One-link integration before Bakry–Émery for SU(2) lattice Yang–Mills

An explicit strong-coupling threshold $\beta < \sqrt{(2/27)}$ in dimension four, and why complete families of links exist only for d $\leq$ 4 — machine-checked in Lean 4

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Abstract

For SU(2) Wilson lattice gauge theory on (Z/L)^4 with action S = $-\beta$ Σ_p (1/2)tr U_p, β = 4/g^2, we integrate out one quarter of the links exactly and apply the Bakry–Émery criterion to the resulting marginal. The set integrated out is a *complete family* I*: a set of links such that every plaquette contains exactly one of them. Since the links of I* share no plaquette, the Haar integral factorises over *stars*, and the marginal density is exp($-\Phi$) with $\Phi = -\Sigma_s tar$ log $F0(\beta^2\|M_s\|^2/4)$, where F0(x) = Σ_k $x^k/(k!(k+1)!)$, so that F0($\kappa^2/4$) = 2I$_1(\kappa)/\kappa$, and M_s is the sum of the six staples of the star s. We prove that the second derivative of Φ along the geodesics $U_e \mapsto U_e$ $e^{\tau X_e}$ is at least $-27\beta^2$ Σ_e $\|X_e\|^2$, for all β and all configurations; with Ric = 2 for the unit S^3 this gives a Bakry–Émery curvature 2 $-$ 27β^2 and a threshold $\beta < \sqrt{(2/27)} = 0.2722$, against $\beta < 1/12 = 0.0833$ from the counting of Shen–Zhu–Zhu. The chain from the Wilson action to the bound on Φ is a theorem of Lean 4 with Mathlib (no `sorry`, no `native_decide`, standard axioms only). The step from there to a log-Sobolev inequality and a mass gap is not formalised: it uses the Bakry–Émery theorem and an adaptation of the decay argument of Shen–Zhu–Zhu that we write out only in outline. A second machine-checked statement is that a complete family exists on Z^d exactly for d $\leq$ 4: restricted to a unit cube the family is a binary code of length d$-$1, size 2^{d-3} and minimum distance 3, and the Hamming bound reads d $\leq$ 4, with equality at d = 4 corresponding to the repetition code {000, 111} being perfect. On the torus $(Z/L)^d$ with d $\geq$ 2 the same existence question is settled with the period included — a complete family of links exists exactly when d $\leq$ 4 and L is even — and for k-cells in place of links, a complete family has density exactly 1/(2(k+1)) and forces d $\leq$ 3k+1 for every period, an odd period admitting none; all of this is machine-checked. For d $\geq$ 5 one can only ask that each plaquette contain at most one link, and the density of such a packing is at most 1/d — also machine-checked — so the gain drops to a factor 1.72. The constant 27 cannot be improved past 21: an explicit configuration, checked in exact arithmetic, gives $-\lambda_min = 21\beta^2$. The character of the result is an improvement by a constant factor inside the Bakry–Émery method; we make no claim about the continuum limit, and none about the Clay problem.

Keywords. Lattice Yang–Mills, strong coupling, one-link integral, Bakry–Émery criterion, log-Sobolev inequality, exact cover, Hamming bound, Lean 4.

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1. Introduction

Shen, Zhu and Zhu [SZZ] study the Langevin dynamics of the Wilson lattice gauge measure on $(Z/L)^d$ with structure group G and action S(Q) = Nβ Re $\Sigma_{p\in P^+}$ $Tr(Q_p)$, the measure being proportional to exp(S). Their Lemma 4.1 bounds the Hessian by |Hess S(v,v)| ≤ 8(d−1)N|β| |v|^2, their (4.8) gives Ric(v,v) = (α(N+2)/4 − 1)|X|^2, and Assumption 1.1 — that the Bakry–Émery curvature K_S = (N+2)/2 − 1 − 8N|β|(d−1) be positive — reads |β| < 1/(16(d−1)) for SU(N) in their (1.3). Under it the invariant measure is unique on the whole lattice, the finite-volume measures converge to it, log-Sobolev and Poincaré inequalities hold with constants uniform in the volume, and correlations of local observables decay exponentially: a strictly positive mass gap.

The threshold is a constant, and the method producing it is local: Ric is fixed by the group, and Hess S is bounded plaquette by plaquette. This note asks how far the constant moves if one performs an exact integration *before* invoking Bakry–Émery. Throughout, G = SU(2) is identified with the unit sphere S^3 in the quaternions $\mathbb{H}$, and we use the physicists' normalisation

$$S = \beta\ \Sigma_p\ (1 - \cos\theta_p),\ \cos\theta_p = (1/2)\ \mathrm{tr}\ U_p,\ \beta = \beta_W = 4/\mathrm{g}^2.$$

The translation to [SZZ] is $\beta_W = 4\beta_S ZZ$, $|X|_S^2 ZZ = 2|\mathrm{X}|^2$, $Ric_S ZZ = 1$ against Ric = 2; for N = 2, d = 4 the threshold (1.3) reads $\beta_S ZZ < 1/48$, that is β_W **< 1/12 = 0.0833**. In this normalisation the infimum of $\lambda_m in(Hess$ S) over configurations is -16β, attained at the Z$_2$ configuration $Q_{(x,\mu)} = (-1)^{x_1+\cdots+x_{\mu-1}}$ where every plaquette equals −1 (the bound $\lambda_m in \le -16\beta$ is elementary; that nothing lies below it is computation, §8), so the counting of [SZZ] is a factor 3/2 loose at d = 4 and Bakry–Émery without any integration cannot pass $\beta < 1/8$.

The object we integrate out is a **complete family**: a set I* of links such that every plaquette contains exactly one link of I*. Counting forces its density to be 1/4. In d = 4 such families exist; the one used here is the period-2 family

$$\mathrm{I}_1 : \mathrm{x}_2 = \mathrm{x}_3 \neq \mathrm{x}_4,\ \mathrm{I}_2 : \mathrm{x}_3 = \mathrm{x}_4 \neq \mathrm{x}_1,\ \mathrm{I}_3 : \mathrm{x}_1 = \mathrm{x}_2 = \mathrm{x}_4,\ \mathrm{I}_4 : \mathrm{x}_1 = \mathrm{x}_3 \neq \mathrm{x}_2\ (\mathrm{mod}\ 2),$$

I_μ being the positions of the links of direction μ. Since the links of I* pairwise share no plaquette, the Haar integral over them factorises; and since *every* plaquette is absorbed, the effective action of the remaining links carries no bare plaquette, so the O(β) term of the Hessian disappears and what is left is O(β^2). This note states three things.

(A) The chain from the Wilson action to the bound `Hess` $\Phi \ge -27\beta^2$ is machine-checked (Theorem 1, §2). With the standard facts that $\tau \mapsto$ U $e^{\tau X}$ is a geodesic of the bi-invariant metric, that the second derivative along a geodesic is the Riemannian Hessian, and that Ric = 2 on the unit S^3, the Bakry–Émery curvature of the marginal is 2 − 27β^2 and the threshold is

$$\beta_W < \sqrt{(2/27)} = \mathbf{0.27217}\ (\beta_S ZZ < 0.06804),$$

a factor 3.27 = 4$\sqrt{}$(2/3) above the 1/12 of [SZZ] and 2.18 above the ceiling 1/8 of the method without integration. For general d, whenever a complete family exists, the same proof gives `Hess` $\Phi \geq$ `−3(d−1)`$^2\beta^2$ and $\beta_W < \surd(2/3)/(\mathrm{d}-1)$, a factor 3.27 above 1/(4(d−1)) independently of d.

(B) A complete family exists on Z^d exactly for d ≤ 4 (Theorem 2, §6), also machine-checked, with no periodicity assumed. The obstruction sits inside a single unit 5-cube, and the proof on paper is a line of coding theory: restricted to a unit cube, the positions of the links of one direction form a binary code of length d−1, size 2^{d-3} and minimum distance at least 3, and the Hamming bound $2^{d-3}(1 + (\mathrm{d}-1)) \leq 2^{d-1}$ is equivalent to d ≤ 4. Four is the last dimension because the binary repetition code of length 3 is perfect. On a torus of period L the existence is settled completely and the period enters: a complete family of links on $(Z/L)^d$, d ≥ 2, exists exactly when d ≤ 4 and L is even (Theorem 2′, §2). Replacing links by k-cells and plaquettes by (k+1)-cells, a complete family has density exactly 1/(2(k+1)) and forces d ≤ 3k+1, for every period; an odd period admits none. In d ≥ 5 one falls back on packings, whose density is at most 1/d (Theorem 6.2, also machine-checked), and the gain over [SZZ] falls to 1.72.

(C) The limits of the method (§8): the constant 27 cannot be pushed below 21, since an explicit frustrated configuration gives $\lambda_m in(Hess\ \Phi) = -21\beta^2$ in exact arithmetic, so the ceiling of this route is $\beta_W \leq \surd(2/21) = 0.3086$; for $\beta \gtrsim 1$ integrating first is worse than not integrating; the iteration does not close, a second one-link integral no longer being Bessel; and the mechanism does not reach SU(N), since the inequality carrying the SU(2) proof — that the covariance of a tilted Haar measure is dominated by that of Haar — is already false for U(2) with M = diag(s, 0), and in the 't Hooft limit the gain disappears.

We do not claim a new regime. The value 0.2722 exceeds every threshold we could find written with an explicit number, but the cluster expansion of Osterwalder–Seiler, the classical source of a mass gap at strong coupling, we could not retrieve, and a careful version of it may well reach further (§9). The honest description of (A) is an improvement by a constant factor inside the Bakry–Émery method.

2. Setting, and what is Lean and what is paper

Lattice. Vertices V(L) = $(\mathrm{Z/L})^4$, links Edge(L) = V(L) × {1,2,3,4}, the link (x, μ) running from x to x + e_μ. A plaquette is (x, $\mu < \nu$) with the four links (x,μ), $(x + e_\nu, \mu)$, (x,ν), $(x + e_\mu, \nu)$. The field is U : Edge(L) → $\mathbb{H}$ with $\|U_e\| = 1$; the Wilson action is, verbatim from `WilsonOneLink.SW` in `LatticeGaugeOneLink.lean`, `SW β U = -β * ∑ p : Plaq L, (hol U p.1.1 p.1.2.1 p.1.2.2).re`, where `hol` is the holonomy around the plaquette and `re` is (1/2)tr. Perturbations are `pert h2 U X τ e = U e * exp (τ • X e)`, that is $U_e \mapsto U_e\ e^{\tau X_e}$ with X_e purely imaginary.

Complete family, stars, staples. For even L the family I* of §1 is `Istar h2`, defined from a mod-2 table; `Star h2` is the subtype of links in I*, `Rest h2` the subtype of the others. For s ∈ Star and i ∈ {1,. . . ,6} the i-th *staple* of s is the product of the three remaining links of the i-th plaquette containing s, with that plaquette's orientation; M_s (`Mst`) is the sum of the six staples, and `Es s` is the set of links of Rest lying on a plaquette of s — the *slots* of the star, 18 of them.

Effective action. With F0(x) = $\Sigma_{k\geq 0}\ x^k/(k!(k+1)!)$, G($\beta$,t) = log F0($\beta^2$t/4) and

```
noncomputable def Phi [NeZero L] (h2 : 2 | L) (hL2 : (2 : ZMod L) ≠ 0) (β
: ℝ)
    (V : Rest h2 → ℍ[ℝ]) : ℝ :=
  ∑ s, -G β (‖∑ i, staple h2 hL2 V s i‖ ^ 2)
```

we have $\Phi = -\Sigma_s \log F0(\beta^2\|M_s\|^2/4)$. On paper $\mathrm{F0}(\kappa^2/4) = 2\mathrm{I}_1(\kappa)/\kappa$ is the one-link integral of SU(2).

Theorem 1 (Lean). *Let L be even, $L \geq 3$, and let μ be any Haar probability measure on the unit quaternions (i.e. any `IsHaarS3` μ). Then: (i) (factorisation) for every field W on Rest with $\|W_e\| = 1$, > $\int$ `u, Real.exp (-SW` β `(glue h2 u W))` ∂`(Measure.pi fun _ : Star h2 => `μ`) = Real.exp (-Phi h2 hL2` β `W)` (ii) (Hessian bound) for every U, X on Rest with $\|U_e\| = 1$ and (X_e).re $= 0$, > `-(27 * `β` ^ 2) * `$\sum$` e, `$\|$`X e`$\|$` ^ 2 `$\leq$` deriv (deriv (fun `τ` => Phi h2 (two_ne_zero_of_le hL) `β` (pert h2 U X `τ`))) 0`*

Both are theorems of Lean 4 with Mathlib, verbatim `WilsonOneLink.integral_star_links` and `StaplePerturbation.hess_gauge` (§11 has the full signatures and the axiom logs). The bound in (ii) holds for every β and every configuration; no smallness is assumed.

Theorem 2 (Lean). *With `PerfectZ J` the property that every plaquette of Z^d contains exactly one link of the family J — verbatim > `def PerfectZ (J : Fin d `$\to$` W d `$\to$` Bool) : Prop := `> $\forall$ `μ ν : Fin d, μ `$\neq$` ν `$\to$ $\forall$` x : W d, `> `(J μ x).toNat + (J μ (shift x ν)).toNat `> `+ (J ν x).toNat + (J ν (shift x μ)).toNat = 1` — we have `no_perfectZ_of_five : 5 `$\leq$` d `$\to$ $\forall$` J, `$\neg$` PerfectZ J`, and complete families exist for d = 2, 3, 4 (`perfect_I2`, `perfect_I3`, `perfect_I4`, each by `decide`). No periodicity is assumed in the negative statement.*

On a torus the same question has an answer in which the period appears, and it extends from links to k-cells: a family of k-cells is *complete* when every (k+1)-cell has exactly one of its 2(k+1) faces in the family (§6; for k = 1 this is the definition above).

Theorem 2′ (Lean). *On $(Z/L)^d$ with $L \geq 1$: (i) for $d \geq 2$, a complete family of links exists **if and only if d $\leq$ 4 and L is even** — verbatim > `theorem perfectK_one_iff [NeZero L] (hd : 2 `$\leq$` d) : `> `(`$\exists$` I : Finset (Fin d) `$\to$` Wp d L `$\to$` Bool, PerfectK 1 I) `$\leftrightarrow$` (d `$\leq$` 4 `$\wedge$` 2 | L)` (ii) for $k \geq 1$ and $k + 1 \leq d$, a complete family of k-cells forces **d $\leq$ 3k+1** with no condition on the period (`perfectK_d_le`), has density exactly **1/(2(k+1))** (`density_of_perfectK`) and satisfies $2(k+1)\cdot|I| = C(d,k)\cdot L^d$ (`card_supportK_of_perfectK`); an odd period admits none, for any k and any $d \geq k+1$ (`no_perfectK_of_odd`).*

Which further cases this decides — d = k+2 for every k, k = 2 with d = 4, and the periods left open at k = 4, d = 9 and at k = 5, d = 8 — is Theorem 6.6.

What replaces a complete family in d $\geq$ 5 is a *packing*, a family meeting every plaquette at most once, and its density is at most 1/d; that too is machine-checked (Theorem 6.2, §6), with the attainment of 1/d checked for d = 4 and d = 5.

The statement we would like to reach is a mass gap, and it is not a Lean theorem. We state it as a corollary and mark the grades.

Corollary 3 (paper; part (a) is standard, part (b) is a sketch). *Let β_W < $\sqrt{(2/27)}$ and put K' := 2 − 27β^2.* ***(a)*** *(standard facts: geodesics of the bi-invariant metric, Ric = 2 on the unit S^3, and the Bakry–Émery theorem.) The marginal measure μ' ∝ exp(−Φ) on $(S^3)^{Rest}$ has Bakry–Émery curvature at least K', hence satisfies log-Sobolev and Poincaré inequalities with constants depending only on K', uniformly in the volume and in the boundary condition.* ***(b)*** *(sketch: the argument of [SZZ, §4] transported to μ'; the iteration has not been written out line by line, see §7.) The infinite-volume Wilson Gibbs measure is unique, and for local observables f, g with disjoint supports > $|Cov_\mu(f,g)| \leq c_1\ e^{-d/2}\ |||f|||_\infty\ |||g|||_\infty + e^{-m\cdot d}\ \|f\|_2\ \|g\|_2$, m = 2K'/(3$e^4$A), A ≤ β^2(99 + 243β^2), with d the distance in the graph in which two links are adjacent when they lie in a common star, which is at least one third of the lattice distance.*

Corollary 3 is *not* machine-checked, and the parts of it are not on the same footing. The following table says which step is which. It should be read as the main disclosure of this note.

Step	**Content**	**Grade**
Wilson action as a sum over stars	`SW_eq`	Lean
one-link integral $\int e^{\beta\langle u,M\rangle}\ d\mu$ = F0($\beta^2\|M\|^2/4$)	`IsHaarS3.one_link`	Lean
I* absorbs every plaquette exactly once	`plaq_unique`	Lean
marginal density is exp(−Φ)	`integral_star_links`	Lean
0 ≤ G' ≤ β^2/8 and G" ≤ 0	`G1_nonneg`, `G1_le`, `G2_nonpos`	Lean
one star: second derivative of $\|\Sigma\ s_i\|^2$ at most 36 $\Sigma\|X\|^2$	`star_S`, `second_deriv_le`	Lean
each remaining link lies in exactly 6 stars, 18 slots per star	`star_count`, `Es_card`	Lean
second derivative of Φ along U $e^{\tau X}$ at least $-27\beta^2\ \Sigma\|X\|^2$	`hess_gauge`	Lean
that second derivative is the Riemannian Hessian; Ric = 2 for the unit S^3	geodesics of a bi-invariant metric; round metric of radius 1	paper, standard
Ric + Hess Φ ≥ K' > 0 ⟹ LSI(K'), Poincaré(K')	Bakry–Émery	paper, textbook
Poincaré + finite range ⟹ exponential decay	[SZZ, Lemma 4.10 and Cor. 4.11] transported to Φ	**paper, outline with constants**
observables containing links of I*	conditional independence, covariance identity	paper, complete
μ = μ' ⊗ ∏ vMF, DLR of the marginal	§7(d)	paper, complete
uniqueness of μ'	decay, uniform in the boundary condition	**paper, outline**

3. The one-link integral

Write F0, F1, F2 for the three series

F0(x) = $\Sigma_{k\geq 0}\ x^k/(k!(k+1)!)$, F1(x) = $\Sigma_{k\geq 0}\ x^k/(k!(k+2)!)$, F2(x) = $\Sigma_{k\geq 0}\ x^k/(k!(k+3)!)$,

so that F1 = F0' and F2 = F1' (`hasDerivAt_F0`, `hasDerivAt_F1`), and Z(κ) := F0(κ^2/4), which on paper is $2I_1(\kappa)/\kappa$.

The integral. The identity $\int_{S^3}$ exp($\beta\langle$u, M$\rangle$) du = F0($\beta^2\|$M$\|^2/4$) is proved from left invariance of Haar alone, with no coordinates on S^3 and no Haar density. `IsHaarS3` μ asks three things — μ is a probability measure, $\|$u$\|$ = 1 almost surely, and $(q\cdot)_*\mu = \mu$ for every unit q. From these: the law of u is invariant under the rotations fixing a unit vector (`IsHaarS3.rot`), the mixed moments vanish (`mixed`), the moments $c_n := \int$ (re $u)^n$ dμ satisfy a two-term recursion (`cm_rec`) with the odd ones vanishing (`cm_odd`), and c_n is the corresponding moment of the density $(2/\pi)$ $\sin^2\theta$ dθ (`cm_eq_mom`); summing the exponential series term by term gives

```
theorem IsHaarS3.one_link (h : IsHaarS3 μ) (β : ℝ) (M : ℍ[ℝ]) :
    ∫ u, Real.exp (β * inner ℝ u M) ∂μ = F0 (β ^ 2 * ‖M‖ ^ 2 / 4)
```

together with `Z_eq_integral : Z κ = 2/π * ∫ θ in (0:ℝ)..π, exp (κ * cos θ) * sin θ ^ 2`, which connects the series to the usual integral representation. The hypothesis is not vacuous: `haarS3`, the normalised push-forward of Lebesgue measure on the unit ball under x $\mapsto$ x/$\|$x$\|$, satisfies it (`isHaarS3_haarS3`) and agrees with Mathlib's `volume.toSphere` normalisation (`haarS3_eq_toSphere`).

The two inequalities for G. Put G(β,t) = log F0(β^2t/4). All the Hessian bound needs about the one-link integral is

Lemma 3.1. *For t $\geq$ 0: 0 $\leq$ G'(t) $\leq$ $\beta^2/8$ and G"(t) $\leq$ 0.*

Proof. G'(t) = $(\beta^2/4)\cdot$F1/F0 and G"(t) = $(\beta^2/4)^2$(F0F2 − F1^2)/F0^2. Positivity of F0 and F1 is termwise. For the upper bound, $a_k - 2b_k = a_k \cdot k/(k+2) \geq 0$ termwise, where a_k, b_k are the coefficients of F0, F1; hence 2F1 $\leq$ F0 for x $\geq$ 0 (`two_F1_le_F0`) and G' $\leq$ $\beta^2/8$. For concavity, the Cauchy product of the series gives the identity

```
theorem turan_series (x : ℝ) :
    F1 x ^ 2 - F0 x * F2 x
      = ∑' n : ℕ, (catalan (n + 2) : ℝ) / ((n.factorial : ℝ) * ((n +
4).factorial : ℝ)) * x ^ n
```

whose coefficients are non-negative, so F1^2 $\geq$ F0F2 for x $\geq$ 0 (`turan_nonneg`) and G" $\leq$ 0. ■

The inequality F1^2 $\geq$ F0F2 is a known Turán-type inequality for modified Bessel functions [BP]; the point of the formalisation is only that it comes without the Hadamard product, the reality of the zeros of J_1 or Rayleigh's sum — the Catalan-number identity is a statement about coefficients that `positivity` closes. The value G'(0) = $\beta^2/8$ corresponds to Rayleigh's $\Sigma_k\ j_{1,k}^{-2} = 1/8$.

4. The star inequality

Fix a star. Its six staples are products of three of the remaining links each; under $U_e \mapsto U_e\ e^{\tau X_e}$ each staple, after transporting the insertions to one end, becomes *exactly*

$$s_i(\tau) = s_i \cdot e^{\tau A_i}\ e^{\tau B_i}\ e^{\tau C_i},$$

with A_i, B_i, C_i purely imaginary and $\|A_i\| = \|X_{i1}\|$ and so on, because Ad of a unit quaternion is an isometry (`staple_fwd`, `staple_bwd`, `qS_pert`). With $Y_i := A_i + B_i + C_i$, $W_i := A_i \times B_i + A_i \times C_i + B_i \times C_i$, M := Σ s_i, t := ‖M‖^2, $p_i := \langle s_i$, M⟩, $u_i := s_i Y_i$ and $v_i := Im(\bar{s}_i M)$, the quaternion relations AB = −⟨A,B⟩ + A×B and A^2 = −‖A‖^2 give $s_i'' = s_i(-\|Y_i\|^2 + 2W_i)$ and hence

$$t''/2 = \|\Sigma_i\ u_i\|^2 - \Sigma_i\ p_i\|Y_i\|^2 + 2\ \Sigma_i\ \langle v_i, W_i\rangle.$$

Theorem 4.1 (Lean, `StarInequality.star_S`). *Let E be a real inner-product space, $s_i \in E$ with $\|s_i\| = 1$ (i = 1,. . . ,6), $M := \Sigma_j\ s_j$, and let $u_i \in E$ and a_i, b_i, c_i, $v_i \in \mathbb{R}^3$ satisfy $\|u_i\|^2 \leq \|a_i + b_i + c_i\|^2$ and $\|v_i\|^2 \leq \|M\|^2 - \langle s_i, M\rangle^2$. Then > $\|\Sigma_i\ u_i\|^2 - \Sigma_i\ \langle s_i, M\rangle \cdot \|a_i + b_i + c_i\|^2 + 2\ \Sigma_i\ \langle v_i,\ a_i \times b_i\ + a_i \times c_i\ + b_i \times c_i\rangle \leq 18\ \Sigma_i\ (\|a_i\|^2 + \|b_i\|^2 + \|c_i\|^2)$.*

These three hypotheses are all that is used: the dimension of E does not enter and no quaternion multiplication appears. The quaternionic form is a separate theorem,

```
theorem second_deriv_le (s A B C : Fin 6 → ℍ[ℝ]) (hs : ∀ i, ‖s i‖ = 1)
    (hA : ∀ i, (A i).re = 0) (hB : ∀ i, (B i).re = 0) (hC : ∀ i, (C i).re
= 0) :
    deriv (deriv (fun τ => ‖∑ i, curve (s i) (A i) (B i) (C i) τ‖ ^ 2)) 0
      ≤ 36 * ∑ i, (‖A i‖ ^ 2 + ‖B i‖ ^ 2 + ‖C i‖ ^ 2)
```

with `curve s A B C τ = s * exp(τ•A) * exp(τ•B) * exp(τ•C)`.

Proof of Theorem 4.1 (explicit sum of squares). If t = 36 then all s_i are equal, $p_i = 6$, $v_i = 0$, and the left side is $\|\Sigma u_i\|^2 - 6\Sigma\|Y_i\|^2 \leq 0$. So assume t < 36; then $|p_i| \leq$ ‖M‖ < 6.

(i) The weights. Put $\lambda_i := (6 - p_i)/(36 - \text{t}) > 0$. Since $\Sigma_i\ p_i = \langle \Sigma\ s_i$, M⟩ = t, the sum $\Sigma\ \lambda_i$ = (36 − t)/(36 − t) = 1 is an identity. Cauchy–Schwarz in the form $\|\Sigma\ u_i\|^2 \leq \Sigma\ \|u_i\|^2/\lambda_i$ (Mathlib's `Finset.sq_sum_div_le_sum_sq_div`, Sedrakyan) gives

$$t''/2 \leq \Sigma_i\ [\ c_i\ \|Y_i\|^2 + 2\langle v_i, W_i\rangle\ ],\ c_i := 1/\lambda_i - p_i = 6 - \nu_i^2/(6 - p_i),\ \nu_i^2 := t - p_i^2.$$

(ii) One staple. Write Y = a+b+c, P := a − c, Z := Y + b. Then W = a×b + a×c + b×c = $\frac{1}{2}$ P×Z, and n := ‖a‖^2+‖b‖^2+‖c‖^2 = ‖b‖^2 + $\frac{1}{2}$‖Y−b‖^2 + $\frac{1}{2}$‖P‖^2. With μ := ‖v‖^2/36 and k := $(9+\mu)/(27-\mu)$ one has the *identity*

$$18n - c\|Y\|^2 - 2\langle v, W\rangle = \|3P - (Z\times v)/6\|^2 + \langle Z,v\rangle^2/36 + (27-\mu)\|b - kY\|^2 + R\|Y\|^2,\ R := 9 - c - \mu - (9+\mu)^2/(27-\mu),$$

obtained from the AM–GM ⟨P, Z×v⟩ ≤ 9‖P‖^2 + ‖Z×v‖^2/36, Lagrange's ‖Z×v‖^2 = ‖Z‖^2‖v‖^2 − ⟨Z,v⟩^{2}, and completing a 2×2 square. For c = c_i one computes

$$(27-\mu)(6-p)\ R = 12\mu(57 - 3\mu + 4p),$$

which is non-negative because p ≥ -6 and $\mu \leq 1$ (from $\|v\|^2 \leq t - p^2 \leq t \leq 36$). Hence $c_i\|Y_i\|^2 + 2\langle v_i, W_i\rangle \leq 18\ n_i$ for each i, and summing finishes. ■

Two remarks. R = 0 is equivalent to c = $54(108 - \nu^2)/(972 - \nu^2)$, which is exactly the largest c for which the 9×9 form $c\|a+b+c\|^2 + 2\langle v, a\times b+a\times c+b\times c\rangle$ has largest eigenvalue 18, so the AM–GM step is not lossy. And the constant 18 (equivalently 36 for t”) is sharp: `star_S_sharp` exhibits E = $\mathbb{R}$, s = (+1,+1,+1,−1,−1,−1), $u_i = 3$, $a_i = b_i = c_i = e_1$, $v_i = 0$, where both sides equal 324, and `second_deriv_sharp` does the same quaternionically. The supremum 36 is attained at t = 0, 4 and 16, that is for $s_i = \pm$s with k pluses, $1 \leq k \leq 5$.

N staples. The same proof with $\lambda_i = (N - p_i)/(N^2 - t)$, $c_i = N - \nu_i{}^2/(N - p_i)$ and the AM–GM $(3N/2)\|P\|^2 + \|Z\times v\|^2/(6N)$ gives 3N in place of 18, that is t” $\leq$ 6N $\Sigma\|X\|^2$ (`star_S_gen`, `second_deriv_le_gen`). In dimension d a star has N = 2(d−1) staples; stars at a free boundary have fewer, and the bound for N’ < N is weaker, so no separate treatment is needed.

5. Counting, and the passage to the lattice

Two combinatorial facts about I* close the gap between one star and the whole lattice.

Lemma 5.1 (Lean). *For even L: every plaquette contains exactly one link of I* (`plaq_unique`, stated as `∃! e, InP e x μ ν ∧ Istar h2 e`). For L $\geq$ 3 (more precisely for $2 \neq 0$ in Z/L): each link not in I* lies in exactly six distinct stars (`star_count`: the filter of stars whose slot set contains the link has cardinality 6), and the 18 slots of one star are 18 distinct links (`Es_card`).*

For L = 2 the forward and backward staples of a star share a link, so a link belongs to three stars, appearing twice in each; the inequality of §4 is stated for arbitrary free vectors $\tilde{X}$ and is therefore insensitive to this, but the counting is, which is why L $\geq$ 3 appears.

With F = −G(t) we have F” = −G’(t)t” − G”(t)(t’)^2 $\geq$ −G’(t)·max(t”, 0) $\geq -(\beta^2/8)\cdot 36$ $\|X\|_s^2 tar$ by Lemma 3.1 and Theorem 4.1 (this is `hess_lower`). Summing over stars and using Lemma 5.1 to rearrange the double sum,

$$\Sigma_s\ \Sigma_{e\in Es(s)}\ x_e = 6\ \Sigma_e\ x_e \ (\texttt{count27}),$$

so the total is $-(\beta^2/8)\cdot 36\cdot 6 = \mathbf{-27}\beta^2$ (`hess_total`, then `hess_star_family` for a general star family, `hess_lattice` and `hess_lattice_slots` for the lattice, and finally `hess_gauge` for the perturbation $U_e \mapsto U_e\ e^{\tau X_e}$). The three factors of 27 = 36·6/8 are: 36 from one star, 6 from the number of stars a link belongs to, 8 from G’ $\leq \beta^2/8$.

In general dimension, with N = 2(d−1) staples per star and N stars per link, the same three factors give

$$\textbf{Hess } \Phi \geq -(1/8)\cdot 6N\cdot N\cdot\beta^2 = -3(d-1)^2\beta^2\textbf{, hence } \beta_W < \sqrt{(2/3)}/(d-1),$$

a factor $4\sqrt{(2/3)}$ = 3.27 above the 1/(4(d−1)) that the counting of [SZZ] gives in this normalisation — the same factor in every dimension, and available only where a complete family exists. That is the subject of the next section.

6. Complete families: links only for d $\leq$ 4, k-cells only for d $\leq$ 3k+1

Let I* be a complete family on Z^d, or on any torus $(Z/p)^d$. Counting incidences of links and plaquettes, 2(d−1)|I*| = $p^d \cdot C(d, 2)$, so the density of I* among all links is 1/4 in every dimension and for every period; counting only the $(\mu\nu)$-plaquettes gives $2|S_\mu| + 2|S_\nu| = p^d$, so for d $\geq$ 3 each direction carries a quarter of its links.

Lemma A (restriction). *Let Q be a unit d-cube. Every 2-face of Q is a plaquette of the lattice and all four of its links are links of Q; hence I* $\cap$ Q covers every 2-face of Q exactly once.*

Index the links of Q of direction μ by their position x $\in F_2^{[d]\setminus\mu}$ and let C_μ be those in I*.

Lemma B (counting). *Q has 2^{d-2} faces of type $(\mu\nu)$ and each link of direction μ lies on exactly one of them, so $|C_\mu| + |C_\nu| = 2^{d-2}$; for d $\geq$ 3, taking three directions, $|C_\mu| = 2^{d-3}$.*

Lemma C (splitting of projections). *Let π_ν delete the coordinate ν. Covering each face ($\mu\nu$, y) exactly once is equivalent to: π_ν injective on C_μ, π_μ injective on C_ν, and $\pi_\nu(C_\mu) \sqcup \pi_\mu(C_\nu) = F_2^{[d]\setminus\mu,\nu}$.*

Lemma D (multiplicities) and Corollary E. *Take three directions μ, ν, ρ, put rest := [d]\{μ,ν,ρ}, and for $z \in F_2^{rest}$ let $\alpha(z)$, $\beta(z)$, $\delta(z)$ count the words of C_μ, C_ν, C_ρ lying over z. Lemma C applied to the three pairs gives $\alpha+\beta = \alpha+\delta = \beta+\delta = 2$; adding, $\alpha+\beta+\delta = 3$, so $\alpha \equiv \beta \equiv \delta \equiv 1$. That is: deleting any two coordinates keeps the 2^{d-3} words of C_μ distinct, so C_μ is a binary code of length n = d−1, size 2^{n-2} and minimum distance at least 3.*

> **Theorem 6.1.** *A complete family exists on Z^d if and only if d $\leq$ 4. On the torus $(Z/L)^d$ with d $\geq$ 2 it exists if and only if d $\leq$ 4* ***and L is even****.*
>
> *Proof.* Apply the Hamming (sphere-packing) bound to the code of Corollary E: balls of radius 1 are disjoint, so $2^{n-2}(1 + n) \leq 2^n$, that is n $\leq$ 3, that is d $\leq$ 4. For d $\leq$ 4 there are families of period 2 — that of §1 for d = 4, and > d = 3: $I_1 : x_2 = x_3 = 0$, $I_2 : x_1 = 0, x_3 = 1$, $I_3 : x_1 = x_2 = 1$; d = 2: $I_1 : x_2 = 0$, $I_2 : \emptyset$ (mod 2) — and a family of period 2 lives on Z^d and on every torus of even period. That an odd period admits none is Theorem 6.6(i). ■

The torus half of Theorem 6.1 is machine-checked as a single equivalence (`PerfectFamily.perfectK_one_iff`, §11.2); the statement on Z^d is `no_perfectZ_of_five` together with `perfect_I2`, `perfect_I3`, `perfect_I4`.

At d = 4 the bound is an equality: n = 3, two words, distance 3 is the repetition code {000, 111}, which is perfect (2·4 = 8). Inside a unit 4-cube the two links of a given direction are always antipodal. For d = 5 the contradiction takes three words by hand: splitting on two coordinates {σ,τ}, the four words read 00ab, 01cd, 10ef, 11gh; distance 3 from 00ab forces cd = ab+11 and ef = ab+11 = cd, and then 01cd and 10ef are at distance 2.

The Lean route is shorter than the counting above and uses no coding theory. Its core is a 3-dimensional subcube: from the conditions on its six faces, exactly one of the four links of a given direction inside it belongs to the family, a decision over 2^12 = 4096 Booleans (`cube3_bool`, by `decide`). From it one gets, for the unit cube, the existence (`edge_ex`) and the uniqueness (`edge_uniq`) of a link of the family agreeing with a given position outside two coordinates — the minimum distance 3 of Corollary E — plus `proj_bijective`. For d $\geq$ 5 one picks four directions besides μ and forces two links of the family agreeing outside two coordinates, contradicting `edge_uniq` (`exists_five`, `no_perfect_of_five`). The passage to Z^d is Lemma A (`restrict_perfect`), and `no_perfectZ_of_five` assumes no periodicity. The positive side is `perfect_I2`, `perfect_I3`, `perfect_I4`, each by `decide`, `I4` being the restriction of the period-2 family of §1 to a unit cube. Three negative controls check that `decide` is not returning truth mechanically (`not_perfect_empty3`, `not_perfect_full3`, `not_perfect_I4_broken`).

What is left in d $\geq$ 5. Without a complete family, the one-link integral still factorises if one only asks that every plaquette contain *at most* one link of the family — a *packing.* Two things are then determined: how large a packing can be, and what the resulting threshold is.

Theorem 6.2 (Lean). *Let I be a family of links on the periodic lattice $(Z/L)^d$ such that every plaquette contains* ***at most*** *one link of I. Then |I| is at most the number of vertices, and the density of I among all links is at most 1/d: > theorem card_edgeSet_le [NeZero L] {I : Fin d → Wp d L → Bool} (h : AtMostOne I) : > (edgeSet I).card ≤ L ^ d > theorem density_le_rat [NeZero L] {I : Fin d → Wp d L → Bool} (h : AtMostOne I) (hd : 0 < d) : > ((edgeSet I).card : ℚ) / (d * L ^ d) ≤ 1 / d*

Proof. Let $\mu \neq \nu$ and let x be a vertex. The links (μ, x) and (ν, x) are the first and the third of the four links of the plaquette with corner x in the directions μ, ν, so they cannot both belong to I (`not_two_dirs`, whose proof is the arithmetic 1 + a + 1 + b $\leq$ 1). Hence at most one link of I leaves any vertex in a positive direction, the map (μ, x) $\mapsto$ x is injective on I (`snd_injOn`), and |I| is at most the number of vertices, which is L^d; the total number of links is $d \cdot L^d$. ∎

The statement is proved for an arbitrary vertex type with an arbitrary "step" map (`Packing`, `card_le_card_vertices`), so the same theorem covers Z^d: there `card_le_of_support` bounds by |S| the number of links of I whose source lies in a finite set S, which gives the density 1/d in any window with no boundary correction and with no periodicity assumed.

A caution the development also records. The family I is *not* a matching on the lattice: two collinear links (μ, x) and (μ, x + e_μ) lie on no common plaquette and may both be taken (`Line_atMostOne`, `Line_not_matching`, an explicit closed line in d = 2). All that is true, and all the proof uses, is that the out-degree in the positive directions is at most one. An argument assuming a matching — restricting to a unit cube, where I *is* one, and averaging over cubes — does more work for the same bound.

The bound 1/d is attained for d $\geq$ 4. Take two disjoint period-2 complete families A, B in dimension 4 (of the eight such families, disjoint pairs exist) and switch between them by the parity of the fifth coordinate, taking no link in the extra directions. For d = 5, L = 2 this is machine-checked: `P5_atMostOne` by `decide`, |P5| = 32 = the number of vertices (`bound_tight_d5`), 5·|P5| = the total number of links (`density_eq_d5`), i.e. density exactly 1/5; and `P5_not_perfect` confirms it is not complete, consistently with Theorem 6.1. At d = 4 the complete family A4 attains the same bound (`bound_tight_d4`: |A4| = 16 = 2^4). For d = 6, 7 the analogous construction gives 0.166667 and 0.142857 with no violated plaquette [computation, not in Lean].

Remark 6.3 (a second proof of d $\leq$ 4 on the torus) [Lean. *] The statements* `four_mul_card_eq` *(a complete family on* $(Z/L)^d$ *has* $4|I| = d \cdot L^d$*),* `perfect_torus_d_le_four` *(hence d* $\leq$ *4) and* `card_cellSet_le` *(a packing of k-cells has* $k|I| \leq L^d \cdot C(d, k-1)$*) are machine-checked in* `PlaquetteCounting`*. On a periodic lattice the two counts already give the dimension bound in two lines: a complete family has density exactly 1/4 (each link lies on 2(d−1) plaquettes, each plaquette has four links), a packing has density at most 1/d (Theorem 6.2), hence d* $\leq$ *4. No code and no Hamming bound are needed for this; what the coding argument of Theorem 6.1 adds is the local statement on* Z^d *with no periodicity, and the reason for equality at d = 4. The same two lines work with k-cells in place of links and (k+1)-cells in place of plaquettes: two k-cells with the same base point x and direction sets S, S′ sharing k−1 directions are faces of one (k+1)-cell, so in a packing the direction sets at x pairwise share at most k−2 directions; their (k−1)-subsets are then pairwise distinct, so k·|F(x)|* $\leq$ *C(d, k−1) and the density is at most C(d,k−1)/(k·C(d,k)) = 1/(d−k+1). Against the exact density of a complete family this closes the dimension, and both halves are now machine-checked for every period (Theorems 6.4 and 6.5).*

Complete families of k-cells

Fix k $\geq$ 1. A *k-cell* of $(Z/L)^d$ is a pair (S, x) with S a set of k directions and x a vertex; the (k+1)-cell (T, x) has the 2(k+1) faces (T\\$\nu$, x) and (T\\$\nu$, x + e_ν), $\nu \in$ T. A family I of k-cells is **complete** when every (k+1)-cell has exactly one of its 2(k+1) faces in I, and a **packing** when it has at most one. For k = 1 the faces of a 2-cell are the four links of a plaquette and the two notions are those used above; the Lean predicate is, verbatim,

```
def PerfectK (k : ℕ) (I : Finset (Fin d) → Wp d L → Bool) : Prop :=
  ∀ T : Finset (Fin d), T.card = k + 1 → ∀ x : Wp d L,
    ∑ ν ∈ T, faceCount I T x ν = 1
```

with `Wp d L = Fin d → ZMod L`, `shiftp x i = x + e_i` and `faceCount I T x ν = (I (T.erase ν) x).toNat + (I (T.erase ν) (shiftp x ν)).toNat`. Replacing `= 1` by $\leq$ `1` gives the packing predicate, which is indexed by the cell size less one (`PackK m` for cells of k = m+1 directions, so that `PerfectK (m+1) I → PackK m I`, `packK_of_perfectK`). The 2(k+1) faces are distinct exactly when L $\geq$ 2 (`card_faceSetK`; at L = 1 the face (T\\$\nu$, x + e_ν) collapses onto (T\\$\nu$, x)), and L $\geq$ 2 need not be assumed anywhere below, since a period of 1 is odd and Theorem 6.6(i) applies.

Theorem 6.4 (Lean; counting). *Let k + 1* $\leq$ *d and L* $\geq$ *1, and let I be a complete family of k-cells on* $(Z/L)^d$*. Then (i) 2(k+1)·|I| =* $C(d,k) \cdot L^d$ *(*`card_supportK_of_perfectK`*), so the density of I among all k-cells is exactly 1/(2(k+1)),* ***independently of d and of L*** *(*`density_of_perfectK`*); (ii) for every (k+1)-set T of directions,* $2 \cdot \Sigma_{\nu \in T} |I_{T\setminus\nu}| = L^d$*, where* I_S *is the set of base points x with (S, x)* $\in$ *I (*`two_mul_sum_card_slice`*); (iii) hence 2(k+1) divides* $C(d,k) \cdot L^d$ *(*`two_mul_succ_dvd_of_perfectK`*), and* ***L is even*** *(*`two_dvd_of_perfectK`*).*

Proof. (i) counts the incidences between k-cells and (k+1)-cells in two ways; the translation x $\mapsto$ x + e_ν is a bijection of the vertices, so each face of each type contributes once, and C(d,k+1)(k+1) = C(d,k)(d−k) clears the factor d − k. (ii) sums the defining identity of a fixed T over all vertices, the same bijection identifying the two faces (T\\$\nu$, x) and (T\\$\nu$, x + e_ν) in the count. (iii) is (i) as a divisibility, and (ii) with 2 prime. ■

Part (ii) is the sharper of the two tests: the total count (i) admits an odd period whenever 2(k+1) happens to divide C(d,k), while the slice identity forces 2 | L for every k and every d ≥ k+1.

Theorem 6.5 (Lean; the dimension bound). *For k ≥ 1, k + 1 ≤ d and L ≥ 1, a complete family of k-cells on $(Z/L)^d$ forces* ***d ≤ 3k+1*** *(`perfectK_d_le`); equivalently, for 3k+1 < d there is none, on a torus of any period (`no_perfectK_of_lt`). At k = 1 the bound is attained (`bound_sharp_k1`).*

Proof. Density exactly 1/(2(k+1)) by Theorem 6.4(i), density at most 1/(d−k+1) by the packing count of Remark 6.3, and a complete family is a packing. ■

One construction and the two tests then decide several cases. The construction covers the one dimension in which a complete family can be written down for every k: for d = k+2 the complement of a k-set of directions is a pair, so a family may be prescribed pair by pair. Split the d directions into blocks of size 2 and 3; on a block {a,b} put $g_{ab}(y) = y_a \oplus y_b$, on a block {a,b,c} put $g_{ab} = y_a\, y_b$, $g_{ac} = (1 \oplus y_a)\, y_c$, $g_{bc} = (1 \oplus y_b)(1 \oplus y_c)$, and let (S, y) belong to the family when the complement of S is a pair {u,v} lying in one block with $g_{uv}(y) = 1$. Every d ≥ 2 admits such a split, and the family is complete for every k (`Layout.perfect_fam`, `exists_perfect`), of size $(k + 2) \cdot 2^k$ (`card_support_fam`). Having period 2 it lifts, through the parity map ZMod L → ZMod 2, to every even period (`perfectK_of_perfect`).

Theorem 6.6 (Lean; cases decided). *On $(Z/L)^d$ with L ≥ 1: (i) if L is odd there is no complete family of k-cells, for every k and every d ≥ k+1, the hypothesis k ≥ 1 being unnecessary here (`no_perfectK_of_odd`); (ii) for k = 1 and d ≥ 2, a complete family exists* ***if and only if d ≤ 4 and L is even*** *(`perfectK_one_iff`); (iii) for k = 2, d = 4, a complete family exists if and only if L is even (`perfectK_k2_d4_iff`); (iv) for d = k+2 and L even, a complete family exists, for every k (`exists_perfect` with `perfectK_of_perfect`); (v) for k = 4, d = 9 a complete family requires 10 | L, and for k = 5, d = 8 it requires 6 | L (`no_perfectK_k4_d9_ten`, `no_perfectK_k5_d8_six`).*

(v) is the divisibility test of Theorem 6.4(iii) together with (i): $10 \nmid 126 \cdot L^9$ unless 5 | L, and $12 \nmid 56 \cdot L^8$ unless 3 | L. The two tests are independent — the test of (iii) of Theorem 6.4 does not reach k = 3, d = 9, where $8 \mid 84 \cdot L^9$ for every even L, and the bound of Theorem 6.5 does not reach it either, 9 ≤ 10.

What is not decided. The bound d ≤ 3k+1 is sharp at k = 1, and its sharpness is open for every k ≥ 2. Machine-checked existence covers d = k+2 for every k together with (k, d) = (1, 2) and (1, 4), and nothing beyond that: for k = 2 the region between the construction at d = 4 and the bound at d = 7 is open, d = 5 being excluded there unless 6 | L by Theorems

6.4(iii) and 6.6(i), and d = 6, 7 untouched. Exhaustive search at period 2 finds complete families of 3-cells for d = 5, 6, 7, 8 [computation], against the bound d $\leq$ 10 for k = 3. For k = 3, d = 9 an exhaustive search reports no complete family of period 2 [computation: three SAT solvers agree after normalising one base point, where the direction sets must form the affine plane AG(2,3) minus one line; the DRAT proof has not been independently checked], but neither the counting nor the bound excludes it, and larger even periods are untouched. For that k, d $\geq$ 11 is excluded by Theorem 6.5 and d = 10 by an integer refinement of it: at a base point the direction sets form a packing of triples with pairwise intersections of size at most 1, of which there are at most $\lfloor(10/3)\cdot\lfloor 9/2\rfloor\rfloor$ = 13 (the Schönheim bound [to be verified]), fewer than the average C(10,3)/8 = 15 that a complete family requires. None of the statements of this subsection is a classification: the number of complete families, and their shape, are not addressed.

So the maximal density is min(1/4, 1/d), the two bounds having different origins: 1/4 is the incidence count (each link lies on 2(d−1) plaquettes), attained by Theorem 6.1 only for d $\leq$ 4, where it rests on a perfect code; 1/d is the vertex count of Theorem 6.2. They meet at d = 4. Both bounds are formalised — 1/4 as `four_mul_card_le`, with equality for a complete family (`four_mul_card_eq`, the case k = 1 of Theorem 6.4(i)); 1/d as Theorem 6.2 — and for d $\leq$ 3 the 1/d side is the loose one — at d = 3, L = 2 the incidence count gives |I| $\leq 6 < 8$, and 6 is the exhaustive maximum [computation]. In a unit cube the maximum is M(3) = 3, M(4) = 8, M(5) = 16, M(6) = 32, the last by the vertex bound together with a search that found the lower bound [computation, exact].

The threshold is set not by the density but by $max_a\ r_a$ [paper and computation], the number of uncovered plaquettes on a single remaining link: with n = 2(d−1), the counting of §5 applied to a link carrying r_a bare plaquettes and n − r_a absorbed ones gives K'(a) = $4\beta \cdot r_a$ + (3/4)·n·(n − r_a) $\cdot \beta^2$ and Hess $\Phi \geq -max_a$ K'(a). For d = 5 the minimum of $max_a\ r_a$ over all families is exactly 4 (period 2 attains it; t = 3 is infeasible on the torus of period 2 and 4 and also in the free-boundary box $0, \ldots, 4^5$) [computation], so K' = 16β + $24\beta^2$ and the threshold is $\beta_W < 0.1076$, a factor 1.72 above [SZZ] instead of 3.27. The uncovered fraction is $\varepsilon = 1/5$, also sharp.

A sharper arithmetic test, on paper only. The slice identity of Theorem 6.4(ii) is a statement about one (k+1)-set of directions at a time; a statement about each direction set separately would be stronger. On a unit cube, for each (k+1)-set T of directions the numbers $a_S = |C_S|$ (S $\subset$ T, |S| = k) satisfy $\Sigma_{S \subset T}\ a_S = 2^{d-k-1}$. As soon as d $\geq$ 2k+1 this system has the unique solution $a_S = 2^{d-k-1}/(k+1)$, since the rank of the inclusion matrix $W_{k,k+1}$ is min(C(d,k), C(d,k+1)) (Gottlieb's theorem [Go], [to be verified]). So k+1 must divide 2^{d-k-1}, that is be a power of two; on a torus the corresponding condition is 2(k+1) | L^d, on the period rather than on $C(d,k) \cdot L^d$. For k = 2 — plaquettes covering every cube exactly once — this fails for every d $\geq$ 5 because 3 never divides 2^{d-3}: the obstruction is arithmetic, not coding-theoretic (exhaustive search confirms it at d = 5, while k = 3 in d = 6 does have a cover, |I| = 20) [computation]. **This is the one step of the subsection above that is not machine-checked**, and it is the step that would close k = 2, d $\geq$ 5: the rank of an inclusion matrix is, in the Mathlib we searched, not available, so what the development has is the weaker test 2(k+1) | $C(d,k) \cdot L^d$ of Theorem 6.4(iii), which for k = 2, d = 5 leaves every period divisible by 6 open. The correspondence "complete family $\leftrightarrow$ perfect code" is special to k = 1.

7. From the marginal to the full measure

This section is an outline. We record what is complete on paper and what is not, together with the constants, since those are what a reader would have to check.

(a) Bakry–Émery for the marginal [paper, complete]. For $\Lambda \subseteq$ Rest finite and any boundary condition η, the measure $\mu'^{\eta}_{\Lambda} \propto \exp(-\Phi)$ on $(S^3)^\Lambda$ satisfies Ric + Hess $\Phi|_\Lambda \geq$ K' $:= 2 - 27\beta^2$, because restricting a quadratic form to a subspace only helps. Bakry–Émery gives LSI(K'), Poincaré(K') and $|\nabla P_t f| \leq e^{-K't}\ P_t|\nabla f|$, with constants independent of Λ and η. Free and periodic boundaries behave the same: a star at the boundary loses staples, and the counting of §5 does not deteriorate.

(b) Exponential decay [paper, outline with constants]. Let ē $\sim$ e mean that the two links lie in a common star; each link lies in six stars and each star has 17 other slots, so at most 102 links are adjacent to a given one. The first identity in the proof of [SZZ, Lemma 4.10] uses only that v_e^i generates an isometry and commutes with Δ_e, not the form of the action; the same use is made in [Ni, Lemma 3.23] for a marginal action. With r := $\mathrm{I}_2/\mathrm{I}_1 \leq \min(\kappa/4,\ 1)$, $\kappa = \beta\|\mathrm{M}\|$, and q := $(\mathrm{I}_2/\mathrm{I}_1)^2 - \mathrm{I}_3/\mathrm{I}_1$ one gets $|\nabla_e\Phi| \leq 6\beta\ \mathrm{r}(\kappa) \leq 9\beta^2$, $a_{e,e} \leq \beta^2(55.5 + 6\mathrm{q})$ and $\Sigma_{\bar{e}\neq e}\ a_{e,\bar{e}} \leq \beta^2(43.5 + 108\mathrm{q})$, hence

$$\mathrm{A} := sup_e\ \Sigma_{\bar{e}}\ a_{e,\bar{e}} \leq \beta^2(99 + 108\mathrm{q}) \leq \beta^2(99 + 243\beta^2)$$

(here q $\leq \kappa^2/16$ is loose; q $\leq \kappa^2/48$ holds, replacing 243 by 81, which does not affect the threshold). The iteration of [SZZ, (4.28)–(4.30)], which goes back to Guionnet–Zegarliński [GZ], then runs with C_0 = 3A and yields Corollary 3 with m = 2K'/(3e^{4}A). The hypotheses of [SZZ, Cor. 4.11] are only Assumption 1.1 — that K > 0 — with no condition tying K to the $a_{e,\bar{e}}$; K enters only the rate. **So the threshold for the decay is the same $\sqrt{}$(2/27) as for the curvature.** What we have not done is write the iteration out line by line for the dynamics of Φ (finite volume, boundary conditions, diffusion on the remaining links only). Everything it uses — uniform bounds on $a_{e,\bar{e}}$, finite range, L^∞ contraction of P_t, and (4.9) — Φ satisfies.

(c) Observables that touch I* [paper, complete]. $Cov_\mu(f,g) = E_{\mu'}[Cov(f,g \mid \mathrm{y})]$ + $Cov_{\mu'}(\bar{f},\ \bar{g})$ with $\bar{f}$ = E[f|y]. Given the remaining field y, the links of I* are independent, $u_{e'}$ being von Mises–Fisher with parameter $\beta M_{e'}(y)$; so if f and g share no link of I* the first term vanishes. The conditional expectation $\bar{f}$ is smooth in y, its support grows only by the slots of the stars met, and

$$|||\bar{f}|||_\infty \leq |||f|||_\infty + 36\beta \cdot n_{I*}(f) \cdot \|f\|_\infty,\ \|\bar{f}\|_2 \leq \|\mathrm{f}\|_2,\ d_\sim(\Lambda_{\bar{f}},\ \Lambda_{\bar{g}}) \geq d_\sim(\Lambda_f,\ \Lambda_g) - 2.$$

(d) Infinite volume [paper; complete, except for the uniqueness]. If μ is a DLR measure for the Wilson action then its y-marginal is a DLR measure for Φ: for $\Lambda \subseteq$ Rest take $\tilde{\Lambda} := \Lambda$ together with the links of I* whose star meets Λ; every plaquette touching $\tilde{\Lambda}$ belongs to the star of exactly one link of I* in $\tilde{\Lambda}$, so integrating $u_{\tilde{\Lambda}}$ gives $\prod\ F0(\beta^2\|M_{e'}(y)\|^2/4)$ and nothing outside. Conversely the conditional law of u given y is the product of the von Mises–Fisher laws, so $\mu = \mu' \otimes \prod_{e'}\ vMF(\beta M_{e'}(y))$ and μ is unique as soon as μ' is. Uniqueness of μ' follows from (b) used uniformly in the boundary condition — moving one boundary link b gives $\partial_{\eta_b}\ \mu'^{\eta}_{\Lambda}(f) = -\mathrm{Cov}(\mathrm{f},\ \partial_{\eta_b}\Phi)$ with $\partial_{\eta_b}\Phi$ local and bounded, so two boundary conditions differ by $\mathrm{O}(\ell^3\ e^{-m\ell/3})$ — but we have not filled in every line, and the coupling argument of [SZZ, §5] transported to Φ would be an alternative we have not written either.

8. Limits of the method

The constant 27 cannot go below 21 [computation, exact arithmetic]. There is an explicit configuration in which the bound is off by exactly 27/21. Take all links commuting, $U_e = exp(\theta_e$ n) with a fixed imaginary unit n and all θ_e multiples of $\pi/3$, given by a period-2 table on $(Z/2)^4$ and tiled to $(Z/4)^4$. For X $\perp$ n each u_a is $\pm e^{i\psi_a} x_a$ in the plane orthogonal to n, so the Hessian lives in the ring Z[ω], $\omega = e^{i\pi/3}$, and there one checks in integer arithmetic that every star has M = 0 and $\|$M'$\|^2$ = 252 = $\|12 + 6\omega\|^2$. Since M = 0 gives t = t' = 0, Hess Φ(X,X) = $-(\beta^2/8) \cdot \Sigma_{e'}$ $2\|M'_{e'}\|^2$ exactly, for every β, and the Rayleigh quotient is 129024/(8·768) = 21. Hence

$$\mathbf{21}\beta^2 \le sup(-\lambda_m in \ \textbf{Hess } \Phi) \le \mathbf{27}\beta^2\textbf{,}$$

and the ceiling of this route is $\beta_W \le \sqrt{(2/21)} = 0.3086$. A gradient search on $(Z/4)^4$ (Hellmann–Feynman derivatives of $\lambda_m in$, L-BFGS, ten Haar starts) reaches 21.000000 from all ten, with the two top eigenvalues degenerate at 21 and the third at 14.7446; the configurations reached are not tilings of the period-2 table, so the worst configurations form a large family, and nothing above 21 was found. The worst star has its forward and backward staples equal in pairs and the three pairs at 120°, so the eighteen u_a lie in the two-dimensional complement of the staple plane, twelve in phase and six rotated by 60° — whence 252 = 324·(7/9) and 27·(7/9) = 21.

The estimate is loose in one place only: the triangle inequality $\|M'_{e'}\| \le \Sigma_a \|u_a\|$, a factor $\sqrt{(7/9)}$. The weighted Cauchy–Schwarz, M = 0 and G' $\le \beta^2/8$ are all equalities there. Capturing the loss on paper would mean bounding sup_Q $\|$dM(Q)$\|^2$ conditionally on M, which has the shape of a frustrated ground-state estimate; and at finite β the weights $G'(t_{e'})$ differ from star to star, so an argument using cancellations must survive a supremum over weights in [0,1]. We do not have it.

Large β [computation]. For $\beta \gtrsim 1$ integrating first is worse: $\lambda_m in(Hess\ \Phi)$ is $-12.1\beta^2$ at $\beta = 1$ and $-43.1\beta^2$ at $\beta = 2$, against -16β and -32β without integration. The reason is structural. Bakry–Émery asks for a bound at *every* configuration and cannot use that for large β the measure concentrates near $Q_p \approx 1$, where Φ is convex ($\lambda_m in = 0$ at the cold configuration). Since Ric = 2 does not move with β, no number of local integrations pushes the threshold beyond O(1).

The iteration does not close. Fixing one remaining link a under Φ, it appears once in each of six stars and each M_k is affine in u_a, so the conditional density is $\prod_{k=1}^{6} F0(\beta^2(\|N_k\|^2 + 1 + 2\|N_k\| t_k)/4)$ with $t_k = \langle u_a, c_k \rangle$ — a product of non-Wilson class functions, which does not return to a Bessel form. To leading order in small β the factors are Wilson-type with $\beta_e f f \le 5\beta^2/4$, so a second round is formally possible, but the range grows ($2 \to 6 \to \dots$), the combinatorial constants grow with it, and a quadratic recursion contracts only when the first term is already small. What one gets is a further constant, which is the language of the strong-coupling expansion.

SU(N) [computation, with the mechanism on paper]. The key input of §3, G' $\le \beta^2/8$ and G" ≤ 0, says that the covariance of the tilted measure $\propto$ exp(h$\langle$u,M$\rangle$)du is dominated by the covariance of Haar. With h := Nβ and $Var_0(\langle u, V\rangle) = \|V\|_H^2 S/(2N)$ for U(N) and for SU(N), N $\ge$ 3 — while SU(2) alone has $\|V\|_H^2 S/2$, twice the U(2) value, because V $\otimes$ V carries an invariant — the domination is **false** already for U(2) with M = diag(s, 0) and V = E_{22}: the ratio exceeds 1 from s = 0.03 on, tends to 2 as s $\to \infty$, and to N for U(N). The mechanism, on paper: a rank-deficient tilt pushes u into the leftover subgroup, where the Haar variance 1/2 is larger than 1/(2N). The same happens for SU(N), N $\geq$ 3. What makes SU(2) work is the accident that the leftover degree of freedom is the whole group.

Measuring the star constant directly instead — the form is exactly quadratic in X, so the inner maximisation is an eigenvalue problem and only the staples are searched over — reproduces the SU(2) value to seven digits (Λ/h^2 = 8.999998 = 3n/2 with n = 6, i.e. the $-27\beta_W{}^2$ above, in an implementation using no quaternions), and for U(N) gives Λ(N,h) = $3n \cdot h^2 \cdot w_N(h)$, with w_N the largest eigenvalue of the covariance at M = h·diag(n,. . . ,n,0); the worst star maximises the violation, and the M” term contributes zero to machine precision. The thresholds are $\beta_S ZZ$ = 0.0939, 0.0933, 0.0931 for N = 2, 3, 4 at d = 4 — a factor 4.5 above 1/48 — but the ratio $r_N(s)$ tends to N, so β*(N) $\sim$ 1/(2(d−1)$\surd$(3N)) = 0.0962/$\surd$N and the ratio to [SZZ]'s N-independent 1/(16(d−1)) is 8/$\surd$(3N), below 1 for N $\gtrsim$ 21. **The gain does not survive the ’t Hooft limit; it is a phenomenon of small N and of d $\leq$ 4.**

9. Comparison with the literature

All entries are converted to $\beta_W = 4\beta_S ZZ$ for SU(2), d = 4. The statements differ, so the column is not an ordering.

Source	**Statement**	β_W
[SZZ]	mass gap, LSI, uniqueness, for SU(N)	< 1/12 = 0.0833
[CNS2]	area law for U(N), SU(N), SO(2N), via the dynamics of [SZZ] and the gap condition of [DF]	< 1/6 = 0.1667
[CNS1]	area law for U(N), master loop equation	$\beta_S ZZ \leq 10^{-10d}/d$
[Ni]	mass gap for U(N), splitting U(1) × SU(N)	$\beta_S ZZ < 10^{-6d}$
[OS], [Se] (cluster expansion)	mass gap, area law, all compact groups	**no explicit value retrieved**
here	Bakry–Émery curvature 2 − 27β^2 [Lean plus standard facts], hence mass gap [paper, outline]	< 0.2722; ceiling of the method 0.3086

Two things must be said about the last two rows. First, Osterwalder–Seiler [OS] is the classical source of a mass gap at strong coupling, and we could not obtain it: the journal is behind a paywall, no scan is attached to the INSPIRE record, and Seiler’s lecture notes [Se] were likewise unavailable. Every secondary text we have read — [SZZ], [CNS1, Remark 1.3], [Ni] — describes its range only as "sufficiently small β" or as a constant deteriorating with N. Second, our own naive Kotecký–Preiss estimates for the expansion give $\beta_W \lesssim$ 0.014 in the Mayer form (activity tanh β, 20 plaquettes adjacent to one, $(e\Delta)^{n-1}$ polymers) and $\beta_W \lesssim$ 0.06 in the character expansion (closed surfaces, minimum six plaquettes, activity I_2/I_1); but these are crude, and the necessary condition for absolute convergence of the character expansion, $u \cdot \mu_s < 1$ with μ_s the growth constant of closed surfaces, allows β_W up to 0.4–0.8 for μ_s between 5 and 10. **We therefore do not claim a new regime.** What we can say is that among the thresholds we found written with an explicit number, none exceeds 0.2722, and that a careful cluster expansion may well.

Adjacent work that is not a comparison: [AC] treats finite groups at weak coupling; [Ch] and [Ja] give 1/N expansions for Wilson loops at strong coupling with no explicit range for a gap; [Ni2], [Br] and [Le] cite [SZZ] for other purposes; [BG] studies the combinatorics of Wilson loop expectations for SO(N), not the Hessian of a one-link integral.

Two ingredients of the present construction do appear in the literature, in other combinations. Applying the dynamics of [SZZ] to a *marginal* measure is the scheme of [Ni], where

U(N) is split as U(1) $\times$ SU(N); there the conditional measure is controlled by a cluster expansion and the conditional expectations by a quasi-locality estimate, whereas here conditional independence makes both steps immediate. *Conditioning* before applying Bakry–Émery is the scheme of [CNS2], which cuts the lattice into slabs of height one. Neither replaces the action by the exact integral of a sub-family of links.

10. What is not claimed

1. **No claim about the continuum limit, and none about the Clay millennium problem.** Everything here is a statement about a fixed lattice at $\beta < \mathrm{O}(1)$. The method cannot reach $\beta \to \infty$: Ric = 2 does not depend on β, and §8 shows that integrating first is *worse* for $\beta \gtrsim 1$.

2. **The mass gap is not a Lean theorem.** What is machine-checked is Theorem 1, Theorem 2 and Theorem 2′, and Theorems 6.2, 6.4, 6.5 and 6.6 — the Hessian bound and the combinatorics of complete families, not the gap. The identification of the second derivative of Φ along U $e^{\tau X}$ with the Riemannian Hessian, the value Ric = 2, the Bakry–Émery theorem and the decay argument are all on paper, and two of those steps — the transported [SZZ, Lemma 4.10]/[SZZ, Cor. 4.11] and the uniqueness of the marginal — are outlines with constants rather than complete proofs (§2, table; §7).

3. **The comparison with the cluster expansion is undecided.** [OS] and [Se] were not obtained, and our own estimates of the convergence radius are crude in both directions (§9). The phrase "an improvement inside the Bakry–Émery method" is the only characterisation we are prepared to defend.

4. **The constant 27 is not optimal.** The truth is between 21 and 27 (§8); 21 is exact for one explicit configuration and the numerical searches did not exceed it, but no proof of 21 exists here.

5. **No claim of priority.** We did not find "complete family", or "integrate a sub-family of links, then Bakry–Émery", in the literature we searched — arXiv searches, the 18 INSPIRE and 19 Semantic Scholar citations of [SZZ] by title, and full-text searches of [CNS2] and [Ni] — but the search is limited, and the idea could be known in the context of the multi-hit / one-link-average variance reduction of Monte Carlo simulation, where sets of links sharing no plaquette are updated simultaneously; the family of density 1/4 in d = 4 may be known on the numerical side [PPR] [to be verified]. MathSciNet and zbMATH were not available to us. The tools of §4 and §6 (weighted Cauchy–Schwarz, a sum-of-squares identity, the Hamming bound, the perfectness of the length-3 repetition code, exact cover) are classical, and the Turán-type inequality of §3 is known [BP].

6. **The bound d $\leq$ 3k+1 is not claimed to be sharp, and none of §6 is a classification.** Sharpness is known only at k = 1; for k = 3 the only evidence against d = 9 is an exhaustive search at period 2 by SAT, which is not machine-checked and says nothing about larger even periods. The number and the shape of complete families are not addressed anywhere. The sharper arithmetic test — 2(k+1) dividing the period power L^d, which would close k = 2 for all d $\geq$ 5 — rests on the rank of an inclusion matrix (Gottlieb's theorem, itself [to be verified]) and is **on paper only**; the machine-checked test is the weaker 2(k+1) $\mid C(d,k) \cdot L^d$.

7. **Everything in §6 beyond Theorems 6.1, 6.2, 6.4, 6.5 and 6.6, and everything in §8, is computation.** The attainment of the packing density 1/d is machine-

checked only for d = 4, 5. The exhaustive searches (exact cover by depth-first search and by SAT, integer linear programming, branch and bound for M(d)), the gradient searches for λ_min and the U(N) measurements are cross-checked by positive and negative controls but not kernel-checked. Where a search did not finish we say so: M(6) = 32 rests on the paper upper bound plus a search that found only the lower bound, and the d = 6 value of min max_a r_a was not determined.

8. **No independent authorship, no external review.** The observation, the paper proofs, the programs, the Lean development and this draft were produced by the same agent (§12), and no mathematician outside the authors has read them. Lean's kernel and the agreement of independent reimplementations are the only checks not internal to the authors, and those reimplementations are procedurally, not conceptually, independent.

11. The Lean development

11.1 Files

Everything checks from `import Mathlib` alone in seven files: `LatticeGaugeOneLink.lean` carries the first six namespaces below, `PerfectFamily.lean` and `PlaquetteCounting.lean` the next three, and four further files carry the namespace `PerfectFamily` again, each self-contained. Since a file that checks from `import Mathlib` alone cannot import another such file, those four repeat the definitions and the lemmas they reuse; §11.3 says how much is repeated and how it was checked to be identical. "Lines" is the extent of the namespace inside its file.

Namespace (file)	Lines	Content
`StarInequality`	586	the algebraic star inequality: `weighted_cs`, `W_eq_half_P_cross_Z`, `staple_sos`, `R_closed`, `R_nonneg`, `staple_ineq`, **`star_S`**, and the N-staple versions ending in `star_S_gen`; the sharpness `star_S_sharp` is in a separate file of the development, outside the three above
`StapleCurve`	394	the quaternionic bridge: `curve`, `D2_vec`, `second_deriv_eq_hessForm`, **`second_deriv_le`** (36), `second_deriv_le_gen` (6N), `second_deriv_sharp`
`OneLinkSeries`	886	the series F0, F1, F2 and Z: `cauchy_coeff_catalan`, **`turan_series`**, `two_F1_le_F0`, `turan_nonneg`, `G1_le`, `G2_nonpos`, `hess_lower`, `count27`, `hess_total`, `hess_star_family`, the moments `cm_rec`, `cm_odd`, `cm_eq_mom`, **`Z_eq_integral`**
`CompleteFamilyLattice`	609	the geometry of I*: `Istar`, **`plaq_unique`**, `Star`, `Rest`, `Es`, **`star_count`**, **`Es_card`**, `slot_injective`, `hess_lattice`, `hess_lattice_slots`
`StaplePerturbation`	386	the perturbation of staples: `pert`, `staple_fwd`, `staple_bwd`, `qS_pert`, `Phi`, **`hess_gauge`**
`WilsonOneLink`	1,010	the Wilson action and the integral: `SW`, **`SW_eq`**, `IsHaarS3` with `rot`, `mixed`, `integral_exp_re`, **`IsHaarS3.one_link`**, `haarS3`, `isHaarS3_haarS3`, `haarS3_eq_toSphere`, `glue`, **`integral_star_links`**
`PerfectFamily` (`PerfectFamily.lean`)	462	the dimension bound on Z^d: `Perfect`, `cube3_bool`, `proj_bijective`, `edge_ex`, `edge_uniq`, `no_perfect_of_five`, `PerfectZ`, `restrict_perfect`, **`no_perfectZ_of_five`**, `perfect_I2/I3/I4`, three negative controls

Namespace (file)	Lines	Content
`PlaquettePacking`	300	the density of a packing: `AtMostOne`, `Packing`, `not_two_dirs`, `snd_injOn`, `card_le_card_vertices`, **`card_edgeSet_le`**, `density_le_rat`, `card_le_of_support` (Z^d), `bound_tight_d4`, `bound_tight_d5`, **`density_eq_d5`**, `P5_not_perfect`, `Line_not_matching`, three negative controls
`PlaquetteCounting`	529	the double count: `plaqCount`, `sum_plaqCount_add`, `four_mul_card_le`, **`four_mul_card_eq`**, **`perfect_torus_d_le_four`**, `four_dvd_of_perfect`, **`density_eq_quarter`**, the k-cell versions `PackK`, `card_cellsAt_le`, **`card_cellSet_le`**, and negative controls
`PerfectFamily` (`PerfectFamilyConstruction.lean`)	1,002	complete families of k-cells at period 2, and the construction in d = k+2: `Perfect`, `faceSet`, `perfect_iff`, `perfect_one_iff`, `support`, `total_incidences`, **`card_support_of_perfect`**, **`density_of_perfect`**, the gadgets `Blk`, `Blk.g`, `Layout`, `local_sum`, **`Layout.perfect_fam`**, `stdLayout`, **`exists_perfect`**, `card_support_fam`, **`fam_invariant`**, the kernel-checked examples `perfect_fam4`, `card_fam4`, `perfect_fam5`, `card_fam5`, and three negative controls
`PerfectFamily` (`PerfectFamilyBound.lean`)	642	the dimension bound at period 2: `PackCell`, `packCell_of_perfect`, `not_share`, `card_cellsAt_le`, `card_support_le_of_packCell`, `density_le_of_packCell`, **`perfect_d_le`**, `no_perfect_of_lt`, `perfect_one_d_le_four`, **`two_mul_succ_dvd_of_perfect`**, `no_perfect_of_not_dvd`, `no_perfect_k4_d9`, `no_perfect_k5_d8`, `perfect_par5` (the case k = 0, which the bound excludes), and the controls `bound_fam4`, `fam4_numbers`, `dvd_pass_k2_d4`, `dvd_pass_k3_d9`
`PerfectFamily` (`PerfectFamilyPeriodic.lean`)	979	the same for an arbitrary period: **`PerfectK`**, `packK_of_perfectK`, `faceSetK`, `card_faceSetK`, `perfectK_iff`, `no_perfectK_L1`, `supportK`, `total_incidencesK`, **`card_supportK_of_perfectK`**, **`density_of_perfectK`**, **`two_mul_sum_card_slice`**, **`two_dvd_of_perfectK`**, **`no_perfectK_of_odd`**, `two_mul_succ_dvd_of_perfectK`, `dvd_iff_of_coprime`, `no_perfectK_k4_d9_ten`, `no_perfectK_k5_d8_six`, `open_moduli_pass`, **`perfectK_d_le`**, `no_perfectK_of_lt`, `vOfPar`, **`perfectK_of_perfect`**, `perfectK_two_iff`, **`perfectK_k2_d4_iff`**, and controls at period 6
`PerfectFamily` (`PerfectFamilyEdges.lean`)	969	the case k = 1 decided: `edge4` with `edge4_eq_A4`, `perfect_edge4` and `perfect_edge4_decide`, `edge3`, `edge2`, `perfectK_one_d4/d3/d2`, `no_perfectK_one_of_five_le`, **`perfectK_one_iff`**, `perfectK_one_iff_range`, **`bound_sharp_k1`**, `exists_perfectK_one_infinite`, `card_support_edges`, `iff_fails_without_hd`, and the controls `edge3bad`, `not_perfect_edge3bad`, `card_support_edge3bad`

11.2 The main statements, verbatim

From `LatticeGaugeOneLink.lean`, `StaplePerturbation.hess_gauge`:

```
theorem hess_gauge [NeZero L] (h2 : 2 ∣ L) (hL : 3 ≤ L) (U X : Rest h2 →
ℍ[ℝ])
    (hU : ∀ e, ‖U e‖ = 1) (hX : ∀ e, (X e).re = 0) (β : ℝ) :
```

```
    -(27 * β ^ 2) * ∑ e, ‖X e‖ ^ 2 ≤
      deriv (deriv (fun τ => Phi h2 (two_ne_zero_of_le hL) β (pert h2 U X
τ))) 0
```

From `LatticeGaugeOneLink.lean`, `WilsonOneLink.integral_star_links`:

```
theorem integral_star_links (hL2 : (2 : ZMod L) ≠ 0) (β : ℝ) (W : Rest
h2 → ℍ[ℝ])
    (hW : ∀ e, ‖W e‖ = 1) {μ : Measure ℍ[ℝ]} (hμ : IsHaarS3 μ) :
    ∫ u, Real.exp (-SW β (glue h2 u W)) ∂(Measure.pi fun _ : Star h2 =>
μ)
      = Real.exp (-Phi h2 hL2 β W)
```

From `PerfectFamily.lean` and `PlaquetteCounting.lean`:

```
theorem no_perfectZ_of_five {d : ℕ} (hd : 5 ≤ d) (J : Fin d → W d →
Bool) : ¬ PerfectZ J
theorem perfect_I4 : Perfect I4 := by decide
theorem card_edgeSet_le [NeZero L] {I : Fin d → Wp d L → Bool} (h :
AtMostOne I) :
    (edgeSet I).card ≤ L ^ d
```

The measure hypothesis `IsHaarS3` is a three-field structure (probability, unit norm almost surely, left invariance) and is not vacuous: `isHaarS3_haarS3`.

From `PerfectFamilyEdges.lean`, the equivalence of Theorem $2'$(i) and the sharpness at $k = 1$:

```
theorem perfectK_one_iff [NeZero L] (hd : 2 ≤ d) :
    (∃ I : Finset (Fin d) → Wp d L → Bool, PerfectK 1 I) ↔ (d ≤ 4 ∧ 2
| L)
theorem bound_sharp_k1 (hL : 2 | L) :
    (3 * 1 + 1 = 4) ∧ ∃ I : Finset (Fin 4) → Wp 4 L → Bool, PerfectK 1
I
```

From `PerfectFamilyPeriodic.lean`, the counting, the slice identity, the dimension bound and the two cases they decide:

```
theorem card_supportK_of_perfectK [NeZero L] {k : ℕ} (hk : k + 1 ≤ d)
    (I : Finset (Fin d) → Wp d L → Bool) (hI : PerfectK k I) :
    2 * (k + 1) * (supportK k I).card = Nat.choose d k * L ^ d
theorem density_of_perfectK [NeZero L] {k : ℕ} (hk : k + 1 ≤ d)
    (I : Finset (Fin d) → Wp d L → Bool) (hI : PerfectK k I) :
    ((supportK k I).card : ℚ) / ((Nat.choose d k : ℚ) * (L : ℚ) ^ d)
      = 1 / (2 * ((k : ℚ) + 1))
theorem two_mul_sum_card_slice [NeZero L] {k : ℕ} {I : Finset (Fin d) →
Wp d L → Bool}
    (hI : PerfectK k I) {T : Finset (Fin d)} (hT : T.card = k + 1) :
    2 * ∑ ν ∈ T, (∑ x : Wp d L, (I (T.erase ν) x).toNat) = L ^ d
```

```
theorem two_dvd_of_perfectK [NeZero L] {k : ℕ} (hk : k + 1 ≤ d)
    (I : Finset (Fin d) → Wp d L → Bool) (hI : PerfectK k I) : 2 ∣ L
theorem no_perfectK_of_odd [NeZero L] {k : ℕ} (hk : k + 1 ≤ d) (hL : ¬
(2 ∣ L)) :
    ¬ ∃ I : Finset (Fin d) → Wp d L → Bool, PerfectK k I
theorem perfectK_d_le [NeZero L] {k : ℕ} (hk1 : 1 ≤ k) (hk : k + 1 ≤ d)
    (I : Finset (Fin d) → Wp d L → Bool) (hI : PerfectK k I) : d ≤ 3 *
k + 1
theorem perfectK_k2_d4_iff [NeZero L] :
    (∃ I : Finset (Fin 4) → Wp 4 L → Bool, PerfectK 2 I) ↔ 2 ∣ L
theorem no_perfectK_k4_d9_ten [NeZero L] (hL : ¬ (10 ∣ L)) :
    ¬ ∃ I : Finset (Fin 9) → Wp 9 L → Bool, PerfectK 4 I
theorem no_perfectK_k5_d8_six [NeZero L] (hL : ¬ (6 ∣ L)) :
    ¬ ∃ I : Finset (Fin 8) → Wp 8 L → Bool, PerfectK 5 I
```

From `PerfectFamilyConstruction.lean` the construction in d = k+2, at period 2, and from `PerfectFamilyPeriodic.lean` the lift of any period-2 family to an even period (`vOfPar` being the parity map ZMod L → ZMod 2 applied coordinatewise):

```
theorem exists_perfect (k : ℕ) :
    ∃ A : Finset (Fin (k + 2)) × V (k + 2) → Bool, Perfect k A
theorem perfectK_of_perfect (h : 2 ∣ L) {k : ℕ} (A : Finset (Fin d) × V d
→ Bool)
    (hA : Perfect k A) : PerfectK k (fun S x => A (S, vOfPar h x))
```

Here `V d = Fin d → Bool` is the vertex set at period 2 and `Perfect` the corresponding predicate, stated on pairs (S, y) rather than curried; `perfectK_two_iff` says that at L = 2 the two predicates carry the same content. The construction itself is `Layout.perfect_fam`, which takes a splitting of the d directions into blocks of size 2 and 3 and returns a complete family; `stdLayout` produces such a splitting for every d ≥ 2.

11.3 Trust base and reproduction

`#print axioms` was run on every declaration and the outputs are kept as `axioms-LatticeGaugeOneLink.txt` (166 lines, 152 distinct declarations), `axioms-PerfectFamily.txt` (22), `axioms-PlaquetteCounting.txt` (65), `axioms-PerfectFamilyConstruction.txt` (83), `axioms-PerfectFamilyBound.txt` (47), `axioms-PerfectFamilyPeriodic.txt` (98) and `axioms-PerfectFamilyEdges.txt` (99). Every line is `[propext, Classical.choice, Quot.sound]` or a subset of it — some purely computational lemmas depend on `[propext, Quot.sound]`, `cube3_bool` on `[propext]` alone, and a few definitions on no axiom at all. There is no `sorryAx` and no `Lean.ofReduceBool`, that is no `native_decide`, anywhere; `sorry` and `admit` were checked absent by `grep`. Where a decision procedure is used it is `decide` or `decide +kernel`. Verbatim, `'StaplePerturbation.hess_gauge' depends on axioms: [propext, Classical.choice, Quot.sound]`.

Seven files reproduce everything from `import Mathlib` alone: `LatticeGaugeOneLink.lean` (3,979 lines: the chain from the algebraic star inequality through the quaternionic bridge, the series, the geometry of I* and the perturbation of staples to the one-link integral), `PerfectFamily.lean` (498 lines), `PlaquetteCounting.lean` (864 lines), `PerfectFamilyConstruction.lean` (1,050), `PerfectFamilyBound.lean` (701), `PerfectFamilyPeriodic.lean` (1,046) and `PerfectFamilyEdges.lean` (1,037). Checking them with `lake env lean` takes 57.7, 7.4, 9.3, 12.8, 9.6, 10.8 and 8.6 s of wall time respectively, with exit code 0 and no errors and no warnings. Toolchain: Lean 4 v4.33.1 with the matching Mathlib. The archive with a pinned toolchain will be placed at `<URL>`, together with the programs behind the computations of §6 and §8.

Because none of these files may import another, the four files on complete families repeat what they reuse: 247 lines in `PerfectFamilyPeriodic.lean` (nine blocks taken from `PlaquetteCounting.lean` and `PerfectFamilyBound.lean`) and 550 lines in `PerfectFamilyEdges.lean` (twelve blocks), the namespace name being the only alteration. Each block was cut by line range and compared with `diff` against its source after assembly; all comparisons are empty. `PerfectFamilyBound.lean` likewise repeats the counting of `PerfectFamilyConstruction.lean` verbatim and carries the k-cell packing bound of `PlaquetteCounting.lean` transposed to the vertex type `Fin d → Bool`, a change of type that the proof does not use. In `PerfectFamilyPeriodic.lean` the packing bound is the one from `PlaquetteCounting.lean` unaltered, that file already stating it for an arbitrary period.

11.4 What the formalisation changed

1. **The star inequality got shorter.** The earlier proof used a 9×9 secular equation for the per-staple form and a convexity-plus-vertices argument for a scalar inequality $\Sigma\, 1/(p_i + c * (\nu_i{}^2)) \leq 1$. Neither is needed: the weights $(6 - p_i)/(36 - \mathrm{t})$ make $\Sigma\lambda_i = 1$ an identity, and the per-staple step is the sum of squares of §4(ii), which Lean closes by `ring` once the identity is supplied. The secular equation survives only as the check that the AM–GM is not lossy.

2. **Two hypotheses had to be weakened.** Lean required $\|v_i\|^2 \leq \mathrm{t} - p_i{}^2$ as an inequality — R is decreasing in c, so stating the per-staple lemma for $\mathrm{c} \leq 6 - 36\mu/(6{-}\mathrm{p})$ suffices — and $\mu \leq 1$ to be derived from $\|\mathrm{v}\|^2 \leq \mathrm{t} \leq 36$, which also removes the need to treat $\mu = 27$ separately.

3. **The one-link integral needed no Haar density.** Stating the hypothesis as the structure `IsHaarS3` and running the moment recursion from left invariance replaced any construction of coordinates on S^3; the identification with `volume.toSphere` came afterwards and is not used in the chain.

4. **The dimension bound became a `decide`.** The paper proof goes through the Hamming bound; the formal proof replaces the code-theoretic step by the $2^12{-}case$ decision on a 3-dimensional subcube plus the injectivity of one projection, needing no coding theory in Mathlib.

5. **The density bound lost its averaging.** On paper the bound 1/d came from observing that inside a unit cube the family is a matching and averaging over the 2^{d-1} cubes containing a link. The formal proof needs neither: the source map is injective (Theorem 6.2). One gain is visible only there — the statement holds for an arbitrary vertex set and step map, so the torus and Z^d are one theorem with no boundary correction.

6. **The parity of the period is a conclusion, not a hypothesis.** The families of §1 have period 2, and an even period could be carried as a standing assumption. It need not be: summing the completeness identity of a single (k+1)-set of directions over all

vertices gives 2 | L for every complete family of k-cells (Theorem 6.4(ii)–(iii)), so 2 | L occurs in no hypothesis on the negative side and only where a family is constructed. The only place a lower bound on the period is still required is the count of the faces of a cell: at L = 1 the face (T\$\nu$, x + e_ν) collapses onto (T\$\nu$, x), and 2 ≤ L is exactly what keeps the 2(k+1) faces distinct (`card_faceSetK`, with the negative control `card_faceSetK_fails_L1`).

7. **What resisted.** The per-staple identity did not go through `field_simp; ring` in one step (the simp set exceeded its recursion depth); it had to be split into a polynomial part closed by `ring` and a one-variable lemma for the division, assembled by `linear_combination`.

12. Disclosure of AI use

Two of the three authors are AI agents. *Shiori* and *Rin* are instances of Claude (Anthropic) run as the coding agent Claude Code, on a laptop and on the server of the website named on the first page respectively; the models used were publicly released Claude models of the Opus 5 and Fable 5 series. The observation, the paper proofs, the programs behind §6 and §8, the Lean 4 development including the formalised statements, the technical reports on which this note is based, and this draft were produced by Shiori. Rin is responsible for rebuilding the Lean development from the distributed archive on a separate machine, for checking the numerical claims of this note against the reports, and for preparing the distributed version; for this draft Rin prepared the web edition (typesetting, the site page and the placement of the archive), and the independent rebuild is still pending. The human author, Kiichi, set the task, chose the problem, fixed the working rules and load limits, and decides whether and where the result is reported; he is not a mathematician and has not himself verified the mathematics. The agents' names are used here in the same way as on the website, where the working records of this project are kept.

Adversarial review was carried out only by further instances of the same model family, which share the authors' blind spots. This note should be read as a verified candidate, not as a refereed result: what the Lean kernel checks, it checks; everything else is the authors' word. In particular Corollary 3, which is the statement a reader is most likely to want, is the part with the least support.

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