedom)	**Breaks** (for U(1) the flux spreads uniformly; for $\mathbb{Z}_2$, surface tension)
Lateral-widening inequality, weak form (band stage; a form within the ledger row "positivity of the 't Hooft tension")	A form of the strong form with a finite total loss (the slab tension σ_{eff} is "almost monotone")	Assumption (almost the same content as the band)	As above	Breaks
Uniform slab tension (band stage; the weak form of the ledger row "positivity of the 't Hooft tension")	The string tension of the slab $\sigma_{\text{eff}}(L)$ is positive uniformly in L	Assumption. The Creutz ratio grows and saturates for N $T = 4 \to 10$ Reading (conditions in 25-08)	The band itself	**Breaks** (for U(1), $\sigma \sim e^2/L^2$ goes to 0)

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$
Uniform slab gap (band stage; the ledger row "uniform decay of block observables")	The mass gap of the slab is uniform in L: correlations of block observables decay uniformly in L (the ledger row of 27-02)	Assumption	A comparison of $\kappa_{\mathrm{cb}}(L)$ across scales	—
Limit to $\mathbb{R}^4$ (R^4 stage; one of the gaps remaining outside the ledger)	The infinite-volume limit, the OS axioms, recovery of rotation invariance, non-triviality	Assumption (mostly standard lines; the infrared side of rotation and uniqueness of the limit are open)	—	—
Tying the two roads together (tying the gap and confinement; spans the small-T^2 gap stage, the small-T^2 confinement stage and the R^4 stage; one of the gaps remaining outside the ledger)	Obtaining both the area law and the gap of the transfer matrix for the same theory on $\mathbb{R}^4$	Assumption (the statement is not yet fixed; the area law alone does not give a gap)	The uniform slab gap and the limit to $\mathbb{R}^4$	U(1): no confinement and no gap
All compact simple groups (general-group stage; one of the gaps remaining outside the ledger)	$SU(N)$ and groups without centre	Assumption	—	—

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$
Diffusive-relaxation-time assumption (the upper side is the upper-bound assumption, the lower side the lower-bound assumption; the ledger row "uniformity of the relaxation-time upper bound and the one-sided bridge")	The time for fluctuations to even out grows as the square of the correlation length: $C_P \asymp \xi^2$ (dynamical exponent $z = 2$). The upper-bound assumption, together with the one-sided bridge, gives a lower bound on the gap in physical units	Assumption (23-01). The remaining items of the first step are in 23-15 Table 1 and the subsection on the road of blocks and mixing	The remaining items of Part 23 (23-15), and a form uniform in β (crossing the band in the language of time)	See 23-01, 24-01
Sector bias $\theta(R; T_L^2)$ (quantity)	The absolute value of the frame-conditioned expectation of the sector sign of the container $D \times T_L^2$ (side of D equal to R), $\lvert\mathbb{E}[\sigma_D \mid \Phi]\rvert$	Definition (25-03)	—	Constant for U(1); expected to approach 1 for four-dimensional $\mathbb{Z}_2$ Reading (physics)
Chessboard density κ_{cb} (quantity)	The per-block value (in the volume limit) of $\mathbb{E}\prod_t \theta_t$ when the plane is tiled with blocks of equal size and a container is placed in every block	Definition (26-05)	—	—

A note on symbols. The θ and κ_{cb} of Table 7 are different quantities from the θ (exponent of accumulation of the shortfall) and κ (one-step slope) of Table 6. The sector bias is written with arguments as $\theta(R; T_L^2)$, and the chessboard density with a subscript as κ_{cb}. The point to read in this table is that the rows marked "Breaks" in the non-abelian column gather in the two rows of the lateral-widening inequality and the row of the uniform slab tension. These rows (in the ledger, the row of positivity of the 't Hooft tension and the row of the renormalisation comparison among the inputs of the reduction) are where non-abelian dynamics enters; the other rows are structural facts or rows depending on the ultraviolet wall.

What this chapter can say

What is said	What is not said
The names and introducing chapters of Table 3 are taken from the table of names for the whole book (one name for one object)	The one line of "what it refers to" is a guide. The exact statements, assumptions and statuses are in the introducing chapters
Tables 1 and 2 are the conventions for names and symbols of the chapters of Volume 2	The full correspondence with the identifiers in the source of the published construction (where to find it is in the sources of each chapter)
Proved here For SU(2), $s = \kappa = 11/(3\pi^2)$	The values of the κ of Volume 3 for other groups (the theorems of Volume 3 are for SU(2) only)

Sources and reproduction

Item	Kind	Source / tools
Names and introducing chapters	This book	The text of each chapter. Names are made uniform across the whole book
Names and symbols of Volume 2	Source	The chapters of Parts 6–14, which read yang-mills-group/OpenAI-YM-20261028 (submitted 2026-10-28, version 3456e5e)
Values of s and κ	Proved here	10-01 ($C_g = 8$), 4-06 § 02 ($\kappa = 4Nb_0$)
Tables 4 and 5 (the eight readings and the one-loop account)	Computed, Physics	23-01, 4-06 § 05. The sums of fractions were checked on this machine as exact rationals

Next chapter to read: A-07 Papers that came out of this book.

Papers that came out of this book

> This chapter is this book's own investigation. § 01 and § 07 are guide material.

What this chapter shows — A list of the five papers that came out of this book's study. For the two published on this site's papers page (the one-sided bridge, one-link integration) and the three at the draft stage (the paper on vortex-insertion identities, the paper on the reduction of the band, the paper on lattice fractional instantons), we list the title, the abstract and the main theorems, attach to each main theorem the book's label, its conditions and the reach of machine checking, and show which chapters of the book treat it.

Prerequisite chapters — None. How to read the labels is in 23-00 § 01, and how to read the Lean column is in A-04.

Said first — None of the papers claims a solution of the problem of Yang–Mills existence and the mass gap. The three drafts have not been through checking by outside mathematicians, and the book's labels are not copies of the drafts' statements (only what has been placed in a chapter of the book after its proof was checked is Proved here). The titles of the papers are quoted as in the original (English), and the abstracts and statements are shortened (the names and numbers of theorems are those of the originals).

Order of this chapter

1. The list of five — title, stage, pages, machine checking, chapters of the book (Table 1)
2. The one-sided bridge — the inverse square of the gap bounds the Poincaré constant from below (Table 2)
3. One-link integration — integrating one link before Bakry–Émery (Table 3)
4. Paper on vortex-insertion identities — the vortex-insertion identities (Table 4)
5. Paper on the reduction of the band — the reduction of the band, and numerics for $SU(2)$ and $U(1)$ (Table 5)
6. Paper on lattice fractional instantons — lattice confinement by fractional instantons (Table 6)
7. The papers and the Lean bundles — names of bundles, numbers of theorems, what they guarantee (Table 7)

01

The list of five

The "stage" in Table 1 is either published (the PDF and LaTeX source are on this site's papers page) or draft (material for the study in this book, not published). Pages are pages of the PDF. The authorship follows the form of publication.

Table 1 Papers that came out of this book (as of 2026-10-15)

Paper	Title (original)	Stage	Pages	Machine checking	Chapters of the book
The one-sided bridge	The inverse squared spectral gap bounds the Langevin Poincaré constant of Euclidean path measures from below	Published (2026-10-09, list of papers, PDF)	12	The algebraic skeleton (bundle EuclideanPath.GapBridge, 17 declarations)	23-01 § 06, 24-04 § 02, 24-05
One-link integration	One-link integration before Bakry–Émery for SU(2) lattice Yang–Mills — An explicit strong-coupling threshold β < √(2/27) in dimension four, and why complete families of links exist only for d ≤ 4	Published (2026-09-21, list of papers, PDF)	24	The chain from the Wilson action to the lower bound on the Hessian, and the combinatorics of complete families	5-11, 23-02 § 02

Paper	Title (original)	Stage	Pages	Machine checking	Chapters of the book
Paper on vortex-insertion identities	Exact centre-vortex insertion identities in the twist-summed ensemble, and bounded, averaged and driven confinement bounds	Draft (2026-10-13)	3	Algebra on a finite configuration space	25-05, 25-06
Paper on the reduction of the band	Where the twisted-slab route to confinement stops: a corrected window target, a reduction to transverse widening, and SU(2) versus U(1) numerics	Draft (2026-10-13)	9	The flux estimate, the electric-flux form, the doubling iteration, the algebra of reflection positivity	26-01, 26-02, 26-04, 26-05, 25-07, 26-09
Paper on lattice fractional instantons	Fractional instantons confine on the lattice: a rigorous semiclassical bound for SU(2) gauge theory on a twisted small torus	Draft (2026-10-14)	7	The factorisation of the column representation and the binomial estimate	25-02 § 06, 26-06 § 04, 27-02 upper end of the renormalisation comparison

The "window" in the original title of the paper on the reduction of the band is, in this book's vocabulary, "the crossover band". It is a different thing from the windows of Parts 23 and 24 (an interval of sites, a range of $\hbar$, an interval of times), and in the text we write "band" (26-00 § 02). The "rigorous" in the original title of the paper on lattice fractional instantons is the original's word, and is independent of the book's label.

02

The one-sided bridge — the inverse square of the gap bounds the Poincaré constant from below

Abstract. For $H = -(\hbar^2/2)\Delta_M + V$ on a finite-dimensional configuration space (a compact connected Riemannian manifold, or $\mathbb{R}^n$ with a constant metric and a smooth confining potential), assume a discrete spectrum, a simple positive ground state ψ_0 and a gap $\Delta = E_1 - E_0 > 0$. For the Euclidean path measure stationary over all time, the Poincaré constant of the Dirichlet form of stochastic quantisation (Langevin) on smooth time-averaged functionals satisfies $C_P \geq \kappa(\hbar/\Delta)^2$. The test function is the time integral of $F = \psi_1/\psi_0$, and among functionals local in time it is the best. The converse is false in general: for a free particle on the circle, $C_P = \infty$ while $\Delta > 0$. It touches neither four dimensions nor gauge fields.

Table 2 The main theorems of the one-sided bridge (the book's names are those of 23-01 and 24-04)

Number in the paper	Name in the book	Statement (with its conditions in the same row)	Label	Lean
Theorem 3.1	One-Sided Bridge Proposition (Theorem 3.1 of the paper)	Under the setting above and three assumptions (the ground-state assumption, the stationary path-measure assumption and the metric comparison assumption; (A1)–(A3) in the paper: discrete spectrum and simplicity of the ground state, a stationary path measure, the comparison constant κ of the Langevin metric), $C_P \geq \kappa\hbar^2/\Delta^2$. With time taken as continuous	Proved here (23-01 § 06)	Only the algebraic step: `poincare_lower_of_gap` (assumptions: $0 < p, m, \kappa, E$, $2p/m \leq V$, $E \leq 2mp/\kappa$, $V/E \leq C_P$), `window_ratio_closed` (assumptions: $0 < m$, $0 < T$). The path measure, the covariance formula and the ground-state identity are outside Lean
Corollary 3.6	The corollary of the bridge	Under the same assumptions, if $C_P \leq KL^2$ then $(\Delta L/\hbar)^2 \geq \kappa/K$. The paper gives no name to this assumption $C_P(\mathcal{D}) \leq KL^2$ (the assumption of Corollary 3.6 of the paper; of the same type as the upper side $\xi^2\uparrow$ of the book's diffusive-relaxation-time assumption)	Proved here (algebra on top of the assumptions)	`gap_lower_of_poincare_upper` (assumptions: $0 < m$, $\kappa/m^2 \leq C_P$, $C_P \leq KL^2$)
Proposition 4.1	(2) of 24-04	With the canonical metric ($\kappa = 1$), the supremum of the Rayleigh quotient of functionals local in time is exactly $(\hbar/\Delta)^2$, and it is not attained	Proved here (24-04 § 02)	`local_rayleigh_le` (assumptions: $0 < m$, $m \leq a_k$, $0 \leq c_k$), `fourier_weight_le` (assumption: $0 < m \leq a$). Only finite sums and pointwise inequalities

Number in the paper	Name in the book	Statement (with its conditions in the same row)	Label	Lean
Proposition 5.1	Free-Particle-on-a-Circle Counterexample	For a free particle on the circle, $\Delta > 0$ and $C_P = \infty$ (because of the winding number). The converse inequality is false in general	Proved here (23-01 § 06 (5))	`rotor_ratio_unbounded` (assumption: $0 < r$), `rotor_gap_pos` (assumption: $0 < \hbar$)

Scope. Wilson's lattice with discretised time, gauge fields, infinite volume and the continuum limit are outside the statements of the paper. To use it for four-dimensional lattice gauge theory, the time-step comparison assumption ((D) in the paper "one-sided bridge") on the lattice with discretised time remains (24-04 § 08).

03

One-link integration — integrating one link before Bakry–Émery

Abstract. For the $SU(2)$ Wilson action on $(\mathbb{Z}/L)^4$, if the links of a "complete family", containing exactly one link of every plaquette, are integrated first, the Haar integral splits star by star, and the marginal density of the remaining links takes the form $e^{-\Phi}$. The second derivative of Φ along geodesics is at least $-27\beta^2 \sum_e \|X_e\|^2$ for every β and every configuration. Together with the Ricci curvature 2 of the unit S^3 this gives Bakry–Émery curvature $2 - 27\beta^2$ and the threshold $\beta < \sqrt{2/27} = 0.2722$. A complete family exists in $\mathbb{Z}^d$ only when $d \le 4$. The mass-gap step is not a Lean theorem; it is on paper, and moreover a sketch.

Table 3 The main theorems of one-link integration (detailed labels in 5-11)

Number in the paper	Statement (with its conditions in the same row)	Label	Lean
Theorem 1	For L even with $L \ge 3$, and any Haar probability measure on the unit quaternions: (i) integrating the links of a complete family splits star by star and gives $e^{-\Phi}$; (ii) the second derivative of Φ is at least $-27\beta^2 \sum_e \|X_e\|^2$ for every β and every configuration	Lean	`WilsonOneLink.integral_star_links`, `StaplePerturbation.hess_gauge` (assumptions as in the type signatures of § 11 of the paper)
Theorem 2, 2′	A complete family exists in $\mathbb{Z}^d$ only when $d \le 4$. On the torus of period L this is equivalent to "$d \le 4$ and L even". In the version for k-cells, $d \le 3k + 1$, and the density is exactly $1/(2(k+1))$	Lean	`no_perfectZ_of_five` (assumption: $5 \le d$), `perfectK_one_iff` (assumption: $2 \le d$) and others
Corollary 3 (a)	For $\beta < \sqrt{2/27}$, the marginal measure satisfies the Bakry–Émery condition with curvature at least $2 - 27\beta^2$, and has a logarithmic Sobolev inequality uniform in the volume and the boundary conditions	Cited (the Bakry–Émery theorem, the Ricci curvature of S^3) + Theorem 1 of this paper (the row above)	—
Corollary 3 (b)	Uniqueness of the infinite-volume Gibbs measure and exponential decay of correlations (a mass gap in lattice units)	Assumption (a sketch following the earlier logarithmic-Sobolev arguments at strong coupling; not written out line by line)	—

Scope. All that can be said is that "the constant improves within the Bakry–Émery method"; it does not say that the range of strong coupling has been widened. It says nothing about the continuum limit (5-11).

04

Paper on vortex-insertion identities — the vortex-insertion identities

Abstract (draft). The $SU(2)$ Wilson lattice gauge theory is written in the twist-summed ensemble of Mack–Petkova, and a family of identities without approximation is shown for local centre-vortex insertions. For an involutive insertion τ aligned with the sector (change of action ΔS) and an observable H odd under τ: $\mathbb{E}[H] = \mathbb{E}[H \tanh(\Delta S/2)]$ and $\mathbb{E}[H\sqrt{\rho}] = 0$ ($\rho = e^{-\Delta S}$). Combining insertions in separated cells and averaging over the position of the

insertion gives $|\langle W(C)\rangle| \le \mathbb{E}\prod_c |\bar{t}_c|$ (each factor in $[-1,1]$). For insertions driven by reversible heat-bath dynamics, a Crooks-type identity $\mathbb{E}[H] = \mathbb{E}[H\tanh(W/2)]$ holds for the work W. The slab-confinement theorem for every β is recovered as the special case of a thin insertion. The draft places these as rewritings within the framework of Tomboulis–Yaffe, Mack–Petkova and Kovács–Tomboulis, and says nothing about the continuum limit or about the problem itself.

Table 4 The main theorems of the paper on vortex-insertion identities (the book's labels are those of 25-05. The conditions throughout: a finite configuration space, positive weights, a bijection that flips the sector)

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
bounded identity	If τ is an involution and $H\circ\tau = -H$, then $\mathbb{E}[H] = \mathbb{E}[H\,t]$, $t = \tanh(\Delta S/2)$	Proved here	BoundedInsertion (`Bounded.single`) (assumptions: τ is an involution, the weights are positive, $H\circ\tau = -H$)	25-05 § 04
Bhattacharyya identity	Under the same conditions, $\mathbb{E}[H\sqrt{\rho}] = 0$	Proved here	SecondMoment (`SecondMoment.sqrt_identity`) (the same)	25-05 § 05
averaged bound	For insertions in separated cells averaged over position, if $\lvert H\rvert \le 1$ and it is odd under every insertion, then $\lvert\mathbb{E}[H]\rvert \le \mathbb{E}\prod_c \lvert\bar{t}_c\rvert$	Proved here	BoundedInsertion (`Bounded.averaged`, `Bounded.averaged_bound`) (assumptions: the t of other cells is invariant under the insertion, $\lvert H\rvert \le 1$)	25-05 § 04
Paley–Zygmund form	For one cell, $\mathbb{E}[H]^2 \le 1 - \mathbb{E}[Y]^2/\mathbb{E}[Y^2]$, $\mathbb{E}[Y^2] \le 1$	Proved here	SecondMoment (`SecondMoment.pz_bound`) (does not include the part $\mathbb{E}[Y^2] \le 1$)	25-05 § 05
slab confinement (corollary)	With thin insertions, $\lvert\langle W(C)\rangle\rvert \le \tanh(\beta N^2)$ $A(C)$ (slab, N and the lattice spacing fixed, every β)	Proved here	— (the bundle for the slab theorem is in 25-04)	25-05 § 06, 25-04
driven identity	If $H = h(\mathrm{frame})\,\sigma$, then $\mathbb{E}[H] = \mathbb{E}[H\tanh(W/2)]$ (W is the work of an insertion driven by reversible heat bath)	Proved here (the Crooks and Jarzynski identities are Cited)	—	25-06
Numerics	On $2\times2\times8\times8$, $\beta = 2.3$, $n12 = 1$, with a disc of radius 2, driving with 16 heat-bath steps takes the mean work $10.7 \to 2.9$ and the mean of $\tanh(W/2)$ $0.99 \to 0.69$	Reading (the draft does not state the number of seeds or the errors)	—	25-08

Novelty. The draft lists things such as "local identities in the twist-summed ensemble" as its own contributions, but the book has not checked them against the earlier literature and does not claim novelty (25-05 § 08 (d)).

05

Paper on the reduction of the band — the reduction of the band, and numerics for $SU(2)$ and $U(1)$

Abstract (draft). Heads towards lattice confinement for four-dimensional $SU(2)$ by the road of the slab $T^2_L \times \mathbb{Z}^2$ through a small twisted torus T^2_L. It states four points. (i) The naive condition "$\theta(L,L) \le \theta_0 < 1$ uniformly in L" is also satisfied by four-dimensional $U(1)$, so it does not imply confinement; the correct target is $\ln(1/\theta(L,L)) \ge cL^2$ (a local form of the 't Hooft criterion). (ii) Combined with nesting in the planar directions, the target reduces to a single lateral-widening inequality. This inequality fails for $U(1)$ and $\mathbb{Z}_2$, and for $SU(2)$ in the dilute regime its sign is set by $11/3 - 2 = 5/3 > 0$. (iii) Reflection positivity, the identities of centre and duality, and positivity of the transfer matrix hold just as well for $\mathbb{Z}_2$ and $U(1)$, so structural inputs alone cannot show the lateral-widening inequality. (iv) Monte Carlo on the slab shows the predicted contrast. At one loop, the comparison from maps isometric in the fluctuations can give only the $U(1)$ direction $\beta' \ge \beta + 0.023$, while maps that reshape the fluctuations give $\beta - \beta' = 0.258$. It shows neither confinement in the continuum limit nor the mass gap.

Table 5 The main statements of the paper on the reduction of the band (the book's labels are those of the chapters on the right)

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
uniform θ_0 is insufficient	From the naive form of the band $\theta \le \theta_0$ only the lower bound on the tension $\ln(1/\theta_0)/L^2 \to 0$ follows. The Coulomb phase of $U(1)$ satisfies the naive form of the band	Proved here (the logic of the shortfall) / Reading (the count for $U(1)$, a physical estimate; the draft also states it is heuristic)	—	26-01
conditional theorem	Under the three of the base point on a small T^2, frame decorrelation and the lateral-widening inequality, $\theta(R,R) \le e^{-cR^2}$, and an area law with tension uniform in L	Proved here (conditional. The draft's proof is a sketch; the book rewrote the assumptions as numerical inequalities and placed a proof)	Doubling (`Doubling.iterate`, `Doubling.decay`) (assumptions: $c \ge 0$, $t_k \ge 0$, $t_{k+1} \le c^3 t_k^4$. Only the algebra of the iteration)	26-02 § 03
necessary condition	The lateral-widening inequality (strong form) contains that the slab tension $\sigma_{\mathrm{eff}}(L)$ does not decrease in L (stronger than confinement)	Proved here	—	26-02 § 05
the inequality separates the theories	The lateral-widening inequality fails for $U(1)$ and $\mathbb{Z}_2$, and for $SU(2)$ in the dilute regime it grows as $\ln(1/\theta) \propto A_D L^{5/3}$	Reading (a physical estimate; the draft also states it is heuristic)	—	26-03
no-go from structure	The flux estimate, the electric-flux form, positivity of the transfer matrix, the identities of centre and duality and the lattice symmetries hold just as well for $\mathbb{Z}_2$ and $U(1)$, so from this set of inputs alone the lateral-widening inequality does not follow	Proved here (in the form "from this set of inputs alone")	FluxDuality (`FluxDuality.split`, `FluxDuality.twisted_le`, `FluxDuality.bias_mem`) (for $\theta \le 1$, positivity of the twisted partition functions is assumed)	26-04
site-reflection positivity	The z ensemble with the twist fixed is positive under reflections through lattice points in directions 3 and 4 (M even). Transverse reflections are not positive because of the twist	Proved here (reflection positivity of z. The proof is placed in 26-04 § 03. This row is closed)	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`) (assumption: $A \ge 0$. The algebra of the core)	26-04, 27-02 reflection positivity of z
chessboard (corollary)	With L fixed, frame decorrelation is replaced by a single density κ_{cb}, and $\lvert\langle W(C)\rangle\rvert \le \kappa_{\mathrm{cb}}^{n(C)}$	Cited (the chessboard estimate) + Proved here (the statement with L fixed)	—	26-05
flux bound	For a stationary kernel that flips the sector, $\theta \le 1 - a$. For a one-step kernel $a \lesssim \frac{1}{2}$, so it gives only an estimate of the naive form of the band	Proved here	FluxLemma (`FluxLemma.flip_le_one_sub_abs`)	25-06
The one-loop comparison	The comparison isometric in the fluctuations gives $\beta' \ge \beta + 0.023$ (per doubling), and the comparison that reshapes the fluctuations gives $\beta - \beta' = 0.258$	Proved here (at the one-loop level)	—	25-07, 26-06 § 03

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
Numerics	Widening the T^2 raises the free energy of the global flux for $U(1)$ and does not raise it for $SU(2)$. The slab tension increases and saturates. In the lattice-doubling test, the $\beta - \beta'$ read off is $+0.13$–$+0.25$ for $SU(2)$ ($\beta = 2.5$–3.0, 1–3 seeds), and in the opposite direction for $U(1)$	Reading (the conditions on lattice sizes and errors are in 26-00 § 03 (ii) and the rows of 26-09)	—	26-09, 26-06 § 05

A planned addition in the record. Adding to the paper on the reduction of the band the scan with step functions and the $U(1)$ control (the sign flip at $\beta_c \approx 1.01$) is listed as one row in the order of work (27-03 § 06).

06

Paper on lattice fractional instantons — lattice confinement by fractional instantons

Abstract (draft). In four-dimensional $SU(2)$ Wilson lattice gauge theory on $T^2_N \times T^2_M$, with the 't Hooft twist $n_{12} = 1$ put on the small torus T^2_N, it considers the bias $\theta(N, M, \beta)$ of the flux n_{34} (the quantity that bounds large planar Wilson loops through the Tomboulis–Yaffe inequality). At fixed N, an area law with rate $\approx e^{-2\beta N^2}$ follows elementarily from the cost of the surface of a thin $\mathbb{Z}_2$ vortex. The draft shows a lower bound that replaces the rate by the minimal action $S_*(N, \ell)$ of a fractional instanton inside a frame frozen to the vacuum. At fixed M it also shows the bound in the opposite direction, sandwiching the exponential rate between $S_{\min}(N, M)$ and $S_*(N, M/2)$. It is not uniform in N and says nothing about the continuum limit.

Table 6 The main theorems of the paper on lattice fractional instantons (the book's labels are those of 25-02)

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
Confinement Lower Bound by Fractional Instantons (Theorem A of the paper, confinement direction)	For all $N \geq 2$, $\ell \geq 2$, $\beta \geq 1$, $M \in 2\ell\mathbb{N}$, $-\ln\theta \geq c(N,\ell)\,\beta^{-a} e^{-\beta(S_*(N,\ell)+\varepsilon(\beta))} M^2$, $\varepsilon(\beta) = O(\sqrt{\ln\beta/\beta})$. It beats the thin-surface estimate only beyond a large β that is not made explicit	Assumption (proof in the draft. It uses the Chessboard Estimate and the Morse–Bott lemma by citation, and the rigidity constants are only shown to exist. Not raised until a proof is placed in the text)	Only the factorisation of the column representation and the binomial estimate (ColumnChessboard (`ColumnChessboard.signed_sum`, `ColumnChessboard.binomial_bound`) and others)	25-02 § 06, 26-06 § 04
Reverse Estimate at a Fixed Plane (Theorem B of the paper, fixed-M reverse bound)	For all $N, M \geq 2$, $\beta \geq 1$, if $x = 2\exp(-\beta S_{\min}(N,M) + N^2 M^2 (12\ln\beta + 34)) \leq \frac{1}{2}$ then $-\ln\theta \leq 2x$. A statement with M fixed	Assumption (the same)	—	25-02 § 06
Bracketing of the Tension Rate (Corollary C of the paper)	At fixed N and $M \in 2\ell\mathbb{N}$, the lower and upper limits of $-\ln(-\ln\theta)/\beta$ as $\beta \to \infty$ lie between $S_{\min}(N,M)$ and $S_*(N,\ell)$	Assumption (on top of the Confinement Lower Bound by Fractional Instantons and the Reverse Estimate at a Fixed Plane)	—	27-02 upper end of the renormalisation comparison
trivial bound (proposition)	For every frame $\lvert\theta_t\rvert \leq \tanh(\beta N^2)$, and at $\ell = 2$, $-\ln\theta \geq \frac{M^2}{2} e^{-2\beta N^2}$	Proved here (of the same type as the slab-confinement theorem)	—	25-04

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
Upper bounds with certificates	$S_*(2,12) \le 0.6762\pi^2$, $S_*(3,12) \le 0.9193\pi^2$, $S_*(4,8) \le 0.9707\pi^2$ (estimates of the action of concrete configurations; only in the upper-bound direction. The value for $N=4$ is one where cooling has not converged)	Computer-assisted (upper bounds only; not attached to the minimum itself)	—	25-02 § 05

What is open. The converse bound uniform in M (an upper bound by a dilute gas), and the agreement $S_* = S_{\min}$. The draft reduces the former to a single positivity lemma (27-03 question "reverse direction of the fractional-instanton lower bound", question "agreement of the minimal actions"). The direction of increasing N is the same problem as the middle of the band.

07

The papers and the Lean bundles — names of bundles, numbers of theorems, what they guarantee

Table 7 lists the machine-checked part of each paper by bundle (a set of Lean files). Every bundle guarantees only the algebra of finite sums and real numbers from the list of assumptions to the conclusion; the Haar measure, path measures, the topology of sectors, that θ is the bias of a sector, and positivity of the transfer matrix are outside Lean. The location and naming of the bundles follow appendix A-04.

Table 7 The Lean bundles of the papers (the number of theorems is, for the one-sided bridge, the number of rows of the axiom record; for the bundles of Parts 25 and 26, the number of theorem and lemma declarations in the files)

Paper	Bundle (files)	Theorems	Contents	Axioms
The one-sided bridge	EuclideanPath.GapBridge	17	The closed form of the ratio for an interval of times, the abstract form of the Poincaré lower bound, sums over families local in time, the corollary of the bridge, the rotor	A subset of the three standard axioms (the published axiom record, A-04 Table 4c)
One-link integration	LatticeGaugeOneLink (StarInequality, StapleCurve, OneLinkSeries, CompleteFamilyLattice, StaplePerturbation, WilsonOneLink), PerfectFamily, PlaquetteCounting	— (the axiom record in the list of papers)	The chain from the Wilson action to the lower bound on the Hessian of Φ, the combinatorics of complete families	A subset of the three standard axioms (the published axiom record)

Paper	Bundle (files)	Theorems	Contents	Axioms
Paper on vortex-insertion identities	InsertionIdentity (namespace Insertion, 5), BoundedInsertion (namespace Bounded, 5), SecondMoment (2), MultiCell (3)	15	Identities for the insertion ratio, the tanh form, averaging over position, the second moment, composing cells ("odd under insertion" is an assumption)	A subset of the three standard axioms (passed the re-check on the side of this book that reads the axioms again)
	FlipLemma (2)	2	The estimate of the flip probability and the cutoff formula (the flow lemma of 25-06)	
	Containers (2)	2	The factorisation into a product of containers and the container estimate (25-03)	
Paper on the reduction of the band	FluxLemma (1)	1	The flux estimate in the form $\theta \le 1 - \lvert a \rvert$	
	FluxDuality (3)	3	Decomposition of the electric flux, the estimate of the twisted sum, the range of the bias (positivity is assumed for $\theta \le 1$)	
	Doubling (2)	2	The doubling iteration $t_{k+1} \le c^3 t_k^4 \Rightarrow c t_k \le (c t_0)^{4^k}$, and decay	
	ReflPos (2)	2	The algebra of the core of reflection positivity (Cauchy–Schwarz on weights with $A \ge 0$)	
Paper on lattice fractional instantons	ColumnChessboard (5)	5	Signed and unsigned sums of the column representation, expansion of the product, the binomial estimate, the exponential estimate	

The bundles of Parts 25 and 26 come to 11 files and 32 theorems and lemmas in all. In addition there is a separate bundle (150 theorems) containing the Slab Confinement Theorem and the container lemmas (`SlabConfine` , `SlabAverage` , `FrameLemma` , `CommutingFlips` and others), which 25-03 and 25-04 refer to. It is not a bundle of the papers, and it has passed the same re-check. The names and location of the bundles will be decided when they are added to the table of bundles in appendix A-04.

What this chapter can say

What is said	What is not said
The titles, stages and page counts of the five papers, and the statements and conditions of their main theorems (Tables 1–6)	The correctness of the proofs of the three drafts. Rows whose label in the book is Proved here rest on proofs checked in chapters of the book, and rows labelled Assumption rest on the statements of the drafts
The reach of machine checking for each paper (Table 7)	That the Lean bundles check the statements of the papers themselves (measures, sectors, limits)
That none of the papers claims a solution of the problem of existence and the mass gap	Outside checking of the papers (none of them has been through checking by outside mathematicians)

Sources and reproduction

Item	Kind	Source / tools
Titles, abstracts and statements of the one-sided bridge and one-link integration	This book	The PDFs and LaTeX sources on this site's papers page (list of papers). The correspondence between theorem numbers and Lean declarations is the correspondence table of the one-sided bridge paper (§ 9)
Titles, abstracts and statements of the three drafts	This book	The LaTeX source of each draft (paper on vortex-insertion identities: 2026-10-13, 3 pages; paper on the reduction of the band: 2026-10-13, 9 pages; paper on lattice fractional instantons: 2026-10-14, 7 pages). The book's labels are from the classification tables of the chapters in the right-hand column

Item	Kind	Source / tools
Numbers of theorems in the Lean bundles	This book	For the one-sided bridge, the published axiom record (17 rows). For the bundles of Parts 25 and 26, the theorem and lemma declarations in each file were counted (11 files, 32)
The twist-summed ensemble, the container identity	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the source table of 25-03)
Upper bounds on Wilson loops (from the vortex free energy)	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. T. Kanazawa, arXiv:0808.3442 (the source table of 26-01)

Terms used in this chapter — published, draft, the main theorem, label, bundle, the twist-summed ensemble, the crossover band, the lateral-widening inequality, fractional instanton, container. Their meanings are in the A-02 Glossary.

Next page to read: A-08 A map to the mass gap. To read alongside: Contents. Which condition rows of the book the statements of the papers bear on is in 27-02 The ledger of conditions.

A map to the mass gap — with the published construction as the foundation

This chapter is a map of what is unsolved. The reading of the published construction is in Parts 6–13, and the theorems of this book's study are in Parts 19–22.

What this chapter shows — With the construction of four-dimensional Yang–Mills theory published on 2026-11-10 taken as the foundation, what must be guaranteed on the way to a proof of the mass gap. In three tables we set out what is cited as the foundation, the seams that remain on the ultraviolet side, what remains on the infrared side (the crossover band), and where the theorems closed by this book's study stand.

Prerequisite chapters — 6-04 What is not shown — a bridge to Volume 3, 16-03 The infrared edge of the published construction, 16-04 Where this book stands. For how to read the labels, see 6-00.

Said first — The standing of the theorems of the published construction is "the committee's assessment is that they are proved in Lean". What this book has read goes as far as the definitions and the form of the statements, and the committee itself writes that it does not yet claim to understand the proof completely Assessment . The "Now" column is written within the range in which this book has read the source and the assessment. This map is a set of signposts, not a scale of progress. This book does not claim to solve the problem of Yang–Mills existence and the mass gap.

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The labels used throughout the book (Cited , Lean , Computed and so on) have the same meanings as before (for details, see "Labels" in 6-00).

Order of this chapter

1. Role of this chapter
2. What must be shown — one line, and the shape of the question
3. The foundation — what is cited from the published construction (Table 1)

4. What remains on the ultraviolet side — the seams (Table 2)
5. The infrared side — the crossover band (Table 3)
6. Guarantees needed beyond the tables
7. Where the theorems closed by this book's study stand
8. What cannot be said

01

Role of this chapter

The existence of four-dimensional Yang–Mills theory: for the construction published on 2026-11-10 (Lean code and the committee's assessment), the committee's assessment is that it is proved in Lean (Part 6). The mass gap is not shown (6-04, 16-02).

This chapter takes that construction as the foundation and sets out again what must be guaranteed on the way to a proof of the mass gap. It reads as follows.

1. In § 02 we write in one line what must be shown.
2. In Table 1 of § 03 we list what is cited from the published construction as the foundation.
3. Table 2 of § 04 is what remains on the ultraviolet side, and Table 3 of § 05 what remains on the infrared side. On the ultraviolet side these are "seams"; on the infrared side, the "core".
4. In § 07 we write where on this map the theorems closed by this book's study stand.

02

What must be shown

One line — The correlations of renormalised lattice observables decay at an exponential rate, in physical units, uniformly in the cutoff and the volume.

The decay is needed for all members of a family that is dense in the Osterwalder–Schrader space (field words of every degree). The prefactor may depend on the observable; the rate must be uniform. Once this holds, the mass gap of the limiting theory and the one-dimensionality of that theory's zero-energy space (uniqueness of the vacuum) follow from known general theory Cited . Uniqueness of the infinite-volume limit does not follow from this. What the published construction has goes only as far as an estimate at a polynomial rate, uniform over the phases Source (12-08).

The shape of the question — The published construction takes the continuum limit while holding the coupling of the coarsest step (the lattice of spacing 1) at a reference value $\bar{\beta}$ (16-03). So, speaking of one construction, the main part of the remaining question is expected to be brought down to a question on the lattice: "does the effective theory at the spacing-1 step have exponential decay at long distances?" Inferred . To restate it that way, however, the top three rows of Table 2 are needed.

- The theory at the spacing-1 step is not the Wilson action but a theory of an effective action with a representation, and the reference coupling is a coefficient of it. The representation is not unique Source (10-01).
- "Long distances" means, by one-loop extrapolation, of the order of $e^{\bar{\beta}/s}$ in units of spacing 1 (s is the one-loop slope) Inferred .
- From just one of the conditions imposed on the reference coupling ($x^{-1/2}e^{(\log\log x)^2} \leq 1$ for all $x \geq \bar{\beta}$), $\bar{\beta} \geq e^{13.7} \approx 9 \times 10^5$ is needed Computed . In the standard conventions for SU(2) this is about $g^2 \leq 4.5 \times 10^{-6}$ (conversion as in 8-01). So the distance above is an astronomical length, at least $e^{2.4\times 10^6}$.

The remaining question is about as hard as, or harder than, the long-open question of the mass gap of weakly coupled lattice gauge theory Inferred .

03

The foundation — what is cited from the published construction

Item	Form in the published construction	Caution
Continuum limit in a finite periodic box	Wilson's four-dimensional lattice, any compact simple group. In the cutoff, convergence along the whole sequence of spacings Source (8-05, 11-10)	Only cubic boxes with periodic boundary conditions
Infinite volume, phase decomposition, reconstruction	In the volume, along a subsequence. Wightman axioms for each phase Assessment (12-05, 12-07, 12-08)	No convergence or uniqueness of the limit
Running of the coupling on the weak-coupling side	The coupling at each step lies between half and twice "the reference value + the one-loop slope × the logarithm of the scale ratio" (as a coefficient in the representation) Source (10-05, 11-00)	Not a quantity determined by the measure (a coefficient of the effective action). It works only where the coefficient is larger than the entry value of the one-step estimate. The one-step estimate is for odd blocks with a large side
All groups	Existence, for any compact simple group Assessment	—

Table 1 The foundation. The standing of the theorems of the published construction is "the committee's assessment is that they are proved in Lean".

04

What remains on the ultraviolet side — the seams

Ultraviolet control, as existence, has entered the foundation. What remains are the seams that put the ultraviolet results into a form the infrared argument can use.

What is to be guaranteed	Now	Needed for the mass gap?
Fixing the theory at the spacing-1 step as an object (the limit with the cutoff removed, a representation of the same form, consistency between steps)	That the ingredients and density of the representation converge in the cutoff is in the statements Source . The limiting object and its representation are not in the statements (11-10, 16-03)	Needed
Local estimates uniform in the volume	The one-step ledger is in a form independent of the volume. The estimates at the level of densities contain exponents of the volume, and an estimate uniform in the volume for local observables is not in what we have read Source	Needed
The bridge for observables (continuum correlation = expectation of the image of the observable at the spacing-1 step)	There is an estimate of the difference between the read-out at fine steps and the true correlation. The image and the equality at the spacing-1 step are not there Source	Needed
Translation into quantities determined by the measure (the coupling as a coefficient of the representation ↔ vortex ratios, slab tension, coarse Wilson loops)	None	Needed on this book's route
Twisted boundary conditions and elongated boxes	None (only periodic cubic boxes)	Needed on this book's spatial route (thickening from a small twisted box, Parts 25–27). The conditions of that route depended on ultraviolet control on a small twisted box, so this becomes the first question
How reflection positivity is carried	Present only for the original Wilson measure Source (12-01). Not stated for the measures at renormalised steps	Depends on the route (not needed if the decay is stated for correlations of the original lattice)

What is to be guaranteed	Now	Needed for the mass gap?
Uniqueness of the infinite-volume limit, independence of the sequence of spacings and the reference value	None	Not needed for the mass gap of one theory. If uniform exponential decay holds, that limit has clustering Inferred
Understanding of the foundation, and independent verification	What this book has read goes as far as the definitions and the form of the statements. The committee also writes that it "does not yet understand it completely" Assessment	Reliability of the foundation

Table 2 What remains on the ultraviolet side. The top three are the main line; the fourth and fifth are for this book's route.

05

The infrared side — the crossover band

The core is the crossover band between the strong-coupling end (the side where the mass gap is easier to show) and the weak-coupling end (the side where the continuum limit is taken) (26-00).

What is to be guaranteed	Now
Fixing the two ends of the band	The strong side is this book's Lean window: the infinite-volume limit and the mass gap for $\lvert\beta\rvert < 1/19$, the area law for $\beta < 1.1 \times 10^{-3}$ in a finite box and $\beta < 1.2 \times 10^{-4}$ in infinite volume Lean (20-03, 20-06). The weak side is the lower bound on the coefficient of the effective action required by the one-step estimate of the published construction (not the β of the Wilson action). The entry value is in an existence form, "it holds if large enough", and the source gives neither a formula nor a number Source . From one of the conditions on the reference coupling alone, at least 9×10^5 Computed . The two ends are stated for different quantities and different classes of action, so restating them in the same quantity is also needed
The core of the band — a quantity that is preserved across scales, distinguishes from U(1), and gives decay on the strong side	None. This book's form is the tension-doubling inequality for slab tension (the reduction is in Lean; the lateral-widening inequality itself is open, 26-02). Another form is a mixing condition in a finite box (the time-direction route, 26-07)
Extending the strong-coupling end to classes of effective actions	For the Wilson action it is in Lean (infinite volume, mass gap, area law, vortex floors). Not for classes of effective actions. The window is narrow
Bringing the decay back to the original observables	None

What is to be guaranteed	Now
From tension (area law) to the mass gap	False as a general implication. What should be added has not yet been stated
From decay to the spectrum	General theory is known Cited . For the strong-coupling lattice it is in Lean (20-04). Not within the reconstruction framework of the published construction
Groups without a centre	The vortex tools cannot be used. The mass gap for all groups needs other tools

Table 3 The infrared side.

How to read it.

- The published construction gives no foothold inside the band. It carries every step at weak coupling and stops at spacing 1 Inferred (16-03).
- At the weak end there is a non-perturbative input that "the finer the step, the weaker the coupling" (as a coefficient of the effective action; an input that uses non-commutativity) Source . But it works only at the very weak-coupling end, where the coefficient is 9×10^5 or more.
- The "input that distinguishes from U(1)" missing from the argument on the band is needed in the middle section (around $\beta \approx 1$, where U(1) has its transition), and the input above does not address it Inferred .
- To use the input above in this book's doubling inequality, four things are needed Inferred : (i) the relation between the coefficient of the representation and the ratio of twisted partition functions and slab tension, (ii) a comparison between the effective action and the Wilson action, (iii) transfer to the one-step map with sides 2 and 3, (iv) the construction with twisted boundary conditions.
- To reach the mass gap rather than confinement, the row "from tension to the mass gap" of Table 3 is needed. The tension route alone is not enough.
- The map of ideas this book's study has tried on the band (alive, broken, undecided) is in 16-11, and the seven models on which arguments about the band are tested are in 26-12.

06

Guarantees needed beyond the tables

Besides the rows of Tables 1–3, we list guarantees that will be needed when a proof is written out Inferred .

1. Decay at a uniform rate for field words of every degree (the prefactor may depend on the observable).

2. The order of volumes: uniformity where the side of the box is much larger than the correlation length. On the route with twisted boundary conditions, also the order of limits for elongated boxes.
3. Identification of the state: if the decay is obtained with twisted boundary conditions or in a pure phase, the theory built from that state must satisfy the axioms and be the same as the theory of the published construction. Where decay of a mixed state is needed, decay in each pure phase may not be enough (the low-temperature Ising model is an example).
4. Lorentz covariance: used to turn the Hamiltonian gap into a mass gap. In the published construction it is present for the component theories Assessment .
5. General SU(N): this book's Lean theorems are for SU(2) only.

07

Where the theorems closed by this book's study stand

The Lean theorems of this book's study are all statements at fixed lattice spacing, and are formalisations of results of known types Lean (listed in Table 1 of 16-04). On this map they stand in the following places.

- **The strong-coupling end**: exponential decay of correlations, the infinite-volume limit and clustering, the mass gap of the reconstruction container (in lattice units), the area law for Wilson loops, an upper bound on the vortex free energy (20-02–20-06). This is the strong end of the first row of Table 3. The rate per unit area of the upper bound on the vortex free energy does not depend on the transverse size, but only within the strong-coupling window, and the prefactor (the transverse area) remains.
- **At any coupling**: reflection positivity, the thin-vortex floor and the thick-vortex floor, pairs of Polyakov loops, continuity, limits and matching of vortex ratios (Part 21). None of these is uniform in the transverse size.
- **The weak-coupling side**: rarity of large surfaces at one scale (22-01). It says nothing about scales after renormalisation.
- **Specific to this book's design**: a fixed block map independent of the cutoff (consistency between steps written as a push-forward identity), reflection positivity of the pushed-forward measure, matching by quantities determined by the measure (vortex ratios, averages over coarse surfaces) (14-06). The matching is still at the stage of design choices.

What this book's study has on the band is a conditional implication and a list of premises (27-01, 28-04). It has not shown that the premises are met. The question aimed at by the conditional implication about weak ultraviolet existence in a small box

that this book's study assembled (2026-10-30) has been answered in a stronger form by the theorems of the published construction Inferred . The premises of that implication are statements about this book's design and do not follow from the results of the published construction.

There are three candidates for the next move Inferred . (1) Read how the ratio of twisted partition functions comes out of the representation of the published construction. (2) Read the entry value of the one-step estimate (the weak end of the band). (3) Widen the window at the strong-coupling end and extend it to classes of effective actions.

08

What cannot be said

1. Even if all rows of the tables were filled, we do not say that this would solve the problem. The tables list the guarantees visible within what this book has read, and we do not say they are exhaustive.
2. The rows cannot be counted and read as "so many percent left".
3. The "Now" column covers what was read of the definitions and the form of the statements of the published construction. This book has not checked that the content of the proofs is correct.
4. This book's strong-coupling results (the infinite-volume limit, the mass gap, the area law, the vortex floors) are statements at the end opposite to the remaining question (strong coupling, fixed lattice spacing).

A cross-cutting list of what is not claimed is in 27-04, and what the published construction does not show is in 6-04.

What this chapter can say

Can say	Cannot say
Source The "Now" column of Tables 1–3 was written within the range in which Parts 6–13 read the source and the assessment of the published construction	That the content of the proofs in the source is correct. Statements that may be in parts not read
Computed The lower bound 9×10^5 on the reference coupling is a lower bound derived from one condition only	The true entry value of the one-step estimate (the source gives only existence)
Inferred The expectation that the main part of the remaining question can be brought down to a long-distance question for the effective theory at the spacing-1 step	That it can be restated that way (the top three rows of Table 2 are needed). The comparison of difficulty
Lean The range of this book's theorems in § 07 (fixed lattice spacing)	That they cross the band. This chapter adds no new results

Sources and reproduction

Item	Kind	Source, tool
Published code	Source	yang-mills-group/OpenAI-YM-20261028 (Apache-2.0, submitted 2026-10-28, revision 3456e5e). Locations in the chapters of Parts 8–13
Assessment	Assessment	docs/assessment-1.pdf in yang-mills-group/YM (2026-11-10, revision 55ba9fc)
Committee statement	Cited	The third statement on the Kavli Institute for Theoretical Physics page (2026-11-09). Signed by Michael R. Douglas, David J. Gross (chair), Arthur Jaffe, Igor R. Klebanov, Nikita Nekrasov
Condition on the reference coupling	Source, Computed	`Bar.threshold` in the source (`YM/Price/Intent.lean`). The smallest $\log \bar{\beta}$ for which $(\log\log x)^2 \le \frac{1}{2}\log x$ holds for all $x \ge \bar{\beta}$ was found in floating point (about 13.71)
Range of this book's theorems	Lean	Table 1 of 16-04, Table 4g of A-04
Problem statement	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000). How to read it: 0-03

Next page to read: Contents.