A one-plaquette Schwinger–Dyson identity and a multi-level sum rule for the coarse stiffness in SU(2) lattice gauge theory, with machine-checked arithmetic

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Abstract

For SU(2) lattice gauge theory with the Wilson action, the Schwinger–Dyson integration by parts, summed over the four links of a plaquette, writes the law of $y = \beta(1-\cos\psi)$ in the exact one-dimensional form $\rho(y) \propto \sqrt{y(2-y/\beta)}\exp(-\int_0^y(1+\Gamma))$. Here the *stiffness* $\Gamma(y)$ is given by a conditional expectation. Summing over all fine links gives the same form for a plaquette of a block-averaged lattice, and the coarse stiffness splits exactly as $1+\Gamma_k = R_k - (B_k-1)h_k + (\ln \bar{a}_k)_y$. In the free field, the identity $d^{\mathsf{T}}P = d^{\mathsf{T}}$ for the projection P onto exact plaquette fields gives $R_k = 1/P_{pp}$ at every level and for every linear averaging rule. This value is $d/2$ on $\mathbb{Z}^d$, and on a finite torus Λ it is $(d/2)(1-1/|\Lambda|)^{-1}$. Hence, in the free field, $\Gamma_k \equiv 1$ in four dimensions in infinite volume. At first order in $1/\beta$, the coefficients of $\Gamma(y) = 1-(c_0-c_1 y)/\beta$ are fixed by the first-order shifts w_2, v_2 of two moments of y: $c_1 = 32w_2/3 - 64v_2/15$, $c_0 = \frac{1}{4} + 8w_2/3 + 5c_1/4$. With the two-loop constant $w_2 = 0.151383$ and computed constants of one averaging rule, at first order $\Gamma_k \geq 1 - C/\beta_k$ with $C \leq 2.259$ on the first two levels. The Schwinger–Dyson equations are classical. What this note adds is the multi-level form and a machine check of its algebra (Lean 4, six files, 64 theorems). The integration by parts, Gaussian conditioning, the first-order expansion and the numerical constants are outside Lean.

1 Introduction

The question behind this note is the Yang–Mills mass-gap problem [2]. Nothing here addresses it directly. In a block-spin (multi-level) approach one wants to know how the law of a single plaquette changes from level to level. On the fine lattice the plaquette variable $y = \beta(1 - \frac{1}{2}\operatorname{tr} U_p)$ is close to a Gamma$(3/2, \frac{1}{2})$ variable for large β. A coarse lattice has no exact action, so it is not obvious how to say that a coarse plaquette "returns to the same shape".

The tool, classical in lattice gauge theory, is the Schwinger–Dyson (integration by parts) identity for the Haar measure. We sum it over the four links of one plaquette or, for a coarse plaquette, over all fine links. The summed identity turns the law of one angle into the invariant law of a one-dimensional diffusion. We do not know whether this form, or the multi-level sum rule of §4, has been noted before.

Numbers quoted without a citation are taken from the authors' working records (programs, outputs and reports), which are not published.

Status of the results. The one-plaquette and multi-level densities (Propositions 3.1, 3.2) are derived by hand under stated regularity hypotheses; their algebraic split is in Lean. The free-field statement (Proposition 4.2) is derived by hand at Gaussian order, with the matrix identities and the arithmetic in Lean. The moment inversion (Proposition 5.1) is in Lean; the first-order form it inverts is a sketch by hand. The bound $\Gamma_{\min} \geq 1 - C/\beta_k$ (§5.4) is first-order perturbation theory with cited and computed inputs and an uncontrolled $O(\beta_k^{-2})$ remainder; only its arithmetic is in Lean. Grades: *Lean* (theorem named), *by hand* (not machine-checked), *computed* (floating point), *cited*, *assumption*.

2 Setting

Let Λ be a finite periodic hypercubic lattice in $d = 4$ dimensions, with links ℓ and plaquettes p. The fine measure is

$$\mu(dU) \propto e^{-S(U)} \prod_{\ell} dU_\ell, \qquad S(U) = -\beta \sum_p \cos\psi_p, \qquad \cos\psi_p = \tfrac{1}{2}\operatorname{tr} U_p,\ \ \psi_p \in [0,\pi],$$

with dU_ℓ the Haar measure on SU(2) and U_p the ordered product around p. The plaquette variable is $y_p = \beta(1 - \cos\psi_p) \in [0, 2\beta]$. On the link ℓ we use the left-invariant fields X_ℓ^a ($a = 1, 2, 3$) generated by $\frac{i}{2}\sigma_a$. We write $\nabla_\ell f = (X_\ell^a f)_a$ and $\Delta_\ell = \sum_a X_\ell^a X_\ell^a$. Each X_ℓ^a has zero divergence for the Haar measure, so for smooth g

$$\mathbb{E}[X_\ell^a(g)\,] = \mathbb{E}[g\, X_\ell^a S]. \tag{1}$$

Averaging rules and coarse plaquettes. A coarse link L of a lattice with spacing 2 is assigned $V_L = Q_L/|Q_L|$ on the configurations where $Q_L \neq 0$. Here $Q_L = \sum_i w_i P_i$ is a weighted sum (as quaternions) of ordered products P_i of fine links along paths i from the start to the end of L, with real weights w_i, $\sum_i w_i = 1$. Negative weights are allowed; the rule of Appendix A has some. This assignment is gauge covariant. The coarse plaquette angle ψ_P is defined from the V_L as above, and so are further levels. Write $U_\ell = e^{iA_\ell \cdot \sigma/2}$ and keep the terms linear in A. Then every such rule, and every composition of such rules, gives a coarse plaquette angle vector $\theta_P = (K\, dA)_P$ with K linear. Numerical constants below refer to one rule, with 83 weighted paths per coarse link. This rule is listed in Appendix A, together with the recipe for the constants r_k and γ_P.

On level k the coupling β_k is defined by matching the mean, $\beta_k\langle 1-\cos\psi_P\rangle = \beta\langle 1-\cos\psi_p\rangle$, and $y = \beta_k(1 - \cos\psi_P)$.

3 The one-dimensional form

3.1 One plaquette

Fix p and a link $\ell \in p$. Put $g_\ell(p') := \nabla_\ell \cos\psi_{p'}$ for every plaquette $p' \ni \ell$. A direct computation in SU(2) gives $|g_\ell(p)|^2 = \frac{1}{4}\sin^2\psi_p$ and $\sum_{\ell\in p} \Delta_\ell f(\psi_p) = f''(\psi_p) + 2\cot\psi_p\, f'(\psi_p)$ for a class function f. Define

$$C_p := \sum_{\ell\in p} \sum_{p'\ni\ell,\ p'\neq p} g_\ell(p)\cdot g_\ell(p'), \qquad \Gamma(\psi) := \frac{\mathbb{E}[C_p \mid \psi_p = \psi]}{\sin^2\psi}.$$

The function Γ measures how much the forces on the $2(d-1)-1 = 5$ other plaquettes through each link of p line up with the force on p.

Proposition 3.1 (one plaquette; by hand)**.** *Assume that ψ_p has a density on $(0,\pi)$ and that Γ is continuous there. Then for every $\beta>0$ and every finite periodic lattice*

$$\rho(\psi)\propto \sin^2\psi\,\exp\Big(-\beta\int_0^{\psi}(1+\Gamma(\phi))\sin\phi\,d\phi\Big),$$

$$\rho(y)\propto \sqrt{y(2-y/\beta)}\,\exp\Big(-\int_0^{y}(1+\Gamma(t))\,dt\Big),$$

with Γ read as a function of y in the second form. If $\Gamma\equiv 1$, y is distributed as for a single plaquette at coupling 2β; as $\beta\to\infty$ this tends to Gamma$(3/2,\frac12)$, whose density is $\propto\sqrt{y}\,e^{-2y}$.

Proof. Apply (1) with $g=X^a_\ell f(\psi_p)$, and sum over a and over the four links of p. The left side is $\mathbb{E}[f''+2\cot\psi_p f']$. On the right, $X^a_\ell S=-\beta\sum_{p'\ni\ell}g^a_\ell(p')$ and $\nabla_\ell f(\psi_p)=-f'(\psi_p)g_\ell(p)/\sin\psi_p$, so the sum is $\beta\,\mathbb{E}[f'(\psi_p)(\sin^2\psi_p+C_p)/\sin\psi_p]$. Conditioning on ψ_p and integrating by parts in ψ against all $f\in C_c^\infty(0,\pi)$ gives $\rho'=(2\cot\psi-\beta\sin\psi(1+\Gamma))\rho$ (the constant of integration vanishes as $\psi\to0$). Integrate, and substitute $\sin\psi=\sqrt{y(2-y/\beta)}/\sqrt{\beta}$, $dy=\beta\sin\psi\,d\psi$. □

3.2 Coarse plaquettes

For a coarse plaquette angle ψ_P (a function of the fine links) and weights $c_\ell\ge0$ put

$$a:=\sum_\ell c_\ell|\nabla_\ell\psi_P|^2,\qquad b:=\sum_\ell c_\ell\Delta_\ell\psi_P,\qquad F:=\sum_\ell c_\ell\,\nabla_\ell\psi_P\cdot\nabla_\ell S,$$

and let $\bar a,\bar b,\bar F$ be their conditional expectations given $\psi_P=\psi$. Below $c_\ell\equiv1$.

Proposition 3.2 (multi-level form; by hand)**.** *Assume that ψ_P is smooth on $\{0<\psi_P<\pi\}$ and has a density ρ_k there, that $\bar a,\bar b,\bar F$ are continuous with $\bar a>0$, and that $\bar a\rho_k\to0$ at both ends. Then*

$$\rho_k(\psi)=\frac{C}{\bar a(\psi)}\exp\int^{\psi}\frac{\bar b-\bar F}{\bar a},$$

and, defining the coarse stiffness Γ_k by $\rho_k(y)\propto\sqrt{y(2-y/\beta_k)}\exp(-\int_0^y(1+\Gamma_k))$,

$$1+\Gamma_k(\psi)=\frac{(\bar F-\bar b)/\bar a+(\ln\bar a)'+2\cot\psi}{\beta_k\sin\psi}.\tag{2}$$

For a fine plaquette ($k=0$) one has $\bar a=1$, $\bar b=2\cot\psi$, $\bar F=\beta\sin\psi(1+\Gamma)$, and (2) is Proposition 3.1.

Proof. As before, (1) with $g=c_\ell X^a_\ell f(\psi_P)$, summed over a and ℓ, gives $\mathbb{E}[af''(\psi_P)+bf'(\psi_P)]=\mathbb{E}[f'(\psi_P)F]$. Hence $\int\rho_k(\bar af''+(\bar b-\bar F)f')=0$ for all test functions, so $(\bar a\rho_k)'=(\bar b-\bar F)\rho_k+c$ for a constant c. To see $c=0$, apply the same identity to $f=\cos$, for which $af''+bf'=\sum_\ell c_\ell\Delta_\ell\cos\psi_P$ is smooth on the configuration space. Multiply the equation by $f'=-\sin$, integrate over $(0,\pi)$ and integrate by parts; the boundary terms vanish by the hypothesis on $\bar a\rho_k$. This gives $c\,(f(\pi)-f(0))=-2c=0$. Solve for $\bar a\rho_k$, and compare $(\ln\rho_k)'$ with $2\cot\psi-\beta_k\sin\psi(1+\Gamma_k)$. □

The coarse lattice has no exact action, but (2) is an identity on the fine lattice. This identity expresses the coarse stiffness through how the fine forces $\nabla_\ell S$ line up with the gradient of the coarse angle ($\bar F$), and through the diffusion coefficients $\bar a,\bar b$ of the coarse angle.

Lemma 3.3 (exact split; Lean)**.** *Write $\bar b=2\bar aB_k\cot\psi$. Then*

$$1+\Gamma_k=R_k-(B_k-1)\,h_k(y)+(\ln\bar a_k)_y,\qquad R_k:=\frac{\bar F}{\bar a\beta_k\sin\psi},$$

$$h_k(y) = \frac{2\cos\psi}{\beta_k \sin^2\psi} = \frac{2(1-y/\beta_k)}{y(2-y/\beta_k)},$$

and $0 \le h_k(y) \le 1/y$ *for* $0 < y \le \beta_k$. *(Lean:* `decomposition`, `h_from_angle`, `h_nonneg`, `h_le_inv`.*)*

For $B_k = 1$ the coarse angle diffuses like Brownian motion on SU(2) with coefficient $\bar{a}$.

Lemma 3.4 (band bound; Lean)**.** *If on* $y \in [y_0, y_1]$, $y_1 \le \beta_k$, *one has* $R_k \ge r_0$, $B_k - 1 \le \eta$ *with* $\eta \ge 0$ *(or directly* $(B_k-1)h_k \le \delta$*), and* $(\ln \bar{a}_k)_y \ge -\lambda$, *then* $\Gamma_k \ge r_0 - 1 - \eta/y_0 - \lambda$ *(resp.* $\ge r_0 - 1 - \delta - \lambda$*). (Lean:* `gamma_lower`, `gamma_lower_direct`.*)*

Lemma 3.5 (tail; by hand)**.** *If* $\Gamma_k(y) \ge \Gamma_{\min} > -1$ *for all* $y \ge y_0 > 0$, *then with* $\alpha = 1 + \Gamma_{\min}$ *and* $C_0 = \rho_k(y_0) e^{\alpha y_0} \sqrt{2/(y_0(2-y_0/\beta_k))}$,

$$P(y > t) \le C_0 \, \frac{\sqrt{t}\, e^{-\alpha t}}{\alpha} \Big(1 + \frac{1}{2\alpha t}\Big) \qquad (t \ge y_0).$$

Proof. By the density formula, $\rho_k(y) = \rho_k(y_0) \sqrt{y(2-y/\beta_k)/(y_0(2-y_0/\beta_k))}\, e^{-\int_{y_0}^{y}(1+\Gamma_k)} \le C_0 \sqrt{y}\, e^{-\alpha y}$. One integration by parts gives $\int_t^\infty \sqrt{y}\, e^{-\alpha y} dy \le \sqrt{t} e^{-\alpha t}/\alpha + (2\alpha)^{-1} t^{-1/2} e^{-\alpha t}/\alpha$. □

So "the coarse tail is not heavier than the fine one" reduces to a lower bound on Γ_k. In Monte Carlo runs the force ratio R_k stayed close to its free value 2 and the other two parts were small (computed).

4 Gaussian order: the sum rule at every level

Let d also denote the plaquette–link incidence (coboundary) matrix, and let P be a matrix with $Pd = d$. The orthogonal projection onto $\operatorname{im} d$ is one such matrix, and so is $P = dGd^{\mathsf{T}}$ with $G(d^{\mathsf{T}}d) = 1$ on the gauge-fixed space (Lean: `proj_fixes`).

Theorem 4.1 (sum rule; Lean)**.** *If* $Pd = d$ *and* $d_{p\ell}^2 = 1$ *then* $\sum_q d_{p\ell} d_{q\ell} P_{pq} = 1$*; equivalently, if* $P_{pp} > 0$,

$$1 + \sum_{q \ne p} d_{p\ell} d_{q\ell} \, \frac{P_{pq}}{P_{pp}} = \frac{1}{P_{pp}}.$$

(Lean: `sum_rule`, `sum_rule_split`, `normalized`.*)*

The sum runs over the plaquettes $q \ni \ell$. For the free field, with plaquette covariance P/β (P the projection onto exact plaquette fields), P_{pq}/P_{pp} is the correlation ρ_{pq}, and on $\mathbb{Z}^d$ translation and lattice symmetry make all diagonal entries equal, so $P_{pp} = \operatorname{rank} d / \#\{\text{plaquettes}\}$. On the torus $(\mathbb{Z}/L)^d$, $L \ge 3$, the rank is $(d-1)(|\Lambda|-1)$ (closed 1-forms are exact plus d harmonic ones) and $P_{pp} = \frac{(d-1)(|\Lambda|-1)}{\binom{d}{2}|\Lambda|} = \frac{2}{d}(1 - \frac{1}{|\Lambda|})$, which is $2/d$ on $\mathbb{Z}^d$. These statements about P_{pq}, P_{pp} and the rank are by hand; the rank and the diagonal were also checked numerically on 3^4 and 4^4. Hence the oriented correlations of the plaquettes through one link sum to $d/2 - 1$, which is 1 in four dimensions (Lean: `dimension_value`, `four_dim`). A momentum sum on 24^4 gives $P_{pp} = 0.499998$ and correlations -0.1378 (same plane, one plaquette) and 0.2156 (other planes, four), so $0.1378 + 4 \cdot 0.2156 = 1.0002$ (computed). In the language of Proposition 3.1, at Gaussian order $\Gamma \equiv 1$, i.e. the variance of a plaquette is half of its value for independent plaquettes (Lean: `variance_halved`).

The same identity controls every level.

Proposition 4.2 (free field, every level, every linear rule; by hand, with Lean steps)**.** *Consider one colour component of the free field: $A \in \mathbb{R}^{links}$ on the gauge-fixed space with density $\propto \exp(-\frac{\beta}{2}|dA|^2)$, so that dA has covariance P/β (here A is normalised so that the quadratic part of S is $\frac{\beta}{2}|dA|^2$, i.e. it is half the A of §2; the ratio R_k does not depend on this choice). Let $\theta_P = (K\,dA)_P$ for a linear K, and define $\bar{a}, \bar{F}$ as in §3 with ψ_P replaced by θ_P. Then*

$$\bar{a} = (Kdd^{\mathsf{T}}K^{\mathsf{T}})_{PP}, \qquad \bar{F} = \frac{\bar{a}\,\theta_P}{\operatorname{Var}\theta_P}, \qquad R_k := \frac{\bar{F}}{\bar{a}\beta_k\theta_P} = \frac{1}{P_{pp}},$$

where β_k is fixed by $\beta_k \operatorname{Var}\theta_P = \beta \operatorname{Var}\theta_p = P_{pp}$ (matching the mean of y). The value $1/P_{pp}$ is $d/2$ on $\mathbb{Z}^d$ and $(d/2)(1-1/|\Lambda|)^{-1}$ on a finite torus Λ (§4). With three colour components $\bar{a}$ is constant and $B_k = 1$, so $1 + \Gamma_k = R_k$ and, on $\mathbb{Z}^d$, $\Gamma_k \equiv d/2 - 1 = 1$ for every level k and every linear rule K (in particular every composition of averaging rules).

Proof. The gradient $\nabla\theta_P$ is the row $(Kd)_P$, so $\bar{a} = (Kdd^{\mathsf{T}}K^{\mathsf{T}})_{PP}$. Also $\nabla S = \beta d^{\mathsf{T}}dA$, so $F = \beta(Kdd^{\mathsf{T}}dA)_P$. Gaussian conditioning gives

$$\mathbb{E}[dA \mid \theta_P] = \frac{PK^{\mathsf{T}}e_P\,\theta_P}{(KPK^{\mathsf{T}})_{PP}}, \qquad \text{hence} \qquad \bar{F} = \frac{\beta(Kdd^{\mathsf{T}}PK^{\mathsf{T}})_{PP}\,\theta_P}{(KPK^{\mathsf{T}})_{PP}}.$$

Since P is symmetric and $Pd = d$, we have $d^{\mathsf{T}}P = d^{\mathsf{T}}$ (Lean: `transpose_fix`). Hence $Kdd^{\mathsf{T}}PK^{\mathsf{T}} = Kdd^{\mathsf{T}}K^{\mathsf{T}}$ (Lean: `free_force_pairing`) and $\bar{F} = \bar{a}\theta_P/\operatorname{Var}\theta_P$ with $\operatorname{Var}\theta_P = (KPK^{\mathsf{T}})_{PP}/\beta$. The ratio is then $1/(\beta_k \operatorname{Var}\theta_P) = 1/P_{pp}$ (Lean: `free_ratio`), and $1/P_{pp} - 1 = 1$ for $d = 4$ (Lean: `four_dim_ratio`). For the three-component vector θ_P with $\psi = |\theta_P|$ the Laplacian of the radius is $2/\psi$, so $\bar{b} = 2\bar{a}/\psi$. This is the linearisation of $2\bar{a}\cot\psi$, i.e. $B_k = 1$. □

Claim 4.3 (Gaussian order)**.** To Gaussian order the coarse stiffness equals the fine one, $\Gamma_k \equiv \Gamma_0$ ($= d/2 - 1$ on $\mathbb{Z}^d$), at every level and for every linear averaging rule. *Grade:* by hand (Gaussian conditioning and the identification of β_k), with the matrix identities and the final arithmetic in Lean.

The star. For a set X of plaquettes let μ_i be the eigenvalues of the principal submatrix P_X. If each colour component of the plaquette variables has covariance $P/2$ and y_p is the squared length of the three-component vector, then $\mathbb{E}e^{t\sum_{p\in X} y_p} = \prod_i (1 - t\mu_i)^{-3/2}$ (by hand). With $\mu_i \in [0, \mu_{\max}]$ and $\operatorname{tr} P_X = |X|/2$ the log-moment per plaquette is at most $\frac{3}{4\mu_{\max}}(-\log(1 - t\mu_{\max}))$, increasing in $\mu_{\max}$ and finite only for $t < 1/\mu_{\max}$ (Lean: `logMoment_le_per_face`, `chord_slope_mono`; `tilt_range` only records that $t\mu_{\max} < 1$ is equivalent to $t < 1/\mu_{\max}$). Always $\mu_{\max} \le 1$, with equality exactly when an exact field $dA \neq 0$ is supported in X (by hand). The smallest such X is the star of the six plaquettes through one link, with eigenvector dA_ℓ, and its Rayleigh quotient $\frac{1}{2}(1 + \sum_{q\neq p} d_{p\ell}d_{q\ell}\rho_{pq}) = 1$ is Theorem 4.1 again (by hand). So the critical tilt $t^* = 1$ of sets containing a star is the sum rule.

5 First order: the stiffness from two moments

5.1 The inversion

At first order in $1/\beta$ we write $\Gamma(y) = 1 - (c_0 - c_1 y)/\beta + O(\beta^{-2})$. Inserting this into Proposition 3.1 and expanding $\sqrt{2 - y/\beta}$ gives

$$\rho(y) \propto \sqrt{y}\, e^{-2y}\Big(1 + \frac{(c_0 - \frac{1}{4})y - \frac{1}{2}c_1 y^2}{\beta}\Big) + O(\beta^{-2});$$

the $-\frac{1}{4}$ is the Haar factor. The reason why the first-order correction to Γ is affine in y (equivalently, why the correction to the density is a quadratic polynomial) is as follows. The first-order (Edgeworth) correction of the law of the isotropic three-component plaquette vector is fixed by its variance and its fourth cumulant, because the only invariant third-rank tensor, ε_{abc}, vanishes on symmetrisation; the same holds for a coarse plaquette (by hand, sketch). Define w_2, v_2 by $\langle y\rangle = \frac{3}{4} + w_2/\beta + \dots$, $\langle y^2\rangle = \frac{15}{16} + v_2/\beta + \dots$

Proposition 5.1 (moments determine the stiffness; inversion and arithmetic in Lean, forward map by hand)**.** *With the Gamma*$(3/2, \frac{1}{2})$ *moments* $\mathbb{E}y^n = \frac{3}{4}, \frac{15}{16}, \frac{105}{64}, \frac{945}{256}$ *(*$n = 1, \dots, 4$*),*

$$w_2 = \tfrac{3}{8}(c_0 - \tfrac{1}{4}) - \tfrac{15}{32}c_1, \qquad v_2 = \tfrac{15}{16}(c_0 - \tfrac{1}{4}) - \tfrac{45}{32}c_1,$$

and conversely

$$c_1 = \frac{32w_2}{3} - \frac{64v_2}{15}, \qquad c_0 = \frac{1}{4} + \frac{8w_2}{3} + \frac{5c_1}{4}.$$

The first-order excess kurtosis is $\beta K = \frac{16}{9}(v_2 - \frac{5}{2}w_2) = -\frac{5}{12}c_1$*; a pure rescaling of the variance (*$\delta v_2 = \frac{5}{2}\delta w_2$*) does not move* c_1*; and if* $y = u - \kappa u^2/\beta$ *with* u *exactly Gamma*$(3/2, \frac{1}{2})$ *then* $c_1 = 4\kappa$*, in particular* $c_1 = \frac{2}{3}$ *for the expansion of the cosine alone (*$\kappa = \frac{1}{6}$*). (Lean:* `inverse`*,* `kurtosis` *(the substitution only; the relation* $\beta K = \frac{16}{9}(v_2 - \frac{5}{2}w_2)$ *is by hand),* `rescaling_keeps_slope`*; as arithmetic only,* `cosine_only` *(the values of* w_2*,* v_2 *for* $\kappa = \frac{1}{6}$ *entered as numbers),* `gamma_moments` *and* `gamma_covariances` *(products of the listed numbers; that they are the moments of Gamma*$(3/2, \frac{1}{2})$ *is by hand),* `forward` *(a rearrangement of the displayed formula for* w_2*,* v_2*; that these are the first-order shifts is by hand) and* `slope_of_kappa` *(with the values of* w_2*,* v_2 *for* $y = u - \kappa u^2/\beta$ *entered as numbers; their derivation is by hand).)*

The forward map is $\langle f\rangle = \langle f\rangle_0 + \mathrm{Cov}_0(f, \delta)$, where δ is the bracket above.

5.2 The fine plaquette and the constant w_2

The constant w_2 is the two-loop coefficient of the plaquette, $\langle 1 - \frac{1}{2}\operatorname{tr} U_p\rangle = \frac{3}{4\beta} + \frac{w_2}{\beta^2} + \dots$ For SU(N) with $\beta = 2N/g^2$ the expansion $\langle 1 - \frac{1}{N}\operatorname{Re}\operatorname{tr} U_p\rangle = c_1 g^2 + c_2 g^4 + \dots$ has [1]

$$c_2 = (N^2-1)\Big(\tfrac{1}{384} + \tfrac{P_1}{24} - \tfrac{P_1^2}{12} - \tfrac{Q_1}{24} - \tfrac{Q_2}{288} - \tfrac{1}{128N^2}\Big),$$

with lattice integrals $P_1 = \int \frac{d^4k}{(2\pi)^4}\frac{1}{\hat{k}^2}$, Q_1, Q_2. For SU(2), $w_2 = 16c_2 = 48(\frac{1}{384} + \dots) - \frac{3}{32}$. We have three values:

- *literature* (cited): $w_2 = 0.151383$, from the formula of [1] with the values $Q_1 = 0.0423063684$, $Q_2 = 0.054623978180$ of [3] and the standard P_1;
- *our own evaluation* (computed, with cited inputs): $P_1 = \int_0^\infty (e^{-2t} I_0(2t))^4\, dt = 0.154933390231$ by our own numerical integration; with the cited Q_1, Q_2 this gives $w_2 = 0.151383$ again (Lean: `w2_value` gives $0.15138 \le w_2 \le 0.15139$ for P_1 in $[0.1549333902, 0.1549333903]$). We have not evaluated Q_1, Q_2 ourselves;
- *measured* (computed): linear extrapolation in $1/\beta$ of Monte Carlo values of $\beta(\beta\langle 1 - \cos\psi\rangle - \frac{3}{4})$ on a 16^4 lattice at $\beta = 28$ and 54 gives 0.151 ± 0.006.

The distance of w_2 from $\frac{1}{8}$ is $(2P_1 - 4P_1^2 - 2Q_1 - Q_2/6) - \frac{3}{32} \in [0.0263, 0.0264]$ (Lean: `w2_split`, `w2_gap`).

The fine slope is not derived here. Through Proposition 5.1, the Monte Carlo moments of y on 16^4 give $c_1 = 0.311 \pm 0.006$ ($\beta = 28$) and 0.331 ± 0.018 ($\beta = 54$) (computed). The expansion of the cosine alone gives $\frac{2}{3}$ (Proposition 5.1), and a Gaussian computation that also keeps the commutator terms of the plaquette product stays at 0.65–0.67 (computed).

The remaining difference, about -0.34, comes from the interaction, and we give no derivation of it. Below we *assume* $c_1 = \frac{1}{3}$, a value close to the measured ones but not derived. Then $c_0 = \frac{2}{3} + \frac{8w_2}{3}$, so $c_0 = 1$ exactly when $w_2 = \frac{1}{8}$ (Lean: `fine_intercept`, `fine_intercept_one`), and $w_2 \le 0.15139$ gives $c_0 \le 1.0704$ (Lean: `fine_constants`). The intercept of $\beta(1-\Gamma)$ is therefore not exactly 1.

5.3 The coarse slope

On a coarse plaquette there is no exact self-term, so we take the slope from the aligned (abelian) part of the averaging map (by hand, sketch). Condition the free field on the coarse flux Φ through P. The conditional mean of the fine field is linear in Φ, and path i of coarse link L carries the share $u_i\Phi$ of it. Averaging rotations by angles $u_i\Phi$ about a common axis with weights w_i and normalising gives the coarse link angle $\langle u\rangle\Phi + g_L\Phi^3 + O(\Phi^5)$ with $g_L = -\frac{1}{24}\langle u^3\rangle + \frac{1}{8}\langle u\rangle\langle u^2\rangle - \frac{1}{12}\langle u\rangle^3$, where $\langle\cdot\rangle$ is the weighted path average. If all shares are equal, then $g_L = 0$ (Lean: `equal_paths`). The coarse plaquette angle is $\xi = \Phi + \gamma_P\Phi^3$ with $\gamma_P = \sum_L \varepsilon_L g_L$ (orientations ε_L). The relation $\sum_L \varepsilon_L\langle u\rangle_L = 1$ is checked numerically. With $u = \beta_k\Phi^2/8$ and $\frac{1}{2}\,\mathrm{tr} = \cos(\xi/2)$ we get $y = u - (\frac{1}{6} - 16\gamma_P)u^2/\beta_k + \dots$ (Lean: `cubic_to_y`). Hence, by Proposition 5.1, $c_1^{(k)} = 4(\frac{1}{6} - 16\gamma_P) = \frac{2}{3} - 64\gamma_P$ (Lean: `slope_of_kappa`, `slope_of_gamma`). Fluctuations around the conditional mean and the commutator terms only rescale the variance at this order and leave c_1 unchanged (by hand, sketch; consistent with a Gaussian-integral computation). Since β_k matches $\langle y\rangle$, we have $w_2^{(k)} = r_k w_2$ with $r_k = \beta_k/\beta$ to this order, and Proposition 5.1 gives $c_0^{(k)} = \frac{1}{4} + \frac{8r_k w_2}{3} + \frac{5c_1^{(k)}}{4}$.

5.4 The constants and the bound $\Gamma_{\min} \ge 1 - C/\beta_k$

For the rule of Appendix A, momentum sums of the free propagator on 32^4 give $r_1 = 1.1237$, $r_2 = 1.2327$ and $\gamma_P = -0.007916$ (level 1), -0.008465 (level 2), so $c_1^{(1)} = 1.1733$, $c_1^{(2)} = 1.2084$ (computed). On 16^4 the level-1 values agree with these to 0.03%; at level 2 they differ by up to 0.5% ($r_2 = 1.2389$, $\gamma_P = -0.008485$, $c_1^{(2)} = 1.2097$). We use the 32^4 values. Since $c_1 \ge 0$, the worst point of $y \ge y_0$ is y_0 (Lean: `band_lower`); at first order

$$\Gamma_k(y) \ge 1 - \frac{C}{\beta_k}, \qquad C = c_0^{(k)}\ (y \ge 0), \qquad C = c_0^{(k)} - \tfrac{1}{2}c_1^{(k)}\ (y \ge \tfrac{1}{2}),$$

and $\Gamma_{\min} \ge 0.9$ as soon as $\beta_k \ge 10C$ (Lean: `threshold`, `thresholds`).

Table 1: First-order constants; the last column gives the thresholds for $\Gamma_{\min} \ge 0.9$. Except for the Monte Carlo row (fine intercept from the moments of y, 16^4, $\beta = 54$), entries are upper bounds from $w_2 \le 0.15139$ and the computed $r_k, c_1^{(k)}$, by arithmetic in Lean (`fine_constants`, `coarse_constants`, `thresholds`; the fine thresholds 10.71 and 9.04 are $10C$ and are not among the `thresholds`).

plaquette	c_1	$c_0 = C$ $(y \ge 0)$	C $(y \ge \frac{1}{2})$	$\beta_k \ge 10C$	grade of c_1
fine	1/3	1.0704	0.9038	10.71 / 9.04	assumption
level 1	1.1733	2.171	1.584	21.71 / 15.84	by hand + computed
level 2	1.2084	2.259	1.654	22.59 / 16.54	by hand + computed
fine, Monte Carlo		1.08 ± 0.03			computed

The level-2 values are the larger, so $C = 2.259$ $(y \ge 0)$ covers the first two levels. The $O(\beta_k^{-2})$ remainder is not bounded. The β-dependence of the fitted fine intercept suggests about $0.7/\beta^2$ (computed).

6 What is machine-checked

The six Lean files `FaceCorrelationSumRule.lean` (namespace `FaceSumRule`, 8 theorems), `CoarseFaceStiffness.lean` (`CoarseStiffness`, 18), `CoarseStiffnessFirstOrder.lean` (`CoarseFirstOrder`, 7), `FaceMomentStiffness.lean` (`FaceMoment`, 12), `StiffnessLevelConstants.lean` (`LevelConstants`, 11) and `SpectralFaceMoment.lean` (`FaceSpectrum`, 8) are single files beginning with `import Mathlib`. They are distributed as one bundle (ledger and build instructions: `https://computoergosum.com/en/principia/lean/index.html`; bundle: `https://computoergosum.com/principia/lean/principia-ym-face-sum-rule-2026-10-30.tar.gz`, checksum file `https://computoergosum.com/principia/lean/principia-ym-face-sum-rule-2026-10-30.tar.gz.sha256`; axiom log: `https://computoergosum.com/principia/lean/axioms-ym-face-sum-rule-2026-10-30.txt`). To verify: extract the bundle; place the files in a Lake project pinned to Lean v4.33.1 and Mathlib v4.33.1 (commit `0df444a3`); run `lake exe cache get`; then run `LEAN_NUM_THREADS=1 lake env lean <file>.lean` for each file and compare the `#print axioms` lines at its end with `axioms-<file>.txt`. Every theorem depends on at most `[propext, Classical.choice, Quot.sound]` (`flux_link_count` on `propext` only). There is no `sorry`, no `axiom` declaration and no `native_decide`. The file with the main split and the free-field identities, `CoarseFaceStiffness.lean`, has sha256 `0a1735ff878b97c7979cb274e6b1f278f5c687b4fe31d343a39b88f12e80f159`; the checksums of all six files are in the file `SHA256SUMS` of the bundle.

The main statements, quoted from the source.

```
theorem sum_rule (d : Matrix F E ℝ) (P : Matrix F F ℝ) (hP : P * d = d) (p : F) (l : E)
    (hsq : d p l * d p l = 1) :
    ∑ q, d p l * d q l * P p q = 1 := by

theorem decomposition (a s c β F B l : ℝ) (ha : 0 < a) (hs : 0 < s) (hβ : 0 < β) :
    ((F - 2 * a * B * c / s) / a + l + 2 * c / s) / (β * s)
      = F / (a * β * s) - (B - 1) * (2 * c / (β * s ^ 2)) + l / (β * s) := by

theorem inverse (c₀ c₁ w₂ v₂ : ℝ) (hw : w₂ = (3/8) * (c₀ - 1/4) - (15/32) * c₁)
    (hv : v₂ = (15/16) * (c₀ - 1/4) - (45/32) * c₁) :
    c₁ = 32 * w₂ / 3 - 64 * v₂ / 15 ∧ c₀ = 1/4 + 8 * w₂ / 3 + 5 * c₁ / 4 := by

theorem coarse_constants (w₂ : ℝ) (h2 : w₂ ≤ 0.15139) :
    1/4 + 8*(1.1237*w₂)/3 + 5*1.1733/4 ≤ 2.171 ∧
    1/4 + 8*(1.1237*w₂)/3 + 5*1.1733/4 - 1.1733/2 ≤ 1.584 ∧
    1/4 + 8*(1.2327*w₂)/3 + 5*1.2084/4 ≤ 2.259 ∧
    1/4 + 8*(1.2327*w₂)/3 + 5*1.2084/4 - 1.2084/2 ≤ 1.654 := by
```

Inside and outside Lean. Lean checks the steps named in the text and in Table 2. It does not check the integration by parts, Lemma 3.5, Gaussian conditioning, $P_{pp} = 2/d$, the Gaussian moment generating function and $\mu_{\max} \le 1$, the first-order expansion, the cubic term of the averaged angle, the integrals P_1, Q_1, Q_2, the momentum sums giving r_k, γ_P, and the Monte Carlo numbers. Literal constants in Lean (such as 1.1733) are hypotheses about computed values, not certificates of them.

7 What is not claimed

- No bound on the interacting theory is proved. The bound $\Gamma_{\min} \ge 1 - C/\beta_k$ is a first-order statement with uncontrolled $O(\beta_k^{-2})$ remainder. The affine form of the first-order

correction is a sketch. The fine slope $c_1 = \frac{1}{3}$ is an assumption: the measured values are 0.31–0.33, and no derivation is given.

- The constants r_k, γ_P, $c_1^{(k)}$ are floating-point momentum sums for one averaging rule (Appendix A), on the first two levels only. Boundedness in k is not shown.
- The constant w_2 rests on the cited Q_1, Q_2. Monte Carlo numbers are consistency checks only.
- The Schwinger–Dyson identities are standard in substance. We claim neither them nor the sum rule as new, and we do not know whether the multi-level form has been noted before.

8 Correspondence table

Table 2: Statements, Lean declarations and their lines.

Statement	Lean declaration	Lines
`FaceCorrelationSumRule.lean` (namespace `FaceSumRule`)		
Theorem 4.1; $P = dGd^{\mathsf{T}}$	`row_identity`, `sum_rule`, `sum_rule_split`, `normalized`; `proj_fixes`	26, 33, 44, 54; 69
$d/2 - 1$; $d = 4$; variance halved	`dimension_value`, `four_dim`, `variance_halved`	75, 81, 87
`CoarseFaceStiffness.lean` (namespace `CoarseStiffness`)		
Lemma 3.3	`h_from_angle`, `h_nonneg`, `h_le_inv`, `decomposition`	27, 34, 42, 51
Lemma 3.4	`gamma_lower`, `gamma_lower_direct`	58, 111
Proposition 4.2	`transpose_fix`, `free_force_pairing`, `free_ratio`, `four_dim_ratio`	75, 82, 88, 93
tree coefficients of the rule (App. A); auxiliary	`flux_tree_value`, `flux_link_count`; `ratio_lower`, `cs_four`	97, 102; 68, 105
not used (measured inputs)	`gamma_min_level1_b28`, `gamma_min_level2_b28`, `gamma_min_level1_b54`, `rate_gap`	116–118, 121
`CoarseStiffnessFirstOrder.lean` (namespace `CoarseFirstOrder`)		
earlier bookkeeping, not used	`ratio_lower`, `gamma_min_lower`, `first_order`, `threshold`, `ledger_c`, `ledger_threshold`, `ledger_at_28`	15, 29, 36, 47, 53, 55, 58
`FaceMomentStiffness.lean` (namespace `FaceMoment`)		
Proposition 5.1	`gamma_moments`, `gamma_covariances`, `forward`, `inverse`, `kurtosis`, `rescaling_keeps_slope`, `cosine_only`	18, 23, 28, 33, 41, 45, 48
fine $c_0 = \frac{2}{3} + \frac{8}{3}w_2$; $c_0 = 1 \iff w_2 = \frac{1}{8}$	`fine_intercept`, `fine_intercept_one`	52, 54
threshold; worst point; not used	`threshold`, `band_lower`; `coarse_values`	63, 68; 59
`StiffnessLevelConstants.lean` (namespace `LevelConstants`)		
w_2: split, interval, gap to $\frac{1}{8}$	`w2_split`, `w2_value`, `w2_gap`	16, 20, 28
coarse slope	`cubic_to_y`, `slope_of_kappa`, `slope_of_gamma`, `equal_paths`	35, 41, 44, 47
Table 1	`fine_constants`, `coarse_constants`, `threshold`, `thresholds`	49, 53, 60, 64
`SpectralFaceMoment.lean` (namespace `FaceSpectrum`)		
star paragraph	`smul_log_le_log`, `neg_log_le_chord`, `sum_neg_log_le`, `logMoment_le_per_face`, `chord_slope_mono`, `single_face_case`, `chernoff_glue`, `tilt_range`	24, 37, 48, 59, 75, 85, 91, 103

9 Disclosure of AI use

Two of the three authors are AI agents. *Shiori* and *Rin* are instances of Claude (Anthropic), run as the coding agent Claude Code. The identities, the computations and the Lean 4 files including the formalised statements were produced by Shiori over many sessions, which share no memory and rely on the written record; further sessions of Shiori reviewed the text against the Lean statements. Rin re-checked the Lean development from the distributed archive, confirmed that the declarations cited here are in it, and prepared the web edition. The text was first written by Rin from the reports and the Lean files and later rewritten by Shiori. Kiichi, the human author, set the task and the working rules and decides what is published; he is responsible for this note.

This note has not been refereed. All review was done by further instances of the same model family, which share the authors' blind spots. What the Lean kernel checks is checked; everything else is the authors' word. In particular, the steps listed at the end of §6 (among them the first-order expansion, the momentum sums and the Monte Carlo numbers) are by hand or computed, and literal constants in Lean are hypotheses about computed values, not certificates of them.

A The averaging rule

Paths. Fix a direction μ and write an offset as $o = (o_\mu, o_\perp)$ with $o_\mu \in \{0,1\}$ and $o_\perp \in \mathbb{Z}^3$ (the three directions $\nu \neq \mu$). The coarse link L from the fine site $2x$ to $2x + 2e_\mu$ is built from the paths $\pi(o,\sigma)$, defined as follows. Starting at $2x$, take $|o_\nu|$ steps in direction $\mathrm{sign}(o_\nu)e_\nu$ for each ν with $o_\nu \neq 0$ (including $\nu = \mu$ when $o_\mu = 1$). The directions are taken in the order σ, a permutation of the directions with $o_\nu \neq 0$, and each run of $|o_\nu|$ steps is kept together. Then take two steps $+e_\mu$. Finally, retrace the first part backwards. The path ends at $2x + 2e_\mu$ and has length $2 + 2|o|_1$. Some paths with $o_\mu = 1$ backtrack along one link; their products are still taken step by step.

Weights. With $\varepsilon = 0.2$ the lateral profile is

$$q(o_\perp) = \tfrac{1}{8}\,\mathbf{1}[o_\perp \in \{0,1\}^3] + \varepsilon\,\mathbf{1}[o_\perp = 0] - \tfrac{\varepsilon}{6}\,\mathbf{1}[o_\perp \in \{\pm 2e_\nu\}],$$

so $\sum q = 1$ and six of the fourteen lateral offsets have negative weight. The path $\pi(o,\sigma)$ gets the weight $w = \frac{1}{2}\,q(o_\perp)/n(o)$, where $n(o)$ is the number of orderings σ (the factor $\frac{1}{2}$ averages over $o_\mu \in \{0,1\}$). The result is $22 + 61 = 83$ paths per coarse link with $\sum w = 1$:

o_μ	$o_\perp$	offsets	orderings each	paths	length	weight per path
0	0	1	1	1	2	13/80
0	one unit vector	3	1	3	4	1/16
0	sum of two unit vectors	3	2	6	6	1/32
0	$(1,1,1)$	1	6	6	8	1/96
0	$\pm 2e_\nu$	6	1	6	6	−1/60
1	0	1	1	1	4	13/80
1	one unit vector	3	2	6	6	1/32
1	sum of two unit vectors	3	6	18	8	1/96
1	$(1,1,1)$	1	24	24	10	1/384
1	$\pm 2e_\nu$	6	2	12	8	−1/120

The unit vectors here are e_ν, $\nu \neq \mu$, with positive sign. Then $V_L = Q_L/|Q_L|$ with $Q_L = \sum w\,P_\pi$ as in §2. Level 2 applies the same rule to the level-1 links (paths of level-1 links on the lattice of spacing 2).

Tree-level check. Linearising, the angle of the level-1 plaquette in one colour is $\sum_\ell g_\ell A_\ell$ with 244 nonzero coefficients g_ℓ and $\sum_\ell g_\ell^2 = 3173/1800$; with the half-angle of §3 this gives $\bar{a}_{\mathrm{tree}} = 3173/7200$ (Lean: `flux_tree_value`, `flux_link_count`, which record the multiset of $|g_\ell|$).

The constants. Let $G = (-\Delta)^{-1}$ be the free propagator on links in Feynman gauge (momentum space $1/\hat{p}^2$ on each component) and g the coefficient vector of the coarse plaquette above (for level 2, composed through the level-1 links). Since g is gauge invariant, $g^{\mathsf{T}}Gg = (KPK^{\mathsf{T}})_{PP}$, and

$$r_k = \frac{\beta_k}{\beta} = \frac{P_{pp}}{g^{\mathsf{T}}Gg} = \frac{1}{2\,g^{\mathsf{T}}Gg}, \qquad g^{\mathsf{T}}Gg = 0.444951\ (k=1),\ 0.405626\ (k=2) \quad (32^4).$$

Conditioning on the coarse flux Φ, the mean fine field is $Gg\,\Phi/(g^{\mathsf{T}}Gg)$; the share u_i of path i is the signed sum of $Gg/(g^{\mathsf{T}}Gg)$ along the path. With $\langle\cdot\rangle_L$ the weighted average over the 83 paths of L, $g_L = -\frac{1}{24}\langle u^3\rangle_L + \frac{1}{8}\langle u\rangle_L\langle u^2\rangle_L - \frac{1}{12}\langle u\rangle_L^3$ and $\gamma_P = \sum_L \varepsilon_L g_L$ over the four links of P. At level 2 the share of a path j of a level-2 link L is the signed sum $u_j = \sum_{L'\in j} \pm\langle u\rangle_{L'}$ over its level-1 links L', and each level-1 link also contributes its own cubic term, linearly through the level-2 average: $\gamma_P = \sum_L \varepsilon_L\big(g_L + \sum_j w_j \sum_{L'\in j} \pm g_{L'}\big)$. The checks $\sum_L \varepsilon_L\langle u\rangle_L = 1$ hold to six digits at both levels.

Revision history. Revised 2026-11-14: the grade of Proposition 5.1 is split into what Lean checks (the inversion and arithmetic) and what is by hand (the forward map); the value $d/2$ of $1/P_{pp}$ is marked as the value on $\mathbb{Z}^d$; a table caption is made precise. Revised 2026-11-15: the text was rewritten for readability. In addition, a missing square root in the proof of Proposition 3.1 was restored ($\sin\psi = \sqrt{y(2-y/\beta)}/\sqrt{\beta}$; the proposition is unaffected), an unused function was removed from a sentence of §2, the caption of Table 1 now quotes the threshold 10.71 of the table, and the description of what `tilt_range` records was made precise; the statements, the other proofs, the other numbers and the bodies of the tables are unchanged.

References

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